Integrodifferential equation for few- and many-body systems

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A complete derivation of a new two-variable integrodifferential equation valid for three- and many-boson systems is given here for the first time, and it is shown to be exact if all correlations higher than those of two-body type can be neglected. Its equivalence to the Faddeev equation for three bodies and its applicability to many-body systems are discussed in detail. Three-body forces are included. It is shown that the three- and four-body binding energies obtained by means of this equation are in good agreement with those obtained from the most sophisticated variational, Faddeev, and Faddeev-Yakubovsky calculations. This indicates that our new two-variable integrodifferential equation should also be useful for larger systems, in particular since unlike other methods it does not suffer from the disadvantage of rapidly increasing complexity with A. We also show that a simple adiabatic method for the solution of this equation (and hence also for the Faddeev equation) is quite sufficient, due to the closeness of the upper and lower bounds obtained in this way. Finally we apply the adiabatic method to nuclear three-body scattering and even include the effect of breakup for spin-dependent forces. It is found that asymptotic behavior is reached for a value of the hyperradius of the order of 35 fm.

I. INTRODUCTION

In a previous paper[1] we proposed a method for introducing the two-body correlations in the wave function, through a hyperspherical harmonic expansion of the wave function in which the potential harmonic basis, which is complete for the expansion of any function of the relative coordinate between two particles, is used. For this purpose the wave function of the A-body system is decomposed into its Faddeev components and subsequently the equation fulfilled by the components is projected on the potential basis for the pair of particles under consideration. This function is given in terms of a series expansion in potential harmonic (PH) polynomials and the original equations are then transformed into an infinite system of second order coupled differential equations in the hyperradius, which is truncated for a numerical solution.

This method is difficult to use when the solution does not converge rapidly enough in terms of the number of PH basis elements used in the truncated expansion of the amplitude.[2] Unfortunately this is precisely what we have to deal with in the case of strongly repulsive core potentials like the realistic forces occurring in nuclear physics.

To circumvent these difficulties we have transformed the infinite system of differential equations into an equivalent two variable integrodifferential equation where one variable, the hyperradius. This is a collective variable which does not generate any correlations. The other coordinate is given by the ratio of r_{ij}, the relative coordinate between two particles, and the hyperradius. This enables us to describe the two-body correlations.[3-5]

This paper is devoted to a presentation of the complete derivation of this equation for bosons and a discussion of its range of applicability followed by a number of applications to few-body systems. The results are compared to those obtained by means of the most sophisticated methods in other recent literature.

In Sec. II we present the hyperspherical background formalism required for the derivation of the above mentioned integrodifferential equation in such a way that the various correlations can be introduced in stages. The states of the A-body system in the hypercentral part of the total interaction are introduced in Sec. III and a comparison is made with the harmonic oscillator independent particle model.

In Sec. IV we introduce the two-body correlations into the Schrödinger equation by means of a Faddeev decomposition of the wave function, restricting ourselves to the case of identical bosons in S states. We thus obtain a partial integrodifferential equation in two variables r_{ij} and r for the amplitude $F(r_{ij},r)$. For the case of three bosons interacting in S states only, a particularly simple derivation of the kernel of this integrodifferential equation is given, which also illustrates the more difficult case of an arbitrary number of particles. This equation can, by means of a simple transformation, be shown to be identical to the Faddeev equation for S-state projected potentials, and is therefore exact in the case we consider here.

For $A > 3$ we then show that if only two-body correlations are retained we again arrive at a two-variable integrodifferential equation of the Faddeev type, but with a more complicated kernel. Using more sophisticated techniques this kernel can also be obtained in analytical form, for bosons interacting in S states, only. The complete derivation of this kernel is given in Appendix A.

It is shown in Sec. V how easy it is to include three-body forces in our formalism. In Sec. VI we explain in detail why our three-boson equations are exact (and identical with the Faddeev equation) for S states, and why this is no longer true for $A \geq 4$. It clearly emerges from this discussion that our two-variable integrodifferential equation for S states for $A > 3$ takes all two-body correla-

tions but nothing else, exactly into account.

In Sec. VII we explain in some detail the application of an adiabatic approximation, in particular the extreme adiabatic approximation (EAA) and the uncoupled adiabatic approximations (UAA), which, respectively, provide lower and upper bounds to the exact binding energies, obtained from the integrodifferential equation. These bounds generally are close and determine the binding energy within narrow limits. Combined with an accurate interpolation formula, they allow us to determine the binding energy in a much simpler way than by a full solution of the partial integrodifferential equation.

The asymptotic behavior of the two-body amplitudes which, in the approximate separation of the variables characteristic of the adiabatic approximation, are obtained as eigenfunctions for each value of r, is discussed in Sec. VIII. Subsequently, in Sec. IX, we consider the problem of three-body scattering in this approximation. In the simplest case, where only one two-body bound state exists and the particles interact by means of pure Wigner forces, we describe the scattering process with the aid of the coupled adiabatic approximation (CAA). In addition, we show how to include the break-up process in our treatment.

Finally in Sec. X we apply our formalism to calculate three- and four-body binding energies for a number of potentials and compare our results with those obtained by means of the most sophisticated methods in the literature. The last point we investigate here is the asymptotic behavior of the eigenpotential corresponding to a three-body bound state.

II. HYPERSPHERICAL BACKGROUND FORMALISM

We eliminate the center-of-mass motion by a suitable chain of Jacobi coordinates given by

$$\begin{cases} \boldsymbol{\xi}_N=\mathbf{x}_1-\mathbf{x}_2=\mathbf{r}_{12}\ , \\ \boldsymbol{\xi}_{N-1}=\sqrt{3}(\mathbf{x}_3-\mathbf{X}_3)\ , \\ \vdots \\ \boldsymbol{\xi}_{N-i+1}=\sqrt{2(i+1)/i}\,(\mathbf{x}_{i+1}-\mathbf{X}_{i+1})=\sqrt{2i/(i+1)}(\mathbf{x}_{i+1}-\mathbf{X}_i)\ , \\ \vdots \\ \boldsymbol{\xi}_1=\sqrt{2A/A-1}(\mathbf{x}_A-\mathbf{X}),\quad \mathbf{X}_i=\dfrac{1}{i}\sum_{j=1}^{i}\mathbf{x}_j,\quad \mathbf{X}_A=\mathbf{X},\quad A=N+1\ . \end{cases} \tag{2.1}$$

for a system of A identical particles. The choice of the pair (1,2) for $\boldsymbol{\xi}_N$ is arbitrary. We could also use the pair (ij) for $\boldsymbol{\xi}_N$. Then $\sqrt{3}(\mathbf{x}_k-\mathbf{X}_k)$ becomes $\boldsymbol{\xi}_{N-1}$, where $\mathbf{X}_k$ is the center of mass of the particles $\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k$ and so on, in a similar way as described in Eq. (2.1). From now on, to be more general, we use the partition $\boldsymbol{\xi}_N,\boldsymbol{\xi}_{N-1},\ldots,\boldsymbol{\xi}_1$ where $\boldsymbol{\xi}_N=\mathbf{r}_{ij}$, $\boldsymbol{\xi}_{N-1}=\sqrt{3}(\mathbf{x}_k-\mathbf{X}_k)$ and so on. Assuming that only two-body potentials occur we write the Faddeev equation

$$(T-E)\phi_{ij}(\mathbf{x})=-V(\mathbf{r}_{ij})\Psi(\mathbf{x}) \tag{2.2}$$

for A particles, where the index ij in this case means that we use the partition where $\boldsymbol{\xi}_N=\mathbf{r}_{ij}$ and where

$$\Psi=\sum_{i<j\le A}\phi_{ij}(\mathbf{x}) \tag{2.3}$$

is a function of all coordinates $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_A)$ of the particles. Obviously by adding Eq. (2.2) for all pairs (ij) we obtain the Schrödinger equation for Ψ.

To solve Eq. (2.2) we will proceed in steps. First we obtain the solution without correlations, then we introduce the two-body correlations generated by the potential. In the next step we would have to consider the correlations where the six degrees of freedom of $\boldsymbol{\xi}_N$ and $\boldsymbol{\xi}_{N-1}$ are involved and so on, until all $\boldsymbol{\xi}_i$ have been taken into account. For obvious reasons we have to stop after inclusion of only a few of the lowest few-body correlations because the problem rapidly becomes too complicated to solve. Nevertheless, the problem has to be treated in this way to successively introduce all the many-body correlations in the wave function. For low energies and for low density systems only the two-body correlations are important, but when the density and/or the energy increase the probability to have more than two particles interacting at the same time increases, and the many-body correlations cannot be neglected.

In order to distinguish between the various correlations, we use the property that any continuous function can be expanded in a series of harmonic polynomials according to

$$\Psi(\mathbf{x})=\sum_{[L]=0}^{\infty}H_{[L]}(\mathbf{x})\Phi_{[L]}(r)\ , \tag{2.4}$$

where $H_{[L]}(\mathbf{x})$ is a harmonic polynomial (HP) in all variables $\mathbf{x}$ characterized by $3N-1$ "quantum" numbers $[L]$, where L stands for the degree of the polynomial and $\Phi_{[L]}(r)$ is a function of the hyperradius

$$r=\left[\sum_{i=1}^{N}\boldsymbol{\xi}_i^2\right]^{1/2}=\left[2\sum_{i=1}^{A}(\mathbf{x}_i-\mathbf{X})^2\right]^{1/2}=\left[\frac{2}{A}\sum_{i<j\le A}r_{ij}^2\right]^{1/2},$$

and where the sum is taken over all quantum numbers

$[L]$, for L running from zero to infinity.

The HP's form a complete orthogonal set over the surface of the unit hypersphere and fulfill the orthonormality condition:

$$\int H^*_{[L]}(\Omega)H_{[L']}(\Omega)d\Omega=\delta_{[L],[L']} , \tag{2.5}$$

where the hyperspherical harmonic (HH) function $H_{[L]}(\Omega)$ is the value of $\mathbf{H}_{[L]}(\mathbf{x})$ over the hypersphere $r=1$. If the angular coordinates over the hypersphere are defined by the spherical coordinates ω_i of each vector ξ_i and by the $\{\phi\}$ coordinates defined by[6]

$$\begin{aligned}&\xi_N=r\cos\phi_N ,\\ &\xi_{N-1}=r\sin\phi_N\cos\phi_{N-1} ,\\ &\xi_i=r\sin\phi_N\ldots\sin\phi_{i+1}\cos\phi_i,\quad(\phi_1=0) ,\end{aligned} \tag{2.6}$$

the surface element $d\Omega$ becomes a product

$$\begin{aligned}d\Omega&=d\omega_1\prod_{j=2}^{N}d\omega_j(\sin\phi_j)^{3j-4}\cos^2\phi_j d\phi_j\\ &=d\omega_1\prod_{j=2}^{N}2^{-3j/2}(1-z_j)^{(3j-5)/2}(1+z_j)^{1/2}dz_j d\omega_j\end{aligned} \tag{2.7}$$

where $z_j=\cos 2\phi_j$. Then by applying the orthonormality condition, Eq. (2.5), and using the notation $Y_{[L]}(\Omega)$ instead of $H_{[L]}(\Omega)$ to specify that we are dealing with the Zernike and Brinkman hyperspherical coordinate system (ω_i,ϕ_i) [see Eq. (2.6)], we find[7] that

$$Y_{[L]}(\Omega)=Y^{m_1}_{l_1}(\omega_1)\prod_{j=2}^{N}Y^{m_j}_{l_j}(\omega_j)\,{}^{(j)}P^{l_j,L_{j-1}}_{L_j} , \tag{2.8}$$

where

$$\begin{aligned}{}^{(j)}P^{l_j,L_{j-1}}_{L_j}(\phi)&=N_{L_j,l_j}(1-z_j)^{L_{j-1}/2}\\ &\times(1+z_j)^{l_j/2}P^{\nu_{j-1},l_j+1/2}_{n_j}(z_i) ,\end{aligned} \tag{2.9}$$

in terms of the Jacobi polynomial $P^{\nu_{j-1},l_j+1/2}_{n_j}$ and

$$N^2_{L_j,l_j}=2^{-(L_{j-1}+l_j)}\frac{2\nu_j\Gamma(\nu_j-n_j)n_j!}{\Gamma(\nu_j-n_j-l_j-\frac{1}{2})\Gamma(n_j+l_j+\frac{3}{2})} ,$$

while

$$\nu_j=L_j+\frac{3j}{2}-1,\quad L_j=l_1+\sum_{k=2}^{j}(2n_k+l_k) ,$$

with

$$L=L_N=\sum_{i=1}^{N}(2n_i+l_i) ,$$

where $n_1=0$ and $n_j=(L_j-L_{j-1}-l_j)/2$.

As a consequence of the product structure of $d\Omega$, the Laplacian (i.e., the kinetic energy operator) expressed in terms of ω_i and z_i is a sum given by

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{3A-4}{r}\frac{\partial}{\partial r}+L^2(\Omega)/r^2 , \tag{2.10}$$

where $L^2(\Omega)$ is the grand orbital operator. Let $L^2_i(\Omega_i)$ be the operator associated with the first vector $\{\xi_1,\ldots,\xi_i\}$. It can be written in terms of the one for the vectors $\{\xi_1,\ldots,\xi_{i-1}\}$ as follows[6]

$$\begin{aligned}L^2_i(\Omega_i)=&\frac{4}{w_i(z_i)}\frac{\partial}{\partial z_i}(1-z_i^2)w_i(z_i)\frac{\partial}{\partial z_i}\\ &+2\frac{l^2(\omega_i)}{1+z_i}+\frac{2}{1-z_i}L^2_{i-1}(\Omega_{i-1}) ,\end{aligned} \tag{2.11}$$

where

$$L^2(\Omega)\equiv L^2_N(\Omega) ,$$

and

$$w_i(z_i)=(1-z_i)^{(3i-5)/2}(1+z_i)^{1/2} , \tag{2.12}$$

where $l^2(\omega_i)$ is the orbital operator with the phase convention

$$[l^2(\omega_i)+l(l+1)]Y^m_l(\omega_i)=0 .$$

It is useful to keep in mind that it is a consequence of Eq. (2.7) that

$$d\Omega=d\omega_1\prod_{j=2}^{N}2^{-3j/2}w_j(z_j)dz_j d\omega_j , \tag{2.13}$$

in such a way that when ξ_i is in an S state (i.e., $l_i=0$), $w_i(z_i)$ is the weight function associated with the Jacobi polynomials $P^{(3i-5)/2,1/2}_{n_i}(z_i)$ occurring in the HH given by Eq. (2.8) when $L_{i-1}=0$.

Let $\Psi(\mathbf{x})$ be a function that is expanded as

$$\Psi(\mathbf{x})=\sum_{[L]}Y_{[L]}(\Omega)\Phi_{[L]}(r) , \tag{2.14}$$

and let $d\Omega_i$ be the part of the surface element $d\Omega$ which contains the coordinates ω_j and z_j for $j\le i$, i.e.,

$$d\Omega_i=d\omega_1\prod_{j=2}^{i}2^{-3j/2}w_j(z_j)dz_j d\omega_j . \tag{2.15}$$

Then if we integrate $\Psi(\mathbf{x})$ over the surface of the unit hypersphere spanned by the vectors $\{\xi_1,\ldots,\xi_i\}$, we eliminate from $\Psi(\mathbf{x})$ the dependence on $\{\xi_1,\ldots,\xi_i\}$. By means of the orthonormality condition

$$\int Y_{[L_i]}(\Omega_i)d\Omega_i=0\quad\text{for } L_i\neq 0 . \tag{2.16}$$

The remainder is then a function of $\{\xi_{i+1},\ldots,\xi_N\}$ only. This property will be used to project a function of the set of all coordinates $\{\mathbf{x}\}$ on a definite subset $\{\xi_{i+1},\ldots,\xi_N\}$ of the Jacobi coordinates associated with the partition, e.g., (2.1) used to define the $Y_{[L]}(\Omega)$ basis.

III. THE STATES OF THE N-BODY SYSTEM

Let us extract that part of the potential $V(\mathbf{r}_{ij})$ in Eq. (1.2), which is independent of the distance r_{ij} between the

particles. It is called the hypercentral part of the potential and it is given by the average of the potential over the whole unit hypersphere $r=1$:

$$V_0(r)=Y_{[0]}\int Y_{[0]}V(r_{ij})d\Omega\ , \tag{3.1}$$

where $Y_{[0]}$ is normalized over the hypersphere as $\int (Y_{[0]})^2 d\Omega=1$. By using Eq. (2.7) and $r_{ij}=r\cos\phi_N=r\sqrt{(1+z)/2}$, where $z=\cos 2\phi$, one finds,

$$V_0(r)=\frac{\Gamma(D/2)}{\sqrt{\pi}2^{(D/2)-2}\Gamma[(D-3)/2]}\int_{-1}^{1}(1-z)^{(D-5)/2}(1+z)^{1/2}V\left[r\left[\frac{1+z}{2}\right]^{1/2}\right]dz\ , \tag{3.2}$$

where $D=3(A-1)$. Let us assume that $V_0(r)$ contains the major part of the interaction, which is in general true.

Inserting the approximation $V(r_{ij})=V_0(r)$ into Eq. (2.2) and summing over all pairs (i,j) we obtain the approximate equation, equation

$$\left[T+\frac{A(A-1)}{2}V_0(r)-E\right]\Psi_0(\mathbf{x})=0\ . \tag{3.3}$$

Its exact solution is the product

$$\Psi_0(\mathbf{x})=H_{[L]}(\mathbf{x})u(r)/r^{\mathcal{L}+1}\ , \tag{3.4}$$

where $\mathcal{L}=L+(D-3)/2$, of a properly symmetrized HP and a function of r which is the solution of the radial equation

$$\left\{\frac{\hbar^2}{m}\left[-\frac{d^2}{dr^2}+\frac{\mathcal{L}(\mathcal{L}+1)}{r^2}\right]+\frac{A(A-1)}{2}V_0(r)-E\right\}u(r)=0\ . \tag{3.5}$$

The state is defined by the quantum numbers $[L]$ which characterize the harmonic polynomial $H_{[L]}(\mathbf{x})$.

In this approximation, which practically does not contain any correlations, because it is identical to the generator coordinate method applied to a harmonic oscillator Slater determinant,[27] the ground state is obtained when the repulsive centrifugal barrier is minimal, i.e., when the degree L of the HP is minimal. In the particular case of a harmonic oscillator potential

$$V(r_{ij})=\frac{2\hbar^2}{Am}\frac{r_{ij}^2}{b^4} \tag{3.6}$$

the total potential $\sum_{i>j}V(r_{ij})$ is hypercentral because $r^2=2/A\sum_{i<j\le A}r_{ij}^2$ and generates, as is a well known, a solution without correlations which can be written as in Eq. (3.4), where

$$u_{\mathcal{L},n}(r)=\left[\frac{2n!}{\Gamma(n+\mathcal{L}+\frac{3}{2})}\right]^{1/2}\left[\frac{r}{b}\right]^{\mathcal{L}+1}L_n^{\mathcal{L}+1/2}\left[\frac{r^2}{b^2}\right]\times e^{-r^2/2b^2}\ , \tag{3.7}$$

and n is the number of nodes of the Laguerre polynomial $L_n^{\mathcal{L}+1/2}(r^2/b^2)$.

Let n_i and l_i be the harmonic oscillator (HO) quantum numbers associated with the particle (i), where the system is described by an independent particle HO model. It is well known that the ground state is obtained when

$$L=\sum_{i=1}^{A}(2n_i+l_i) \tag{3.8}$$

is a minimum. The quantum number L is also the degree of the HP in Eq. (3.4), therefore, the quantum numbers which describe the ground state in an HO independent particle model are those occurring in the HP description of this state. The only difference with an HO model is in the radial function $u(r)$, which is determined by the central potential $V_0(r)$, and in the translational invariance of the solution of Eq. (3.4). For an HO interaction we have for the ground state

$$u(r)/r^{\mathcal{L}+1}=C\exp(-r^2/2b^2)\ , \tag{3.9}$$

where b is the HO parameter defined above and C a normalization constant.

When excited states are considered the HO independent particle model generates spurious center-of-mass excited states together with hyper-radial excited states (breathing modes), which are admixed to the solution. In contrast to this the center-of-mass motion is eliminated and the admixture with the breathing mode is avoided in the hyperspherical harmonic (HH) description of the excited states, which provides an unambiguous definition of the "state" of the N-body system.

IV. INCLUSION OF THE TWO-BODY CORRELATIONS AND THE INTEGRODIFFERENTIAL EQUATION

In order to introduce the correlations we have to take into account the degrees of freedom that we have neglected so far. For this purpose we rewrite Eq. (2.2) by introducing on both sides the hypercentral part $V_0(r)$:

$$\left[T+\frac{A(A-1)}{2}V_0(r)-E\right]\phi_{ij}(\mathbf{x})=-[V(r_{ij})-V_0(r)]\Psi(\mathbf{x})\ , \tag{4.1}$$

where $V(r_{ij})-V_0(r)$ is the residual interaction and where, for the sake of simplicity, we assumed a central two-body potential $V(r_{ij})$. The wave components $\phi_{ij}(\mathbf{x})$ now represent modified Faddeev components for the A-body problem. By substituting the first order approximation of Eq. (3.4) for $\Psi(\mathbf{x})$ in Eq. (4.1) we find that $\phi_{ij}(\mathbf{x})$

can be written as the product

$$\phi_{ij}(\mathbf{x})=H_{[L]}(\mathbf{x})F(r_{ij},r) , \tag{4.2}$$

where the function $F(r_{ij},r)$ is a solution obtained by substitution of Eq. (4.2) in Eq. (4.1)

$$\left[T+\frac{A(A-1)}{2}V_0(r)-E\right]H_{[L]}(\mathbf{x})F(r_{ij},r) = -[V(r_{ij})-V_0(r)]H_{[L]}(\mathbf{x})\sum_{k<l\le A}F(r_{kl},r) . \tag{4.3}$$

The general case where $L\neq 0$ has been considered elsewhere.[8] Here we will only discuss the case of the bosons in S states, for which $L=0$ and the HH of degree zero, a constant, normalized according to Eq. (2.5) is

$$H_{[0]}(\mathbf{x})=Y_{[0]}(D)=\left[\frac{\Gamma(D/2)}{2\pi^{D/2}}\right]^{1/2} = \left[\frac{1}{\pi^{3/2}}\frac{\Gamma(D/2)}{\Gamma[(D-3)/2]}\right]^{1/2}Y_{[0]}(D-3) .$$

In this paper the full details of the derivation are given for the first time, however.

In order to obtain an equation for $F(r_{ij},r)$ only, we have to eliminate the other degrees of freedom. For this purpose we write the surface element over the unit hypersphere, using Eq. (2.7), as

$$d\Omega=2^{-D/2}(1-z)^{(D-5)/2}(1+z)^{1/2}d\omega_{ij}dz\,d\Omega_{N-1} , \tag{4.4}$$

where $z=\cos 2\phi_N$, $\cos\phi_N=r_{ij}/r$, and where $d\Omega_{N-1}$ is the surface element for the $3(N-1)$-dimensional space spanned by the vectors $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{N-1}$ with our standard choice $\boldsymbol{\xi}_N=\mathbf{r}_{ij}$. Then we use the property that $d\Omega_{N-1}$ is the weight function from which the HH are constructed. Let $[L_{N-1}]$ be the set of quantum numbers associated with the HH in the space of $(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{N-1})$. Then the integral

$$\int Y_{[L_{N-1}]}(\Omega_{N-1})d\Omega_{N-1}=0 , \tag{4.5}$$

for $L_{N-1}\neq 0$. The equation which determines $F(r_{ij},r)$ is, after multiplication of Eq. (4.3) by $H_{[0]}$ and integration over $d\Omega_{N-1}d\omega_{ij}$ given by

$$\left[T+\frac{A(A-1)}{2}V_0(r)-E\right]F(r_{ij},r)=-[V(r_{ij})-V_0(r)]\int\left[\sum_{k<l\le A}F(r_{kl},r)\right][Y_{[0]}(D-3)]^2d\Omega_{N-1}\frac{d\omega_{ij}}{4\pi} , \tag{4.6}$$

where

$$\int[Y_{[0]}(D-3)]^2d\Omega_{N-1}=1 . \tag{4.7}$$

Equation (4.6) is a partial integrodifferential equation in two variables, which can be routinely solved for three bodies. For more than three particles another approach is needed to integrate over the other variables under the integral sign on the right-hand side of Eq. (4.6) and thus to eliminate them.

Instead of r_{ij} and r let us use the variables $z=\cos 2\phi_N=2r_{ij}^2/r^2-1$ and r and let us set

$$F(r_{ij},r)=P(z,r)/r^{(D-1)/2} . \tag{4.8}$$

The kinetic energy operator applied to $F(r_{ij},r)$ becomes according to Eq. (2.11):

$$TF(r_{ij},r)=r^{-(D-1)/2}\frac{\hbar^2}{m} \times\left[-\frac{d^2}{dr^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2} -\frac{4}{r^2}\frac{1}{w_\alpha(z)\partial z}(1-z^2)w_\alpha(z)\frac{\partial}{\partial z}\right]P(z,r) , \tag{4.9}$$

where

$$w_\alpha(z)=(1-z)^\alpha(1+z)^{1/2},\quad \alpha=(D-5)/2 ,$$

and

$$\mathcal{L}_0=(D-3)/2=3A/2-3 . \tag{4.10}$$

In the simplest case, i.e., three particles in S states, only three variables are involved: (1) the hyperradius r; (2) the angle θ between the two vectors $\boldsymbol{\xi}_1=\sqrt{3}(\mathbf{x}_k-\mathbf{X})$ and $\boldsymbol{\xi}_2=\mathbf{r}_{ij}$; (3) the ratio $\xi_1/\xi_2=\tan\phi$ with $z=\cos 2\phi=2r_{ij}^2/r^2-1$. The wave function is then given by

$$\Psi=\sum_{m<n\le 3}F(r_{mn},r) , \tag{4.11}$$

and can be expanded in terms of the Legendre polynomials

$$\Psi=r^{-5/2}\sum_l P_{(l)}(z,r)P_l(\cos\theta) . \tag{4.12}$$

Starting from Eqs. (4.6) and (4.8), for $A=3$ and $D=6$, we have $Y_{[0]}(D-3)=Y_{[0]}(3)=1/\sqrt{4\pi}$, $d\Omega_{N-1}=d\omega_k$, where ω_k represents the spherical coordinates of the spectator particle, and $d\omega_{ij}=d\varphi_{ij}\sin\theta d\theta$ when the z axis of the coordinate systems for $\mathbf{r}_{ij}$ is taken along $\boldsymbol{\xi}_1$. Then

$$F(r_{kl},r)=r^{-5/2}P[2r_{kl}^2/r^2-1,r] , \tag{4.13}$$

which leads to

$$\Psi=r^{-5/2}\sum_{k<l\le 3}P[2r_{kl}^2/r^2-1,r] . \tag{4.14}$$

For the calculation of the integral in Eq. (4.6) we introduce the kinematic rotation vector $\mathbf{z}(\varphi)$ which expresses the relative coordinates $\mathbf{r}_{mn}$ in terms of the Jacobi coordinates defined in the partition where $\xi_N=\mathbf{r}_{ij}$ and $\xi_{N-1}=\sqrt{3}(\mathbf{x}_k-\mathbf{X})$ (Ref. 6) as follows:

$$\mathbf{z}(\varphi)=\xi_N\cos\varphi_N+\xi_{N-1}\sin\varphi_N\cos\varphi_{N-1}+\cdots$$
$$+\xi_2\sin\varphi_N\cdots\sin\varphi_3\cos\varphi_2+\xi_1\sin\varphi_N\cdots\sin\varphi_3\sin\varphi_2 . \tag{4.15}$$

The set of angular parameters $\{\varphi\}$ determines the linear combination $\mathbf{z}(\varphi)$ and one notices that for $\varphi_N=0$, $\mathbf{z}(0)=\xi_N=\mathbf{r}_{ij}$, while for $\varphi_N=\pm 2\pi/3$, and $\varphi_{N-1}=0$, $\mathbf{z}(\varphi)$ corresponds to $\mathbf{r}_{ik}$ or $\mathbf{r}_{jk}$, where one of the particles is either (i) or (j) (connected pairs). On the other hand, for $\varphi_N=\pi/2$ one can obtain only pairs $\mathbf{r}_{mn}$, where both m and $n\neq i$ and j (disconnected pairs).

For $A=3$ we have only two Jacobi coordinates and

$$2r_{kl}^2/r^2-1=2z^2(\varphi_2)/r^2-1$$

where φ_2 is the appropriate angle, i.e., $\phi_N=0$ or $\pm 2\pi/3$ according to whether we have to deal with the pair (ij), (jk) or (ki). The integrand in Eq. (4.6) is given by

$$\sum_{k<l\le 3}F(r_{kl},r)\left[\frac{1}{\sqrt{4\pi}}\right]^2\frac{d\omega_k}{4\pi}d\varphi_{ij}\sin\theta\,d\theta ,$$
$$\omega_{ij}\equiv(\theta,\varphi_{ij}) , \tag{4.16}$$

while in Eq. (4.13) we have

$$P(2r_{kl}^2/r^2-1,r)=P(2[\mathbf{z}(\varphi)]^2/r^2-1,r)=P[z\cos2\varphi+(1-z^2)^{1/2}\sin2\varphi\cos\theta,r],\ \phi=\phi_2 . \tag{4.17}$$

The integral in Eq. (4.6) then becomes $(z=2r_{ij}^2/r^2-1)$

$$r^{-5/2}\left\{P(z,r)+\int_0^{\pi}P[z\cos2\varphi+(1-z^2)^{1/2}\sin2\varphi\cos\theta,r]\sin\theta\,d\theta\right\}=r^{-5/2}\left[P(z,r)+\frac{1}{\sin2\varphi(1-z^2)^{1/2}}\int_{z_-}^{z_+}P(x,r)dx\right] , \tag{4.18}$$

where $z_{\pm}=z\cos2\varphi\pm(1-z^2)^{1/2}\sin2\varphi$. Using this result and Eq. (4.9) for $D=6$ we obtain the integrodifferential equation for $P(z,r)$

$$\left\{\frac{\hbar^2}{m}\left[-\frac{\partial^2}{\partial r^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}-\frac{4}{r^2(1-z^2)^{1/2}}\frac{\partial}{\partial z}(1-z^2)^{3/2}\frac{\partial}{\partial z}\right]+3V_0(r)-E\right\}P(z,r)$$
$$=-\{V[r\sqrt{(1+z)/2}]-V_0(r)\}\left[P(z,r)+\frac{1}{\sin2\varphi(1-z^2)^{1/2}}\int_{z_-}^{z_+}P(x,r)dx\right] , \tag{4.19}$$

where $\sin2\varphi=\sqrt{3}/2$ and $\cos2\varphi=-\frac{1}{2}$ for equal mass particles.

The change of variable from z to Θ with $z=\cos\Theta$ and $P(z,r)=\sqrt{r}\,U(r,\Theta)$ transforms Eq. (4.19) into the well-known Faddeev equation in terms of $U(r,\Theta)$ when for S-state projected potentials $V_0(r)$ is cancelled on both sides of this equation.[9] This proves that our approximate derivation of Eq. (4.6) for arbitrary A, leads to an exact result for three bodies.

For more than three particles and at the level, where only two-body correlations are included, we also obtain a two-variable partial integrodifferential equation, but we have to employ another method to obtain the kernel of the integral part. Even though in this case the equation is not exact any more for S-state projected local potentials, it still takes all two-body correlations into account exactly.

In order to project $F(r_{kl},r)$ in Eq. (4.3) on the $\mathbf{r}_{ij}$ space for $L=0$ we expand this function by using the potential harmonic (PH) basis $\mathcal{P}_{2K+l}^{l,m}(\Omega_{ij})$ which enables one to make a complete expansion of any function of $\mathbf{r}_{ij}$. The general procedure for constructing this basis for any state is explained in Ref. 1. Here we restrict our comments to the simplest case of bosons in the ground state.

By using the partition, where $\xi_N=\mathbf{r}_{ij}$, it is obvious that the HH basis (2.8) needed to expand any function of $\mathbf{r}_{ij}$ only should not depend on the other Jacobi coordinates involved in the partition. It is then given according to Eqs. (2.8) and (2.9) by the HH for which $L_{N-1}=0$, i.e., by

$$\mathcal{P}_{2K+l}^{l,m}(\Omega_{ij})=Y_l^m(\omega_{ij})^{(N)}P_{2K+l}^{l,0}(\phi_N)Y_{[0]}(D-3) ,$$

where $r_{ij}=r\cos\phi_N$ and $z_N=\cos2\phi_N=2r_{ij}^2/r^2-1$. For S states we used the simplified notation

$$\mathcal{P}_{2K}^0(\Omega_{ij})\equiv\mathcal{P}_{2K}^{0,0}(\Omega_{ij}) .$$

Then we write

$$F(r_{kl},r)=\sum_K\mathcal{P}_{2K}^0(\Omega_{kl})\Phi_K(r) , \tag{4.20}$$

and project each PH on the $\mathbf{r}_{ij}$ space leading to

$$\langle\mathbf{r}_{ij}\,|\,F(r_{kl},r)\rangle$$
$$=\sum_K\langle\mathcal{P}_{2K}^0(\Omega_{ij})\,|\,\mathcal{P}_{2K}^0(\Omega_{kl})\rangle\mathcal{P}_{2K}^0(\Omega_{ij})\Phi_K(r) , \tag{4.21}$$

where according to Ref. 6

$$\Phi_K(r)=[Y_0^0 Y_{[0]}(D-3)]^{-1}\int_0^{\pi/2} {}^{(N)}P_{2K}^{00}(\phi)F(r\cos\phi,r)(\sin\phi)^{D-4}\cos^2\phi\, d\phi\ , \tag{4.22}$$

Interchanging the sum and the integral in (4.21) we obtain

$$\langle \mathbf{r}_{ij} \mid F(r_{kl},r)\rangle=[Y_0^0 Y_{[0]}(D-3)]^{-1}\int_0^{\pi/2}\left[\sum_K \langle \mathcal{P}_{2K}^0(\Omega_{ij}) \mid \mathcal{P}_{2K}^0(\Omega_{kl})\rangle \mathcal{P}_{2K}^0(\Omega_{ij})\,{}^{(N)}P_{2K}^{00}(\phi)\right] \times F(r\cos\phi,r)(\sin\phi)^{D-4}\cos^2\phi\, d\phi\ , \tag{4.23}$$

where ${}^{(N)}P_{2K}^{00}(\phi)$ can be obtained from Eq. (2.9)

$${}^{(N)}P_{2K}^{00}(\phi)=\left[\frac{(4K+D-2)\Gamma(K+D/2-1)K!}{\Gamma[K+(D-3)/2]\Gamma(K+\frac{3}{2})}\right]^{1/2} P_K^{\alpha,1/2}(\cos 2\phi)\ . \tag{4.24}$$

The $P_K^{\alpha,1/2}(z)$ are the Jacobi polynomials associated with the weight function $w_\alpha(z)$.

For the sake of simplicity we use the variable $z=\cos 2\phi=2r_{ij}^2/r^2-1$ for ϕ and write

$$F(r_{ij},r)=r^{-(D-1)/2}P(z,r)\ .$$

By using the matrix elements which project the potential harmonics of the pair (k,l) on the one of the pair (i,j)[6]

$$\langle \mathcal{P}_{2K}^0(\Omega_{ij}) \mid \mathcal{P}_{2K}^0(\Omega_{kl})\rangle = P_K^{\alpha,1/2}(\cos 2\varphi_N)/P_K^{\alpha,1/2}(1) \tag{4.25}$$

where $\varphi_N=2\pi/3$ for connected pairs like (i,j) and (j,k) and $\varphi_N=\pi/2$ for disconnected pairs, we find after some algebra, that the projection of $P(2r_{kl}^2/r^2-1,r)$ on the pair (i,j) is given by

$$\int_{-1}^{1} f_{[0]}(z,z',\cos 2\varphi_N)P(z',r)dz'\ , \tag{4.26}$$

where

$$f_{[0]}(z,z',\cos 2\varphi_N)=w_\alpha(z')\sum_{K=0}^{\infty} P_K^{\alpha,1/2}(z)P_K^{\alpha,1/2}(z')P_K^{\alpha,1/2}(\cos 2\varphi_N)/[h_K^{\alpha,1/2}P_K^{\alpha,1/2}(1)]\ , \tag{4.27}$$

and φ_N has a suitable value as discussed above.

The kernel $f_{[0]}(z,z',\cos 2\varphi_N)$ is the projection function *for one pair* in S states, and $h_K^{\alpha,1/2}$ is the normalization constant of the Jacobi polynomials. For other orbital states of the interacting pair the problem has been treated in Ref. 5.

If we now use Eqs. (4.9), (4.10), and (4.27), it can be seen that Eq. (4.6) becomes

$$\left\{\frac{\hbar^2}{m}\left[-\frac{\partial^2}{\partial r^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}-\frac{4}{r^2}\frac{1}{w_\alpha}\frac{\partial}{\partial z}(1-z^2)w_\alpha\frac{\partial}{\partial z}\right]+\frac{A(A-1)}{2}V_0(r)-E\right\}P(z,r) = -\{V[r\sqrt{(1+z)/2}]-V_0(r)\}\left[P(z,r)+\int_{-1}^{1} f_{[0]}(z,z')P(z',r)dz'\right] \tag{4.28}$$

where the kernel

$$f_{[0]}(z,z')=2(A-2)f_{[0]}(z,z',-\tfrac{1}{2})+\frac{(A-2)(A-3)}{2}f_{[0]}(z,z',-1) \tag{4.29}$$

is called the projection function. It sums up the contribution of all pairs but (i,j). The first term of Eq. (4.29) refers to the $2(A-2)$ connected pairs for which $\varphi_N=2\pi/3$ and the last one to the $(A-2)(A-3)/2$ disconnected pairs for which $\varphi_N=\pi/2$.

The projection function *for one pair* is given in Eq. (4.27) by a series which can be summed up analytically, the details of which are given in Appendix A.

The final result is

$$\int_{-1}^{1} f_{[0]}(z,z')P(z',r)dz'=2(A-2)\frac{1}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\frac{(\sin\phi\sin\delta)^{1-2\lambda}}{\cos\phi\cos\delta}\int_{u_-}^{u_+}[(u-u_-)(u_+-u)]^{\lambda-1}P(2u^2-1,r)u du + \frac{(A-2)(A-3)}{2}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-1)}(1-z)^{1/2-\lambda}\int_{-1}^{-z}[-(z+z')]^{\lambda-2}\sqrt{1+z'}P(z',r)dz' \tag{4.30}$$

where $\delta=2\pi/3$ (or $\pi/3$), $\lambda=D/2-2$, $\cos\phi=\sqrt{(1+z)/2}$, $\sin\phi=\sqrt{(1-z)/2}$, $u_\pm=\cos(\phi\mp\delta)$.

As expected, we recover from Eq. (4.28)–(4.30) the Faddeev Eq. (4.19) by using the variable $z'=2u^2-1$ for $A=3$, i.e., $\lambda=1$. We should note that in Ref. 4 where the kernel of this integrodifferential equation was obtained in closed form for the first time, Eq. (7) has to be corrected by substituting u for $\sqrt{(1+z')/2}$, $4u du$ for dz',

and $\eta_{\pm}$ for $z_{\pm}$ in the first integral where the limits are defined without the absolute value symbol. In Ref. 10 this equation has already been given correctly however, but without any details of its derivation.

V. INCLUSION OF A THREE-BODY FORCE

We will show below that our equations give an accurate estimate of the binding energy of the few-body ground state ($A \leq 4$) for a sample of various central potentials, and that our results are in agreement with those obtained by using other sophisticated methods for solving the Schrödinger equation.

The question we now consider is whether we have to restrict our method only to two-body potentials or whether the same scheme can be used for solving Schrödinger equations with three-body forces. The commonly used total three-body potentials have the structure

$$\mathbf{W}(\mathbf{x}) = \sum_{i<j<k\leq A} V^{(3)}(r_{ij}) V^{(3)}(r_{jk}) , \tag{5.1}$$

where $V^{(3)}(r_{ij})$ is a two-body potential, but they can have any other structure, the crucial point being that they are symmetrical functions, which can be expanded by using the fully symmetrical hyperspherical basis:

$$\mathbf{W}(\mathbf{x}) = \sum_{K, i<j\leq A} \mathcal{P}^0_{2K}(\Omega_{ij}) W_K(r) + \text{residual contributions} , \tag{5.2}$$

where $[\mathcal{P}^0_{2K}(\Omega_{ij})]$ is the potential harmonic basis for S states, which is complete for the expansion of any function of r_{ij} and r, and where the $W_K(r)$, $K=0,1,2,\ldots$ are the potential multipoles.[10] Since the hyperspherical basis for symmetric S states contains only one element for $K<6$ in the six-dimensional space, the residual part contains a negligible amount of the interaction as has been shown previously.[14] The total three-body potential can now be written as

$$\mathbf{W}(\mathbf{x}) = \sum_{i<j\leq A} W(r_{ij}, r) ,$$

where

$$W(r_{ij}, r) = \sum_K \mathcal{P}^0_{2K}(\Omega_{ij}) W_K(r) . \tag{5.3}$$

In order to take into account the three-body interaction, the two-body potential has therefore, to be modified in Eq. (4.27) by adding $W_0(r)\mathcal{P}^0_0$ to $V_0(r)$ and $W(r_{ij}, r)$ to $V(r_{ij})$.

The case where $V^{(3)}(r_{ij})$ is a Gaussian potential is especially simple in the three-body problem, because for $V^{(3)}(r_{ij}) = e^{-\alpha(r_{ij})^2}$ the product

$$V^{(3)}(r_{jk}) V^{(3)}(r_{ki}) = \exp[-\alpha(r^2_{jk} + r^2_{ki})] = \exp[-(\tfrac{3}{2})\alpha r^2] \exp[\alpha r^2_{ij}] ,$$

where $r^2 = \frac{2}{3}\sum_{i<j} r^2_{ij}$, is already a function of r_{ij} and r only.

VI. DISCUSSION OF THE CORRELATIONS DESCRIBED BY THE INTEGRODIFFERENTIAL EQUATION

From our derivation it is clear that in the case of the Faddeev equation for three bodies our results are exact for S states and S-state projected potentials. To understand this let us examine the various quantum numbers associated with the spatial degrees of freedom. For three particles, when the center-of-mass motion is eliminated, only two variables remain, the two Jacobi coordinates $\xi_1 = \sqrt{3}(\mathbf{x}_k - \mathbf{X})$ and $\xi_2 = \mathbf{x}_i - \mathbf{x}_j$, i.e., six degrees of freedom. In the six-dimensional space the complete harmonic polynomial basis is characterized by the five quantum numbers associated with the five angular coordinates which are: (a) the degree L of the harmonic polynomial; (b) the orbitals l_1 and l_2 of the two vectors ξ_1 and ξ_2; (c) the total angular momentum l and its projection m.

For S states we have $l=m=0$ and $l_1=l_2=\lambda$, and we only have to consider the two quantum numbers L (even) and λ. The quantum number L is even because S states are even. In Eq. (4.18) we projected the wave function Ψ on the S state for the pair (i,j), i.e., for $\lambda=0$. The residual part of the wave function contains only $\lambda\neq 0$ terms and when we only consider an S-state projected potential, the product of the potential and the residual part, for which the orbitals $\lambda = l_1 = l_2 \neq 0$, vanishes. For more than three particles the residual part of the wave function contains all the neglected quantum numbers however.

For instance, for four-particles the harmonic polynomials are characterized by eight quantum numbers which are: (a) the degree L of the harmonic polynomial; (b) the six orbital quantum numbers l_i, m_i; (c) the quantum number n related to the length of one of the three Jacobi vectors. For a total angular momentum l,m the orbitals are coupled to give l,m as follows

$$[(l_1,l_2)l_{12}, l_3]_{l,m} .$$

For an S-state $l=m=0$ and we have only three quantum numbers $l_3 = l_{12}$ and l_1, l_2.

When Ψ is projected on the S state for $\xi_3 = \mathbf{r}_{ij}$, the orbital $l_3=0$ is selected and we still have the residual quantum number $l_1 = l_2 = \lambda$. This manifests itself in such a way that the product of an S-state projected potential and the residual wave function still contains the two quantum numbers λ and n. These quantum numbers are related to the degrees of freedom generated by the coordinates different from $\mathbf{r}_{ij}$ and hence, to the many-body ($A>2$) correlations. Our equation is therefore only exact to the extent to which all correlations higher than the two-body correlations can be neglected.

The claim in Ref. 11 that our integrodifferential equation is exact for S-state projected potentials is consequently only correct in so far as the two-body correlations are concerned. More than two-body correlations cannot be taken into account by two variables only, except when we are dealing with a system of three particles in S states, as we have shown above.

We have nevertheless, to point out that our definition of the two-body correlations might not coincide with those resulting from solutions of the Schrödinger equa-

tion obtained by other methods, e.g., perturbation methods or by means of Jastrow functions. In the last case the two-body correlations are introduced in the wave function as a product of Jastrow two-body correlation factors in such a way that when a strong repulsive core occurs in the potential, the Jastrow factor $f(r_{ij})$ vanishes when $r_{ij}\to 0$. In our scheme, the wave function is constructed as a sum of amplitudes $\sum_{i<j} F(r_{ij},r)$ and it is the projection of the sum on the $\mathbf{r}_{ij}$ space which cancels out when one of the r_{ij} goes to zero. Exactly the same thing happens in the case of the Faddeev solution, which provides an exact solution for S states when S-state projected potentials are used.

VII. ADIABATIC APPROXIMATION METHOD FOR THE SOLUTION OF THE TWO VARIABLE INTEGRODIFFERENTIAL EQUATION

Equation (4.28) can be solved either directly as a two variable partial integrodifferential equation or by an approximate separation of the variables in the adiabatic approximation.

For this purpose we assume that the radial motion (coordinate r) and the orbital motion (coordinate z) are nearly decoupled as for instance in the adiabatic approximation applied to molecular systems. In that case the velocity of the electrons is large compared to the velocities of the nuclei in such a way that for each distance r between two nuclei the total energy of the electronic cloud determines the interaction between two atoms. Here we assume that the orbital motion is very rapid compared to the radial motion and contains most of the energy, as for electrons in atoms, in such a way that the amplitude can be written as the product

$$P(z,r)=P_\lambda(z,r)u_\lambda(r)\ , \tag{7.1}$$

where $P_\lambda(z,r)$ is assumed to vary slowly with r and is a solution of a one variable integrodifferential equation in z with r regarded as a parameter.[3] Hence,

$$\frac{4\hbar^2}{mr^2}\frac{1}{w_\alpha}\frac{\partial}{\partial z}(1-z^2)w_\alpha\frac{\partial}{\partial z}P_\lambda(z,r)+U_\lambda(r)P_\lambda(z,r)=\{V[r\sqrt{(1+z)/2}]-V_0(r)\}\left[P_\lambda(z,r)+\int_{-1}^{1}f_{[0]}(z,z')P_\lambda(z',r)dz'\right]\ , \tag{7.2}$$

Eq. (7.2) has normalizable solutions only for a definite set of eigenvalues $U_\lambda(r)$ associated with the eigenamplitudes $P_\lambda(z,r)$. Each $U_\lambda(r)$ is generally associated with a $P_\lambda(z,r)$ which has a definite number of nodes for $-1<z<1$. Each element of the infinite set $\{U_\lambda(r)\}$ is called an eigenpotential. The radial function is now a solution of the radial equation:

$$\left\{\frac{\hbar^2}{m}\left[-\frac{d^2}{dr^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}\right]+\frac{A(A-1)}{2}V_0(r)+U_\lambda(r)-E^{ea}_{\lambda,n}\right\}u_{\lambda,n}(r)=0\ , \tag{7.3}$$

where the derivatives of $P_\lambda(z,r)$ with respect to r have been neglected. In this equation n is the number of nodes ($n\neq 0$ for the breathing mode) of the radial wave for bound states, and the wave function is given by

$$\Psi_{\mathrm{EAA}}(\mathbf{x})=u_{\lambda,n}(r)/r^{3A/2-2}\sum_{i<j\le A}P_\lambda(2r_{ij}^2/r^2-1,r)\ . \tag{7.4}$$

It has been shown elsewhere,[12] by means of a hyperspherical harmonic expansion of the wave function, that the solution obtained in this way provides a lower bound E^{EAA} for the binding energy (EAA for extreme adiabatic approximation). An upper bound is obtained when the variation with r of P_λ is taken into account. For this purpose we consider Ψ_{EAA} as a variational solution and we write

$$B_\lambda(r,\Omega)=\sum_{i<j\le A}P_\lambda(2r_{ij}^2/r^2-1,r)\ , \tag{7.5}$$

where $B_\lambda(r,\Omega)$ is normalized to one for any r, i.e.,

$$\int|B_\lambda(r,\Omega)|^2 d\Omega=\langle B_\lambda\,|\,B_\lambda\rangle=1\ , \tag{7.6}$$

and $d\Omega$ stands as usual for the surface element over the unit hypersphere $r=1$. The solution

$$\Psi(\mathbf{x})=B_\lambda(r,\Omega)u_{\lambda,n}(r)/(r^{3A/2-2}) \tag{7.7}$$

would be the exact solution of our integrodifferential equation if $B_\lambda(r,\Omega)$ were independent of r. To take this dependence into account variationally, we introduce B_λ in the right-hand side of the radial equation for $u_{\lambda,n}(r)$ we premultiply by $B^*_\lambda(r,\Omega)$ and we integrate over Ω to obtain the new variational equation for $u_{\lambda,n}(r)$:

$$\int d\Omega B^*_\lambda(r,\Omega)\left\{\frac{\hbar^2}{m}\left[-\frac{d^2}{dr^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}\right]+\frac{A(A-1)}{2}V_0(r)+U_\lambda(r)-E\right\}B_\lambda(r,\Omega)u_{\lambda,n}(r)=0\ .$$

The radial wave function is the solution of the radial equation

$$\left\{\frac{\hbar^2}{m}\left[-\frac{d^2}{dr^2}+\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}\right]+\frac{A(A-1)}{2}V_0(r)+U_\lambda(r)-\frac{\hbar^2}{m}\left\langle B_\lambda\left|\frac{d^2B_\lambda}{dr^2}\right.\right\rangle-E_\lambda^{\mathrm{UAA}}\right\}u_{\lambda,n}^{(\mathrm{UAA})}(r)=0\ , \tag{7.8}$$

where the orthogonality of dB_λ/dr and B_λ has been taken into account. The solution obtained in this way is called the uncoupled adiabatic approximation (UAA).[13,14] As it is variational it provides an upper limit for the exact binding energy.

It is easy to show that the additional potential $-\hbar^2/m\langle B_\lambda \mid d^2B_\lambda/dr^2\rangle$ is always positive. Starting from the normalization $\langle B_\lambda(r,\Omega)\mid B_\lambda(r,\Omega)\rangle=1$ where r is a parameter we obtain, taking derivatives,

$$\left\langle B_\lambda\left|\frac{d^2B_\lambda}{dr^2}\right.\right\rangle=-\left\langle\frac{dB_\lambda}{dr}\left|\frac{dB_\lambda}{dr}\right.\right\rangle\ ,$$

which is always negative, in such a way that

$$\frac{A(A-1)}{2}V_0(r)+U_\lambda(r)+\frac{h^2}{m}\left\langle\frac{dB_\lambda}{dr}\left|\frac{dB_\lambda}{dr}\right.\right\rangle$$

is always larger than $[A(A-1)/2]V_0(r)+U_\lambda(r)$.

By using the adiabatic approximation we simplify the numerical calculations for the solution of Eq. (4.19) and we obtain upper and lower bounds for the binding energy. The interpolation formula

$$E_{\mathrm{exact}}\simeq E^{\mathrm{UAA}}+0.2(E^{\mathrm{EAA}}-E^{\mathrm{UAA}}) \tag{7.9}$$

provides a very accurate estimate of the exact binding energy as has been demonstrated previously.[12]

For the normalization of $B_\lambda(r,\Omega)$ we use the property that when a function $F(r_{kl},r)$ is projected on the $\mid \mathbf{r}_{ij}\rangle$ space, the remainder is orthogonal to any function of $\mathbf{r}_{ij}$ and r. By using this property one finds

$$\langle B_\lambda\mid B_\lambda\rangle=\frac{A(A-1)}{2}\int_{-1}^{1}P_\lambda^*(z,r)w_\alpha(z)dz\left[P_\lambda(z,r)+\int_{-1}^{1}f_{[0]}(z,z')P_\lambda(z',r)dz'\right] \tag{7.10}$$

from which B_λ can be normalized, and

$$\left\langle\frac{dB_\lambda}{dr}\left|\frac{dB_\lambda}{dr}\right.\right\rangle=\frac{A(A-1)}{2}\int_{-1}^{1}dz\frac{dP_\lambda(z,r)}{dr}w_\alpha(z)\left[\frac{dP_\lambda(z,r)}{dr}+\int_{-1}^{1}f_{[0]}(z,z')\frac{dP_\lambda(z',r)}{dr}dz'\right]\ . \tag{7.11}$$

VIII. ASYMPTOTIC BEHAVIOR OF $P_\lambda(z,r)$

In order to calculate the asymptotic behavior of $P_\lambda(z,r)$ for $r\to\infty$ we transform the kinetic energy operator by using, instead of the variables z and r, the variables $r_{ij}=r[(1+z)/2]^{1/2}$ and r. The z-dependent part of the kinetic energy operator for S states

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}+\frac{4}{r^2}\frac{1}{w}\frac{\partial}{\partial z}(1-z^2)w\frac{\partial}{\partial z}\ ,$$

where

$$w(z)=(1-z)^{(D-5)/2}(1+z)^{1/2} \tag{8.1}$$

is modified. It becomes

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}+\nabla_{ij}^2\ , \tag{8.2}$$

with

$$\nabla_{ij}^2=\left[1-\frac{r_{ij}^2}{r^2}\right]\frac{\partial^2}{\partial r_{ij}^2}-\left[\frac{3A-4}{r^2}r_{ij}^2-2\right]\frac{1}{r_{ij}}\frac{\partial}{\partial r_{ij}}\ . \tag{8.3}$$

The limit for $r\to\infty$ for fixed r_{ij} is obviously given by

$$\lim\nabla_{ij}^2\underset{r\to\infty}{\to}\frac{\partial^2}{\partial r_{ij}^2}+\frac{2}{r_{ij}}\frac{\partial}{\partial r_{ij}}\ . \tag{8.4}$$

The right-hand side of the integrodifferential equation (4.6) is given by

$$[V(r_{ij})-V_0(r)]\left[F_\lambda(r_{ij},r)+\int_{-1}^{1}f_{[0]}(2r_{ij}^2/r^2-1,z')F_\lambda[r\sqrt{(1+z')/2},r]dz'\right]\ ,$$

where $F_\lambda(r_{ij},r)$ is normalized for each value of r as in Eq. (7.2) and

$$F_\lambda(r_{ij},r)=r^{-(D-1)/2}P_\lambda[2r_{ij}^2/r^2-1,r]\ .$$

Let $\Phi_\lambda(r_{ij})$ be the asymptotic limit of $F_\lambda(r_{ij},r)$ when $r\to\infty$ for fixed r_{ij}. The function $\Phi_\lambda[r\sqrt{(1+z)/2}]$ tends to a Dirac delta function $\delta(z+1)$ in the z coordinate for $r\to\infty$, and the integral part of the integrodifferential

equation vanishes for $r\to\infty$. Since $\lim_{r\to\infty} V_0(r)=0$ (Ref. 6) the asymptotic equation for $r\to\infty$ becomes

$$\left\{-\frac{\hbar^2}{m}\left[\frac{d^2}{dr_{ij}^2}+\frac{2}{r_{ij}}\frac{d}{dr_{ij}}\right]+[V(r_{ij})-U_\lambda(r)]\right\}\Phi_\lambda(r_{ij})=0\ , \quad (8.5)$$

in such a way that $U_\lambda(r)$ tends for $r\to\infty$ to one of the energy eigenvalues of the two-body bound states or to zero for two-body scattering states.

IX. THREE-BODY SCATTERING IN THE ADIABATIC APPROXIMATION

The adiabatic approximation can be used for solving the scattering problem in the three-body case. We have already seen that the "channel basis" function $P_\lambda(z,r)$ tends to one of the two-body eigenstates for $r\to\infty$. In particular when we have to deal with a bound state $\Phi_\lambda(r_{ij})$ in the asymptotic region, the normalized function $P_\lambda(z,r)$ is asymptotically given by

$$P_\lambda(z,r)\underset{r\to\infty}{\to}(Y_0^0)^2 r^{3/2}\Phi_\lambda(r_{ij})\ , \quad (9.1)$$

where

$$\int[P_\lambda(z,r)]^2 d\Omega=1,\quad z=2r_{ij}^2/r^2-1\ ,$$

and the two-body bound state wave function is normalized, i.e.,

$$\int_0^\infty[\Phi_\lambda(r_{ij})]^2 r_{ij}^2 dr_{ij}=1\ . \quad (9.2)$$

In the uncoupled adiabatic approximation the amplitude is given by

$$\phi_{ij}(\mathbf{x})=P_\lambda(z,r)u_\lambda(r)/r^{5/2}\ , \quad (9.3)$$

where $u_\lambda(r)$ is a solution of the radial equation (7.8).

The effective potential in Eq. (7.8)

$$V_{\text{eff}}(r)=\frac{\hbar^2}{m}\frac{\mathcal{L}_0(\mathcal{L}_0+1)}{r^2}+3V_0(r)+U_\lambda(r)+\frac{\hbar^2}{m}\langle B'_\lambda\mid B'_\lambda\rangle \quad (9.4)$$

goes asymptotically to the eigenenergy E_λ (<0) and the radial wave function has the asymptotic behavior

$$u_\lambda(r)\underset{r\to\infty}{\to}\sin(q_\lambda r+\delta_\lambda)\ , \quad (9.5)$$

where $(\hbar^2/m)q_\lambda^2=E-E_\lambda$ and, according to Eq. (9.3), the asymptotic amplitude becomes

$$\phi_{ij}(\mathbf{x})\underset{r\to\infty}{\to}(Y_0^0)^2\Phi_\lambda(r_{ij})\frac{\sin(q_\lambda r+\delta_\lambda)}{r}\ . \quad (9.6)$$

But the distance between the particles (i) and (j) is confined by the two-body bound state wave function. Hence for $r\to\infty$ we must also have $r\to r_k$,

$$\lim_{r\to\infty}\phi_{ij}(\mathbf{x})=(Y_0^0)^2\Phi_\lambda(r_{ij})\frac{\sin(q_\lambda r_k+\delta_\lambda)}{r_k}\ ,$$

where $\mathbf{r}_k=\sqrt{3}(\mathbf{x}_k-\mathbf{X})$. If r is replaced by r_k in Eq. (9.6), it describes the scattering of the spectator particle k in the S state by the two-body bound system, which is also in an S state and given by $Y_0^0\Phi_\lambda(r_{ij})$. The δ_λ is the corresponding S-wave phase shift. In the UAA all the $P_\lambda(z,r)$ are orthogonal and each channel is described by only one integrodifferential equation (7.2). Hence, we can only have elastic scattering when we are dealing with Wigner forces. When spin-dependent potentials are considered and when only one bound (triplet) state is present, as is the case for nucleons, a transition can occur between the triplet two-body bound state, and the free singlet state, resulting in a break-up reaction. In this case the system is described by two coupled integrodifferential equations

$$\left\{\frac{\hbar^2}{m}\left[-\frac{d^2}{dr^2}+\frac{15}{4r^2}-\frac{4}{r^2}\frac{1}{(1-z^2)^{1/2}}\frac{\partial}{\partial z}(1-z^2)^{3/2}\frac{\partial}{\partial z}\right]-E\right\}P^{(\alpha)}(z,r)$$
$$=-V^{(\alpha)}[r\sqrt{(1+z)/2}]\left\{P^{(\alpha)}(z,r)+\frac{1}{[3(1-z^2)]^{1/2}}\int_{z_-}^{z_+}[P^{(\alpha)}(z',r)-3P^{(\beta)}(z',r)]dz'\right\},\quad \alpha\neq\beta \quad (9.7)$$

with each $P^{(\alpha)}(z,r)$ related to the two-body system in either the triplet ($\alpha=3^+$) or the singlet ($\alpha=1^+$) S state.

To include inelastic (break-up) processes in the case of simple Wigner forces we have to go beyond the uncoupled adiabatic approximation, i.e., take account of the derivatives of $P(z,r)$ with respect to r by means of the coupled adiabatic approximation (CAA).[12,15]

We construct the vector $B_0(r,\Omega)$ as in Eq. (7.5). It is normalized according to Eq. (7.10). The index $\lambda=0$ indicates that $U_0(r)$ is the lowest eigenpotential tending to the two-body eigenenergy $E_0<0$ when $r\to\infty$. It describes the elastic channel where the spectator particle is scattered by the two-body bound state of the interacting particles.

To include the break-up in our description we start again from the Schrödinger equation

$$\left\{-\frac{\hbar^2}{m}\left[\frac{\partial^2}{\partial r^2}+\frac{\mathcal{L}^2(\Omega)}{r^2}\right]+V(r,\Omega)-E\right\}u(r,\Omega)=0\ , \quad (9.8)$$

where

$$\mathcal{L}^2(\Omega)=L^2(\Omega)-(D-1)(D-3)/4,\quad D=3(A-1)\ ,$$

and where

$$\Psi(r,\Omega)=u(r,\Omega)/r^{(D-1)/2} .$$

When $V(r,\Omega)=\sum_{i<j}V(r_{ij})$, where $V(r_{ij})$ is a central local potential, the uncoupled adiabatic solution $u(r,\Omega)$ for the channel $\lambda=0$ is given by $B_0(r,\Omega)u_0(r)$, where $u_0(r)$ is a solution of the radial equation (7.8) and where $B_0(r,\Omega)$ satisfies, according to Eq. (7.2),

$$\left[\frac{\hbar^2}{m}\frac{\mathcal{L}^2(\Omega)}{r^2}+U_\lambda(r)\right]B_\lambda(r,\Omega) =[V(r,\Omega)-V_0(r)]B_\lambda(r,\Omega) , \quad (9.9)$$

with $\lambda=0$.

The hypercentral potential $V_0(r)$ is set equal to zero for an S-projected potential. In this solution the derivatives of $B_0(r,\Omega)$ with respect to r in Eq. (9.8) have been taken into account only by retaining the term $\langle B_0 \mid d^2B_0/dr^2\rangle$ occurring in Eq. (7.8), which corresponds to the UAA. Let us now investigate the full effect of the operator d^2/dr^2 occurring in Eq. (9.8) on the UAA solution $u(r,\Omega)=B_\lambda(r,\Omega)u_\lambda(r)$. It generates the components

$$\frac{d^2}{dr^2}u(r,\Omega)=B_\lambda\frac{d^2u_\lambda}{dr^2}+2\frac{dB_\lambda}{dr}\frac{du_\lambda}{dr}+u_\lambda\frac{d^2B_\lambda}{dr^2} . \quad (9.10)$$

By using the normalization relation, Eq. (7.6), one can write[15]

$$\frac{dB_\lambda}{dr}=C_1(r)B_\lambda^{(1)} , \quad (9.11)$$

$$\frac{d^2B_\lambda}{dr^2}=C_1(r)C_2(r)B_\lambda^{(2)}+\frac{dC_1}{dr}B_\lambda^{(1)}-C_1^2(r)B_\lambda , \quad (9.12)$$

where the $C_i(r)$ are functions of r only and where the $(B_\lambda^{(i)})$ satisfy the orthonormality condition

$$\langle B_\lambda^{(i)} \mid B_\lambda^{(j)}\rangle=\delta_{ij} \quad \text{while } B_\lambda^{(0)}\equiv B_\lambda .$$

The derivatives with respect to r generate the two new components $B_\lambda^{(1)}$ and $B_\lambda^{(2)}$ orthogonal to B_λ. The $[B_\lambda^{(i)}(r)]$ are called the coupled adiabatic basis functions. The coefficient C_1 and C_2 are calculated from Eq.(7.10) where the normalization of $P_\lambda(z,r)$ fulfills Eq. (7.6). Then from Eq. (9.11) the first coefficient $C_1^2=\langle dB_\lambda/dr \mid dB_\lambda/dr\rangle$ is given by Eq. (7.11), and the second coefficient is obtained from Eq. (9.12) as

$$C_2^2=\frac{1}{C_1^2}\left\langle\frac{d^2B_\lambda}{dr^2}\middle|\frac{d^2B_\lambda}{dr^2}\right\rangle-C_1^2-\left[\frac{d}{dr}\ln C_1\right]^2 ,$$

where the matrix element is calculated from Eq. (7.10) by substitution of d^2B_λ/dr^2 and d^2P_λ/dr^2 respectively for B_λ and P_λ. The components $B_\lambda^{(1)}$ and $B_\lambda^{(2)}$ being orthogonal to B_λ can be expanded only in terms of $B_{\lambda'}$ with $\lambda'\neq\lambda$:

$$B_\lambda^{(i)}(r,\Omega)=\sum_{\lambda'\neq\lambda}b_{\lambda'}^{(i)}(r)B_{\lambda'}(r,\Omega) \quad i\neq 0 . \quad (9.13)$$

Therefore, they contain only channels orthogonal to $B_\lambda(r,\Omega)$.

When $\lambda=0$ and only one bound state exists, $B_0^{(1)}$ and $B_0^{(2)}$ are associated with inelastic scattering, i.e., with break up in this case. In the coupled adiabatic approximation the solution is a sum of three terms only, i.e.,

$$u(r,\Omega)=\sum_{i=0}^{2}B_\lambda^{(i)}(r,\Omega)u_\lambda^{(i)}(r) .$$

The coupled equations for the three radial functions $u_\lambda^{(i)}(r)$ have already been given.[12] They are

$$-\left[\frac{\hbar^2}{m}\frac{d^2}{dr^2}+E\right]u_\lambda^{(i)}(r)+\sum_{j=0}^{2}U_\lambda^{(i,j)}(r)u_\lambda^{(j)}(r)=0 , \quad (9.14)$$

where

$$U_\lambda^{(i,j)}(r)=\left\langle B_\lambda^{(i)}\middle|-\frac{\hbar^2}{mr^2}\mathcal{L}^2(\Omega)+V(r,\Omega)\middle|B_\lambda^{(j)}\right\rangle + W_\lambda^{(i)}\delta_{ij}+\frac{\hbar^2}{m}P_j^{(i)} . \quad (9.15)$$

The operators $P_j^{(i)}$

$$P_1^{(0)}=\frac{dC_1}{dr}+2C_1\frac{d}{dr} ,$$
$$P_2^{(0)}=P_0^{(2)}=-C_1C_2 , \quad (9.16)$$
$$P_2^{(1)}=-P_1^{(2)}=\frac{dC_2}{dr}+2C_2\frac{d}{dr} ,$$

are responsible for the occurrence of a break up for $\lambda=0$.

The diagonal terms $W_\lambda^{(i)}(r)$ in the potential matrix are given by

$$W_\lambda^{(0)}=\frac{\hbar^2}{m}C_1^2, \quad C_1^2=\left\langle\frac{dB_\lambda}{dr}\middle|\frac{dB_\lambda}{dr}\right\rangle , \quad (9.17)$$

which is the additional term occurring in the UAA, and

$$W_\lambda^{(2)}=\frac{\hbar^2}{m}C_2^2 \text{ and } W_\lambda^{(1)}=W_\lambda^{(0)}+W_\lambda^{(2)} . \quad (9.18)$$

For $i=j=0$ the matrix element in Eq. (9.15) is easily calculated, using the symmetry property of the grand orbital operator $\mathcal{L}^2(\Omega)$. We have

$$U_\lambda^{00}(r)=3\left\langle B_\lambda\middle|-\frac{\hbar^2}{m}\mathcal{L}^2(\Omega)\middle|P_\lambda(z,r)\right\rangle+3\langle B_\lambda \mid V(r_{ij}) \mid B_\lambda\rangle+W_\lambda^{(0)}$$
$$=3\left\langle B_\lambda\middle|-\frac{\hbar^2}{mr^2}\mathcal{L}^2(\Omega)\middle|P_\lambda(z,r)\right\rangle+V(r_{ij})\left[P_\lambda(z,r)+\int_{-1}^{+1}f_{[0]}(z,z')P_\lambda(z',r)dz'\right]+W_\lambda^{(0)}=U_\lambda(r)+W_\lambda^{(0)} , \quad (9.19)$$

where $U_\lambda(r)$ is given by Eq. (7.2) with $V_0(r)=0$ for an S-projected potential. For a local potential acting in all partial waves we have

$$U_\lambda^{00}(r)=3V_0(r)+U_\lambda(r)+W_\lambda^{(0)} , \tag{9.20}$$

where $U_\lambda(r)$ is given by Eq. (7.2).

Here we should note that according to Eq. (9.7) the matrix element in $U_\lambda^{(0,j)}(r)$, $j\neq 0$, does not exist and the coupling between the main partial wave $u_\lambda^{(0)}(r)$ and the two others is taken care of by the operators $P_1^{(0)}$ and $P_2^{(0)}$.

The system of the three coupled equations (9.14) has been solved[12] for the trinucleon bound state with typical central potentials like the Volkov,[16] Gogny Pires de Tourreil,[17] Eikemeier-Hackenbroich[18] and Afnan-Tang[19] potentials. The norm of the partial radial wave functions in the solution have been found not to exceed 4.10^{-4} and 10^{-4}, respectively, for $u_0^{(1)}(r)$ and $u_0^{(2)}(r)$, and the exact binding energy was nearly fully obtained. This means that truncation after only three terms of the solution leads to a very accurate result for ground states. When the total energy E in Eq. (9.14) becomes larger than the two-body binding energy, which is the asymptotic limit of $U_0^{00}(r)$ for $r\to\infty$, Eq. (9.14) describes the scattering of one particle in S states by a two-body bound S state. When E becomes positive, break-up becomes possible and is described by the partial wave solutions $u_0^{(1)}(r)$ and $u_0^{(2)}(r)$ of Eq. (9.14) when suitable asymptotic boundary conditions are imposed.

We notice that $U_0^{00}(r)$ is the potential from which both the three-body bound state and the two-cluster scattering in an S state are calculated. It tends to the two-body binding energy when $r\to\infty$. On the other hand, both $U_0^{(1,1)}(r)$ and $U_0^{2,2}(r)$ vanish at infinity, as should be the case for a three-body scattering state (break-up state), when only one bound state exists.

X. APPLICATION TO THREE- AND FOUR-NUCLEON GROUND STATES

As a first step and to establish the accuracy of the solutions obtained, we used our equations to solve the few-body ground state problem. For comparison purposes we choose the two-body local central potentials, which have already been treated to a high accuracy by variational methods like the amalgamation of two-body correlations into the multiple scattering process (ATMS), correlated harmonic oscillator expansion, hyperspherical expansion, Monte Carlo and Green's function Monte Carlo methods, and also with the exact solutions obtained by means of the Faddeev-Yakubovsky equation.

The potentials used for comparison purposes are: (i) the Volkov[16] and Gogny-Pires-de Tourreil[17] super-soft core potentials; (ii) the Eikemeier-Hackenbroich potential[18] which reproduces the S-wave phase shifts without a tensor force component; (iii) the Afnan-Tang $S3$ potential[19] which exhibits a rather strong repulsive core; (iv) and finally the Malfliet-Tjon V in the three versions used by Erens *et al.*,[20] Payne *et al.*,[21] and Zabolitzky.[22]

We solved Eq. (4.28) by using the extreme and uncoupled adiabatic approximations, which allow us to separate the variables and interpolate to obtain the binding energy

$$E_I=E^{\mathrm{UAA}}+0.2(E^{\mathrm{EAA}}-E^{\mathrm{UAA}}) ,$$
$$(E^{\mathrm{EAA}}<E^{\mathrm{exact}}<E^{\mathrm{UAA}}<0) . \tag{10.1}$$

It has been shown[12] previously that this empirical formula is very accurate.

In Table I we give the binding energy of the trinucleon bound state ^{3}H obtained by using the adiabatic approximations with a set of central nuclear potentials (first column). The second and third columns refer respectively to the extreme and uncoupled adiabatic approximations. The interpolated energy E_I is shown in the fourth column, and a sample of the results obtained with the best other methods is exhibited in the last columns where HH stands for hyperspherical harmonic expansion,[23] ETBM, for the variational method with correlation factors,[24] and F for the full solution of the Faddeev equation (all partial waves included).[25] For illustrative purposes we display the adiabatic potential $3V_0(r)+U_\lambda(r)$ occurring in Eq. (7.3) for three of the nucleon-nucleon potentials we employed in our calculations in Fig. 1. It is seen that the degree of repulsiveness of the nucleon-nucleon potentials is reflected in the effective potential.

Our results for ^{3}H are in all cases in agreement with those obtained with the most accurate methods. The differences between the upper and lower bound range from 0.25 MeV for the softest potentials to 0.7 MeV for

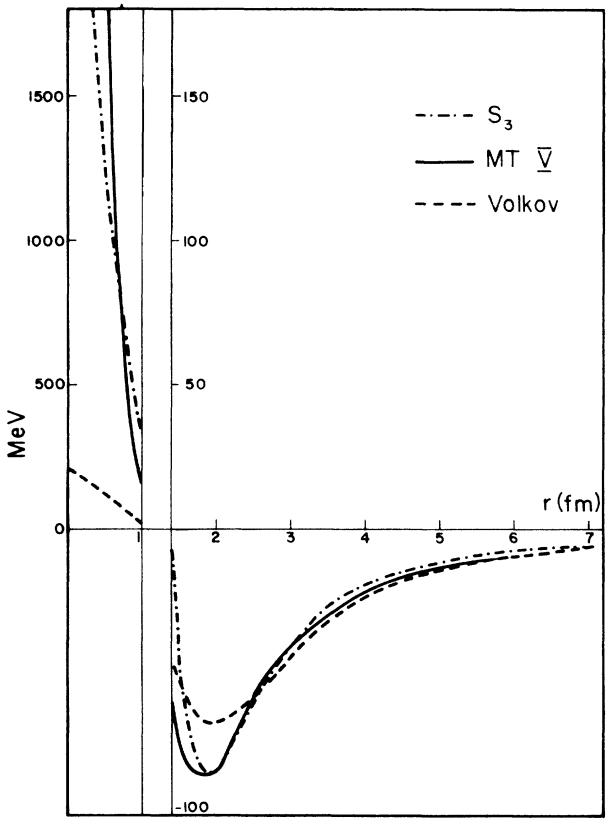

FIG. 1. The effective potential $V_{\mathrm{eff}}(r)=3V_0(r)+U_0(r)$ in Eq. (7.3) for $A=3$ and the potentials: Volkov (Ref. 16) (— — —), S_3 (Ref. 19) (—·—·—), and MTV (Ref. 22) (——).

TABLE I. Three-nucleon binding energies in the EAA and UAA to Eq. (4.28) and the interpolated value (I) according to Eq. (10.1) compared to results in the literature.

Potential	EAA	UAA	I	HH[a]	ETBM[b]	F	Others
Volkov[d]	8.67	8.42	8.47	8.465	8.460		
GPDT[e]	3.89	3.62	3.67				
EH($S4$)[f]	7.372	7.013	7.08	7.05	7.04		
AT($S3$)[g]	7.235	6.557	6.69	6.695	6.677	6.696[c]	
MTV Erens[h]	8.07	7.70	7.78	7.783	7.778		
MTV Friar[i]	8.03	7.65	7.73			7.736[k]	
MTV Zabol[j]	8.570	8.171	8.25			8.253[c]	8.26±0.01[j,k]

[a] Reference 23.
[b] Reference 24.
[c] Reference 25.
[d] Reference 16.
[e] Reference 17.
[f] Reference 18.
[g] Reference 19.
[h] Reference 20.
[i] Reference 21.
[j] Reference 22.
[k] Reference 31.

the strongly repulsive core Malfliet-Tjon potentials. Our interpolated energies are expected to be accurate within at least 0.03 MeV. We see that the interpolation formula gives a binding energy in good agreement with those obtained with the other methods.

The equation that we solved Eq. (4.28), differs from the usual form of the Faddeev equation by the incorporation of the hypercentral part $V_0(r)$ of the interaction which already takes the major part of the effect of the interaction into account. This hypercentral potential appears in the diagonal part of the potential matrix when the Schrödinger equation is solved by a hyperspherical expansion of the wave function, but must be cancelled out on both sides of Eq. (4.19) for the exact solution of the Faddeev equation, when an S-state projected potential is used. The right-hand side of Eq. (4.28), which contains the residual interaction, generates the two-body correlations. It contributes a binding energy which ranges from about 1 MeV for the softest potentials (e.g., Volkov) to nearly the whole binding energy for the hardest core interactions (e.g., $S3$ and Malfliet-Tjon V). It is gratifying that using a single integrodifferential equation we obtain the same binding energy as calculated by Friar and collaborators[25] by solving the Faddeev equation for the S amplitude, using its full partial wave decomposition. This is a consequence of the inclusion of $V_0(r)$, the hypercentral potential, in Eq. (4.28).

The results for the ^{4}He ground state without the Coulomb interaction are given in Table II. It is seen that the difference between the upper and the lower bounds is two to three times larger than for ^{3}H (although it is smaller relative to the binding energy). The interpolated binding energy is in agreement with the variational results for soft core potentials but differs significantly (around 0.4–0.5 MeV) for strong core potentials. One possibility is that this difference might be a consequence of the particular structure of our wave functions which are obtained as a sum of two-body amplitudes, while in the variational wave functions the correlations are described by Jastrow factors. Obviously for a three-body bound S state such variational wave functions are different from the exact Faddeev wave function. The effective potentials $6V_0(r)+U_\lambda(r)$ are displayed in Fig. 2 for the S_3 and Malfliet-Tjon V potentials, and show much more repulsion followed by a deeper and longer ranged attraction than in the three-body case.

Finally we investigated the asymptotic behavior of the total effective potential for three bodies,

$$V_T(r)=\frac{15}{4}\frac{\hbar^2}{m}\frac{1}{r^2}+3V_0(r)+U_0(r)\ . \tag{10.2}$$

Two different potentials have been employed for this pur-

TABLE II. Four-nucleon binding energies in the EAA and UAA to Eq. (4.28) and the interpolated value (I) according to Eq. (10.1) compared to results in the literature.

Potential	EAA	UAA	I	HH[a]	ETBM	ATMS	GFMC
Volkov	30.73	30.32	30.40	30.40	30.32		
GPDT	18.85	18.19	18.32	18.29			
EH($S4$)	29.33	28.55	28.71	27.9	28.18		
AT($S3$)	28.09	26.63	26.92	26.0	26.47		
MTV[b]	29.97	29.28	29.42				
MTV[c]	29.91	29.20	29.34				
MTV[d]	31.22	30.48	30.63			31.36[e]	31.3±0.2[d]

[a] Reference 26.
[b] Reference 20.
[c] Reference 21.
[d] Reference 22.
[e] Reference 31.

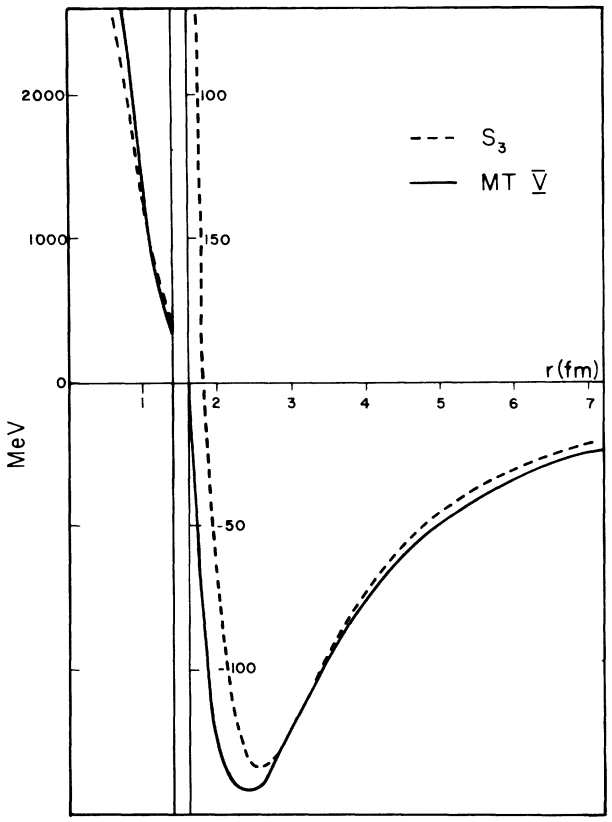

FIG. 2. The effective potential $V_{\text{eff}}(r)=6V_0(r)+U_0(r)$ in Eq. (7.3) for $A=4$ and the potentials: S_3 (Ref. 19) ($-\cdot-\cdot-\cdot$) and MTV (Ref. 22) (——).

pose: the Volkov potential with two-body binding energy $E_2=0.56$ MeV, and the modified Gaussian potential of Baker *et al.*[10]

$$V_B(r)=V_0e^{-(dr)^2}, \tag{10.3}$$

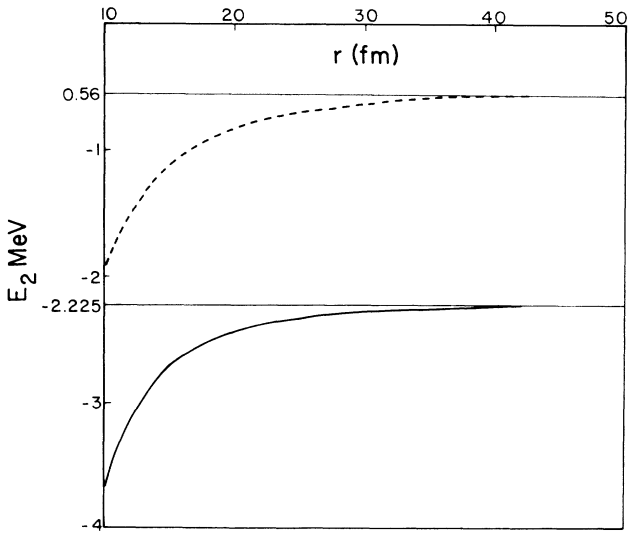

FIG. 3. The asymptotic behavior of the total effective potential of Eq. (10.3) $V_T(r)=\frac{15}{4}\hbar^2/(mr^2)+3V_0(r)+U_0(r)$ in three-body scattering: Volkov (Ref. 16) (— — —) and Baker (Ref. 10) (——).

with $V_0=-66.327$ MeV, $d=0.64041$ fm^{-1}, and $E_2=2.225$ MeV. The results are shown in Fig. 3. It is seen that the asymptotic value is reached for distances of the order of 35 fm in both cases.

XI. CONCLUSION

In this paper we have presented for the first time the full derivation of a two-variable integrodifferential equation of the Faddeev type describing a system of A identical bosons interacting by means of central two-body forces in S states, when only two-body correlations are taken into account. For a three-boson system interacting by means of S-state projected potentials this equation can be transformed into the Faddeev equation and is therefore exact. We have shown here that for $A\geq 4$ this equation does indeed take the two-body correlations into account exactly, while neglecting all higher order correlations. For bosons and S states the kernel has been obtained in analytical form and its full derivation has also been given here for the first time.

The two-body correlations have previously been described by means of an infinite system of one variable coupled differential equations. However, if these equations are truncated they do not converge rapidly enough for potentials with strong repulsive cores. Their transformation to a single two-variable integrodifferential equation with a Faddeev-like structure was performed to overcome this difficulty.

In this way it is demonstrated that the many-boson problem for central potentials can be described exactly by such an equation if only two-body correlations are taken into account. Hence the inclusion of the most important and in most cases dominant correlations, the two-body correlations, can be achieved by solving an equation which is very similar to and not much more difficult to solve than the three-body Faddeev integrodifferential equation in configuration space.

Since the most important change in the kernel of the integrodifferential equation occurs when we go from three to four-boson systems (which is related to the fact that we can have disconnected pairs for $A\geq 4$ but not for $A=3$), it is clear that one should first test the validity of our assumption that all correlations higher than two-body correlations can be neglected in the four-body system. In this case accurate calculations by means of variational methods, like the ATMS, correlated harmonic oscillations expansion, hyperspherical expansion, Monte Carlo and Green's function Monte Carlo (GFMC) methods and the Faddeev-Yakubovsky equations, are available for comparison purposes.

In the present paper, however, we solved the two-variable integrodifferential equation in a much simpler way by means of adiabatic approximations, the extreme adiabatic approximation providing a lower bound and the uncoupled adiabatic approximation an upper bound. In this way only two one variable equations have to be solved. Our calculations showed that these bounds are in general quite close, their difference ΔE varying from 0.2 to 0.7 MeV for three-body systems and from 0.4 to 1.3 MeV for four-body systems. In general the harder the

potential the larger is ΔE.

The form of the integrodifferential equation we employed in our three-body calculations differs from the usual Faddeev equation by the explicit inclusion of the hypercentral part of the interaction, which must be cancelled on both sides of the equation to obtain the exact Faddeev equation for S-state projected potentials. This has the advantage that, while still using a single equation we go beyond S-state projected potentials in the three-body problem and obtain the same binding energy as Friar using the full Faddeev equation for the Malfliet-Tjon potential. In fact all our three-body energies are in good agreement with the results of other groups using variational, GFMC, hyperspherical and Faddeev methods. Consequently, any discrepancies between our four-body calculations and those using other methods would most likely be due to the approximation involved in our four-body equation. The discrepancies between our four-body binding energies and those obtained by other methods are at most of the order of 0.5 MeV, even for the hardest potential, indicating that the neglect of three- and more-body correlations in our integrodifferential equation is justified. We therefore, expect that the integrodifferential equation [Eq. (4.28)] should be a good approximation for the ground state of a system of more than four bosons when the density is low enough that N-body correlations ($N>2$) do not generate a significant contribution to the binding energy. An advantage of our integrodifferential equation approach is that, while most variational wave functions are of the form of a sum of products with Jastrow factors, our wave functions are Faddeev-like and exact for $A=3$. We have also shown how three-body forces can be incorporated in the integrodifferential equation approach.

Finally, we have shown that the adiabatic method is particularly suitable for a simple treatment of three-body scattering in nuclear physics. In the simplest case of pure Wigner forces the scattering process can be described by means of the coupled adiabatic approximation. We demonstrated that in this approximation the eigenpotential corresponding to a three-body bound state tends to the energy eigenvalue of the two-body bound state for the simple cases of the Volkov and the modified Baker potentials at rather large distances of the order of $r\simeq 35$ fm in the hyperradius. In addition the formalism for the treatment of breakup with spin-independent central forces has been given here in the coupled adiabatic approximation leading to a set of three coupled equations in the case that only one two-body bound state exists.

We can therefore conclude that our two variable integrodifferential approach to the many-body problem, has been shown to be accurate not only in three-body, but also in four-body bound systems notwithstanding the omission of correlations higher than two-body ones in the latter case. Since this approach does not lead to a rapid increase in complexity with increasing number of particles, unlike other methods applied to few-body systems, it appears to be a very promising method to apply to many-body bound systems, as long as the two-body correlations remain dominant over the higher order ones as for $A=4$. Although we have only used the adiabatic approximation in the present calculations, which was quite adequate for our purposes and in fact was shown to represent a great simplification of the usual methods of solution of the Faddeev equation, the new equation can of course also be solved by means of the three-body methods developed for two variable integrodifferential equation to obtain higher accuracy where needed.

APPENDIX A

1. Derivation of the projection function

To obtain the projection function $f_{[0]}(z,z')$ we substitute Gegenbauer polynomials for Jacobi polynomials in Eq. (4.27) by using

$$P_K^{\lambda-1/2,1/2}(2x^2-1)=\frac{\Gamma(K+\frac{3}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(\lambda)}{\Gamma(K+\lambda+1)}C_{2K+1}^{\lambda}(x)/x \quad \text{(A1)}$$

to obtain

$$f_{[0]}(z,z',\cos 2\delta)dz'=\frac{1}{\sqrt{\pi}}(1-x'^2)^{\lambda-1/2}x'dx'\frac{1}{\cos\delta\cos\phi}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}$$
$$\times\sum_{(n\ \text{odd})}\left[\frac{n!}{\Gamma(n+2\lambda)}\frac{\sqrt{\pi}\Gamma(2\lambda)\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}C_n^{\lambda}(\cos\phi)C_n^{\lambda}(\cos\delta)\right]C_n^{\lambda}(x')/h_n^{\lambda}\,, \quad \text{(A2)}$$

where $z=\cos 2\phi$, $x=\cos\phi$, $\lambda=\alpha+\frac{1}{2}=D/2-2$, $-1\le x'\le 1$ and $z'=2x'^2-1$ while h_n^{λ} is the normalization constant of $C_n^{\lambda}(x)$. We note that as $(1-x'^2)^{\lambda-1/2}$ is the weight function associated with $C_n^{\lambda}(x')$ and $P(z',r)$ is an even function of x' any $C_n^{\lambda}(x')$ in Eq. (A2) with even n will not contribute to the integral in Eq. (4.26) for $-1\le x'\le 1$. Therefore, the sum over n can be extended to all integer values of n.

Equation (A2) can be simplified by means of the addition theorem for Gegenbauer polynomials. The expression in brackets can be written as the integral

$$\int_0^{\pi}C_n^{\lambda}(\cos\Theta)(\sin\varphi)^{2\lambda-1}d\varphi\,,$$

where $\cos\Theta=\cos\phi\cos\delta+\sin\phi\sin\delta\cos\varphi$, leading to the expression:

$$f_{[0]}(z,z',\cos 2\delta)dz'=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}(1-x'^2)^{\lambda-1/2}\cos\delta\cos\phi x'dx'\int_0^{\pi}\left[\sum_n C_n^{\lambda}(\cos\Theta)C_n^{\lambda}(x')/h_n^{\lambda}\right](\sin\varphi)^{2\lambda-1}d\varphi\,. \quad \text{(A3)}$$

After substitution of Eq. (A3) into Eq. (4.26) we exchange the order of integration of x' and φ and note that we can employ the relation

$$(1-x'^2)^{\lambda-1/2}\sum_n C_n^\lambda(\cos\Theta)C_n^\lambda(x')/h_n^\lambda=\delta(x'-\cos\Theta) \tag{A4}$$

to obtain

$$\int_{-1}^{1} f_{[0]}(z,z',\cos 2\delta)P(z',r)dz'=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\frac{1}{\cos\delta\cos\phi}\int_0^\pi P(2\cos^2\Theta-1,r)\cos\Theta(\sin\varphi)^{2\lambda-1}d\varphi\ . \tag{A5}$$

This can be further simplified if we substitute the variable $u=\cos\Theta$ for φ. Then

$$\int_{-1}^{1} f_{[0]}(z,z',\cos 2\delta)P(z',r)dz'=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)}\frac{1}{\cos\phi\cos\delta}\left[\frac{1}{\sin\delta\sin\phi}\right]^{2\lambda-1}\int_a^b[(u-a)(b-u)]^{\lambda-1}P(2u^2-1,r)u\,du\ , \tag{A6}$$

where $\cos\phi=\sqrt{(1+z)/2}$, $\sin\phi=\sqrt{(1-z)/2}$, $\lambda=D/2-2$, and $\delta=2\pi/3$, or $\pi/3$ for connected pairs. The limits a and b are given by $a=\cos(\phi+\delta)$ and $b=\cos(\phi-\delta)$. For disconnected pairs we have $\delta=\pi/2$ and $\cos\pi/2=0$. Therefore, we have to take the limit of Eq. (A6) for $\delta\to\pi/2$. One finds

$$\int_{-1}^{1} f_{[0]}(z,z',-1)P(z',r)dz'=\frac{2}{\sqrt{\pi}}\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-1)}(1-z)^{1/2-\lambda}\int_{-1}^{-z}[-(z+z')]^{\lambda-2}\sqrt{1+z'}P(z',r)dz'\ . \tag{A7}$$

Consequently, the projection function of Eq. (4.28) is given analytically by Eq. (A6) with $\delta=\pi/3$ (connected pairs) and Eq. (A7) when $\delta=\pi/2$ (disconnected pairs).

APPENDIX B

1. Numerical treatment

We now give details of the numerical methods employed first for the solution of the integrodifferential equation, Eq. (7.2), in order to obtain the eigenpotential $U_\lambda(r)$, secondly, for the treatment of the competing zeros and apparent singularities appearing in the equation, and thirdly for the mapping $r_{ij}r\sqrt{(1+z)/2}$, which for $z\to-1$ give rise to numerical problems for potentials with $1/r_{ij}$ behavior.

A natural method to solve the integrodifferential equation is to expand the eigenfunctions $P_\lambda(z,r)$ in terms of some interpolating polynomials, thereby reducing the equation to a standard eigenvalue problem. The B splines of order N have been chosen for this purpose. These are nonoscillatory piecewise polynomials with continuous first and second derivatives, up to $(N-2)$. They are known to provide a stable and accurate interpolation. For this purpose, the interval $[-1,+1]$ is divided into M knots $(z_1,z_2,\ldots,z_M)$ with $2N$ additional knots $(z_{-N+1}, z_{-N+2}, z_0)$, and $(z_{M+1}, z_{M+2}, z_{M+N})$ outside this interval. To define and evaluate the B splines, we follow the conventions of Cox[28] and employ the following algorithm:

$$M_j^1=\begin{cases}\dfrac{1}{z_j-z_{j-1}}, & z_{j-1}\le z<z_j\ ,\\ 0, & \text{otherwise}\ ,\end{cases}$$

$$j=i-N+1,\ i-N+2,\ldots,i\ . \tag{B1}$$

Then compute

$$M_j^r(z)=\frac{(z-z_{j-r})M_{j-1}^{r-1}+(z_j-z)M_j^{r-1}}{j^z\bar{j}^z_{-r}}\ ,$$

$$r=2,3,\ldots,N,\quad j=i-N+r,\quad i-N+r+1,\ldots,i\ . \tag{B2}$$

The $B_i^N(z)$ is then simply given by

$$B_i^N(z)=M_i^N(z)\ . \tag{B3}$$

Apart from the B splines one requires also their first and second derivatives. These are discussed in Ref. 29. However, they can easily be obtained by differentiating the above algorithm. Thus, for example,

$$[M_j^r]'=\frac{M_{j-1}^{r-1}M_j^{r-1}}{z_j-z_{j-r}}+\frac{(z-z_{j-r})(M_{j-1}^{r-1})'+(z_j-z)(M_j^{r-1})'}{z_j-z_{j-r}}\ , \tag{B4}$$

with $[M_j^1]=0$. The second derivatives can be obtained in similar fashion starting from $[M_j^r]''=0$. All three algorithms are very stable numerically.

The reduction of the integrodifferential equation to a standard eigenvalue problem can be achieved by using either the collocation method as discussed in Ref. 30 or the Galerkin method. The former is in terms of computing time, more efficient than the latter; however it is sensitive to the choice and density of the knots especially in our case, where we have competing zeros and apparent singularities. Thus the choice of the mesh points is always a problem, especially for the Malfliet-Tjon potential. In contrast, the Galerkin method is not as sensitive to the way the points are chosen. In this method one expands

$$P_\lambda(z,r)=\sum_{n=1}^{M+N}a_n^\lambda(r)B_n^N(z)\ , \tag{B5}$$

and the expansion coefficients $a_n^\lambda(r)$ are evaluated by projecting the whole equation on $B_m^N(z)$, i.e., by writing

$$\int_{-}^{+1} B_m^N(z) P_\lambda(z;r)dz = \sum_{n=1}^{M+N} a_n^\lambda(r) \int_{-1}^{+1} B_m^N(z) B_n^N(z)dz \ ,$$

or (B6)

$$\int_a^b B_m^N(z) P_\lambda(z;r)dz = \sum_{n=1}^{M+N} a_n^\lambda(r) \int_c^d B_m^N(z) B_n^N(z)dz \ ,$$

where the integration intervals are changed from $(-1,+1)$ to (a,b) and (c,d) in which the B_m^N and $B_m^N B_n^N$ are nonzero, respectively. Obviously, the method is not as sensitive to the way the mesh points are chosen as the collocation method.

The second problem we were concerned with was the handling of the critical integrals of the form $\int_{u_-}^{u_+}$, $\int_{-1}^{-z}$, $\int_{-1}^{u_-}$, and $\int_{-1}^{u_+}$, with integrands which have limits of the form 0/0. Fortunately in such cases we obtain finite results. Consider, for example, the integral, (which appears in the case $A=4$)

$$I = \frac{1}{(1-z)^{\lambda-1/2}} \int_{-1}^{-z} \sqrt{1+z'}[-(z+z')]^{\lambda-2} B_n^N(z')dz' \ . \tag{B7}$$

For $A=4$ we have $\lambda=\frac{5}{2}$. Thus, for $z=1-\epsilon$ we obtain

$$I \simeq \frac{1}{\epsilon^2} \int_{-1}^{-1+\epsilon} \sqrt{1+z'}[-(z+z')]^{1/2} \times B_n^N(z')dz' \underset{\epsilon\to 0}{\to} \frac{\pi}{8} B_n^N(-1) \ . \tag{B8}$$

In practice we found that the odd B splines gave better results than the even ones, while $N=7$ appears to be the optimum choice. Convergence with respect to number of knots is obtained with about 15 knots, after which any change in number, density and distribution gives results within ± 0.001 MeV for the soft core and within ± 0.005 MeV for the hard-core potentials when $A=3$. For the UAA the corresponding accuracies are ± 0.005 and ± 0.01 MeV, due to inaccuracies in the derivative. In general, however, fewer mesh points required more integration points for convergence. All integrals can be calculated using Gauss-Legendre quadratures, where 30 integration points and 15 mesh points are sufficient to obtain results within ± 0.001 MeV for the soft core potentials, while for the Malfliet-Tjon potential, these numbers are increased to 40 and 20, respectively.

For $A=4$ we had to use the Gauss-Jacobi quadratures, however. This is a consequence of the appearance in the denominator of terms of the form $(1-z)^\alpha(1+z)^\beta$. In this case we multiply the whole equation by these factors and then use the appropriate Gauss-Jacobi quadratures. The accuracy of the results obtained with 40 integration points and 20 mesh points in the case of the UAA is then better than ± 0.01 MeV for soft-core potentials and ± 0.02 MeV for the hard ones and better than ± 0.01 MeV for the EAA in all cases.

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[1]M. Fabre de la Ripelle, Phys. Lett. **135**, 5 (1984).
[2]J. L. Ballot, M. Fabre de la Ripelle, and J. Navarro, Phys. Lett. **143B**, 19 (1984).
[3]M. Fabre de la Ripelle, C. R. Acad. Sci. Ser. B **299**, 839 (1984).
[4]M. Fabre de la Ripelle and H. Fiedeldey, C. R. Acad. Sci. **302**, 851 (1986).
[5]M. Fabre de la Ripelle, Few-Body Syst. **1**, 181 (1986).
[6]M. Fabre de la Ripelle, Ann. Phys. (N.Y.) **147**, 231 (1983).
[7]F. Zernike and H. C. Brinkman, Proc. K. Ned. Akad. Wet. **33**, 3 (1935).
[8]M. Fabre de la Ripelle, *Models and Methods in Few-body Physics,* Volume 273 of *Lecture Notes in Physics,* edited by L. S. Ferreira, A. C. Fonseca, and L. Streit (Springer-Verlag, New York, 1987), p. 283–323.
[9]M. Fabre de la Ripelle and H. Fiedeldey, Phys. Lett. B **171**, 325 (1986).
[10]M. Fabre de la Ripelle, H. Fiedeldey, and S. A. Sofianos, *Supplement to Research Report,* edited by T. Sasakawa *et al.* (Tohoku University Press, Sendai, 1986), Vol. 19, p. 360.
[11]M. Fabre de la Ripelle and H. Fiedeldey, C. R. Acad. Sci. Ser. B **302**, 59 (1986).
[12]J. L. Ballot, M. Fabre de la Ripelle, and J. S. Levinger, Phys. Rev. C **26**, 2301 (1982).
[13]J. H. Macek, J. Phys. B **1**, 831 (1968).
[14]T. K. Das, H. T. Coelho, and M. Fabre de la Ripelle, Phys. Rev. C **26**, 2281 (1982).
[15]M. Fabre de la Ripelle, C. R. Acad. Sci. **244B**, 104 (1972).
[16]A. B. Volkov, Nucl. Phys. **74**, 33 (1965).
[17]D. Gogny, P. Pires, and R. de Tourreil, Phys. Lett. **32B**, 591 (1970).
[18]H. Eikemeier and H. H. Hackenbroich, Z. Phys. **195**, 412 (1966).
[19]I. R. Afnan and Y. C. Tang, Phys. Rev. A **175**, 1337 (1968).
[20]G. Erens, J. L. Visschers, and R. van Wageningen, Ann. Phys. (N.Y.) **67**, 461 (1971).
[21]G. L. Payne, J. L. Friar, B. F. Gibson, and I. R. Afnan, Phys. Rev. C **22**, 823 (1980) (the binding energy is given for *S*-state projected potentials in the Faddeev equation).
[22]J. G. Zabolitzky, Phys. Rev. C **25**, 1111 (1982); and in *Progress in Particle and Nuclear Physics,* edited by A. Faessler (Pergamon, New York, 1986), Vol. 16, p. 103.
[23]G. Erens, J. L. Visschers, R. van Wageningen, Ann. Phys. (N.Y.) **67**, 461 (1971).
[24]S. Fantoni, L. Panattoni, S. Rosati, Nuovo Cimento A **69**, 81 (1970).
[25]J. L. Friar, Few-Body Syst. Suppl. **2**, 51 (1987); private communication.
[26]J. L. Ballot, Z. Phys. A **302**, 347 (1981).
[27]M. Fabre de la Ripelle and J. Navarro, Ann. Phys. (N.Y.) **123**, 185 (1979).
[28]M. G. Cox, J. Inst. Math. Its Appl. **10**, 134–149 (1972).
[29]K. R. Rutterfield, J. Inst. Math. Its Appl. **17**, 15–25 (1976).
[30]H. J. Reinhardt, *Analysis of Approximation Methods for Differential and Integral Equations,* Vol. 57 of *Applied Mathematical Science* (Springer-Verlag, New York, 1985); C. de Boor, *A Practical Guide to Splines* (Springer-Verlag, New York, 1978), No. 27.
[31]Y. Akaishi, in *Models and Methods in Few-Body Physics,* Vol. 273 of *Lecture Notes in Physics,* edited by L. S. Ferreira, A. C. Fonseca, and L. Streit (Springer-Verlay, New York, 1987), p. 324–362.