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Self-consistent retardation in a three-dimensional relativistic equation

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A new technique for approximating solutions of the two-body Bethe-Salpeter equation is presented. Coupled equations for the relative energy dependence and the relative three-momentum dependence of the relativistic T matrix are derived. These equations are solved self-consistently for the Wick-rotated T matrix in a simple model problem and the numerical results are compared with exact as well as usual three-dimensional reduction results.

The Bethe-Salpeter equation[1] is the fundamental equation for the relativistic two-nucleon problem in a meson exchange theory. Its solution depends on both the relative three-momentum and the relative energy variables. Retardation effects are included through the relative energy dependence, since relative energy is conjugate to relative time. Because the equation depends on relative *four*-momentum, it is referred to as a four-dimensional equation. The most commonly used approximation techniques for the Bethe-Salpeter equation involve two steps. First, the interaction (or irreducible kernel) is truncated to some low order in the coupling constant. For example, the so-called ladder Bethe-Salpeter equation truncates the kernel to second-order Feynmann diagrams so that only the one-boson exchange diagram is included. The exact solution to the ladder Bethe-Salpeter equation would then consist of all iterations of this kernel (all ladded diagrams). The second step one usually takes is to approximate the solution of the truncated Bethe-Salpeter equation by making a three-dimensional reduction.[2-4] These reductions use a fixed value of the relative energy variable to reduce the dimensionality of the equation from four to three. Clearly, such an approach limits the types of retardation effects which are included. It is important to distinguish between approximations to the full Bethe-Salpeter equation (a four-dimensional equation) via a truncated Bethe-Salpeter equation (also four-dimensional) and approximation to the truncated Bethe-Salpeter equation by a three-dimensional reduction.

It is possible for one to avoid making the three-dimensional reduction. However, for meson-exchange theories, exact solutions to the truncated Bethe-Salpeter equations have only been obtained for the ladder Bethe-Salpeter equation.[5,6] Unfortunately, the ladder Bethe-Salpeter equation is not necessarily a good approximation to the full Bethe-Salpeter equation. In fact, several authors[2,3] have pointed out that some three-dimensional approximates to the ladder Bethe-Salpeter equation are probably better than the exact solution of the ladder Bethe-Salpeter equation as an approximation to the exact Bethe-Salpeter equation. It should be emphasized that some of the three-dimensional approximations to the ladded Bethe-Salpeter equation are designed to be an approximation to the fourth-order truncated Bethe-Salpeter equation. However, since no exact solutions of higher-order Bethe-Salpeter equations are yet available, one has difficulty choosing the best three-dimensional approximation.[2,3] Thus, higher-order truncated Bethe-Salpeter equations can only be examined with a single class of approximation techniques, all of which limit retardation effects in a similar, restrictive fashion. In this Rapid Communication we examine a new approximation technique which includes retardation effects in a self-consistent fashion. This technique is general enough to be applicable to higher-order truncated Bethe-Salpeter equations. However, as a preliminary study, we present here the results of the application of our technique to the ladder Bethe-Salpeter equation and compare it with exact ladder solutions as well as with usual three-dimensional approximations. We make the comparisons with the three-dimensional approximations not to show their disagreement with the exact ladder results (which they are not necessarily designed to reproduce), but to show the variability of results possible with that method. First, let us present a brief formal description of our technique, then show the results of our numerical study.

The procedure we advocate consists of approximating the Bethe-Salpeter equation with a pair of coupled equations. The first equation is a three-dimensional equation of the form of the usual three-dimensional approximation equations, except that retardation effects in the kernel are included by integration over a relative-energy-dependent factor. The second equation is a one-dimensional equation which determines the relative-energy-dependent factor consistent with the first equation. These two coupled equations are solved self-consistently. Since this technique involves the solution of, at most, three-dimensional equations, some of the difficulties associated with using higher-order kernels can be avoided.

Let us focus on the half-off-shell T matrix. For simplicity, we assume that an uncoupled partial-wave expansion is possible; thus, we write[5] the Bethe-Salpeter equation for the half-off-shell T matrix for a single partial wave as

$$T_{\omega p}=V_{\omega p,0\hat{p}}+4\pi i\int d\omega'\int q^2dq V_{\omega p,\omega' q}G_{\omega' q}T_{\omega' q}\ , \qquad (1)$$

where ω denotes relative energy and p, relative momentum. The interaction V and propagator G will be given explicitly for a model problem later in this paper. The on-shell relative energy is $\omega=0$ and the on-shell relative momentum is defined to be $p\equiv\hat{p}$. The functions in Eq. (1) have an implicit dependence on the square of the total four-momentum s and the orbital angular momentum l. The range for all integrals in ω (or, later, in y) is $[-\infty,+\infty]$, and in q is $[0,+\infty]$, unless otherwise specified. Now let us derive a pair of coupled equations to approximate Eq. (1). We define two functions τ_p and $\tilde{\tau}_\omega$ using some given weight functions as

$$\tau_p\equiv\Omega^{\omega}T_{\omega p}\equiv\int d\omega w_\omega T_{\omega p}$$

and

$$\tilde{\tau}_\omega\equiv\Omega^{p}T_{\omega p}\equiv\int p^2dp\tilde{w}_pT_{\omega p}\ .$$

We can define two projectors in terms of $\tilde{\tau}_\omega$ and τ_p as follows: $\Pi_\omega\equiv\tilde{\tau}_\omega\Omega^{\omega}/t_s$ and $\Pi_p\equiv\tau_p\Omega^{p}/t_s$, where $t_s\equiv\Omega^{\omega}\Omega^{p}T_{\omega p}$ is a number depending only on s through the implicit dependence of $T_{\omega p}$ on s. The projector Π_ω is defined to act on the ω dependence of the function upon which it operates, viz.,

$$\Pi_\omega\mathcal{O}_{\omega p,\omega' q}=\frac{\tilde{\tau}_\omega}{t_s}\int d\omega'' w_{\omega''}\mathcal{O}_{\omega'' p,\omega' q}\ ,$$

with Π_p similarly defined. It is easily shown that $\Pi^2=\Pi$, for either projector. We also note that

$$\Pi_pT_{\omega p}=\Pi_\omega T_{\omega p}=\frac{\tau_p\tilde{\tau}_\omega}{t_s}\ .$$

Thus, either of the projectors when operating on T yields a *factorized approximation* to T, with one factor τ_p depending only on relative momentum and the other factor $\tilde{\tau}_\omega$ depending only on relative energy. Our approximation technique is easily defined in terms of either Π.

If we approximate T by the factorized form ΠT in the integral term of Eq. (1) and then operate on that equation with Ω^{ω} we obtain

$$\tau_p=\Omega^{\omega}V_{\omega p,0\hat{p}}+4\pi i\int q^2dq\left\{\int d\omega'\Omega^{\omega}V_{\omega p,\omega' q}G_{\omega' q}\frac{\tilde{\tau}_{\omega'}}{t_s}\right\}\tau_q\ . \qquad (2a)$$

Likewise, if we operate with Ω^{p}, we obtain

$$\tilde{\tau}_\omega=\Omega^{p}V_{\omega p,0\hat{p}}+4\pi i\int d\omega'\left\{\int q^2dq\,\Omega^{p}V_{\omega p,\omega' q}G_{\omega' q}\frac{\tau_q}{t_s}\right\}\tilde{\tau}_{\omega'}\ . \qquad (2b)$$

These are the two coupled integral equations for τ_p and $\tilde{\tau}_\omega$ which define our approximation technique in the simplest possible situation. Note that these equations can be made manifestly Lorentz invariant by writing the center-of-mass relative energy, $\omega_{\rm cm}$, and the magnitude of the center-of-mass relative three-momentum, $p_{\rm cm}$, in terms of Lorentz scalars as follows: $\omega_{\rm cm}=P^{({\rm tot})\mu}p_\mu^{({\rm rel})}/\sqrt{s}$ and $p_{\rm cm}^2=\omega_{\rm cm}^2-p^{({\rm rel})\mu}p_\mu^{({\rm rel})}$, where $P^{({\rm tot})}$ and $p^{({\rm rel})}$ are the total and relative momentum four-vectors, respectively.

Before discussing the model problem, let us examine the structure of the approximate equations [Eqs. (2)]. Equation (2a) is a one-dimensional integral equation for the relative momentum dependence of the T matrix. It has a kernel (enclosed in curly brackets) that depends only on the magnitude of the three-momentum and, in this sense, has the form of a usual three-dimensional reduction of the Bethe-Salpeter equation. However, the kernel of this equation is obtained by integration over the relative-energy-dependent factor $\tilde{\tau}_\omega$. Thus, relative energy (retardation) effects are included in a self-consistent fashion. For the usual three-dimensional approximations the relative energy is fixed. Equation (2b) is an equation for the relative-energy-dependent factor in which relative momentum (retardation in space) effects are included in a similar self-consistent fashion. The usual three-dimensional approximations have no equation corresponding to Eq. (2b).

As a first application of our technique, we examine the ladder Bethe-Salpeter equation for two scalar "nucleons" interacting via the exchange of a scalar "meson."[5] For this problem the interaction and propagator for S waves are

$$V_{\omega p,\omega' q}=\frac{\lambda}{2\pi^2pq}Q_0(x)\ ,$$

where

$$x\equiv[p^2+q^2-(\omega-\omega')^2+\mu^2-i\epsilon]/2pq$$

and

$$G_{\omega q}=(-1)[(\sqrt{s}/2+\omega)^2-E_q^2+i\epsilon]^{-1}\times[(\sqrt{s}/2-\omega)^2-E_q^2+i\epsilon]^{-1}\ ,$$

where $E_q^2=q^2+m^2$ and Q_0 is the zeroth-order Legendre function of the second kind. Note that $\sqrt{s}\equiv2E_{\hat{p}}$. This problem is sufficiently simple to allow us to solve it exactly by performing a Wick rotation to imaginary relative energy and solving a two-dimensional integral equation.

Details of the Wick rotation procedure[7] can be found in the literature.[5] Let us briefly describe the procedure and final results here. To perform a Wick rotation, a contour in the complex ω' plane is added to the ω' integral in the original Bethe-Salpeter equation. The new contour is chosen so that (1) it exactly cancels the contribution of the original contour along the real ω' axis, (2) it adds a contribution along the imaginary ω' axis, and (3) it is deformed around all singularities of the integrand so that it makes zero net contribution to the integral. This contour closes at infinity in the first and third quadrants. After the Wick rotation, the Bethe-Salpeter equation, Eq. (1), (evaluated at $\omega=\xi$) takes the form

$$T_{\xi p}=V_{\xi p,0\hat{p}}-\frac{2\pi^2}{\sqrt{s}}\int_0^{\hat{p}}q^2dq(V_{\xi p,\omega_q q}+V_{\xi p,-\omega_q q})\frac{T_{\omega_q q}}{E_q\omega_q}-4\pi^2\int dy\int q^2dqV_{\xi p,i\cdot yq}G_{i\cdot yq}T_{i\cdot yq}\ , \qquad (3)$$

where $\omega_q \equiv \sqrt{s}/2 - E_q + i\epsilon$. The one-dimensional integral term in Eq. (3) comes from the residues of the poles of the propagator and the two-dimensional integral term comes from the contribution of the contour on the imaginary ω' axis. A pair of coupled linear integral equations is obtained by evaluating Eq. (3) at $\xi = \omega_p$ and $\xi = i \cdot z$. For the one-boson exchange interaction Eq. (3) no longer has any singularities due to the interaction. The remaining pinch singularity at the fully on-shell point can be removed by performing a Kowlaski-Noyes subtraction[8,9] leaving one with nonsingular Fredholm equations which can be solved numerically. Let us now discuss the numerical results.

We solve the model problem in three different ways.

(1) We solve the ladder Bethe-Salpeter equation exactly (to within controllable numerical uncertainties) by solving the Wick-rotated equation [Eq. (3)]. Those results will be referred to as the *exact ladder* results.

(2) We use our self-consistent retardation approximation technique [Eqs. (2)] as it applies to the Wick-rotated Bethe-Salpeter equation. We choose the weight functions in the $(p, i\cdot z)$ space to be delta functions at the on-shell points, i.e., $w_{i\cdot z} = \delta(z)$ and $\tilde{w}_p = \delta(p - \hat{p})$, so that elastic unitarity is satisfied.[10] For definiteness, the equations for τ_p and $\tilde{\tau}_{i\cdot z}$ are

$$\tau_p = V_{0p,0\hat{p}} - \frac{2\pi^2}{\sqrt{s}} \int_0^{\hat{p}} q^2 dq (V_{0p,\omega_q q} + V_{0p,-\omega_q q}) \frac{T_{\omega_q q}}{E_q \omega_q} - 4\pi^2 \int q^2 dq \left\{ \int dy V_{0p,i\cdot yq} G_{i\cdot yq} \frac{\tilde{\tau}_{i\cdot y}}{t_s} \right\} \tau_q$$

and

$$\tilde{\tau}_{i\cdot z} = V_{i\cdot z\hat{p},0\hat{p}} - \frac{2\pi^2}{\sqrt{s}} \int_0^{\hat{p}} q^2 dq [V_{i\cdot z\hat{p},\omega_q q} + V_{i\cdot z\hat{p},-\omega_q q}] \frac{T_{\omega_q q}}{E_q \omega_q} - 4\pi^2 \int dy \left\{ \int q^2 dq V_{i\cdot z\hat{p},i\cdot yq} G_{i\cdot yq} \frac{\tau_q}{t_s} \right\} \tilde{\tau}_{i\cdot y} .$$

The equation for $T_{\omega_p p}$ is given by Eq. (3) with $\xi = \omega_p$ and $T_{i\cdot yq} \approx \tau_q \tilde{\tau}_{i\cdot y}/t_s$. This comprises three coupled *one-dimensional* integral equations. Because they are nonlinear, we solve them iteratively and find that convergence is extremely fast for all parameter sets examined. These results are referred to as the *factorized approximation*, since the Wick-rotated T matrix is approximated by a factorized form.

(3) Last, we solve the model problem using usual three-dimensional reductions of the ladder Bethe-Salpeter equation. We calculated phase shifts using the same *six* different three-dimensional approximations of the usual kind which were compared in Ref. 2. These included the Gross equation, the equation of Erkelenz and Holinde, the Kadyshevksy equation, and the Blenkenbecler-Sugar equation. However, we show here only the largest and smallest phase shifts from those six. The largest phase shifts were given by the Gross equation in every case and are labeled by A in Fig. 1, the smallest phase shifts were given by an equation which is similar to the Kadyshevsky equation and, following the notation of Ref. 2, is labeled by F. These equations are solved in the usual way,

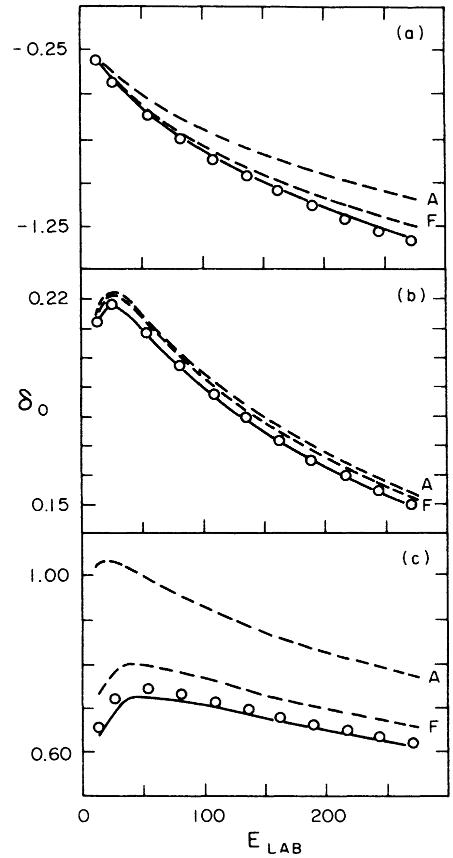

FIG. 1. S-wave phase shifts δ_0 in radians vs lab energy E_{LAB} in MeV with (a) $\mu = 4.9$ fm^{-1} and $\lambda = -338.27$ fm^{-2}, (b) $\mu = 0.7$ fm^{-1} and $4\pi\sqrt{\lambda} = 9.2$ fm^{-1}, and (c) $\mu = 2.1$ fm^{-1} and $4\pi\sqrt{\lambda} = 27.6$ fm^{-1}. See text for legend.

without performing any further iterations to find effective interactions. (See Ref. 2 for exact definitions as well as for references to the original work.) We show these results to emphasize the variability possible with three-dimensional approximations.

Our numerical results for three different parameter sets are shown in Figs. 1(a), 1(b), and 1(c). In all three plots, the S-wave phase shifts δ_0 in radians are plotted versus the laboratory energy, E_{LAB}, in MeV. We used a nucleon mass of $m = 4.758$ fm^{-1}. The meson mass μ and coupling constant λ differ for each parameter set and are listed in the figure caption. In each plot the exact ladder results are represented by open circles; the three-dimensional approximation results are represented by the labeled dashed lines. Our factorized approximation results are indicated by the solid line. It can be seen that for the three parameter sets examined the factorized approximation results are very good at reproducing the exact ladder results. For comparison, one can see that the difference between the factorized approximation and the exact ladder results is small compared to the differences between the various three-dimensional approximation results. This agreement is due to the improved handling of retardation effects. Note that four other three-dimensional reductions were calculated and that those results all lie between the results labeled A and F. The parameter sets examined here correspond to three parameter sets which were previously examined in Refs. 2 and 11.

As we have just seen, the factorized approximation results agree very well with the exact ladder results. This represents an important development since our approximation scheme can, in principle, be applied to higher-order Bethe-Salpeter equations as well. Thus, in the future one should be able to examine retardation effects in the fourth-order Bethe-Salpeter equation in a self-consistent approximation scheme. We are presently applying our factorized approximation to the non-Wick-rotated ladder Bethe-Salpeter equation and examining applications to higher-order equations. That work will be described in detail elsewhere.

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[10]One can show that *at the on-shell point* $\Pi T_{0\hat{p}} = T_{0\hat{p}}$, for either Π. Since the elastic unitarity relation involves only the on-shell T-matrix elements, one can replace $T_{0\hat{p}}$ by $\Pi T_{0\hat{p}}$ in that relation and prove that $\Pi T_{0\hat{p}}$ also satisfies elastic unitarity. If one uses delta functions at the on-shell points for weight functions for the non-Wick-rotated Bethe-Salpeter equation, then elastic unitarity is satisfied in that case as well. However, in that case, the $\tilde{\tau}_\omega$ equation has singularities in the region of integration. Thus, nonsingular weight functions obeying a unitarity constraint should be used to smooth the singularities for the non-Wick-rotated Bethe-Salpeter equation. Since our model problem equations satisfy unitarity, a general discussion of the unitarity constraint will not be included in this paper.

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