Interaction strengths in the interacting boson-fermion model

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Although a number of different types of boson-fermion interactions are possible in the interacting boson-fermion model, empirical results have shown that the exchange term is of cardinal importance. A recently constructed similarity transformation which reexpresses the Dyson boson images of the single-j shell fermion operators in terms of seniority bosons is used to confirm that the origin of the exchange term in spherical nuclei is linked to the quadrupole pairing interaction between identical nucleons. The relative importance of the exchange and direct boson-fermion interactions is also discussed and it is indicated that the direct term is essential for a complete description of that excited state which carries the j value of the shell.

I. INTRODUCTION

The interacting boson model (IBM) provides a successful description of collective states in even-even nuclei in terms of $J=0^+$ (s) and $J=2^+$ (d) bosons.[1] Furthermore the model can be extended to describe odd-A nuclei by coupling an odd nucleon to the system of bosons.[2]

In its most general form the boson-fermion interaction which appears in this interacting boson-fermion model (IBFM) contains a large number of free parameters which complicates a meaningful phenomenological analysis of observed spectra. Iachello and Scholten[3] have subsequently illustrated that a simplified version of the boson-fermion interaction, in which an exchange term plays a crucial role, is sufficient to describe a variety of observed odd-A spectra. For the case where the odd fermion is considered to be restricted to a single-j shell, this exchange term is taken to be[3]

$$-\frac{1}{\sqrt{2\Omega}}\Lambda{:}[(d^{\dagger}\tilde{a}_j)^j(a^j\tilde{d})^j]^{(0)}{:}\ , \tag{1}$$

where Λ denotes the coupling strength. Here $2\Omega=2j+1$ while normal ordering (::) has been introduced. The d boson and odd fermion creation (modified annihilation) operators are denoted by d^{μ} $[\tilde{d}_{\mu}\equiv(-1)^{\mu}d_{-\mu}]$ and $a^{jm}[\tilde{a}_{jm}\equiv(-1)^{j-m}a_{j-m}]$, respectively.

Kaup *et al.*[4] have tested the simplified version of the IBFM in a region where the boson part of the Hamiltonian possesses $U(5)$ dynamical symmetry to a good approximation (i.e., for approximately vibrational nuclei). Using a single-j shell approximation, in which the exchange term appears as in Eq. (1), they determined Λ to be about 10 MeV (see also Ref. 5). The initial attempts[6,7] to derive this strength of the exchange term from an underlying fermion-fermion interaction (by focusing on proton-neutron interactions) have not been able to yield quantitatively the above empirical value of Λ. Recently, however, Otsuka *et al.*[8] have illustrated that the origin of the exchange term in spherical nuclei can be traced quantitatively to the fermion quadrupole pairing interaction between identical nucleons. Although not elaborated at the time, this possibility was already pointed out by Geyer and Hahne.[9]

We note that in the single-j shell approximation given above, the exchange term is characterized by the fact that the boson operators and odd fermion operators are only coupled to the angular momentum of the chosen j shell.[3,4] Although a cursory reference to a similar structure in nuclear field theory (NFT) can be found in Ref. 3, no thorough motivation for the neglect of coupling to the other allowed angular momentum values has yet been given, either within the framework of IBFM or NFT. In this paper a satisfactory resolution is presented, at least for the specific fermion-fermion interaction considered. It should be mentioned that for $j=\frac{3}{2}$, the above-mentioned restricted coupling leads to a spin (6) symmetry as discussed by Iachello and Kuyucak.[10] The transition from fermion microscopy to boson phenomenology requires a transcription from fermion to boson space and the identification of the appropriate collective bosons. The actual mapping procedure we prefer, namely the generalized Dyson boson mapping (DBM), characterized by finite boson images, has been demonstrated[9,11–15] to be a very efficient tool for investigating boson models directly linked to fermion microscopy. However, the direct application of the DBM leads to a description in terms of pair bosons, which in general presents an unfavorable point of departure when a truncated description in terms of the physically relevant bosons, such as s and d bosons of the IBM is considered.[11]

Among other considerations, the numerical results given by Otsuka, Arima, and Iachello (OAI)[16] and by Halse[17] seem to indicate that an association between (fermion) seniority and the number of non-s bosons presents a more favorable point of departure as far as the identification of the "physical" bosons is concerned. (This seems to be the case for vibrational-type situations, at least. Halse's results seem to indicate that the seniority association has wider validity, but this requires further investigation.)

For even-A nuclei the OAI formalism[16] involves a Marumori-type mapping of fermion seniority states onto the IBM bosons states, from which the boson images of fermion operators are subsequently constructed by requiring equality between boson matrix elements and the corresponding fermion matrix elements. The mapping of the states in the OAI scheme implies that the seniority in the boson space should be associated with twice the number of non-s bosons. This requirement has recently led to the construction of a similarity transformation[14] which reexpresses the Dyson pair boson images of bifermion operators in terms of the physical seniority bosons.

The seniority transformation of Ref. 14 can simply be extended to odd-A nuclei by associating seniority with the sum of twice the number of non-s bosons and the number of unpaired fermions. This leads to a structure analogous to IBFM where basis states are most conveniently described in the weak coupling scheme.[18] After a short introduction of the DBM we discuss the above construction in Sec. II. In Sec. III we show the results when the formalism is applied to the monopole pairing plus quadrupole pairing Hamiltonian previously discussed by Otsuka *et al.*[8], while Sec. IV contains some concluding remarks.

II. DYSON MAPPING AND CONSTRUCTION OF THE SENIORITY TRANSFORMATION

The generalized DBM for an odd system is defined by the mapping[9]

$$
\begin{aligned}
b^{\alpha\beta} &\equiv c^{\alpha}c^{\beta} \\
&\to R^{\alpha\beta} \\
&\equiv B^{\alpha\beta} - B^{\alpha\theta}B^{\beta\rho}B_{\theta\rho} - B^{\alpha\lambda}a^{\beta}a_{\lambda} - B^{\lambda\beta}a^{\alpha}a_{\lambda} \,, \\
b_{\alpha\beta} &\equiv c_{\beta}c_{\alpha} \to R_{\alpha\beta} \equiv B_{\alpha\beta} \,, \\
b^{\alpha}_{\ \beta} &\equiv c^{\alpha}c_{\beta} \to R^{\alpha}_{\ \beta} \equiv B^{\alpha\theta}B_{\beta\theta} + a^{\alpha}a_{\beta} \,,
\end{aligned}
\tag{2}
$$

between the bifermion operators b and the ideal boson and fermion operators, denoted by B and a, respectively. Here we have used a summation convention over repeated indices which represent the shell-model quantum numbers jm. [Creation (annihilation) operators are distinguished by superscripts (subscripts), e.g., the fermion creation (annihilation) operators in the original space are denoted by c^{α} (c_{α}).] The ideal fermion operators satisfy the usual fermion anticommutation relations and are defined to commute with the ideal bosons which satisfy the boson algebra

$$[B_{\alpha\beta}, B^{\mu\nu}] = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha} \,, \tag{3}$$

$$[B_{\alpha\beta}, B_{\mu\nu}] = [B^{\alpha\beta}, B^{\mu\nu}] = 0 \,, \tag{4}$$

$$[B_{\alpha\beta}]^{\dagger} = B^{\alpha\beta} \,, \tag{5}$$

$$B^{\alpha\beta} = -B^{\beta\alpha} \,. \tag{6}$$

Ideal fermions represent the unpaired fermions in the IBFM and by construction the operators R constitute a complete realization of the bifermion algebra.

For a single-j shell it is convenient to introduce spherical pair boson operators

$$B^{JM} = (1+\delta_{j_1 j_2})^{-1/2} \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 \mid JM \rangle B^{j_1 m_1 j_2 m_2} \,. \tag{7}$$

From the generalized DBM and the transformation to spherical pair bosons the pair boson images of the spherical fermion operators

$$
\begin{aligned}
A^{JM} &= \frac{1}{\sqrt{2}}[c^{j}c^{j}]^{J}_{M} \,, \\
\tilde{A}_{JM} &= -\frac{1}{\sqrt{2}}[\tilde{c}_{j}\tilde{c}_{j}]^{J}_{M} \quad (\tilde{A}_{JM} = (-1)^{J-M}[A^{J-M}]^{\dagger}) \,, \\
U^{J}_{M} &= [c^{j}\tilde{c}_{j}]^{J}_{M} \,,
\end{aligned}
\tag{8}
$$

[with $\tilde{c}_{jm} = (-1)^{j-m}c_{j-m}$] can be obtained by using standard angular momentum coupling. These images are

$$(A^{JM})_D = B^{JM} - 2\sum_{J_1 J_2 J_3 L} \hat{J}_1\hat{J}_2\hat{J}_3\hat{L} \begin{Bmatrix} j & J_1 & j \\ j & J_2 & j \\ J_3 & L & J \end{Bmatrix} [(B^{J_1}B^{J_2})^{L}\tilde{B}_{J_3}]^{J}_{M} + 2\sum_{J'j'} (-1)^{j'+j}\hat{J}'\hat{j}' \begin{Bmatrix} j & j & J' \\ j & j' & J \end{Bmatrix} [(B^{J'}a^{j})^{j'}\tilde{a}_{j}]^{J}_{M} \,, \tag{9}$$

$$(A_{JM})_D = B_{JM} \,, \tag{10}$$

$$(U^{J}_{M})_D = 2\sum_{J_1 J_2} (-1)^{J}\hat{J}_1\hat{J}_2 \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix} [B^{J_1}\tilde{B}_{J_2}]^{J}_{M} + [a^{j}\tilde{a}_{j}]^{J}_{M} \,. \tag{11}$$

Here the standard notation $\hat{J} = \sqrt{2J+1}$ is used.

We define the s-, d-, g-boson creation (annihilation) operators as $s^{\dagger}(s) \equiv B^{00}(B_{00})$, $d^{\mu}(d_{\mu}) \equiv B^{2\mu}(B_{2\mu})$, $g^{\mu}(g_{\mu}) \equiv B^{4\mu}(B_{4\mu})$, etc. The pair boson image of the monopole pairing interaction $H^{F}_{0} = -G_0\Omega A^{00}A_{00}$, is of particular importance in obtaining the seniority transformation. From expressions (6) and (7) we obtain the pair boson image H^{D}_{0} of H^{F}_{0} as

$$H^{D}_{0} = H_0 + W \,, \tag{12}$$

with

$$H_0=-G_0\left[\Omega n_s-n_s(n_s-1)-n_s\left[2\sum_{J=1}^{\Omega-1}n_{2J}+n_a\right]\right], \tag{13}$$

$$W=G_0{\sum_J}'B^J\cdot B^J ss-G_0\sqrt{2\Omega}{\sum_J}'\hat{J}[(B^Ja^j)^j\tilde{a}_j]^0 s-G_0\sqrt{2\Omega}\sum_{J_1J_2J_3}{}'\hat{J}_1\hat{J}_2\hat{J}_3\begin{Bmatrix}J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}[(B^{J_1}B^{J_2})^{J_3}\tilde{B}_{J_3}]^0 s\ . \tag{14}$$

Here n_s and $n_{2J=2}\equiv n_d$ denote boson number operators while n_a denotes the fermion number operator. The primed summation implies a summation over J values other than zero. We note that when H_0^D is represented in an ideal boson basis, its matrix representation is of triangular form. [In Ref. 13 it has been proved that the use of a (complete) ideal boson basis in conjunction with a mapped operator, such as H_0^D, is always permissible.] This means that H_0^D has the same spectrum as its diagonal part H_0. However, when the ideal space equivalent of seniority is (naturally) chosen as $v=2\sum' n_{2J}+n_a$, then H_0^D, in contrast with its fermion counterpart H_0^F does not conserve seniority. In order to retain the above ideal space association for the seniority and have a boson counterpart of H_0^F which conserves this seniority, a similarity transformation is needed which will transform away the seniority breaking part W of H_0^D. This similarity transformation, denoted by Z, should therefore satisfy

$$Z(H_0+W)Z^{-1}=H_0\ . \tag{15}$$

Operating from the left with Z^{-1}, and reordering, one has

$$[Z^{-1},H_0]=WZ^{-1}\ . \tag{16}$$

The solution of this commutator equation is given by (see Appendix A)

$$Z^{-1}\equiv\sum_{k=0}^{\infty}\left[\frac{1}{\hat{H}_0-H_0}W\right]^k\wedge\ , \tag{17}$$

where a positional operator $\hat{H}_0$ is introduced.[15] The position of the hat indicates where in an expression a number-dependent hat operator is to be evaluated.

By construction Z is therefore the seniority transformation which transforms the pair bosons into the seniority (physical) bosons (of the vibrational limit) of the IBM. (When states with more than one d-boson are involved, one actually needs a further similarity transformation to retain the precise association between seniority and non-s bosons. Since we deal here with states with one d boson at most, a discussion of this further transformation is postponed to a sequel paper.) From the structure of Z^{-1} it can be inferred that the inclusion of terms with $k>1$ in the sum will only introduce higher-order multiboson contributions to the seniority images of bifermion operators without changing the structure of the lower-order contributions which resulted from $k=0$ and $k=1$. (When the similarity transformation is applied to many-body operators such as the Hamiltonian, this observation does not necessarily hold and terms with $k>1$ may contribute to terms of the same order as those generated by the $k=0$ and $k=1$ terms. In the present discussion, however, this aspect is not crucial, since we eventually focus on states with one boson at most.) In what follows, we adopt the lowest-order approximation to Z^{-1} which retains only the $k=0$ and $k=1$ terms:

$$Z^{-1}\approx 1-Z_1+Z_2+Z_3\ , \tag{18}$$

where

$$Z_1=\frac{1}{2}\frac{1}{\Omega+3-2N+2n_s-n_a}{\sum_J}'(B^J\cdot B^J)ss\ , \tag{19}$$

$$Z_2=\frac{\sqrt{2\Omega}}{\Omega+2-2N+2n_s-n_a}{\sum_J}'\hat{J}[(B^Ja^j)^j\tilde{a}_j]^0 s\ , \tag{20}$$

$$Z_3=\frac{\sqrt{2\Omega}}{\Omega+2-2N+2n_s-n_a}\sum_{J_1J_2J_3}{}'\hat{J}_1\hat{J}_2\hat{J}_3\begin{Bmatrix}J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}\times[(B^{J_1}B^{J_2})^{J_3}\tilde{B}_{J_3}]^0 s \tag{21}$$

with $N=\sum' n_{2J}+n_s$. In the same approximation Z and the seniority image of a pair boson operator Θ are given, respectively, by

$$Z\approx 1+Z_1-Z_2-Z_3 \tag{22}$$

and

$$\Theta_s\equiv Z\Theta Z^{-1}\approx\Theta-[\Theta,Z_1]+[\Theta,Z_2]+[\Theta,Z_3]\ . \tag{23}$$

III. APPLICATION AND RESULTS

We now turn to the seniority image of the quadrupole pairing Hamiltonian

$$H^Q=-G(A^2\cdot\tilde{A}_2) \tag{24}$$

which has the pair boson image

$$H_B^Q=-G(d^\dagger\cdot\tilde{d})+2G\sum_{J_1J_2J_3L}\hat{J}_1\hat{J}_2\hat{J}_3\hat{L}\begin{Bmatrix}j & J_1 & j\\ j & J_2 & j\\ J_3 & L & 2\end{Bmatrix}\{[(B^{J_1}B^{J_2})^L\tilde{B}_{J_3}]^2\cdot\tilde{d}\}-2G\sum_{Jj'}(-1)^{j'+j}\hat{J}\hat{j}'\begin{Bmatrix}j & j & J\\ j & j' & 2\end{Bmatrix}\{[(B^Ja^j)^{j'}\tilde{a}_j]^2\cdot\tilde{d}\}\ . \tag{25}$$

G is taken to be 1.043 MeV as used by Otsuka *et al.*[8]

The seniority image of H_B^Q is now obtained by normal ordering the creation and annihilation operators appearing in the various terms of Eq. (23) and recoupling the angular momentum of the resulting terms. Number operators that appear are absorbed into number-dependent coefficients. We only retain those terms which, without their number-dependent coefficient, are one- or two-body operators. We then truncate to s and d bosons, i.e., retain those terms comprised of only s- and d-boson creation or annihilation operators with coefficients which may depend on n_s, n_d, or $N=n_s+n_d$. The seniority image of H_B^Q is then finally obtained as

$$H_s^Q=H_s^B+H_s^{BF} \tag{26}$$

with

$$H_s^B=\epsilon_s(s^\dagger s)+\epsilon_d(d^\dagger\cdot\tilde{d})+\sum_{J=0,2,4}c_J[(d^\dagger d^\dagger)^J(\tilde{d}\tilde{d})^J]^{00}+v_0(d^\dagger\cdot d^\dagger)ss+v_0's^\dagger s^\dagger(\tilde{d}\tilde{d})$$
$$+v_2[(d^\dagger d^\dagger)^2(\tilde{d}s)^2]^{00}+v_2'[(d^\dagger s^\dagger)^2(\tilde{d}\tilde{d})^2]^{00} \tag{27}$$

and

$$H_s^{BF}=A_0[(s^\dagger s)(a^j\tilde{a}_j)^0]^{00}+\Gamma[(s^\dagger\tilde{d})^2(a^j\tilde{a}_j)^2]^{00}+\Gamma'[(d^\dagger s)^2(a^j\tilde{a}_j)^2]^{00}+A_2[(d^\dagger\tilde{d})^2(a^j\tilde{a}_j)^2]^{00}+V_{\rm ex}+V_{\rm dir}\,. \tag{28}$$

Here $V_{\rm ex}$ denotes the exchange term which has been obtained microscopically from the fermion quadrupole pairing interaction (there is no contribution to $V_{\rm ex}$ from the monopole pairing interaction) and is given by

$$V_{\rm ex}=-\frac{10}{\sqrt{2\Omega}}G\left[\frac{\Omega-N-n_d-n_a}{\Omega-2n_d-n_a}\right]:[(d^\dagger\tilde{a}_j)^j(a^j\tilde{d})^j]^{00}:\,. \tag{29}$$

The final term of H_s^{BF} is a direct term which can be rewritten as an exchange term where the boson and fermion operators are coupled to angular momenta other than only the angular momentum of the j shell (the possible role of such terms was also pointed out in Ref. 19). This direct term is given by

$$V_{\rm dir}=10G\sum_{j'}(-1)^{j'+j}\hat{j}'\left[\frac{1}{2\Omega}\left[\frac{N-n_d+1}{\Omega+2-2n_d-n_a}\right]\delta_{j,j'}+18\left[\frac{N-n_d}{\Omega-2n_d-n_a}\right]\begin{Bmatrix}2&2&4\\ j&j&j\end{Bmatrix}\begin{Bmatrix}2&2&4\\ j&j&j'\end{Bmatrix}\right][(d^\dagger a^j)^{j'}(\tilde{d}\tilde{a}_j)^{j'}]^{00} \tag{30}$$

$$=10G\sum_{j'}(-1)^{j'+j}\hat{j}'\left[\left[\frac{N-n_d+1}{\Omega+2-2n_d-n_a}\right]\begin{Bmatrix}j&2&j\\ j&2&j'\end{Bmatrix}+18(-1)^{j'+j}\left[\frac{N-n_d}{\Omega-2n_d-n_a}\right]\begin{Bmatrix}2&2&4\\ j&j&j\end{Bmatrix}\begin{Bmatrix}2&2&4\\ j&j&j'\end{Bmatrix}\right]$$
$$\times:[(d^\dagger\tilde{a}_j)^{j'}(a^j\tilde{d})^{j'}]^{00}:\,. \tag{31}$$

The (number-dependent) coefficients in Eqs. (27) and (28) are given in Appendix C.

In order to illustrate the role of the direct term (31), we compare for ${}^{93}_{43}$Tc the spectrum of the interaction H_s^{BF} within our formalism with the one obtained in Ref. 8 for states with one d boson. An inspection of the matrix elements of the terms in H_s^Q reveals that the only terms which could contribute to the matrix elements of the states $|dj;J)$ (which are considered in Ref. 8), are those in the combination

$$\epsilon_d(d^\dagger\cdot\tilde{d})+V_{\rm ex}+V_{\rm dir}\,. \tag{32}$$

Since

$$(dj;J\,|\,V_{\rm dir}\,|\,dj;J)=\frac{5G}{\Omega(\Omega-1)}\delta_{J,j}\,, \tag{33}$$

it follows that only the first two terms in Eq. (32) can contribute when $J\neq j$ and

$$(dj;J\neq j\,|\,H_s^Q\,|\,dj;J\neq j)=-G+10GW(j22j;jJ) \tag{34}$$

which is exactly the same as the result obtained by Otsuka *et al.*[8] Here, however, the similarity transformation yields, without any additional considerations, a direct term $V_{\rm dir}$ in addition to the exchange term which is the only one mentioned in Ref. 8. [The term H_3^{BF} mentioned in Ref. 8 can only couple states with respectively one s and one d boson. It has been indicated[20] that a direct term can also be obtained in the OAI formalism. In this case the simple result (33) is not immediately obvious. Furthermore the special role of $V_{\rm dir}$, to be discussed presently, is not mentioned at all in Ref. 8.]

A comparison of the result (34) with a matrix element of the phenomenological exchange term (1) now clearly implies the relationship $\Lambda=10G$. Taking $G=1.043$ MeV from a fit[8] to ${}^{92}_{42}$Mo, this leads to $\Lambda=10.43$ MeV, which compares favorably with the strength of about 10 MeV required by phenomenological fits[4] (as already observed in Ref. 8).

A diagonalization of $H_s^Q+H_0$, expressions (26) and (13) with the strength parameters of Ref. 8, in the $j=\frac{9}{2}^+$ subspace of one boson plus one fermion yields the ${}^{93}_{43}$Tc spectrum shown in Fig. 1(a). This is identical to the spectrum given by Otsuka *et al.*[8] in their Fig. 1(b). If the direct term $V_{\rm dir}$ is *not* included, the resulting spectrum is shown in Fig. 1(c). As expected from expression (33), it is only the excited $j=\frac{9}{2}^+$ state which is significantly influenced [the matrix element in Eq. (33) contributes about 0.25 MeV]. (It is therefore at present not quite clear how the spectrum given in Ref. 8 could have been obtained without a direct term, as seems to be implied by the discussion given by Otsuka *et al.*[8]) For comparison we also show in Fig. 1(d) the spectrum obtained by retaining $k=2$ terms in Z^{-1}. For states with one d boson at most, terms with $k>2$ will have no influence and, as can easily be checked, Fig. 1(d) therefore also corresponds to the exact shell-model calculation for the monopole plus quadrupole pairing interaction.

The above analysis in a single-j shell indicates that the (phenomenological) choice of retaining only an exchange term in the IBFM Hamiltonian can be partly justified. For a complete description of the first excited state which has the j value of the shell involved, a direct term seems to be called for too. [For $J\approx j$, $W(j22j;jJ)\sim 1/\Omega$; a comparison of Eqs. (33) and (34) therefore indicates that there is also a suppression $\sim 1/\Omega$ of the direct term relative to the exchange term. This will obviously be more significant for larger values of Ω, but even then specific numerical coefficients and details of the fermion interaction will play a role in determining the relative importance of exchange and direct terms. When states with more than one boson have to be considered, the present analysis will also have to be extended to determine the relative role of these two interaction terms.]

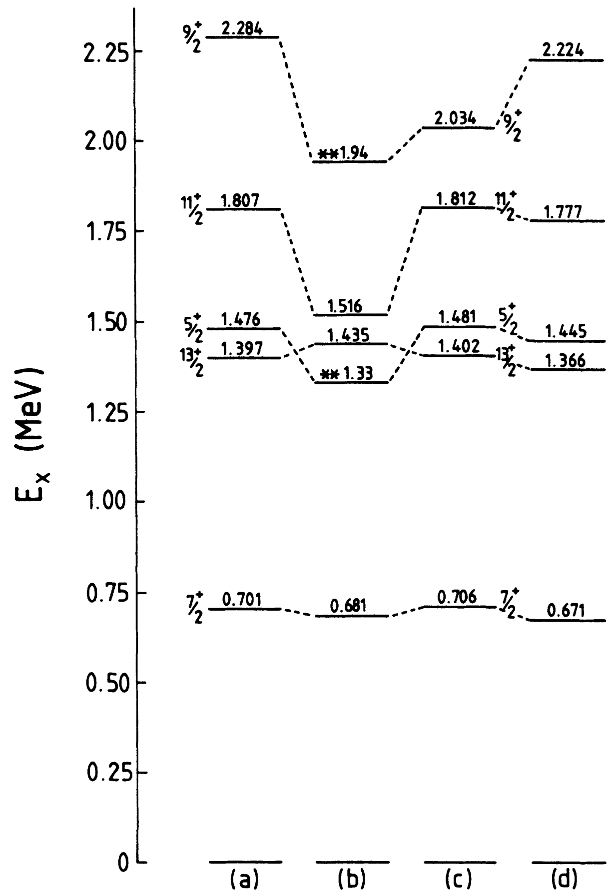

FIG. 1. A comparison of IBFM calculations for ${}^{93}_{43}$Tc with measured levels (b), supplemented by two calculated values (**) from Ref. 21. The spectrum (a) includes the direct term (30) as described in the text, while (c) was obtained without the direct term. The spectrum (d) corresponds to an exact shell-model calculation and was obtained by retaining higher-order terms in Z (see text).

As a final example of the implementation of the seniority transformation (23) we consider the seniority image of the quadrupole operator U^2_μ in Eq. (8). We use the same method as in the case of the quadrupole pairing Hamiltonian, which now means that we eventually only retain terms that are one-body boson operators with possible number dependent coefficients. The seniority image of the quadrupole operator we obtain in this way is given by

$$
\begin{aligned}
(U^2_\mu)_{\rm s}= &-\left[\frac{2}{\Omega}\right]^{1/2}\left[s^\dagger\tilde{d}_\mu+d^\mu s\left[\frac{\Omega-N-n_d-n_a}{\Omega-1-2n_d-n_a}\right]\right]+10\begin{Bmatrix}2&2&2\\ j&j&j\end{Bmatrix}\left[\frac{\Omega-2N-n_a}{\Omega-2n_d-n_a}\right][d^\dagger\tilde{d}]^2_\mu \\
&+\left[\frac{\Omega-2N-n_a}{\Omega-2n_d-n_a}\right]\left[[a^j\tilde{a}_j]^2_\mu+\frac{2}{\Omega+2-2n_d-n_a}\sum_L\frac{1}{\sqrt{5}}\hat{L}[[d^\dagger\tilde{d}]^L[a^j\tilde{a}_j]^2]^2_\mu\right] \\
&+\frac{2\sqrt{10\Omega}}{\Omega+2-2n_d-n_a}\sum_J{}'(-1)^J\hat{J}\begin{Bmatrix}2&2&J\\ j&j&j\end{Bmatrix}[[d^\dagger s]^2[a^j\tilde{a}_j]^J]^2_\mu s\ .
\end{aligned}
\qquad (35)
$$

It is evident that the procedure based on the DBM leads to seniority images that violate Hermiticity relationships in the ideal space which exist in the original fermion space. As discussed, however, in Ref. 14, there arises from the formalism a well-defined prescription for calculating ideal space matrix elements and transition amplitudes for such non-Hermitian operators. Using this prescription, we can, e.g., show that all quadrupole transition amplitudes between three-particle states in the $j=\frac{9}{2}^+$ subspace considered above, are reproduced by $(U^2_\mu)_{\rm s}$ in the ideal space of one boson plus one fermion.

In Ref. 14 it was also indicated that if the individual terms in a non-Hermitian seniority image coupled different ideal space states, a *Hermitian equivalent* operator could be written down for which transition amplitudes are simply calculated in the usual way. From $(U^2_\mu)_{\rm s}$ it is however clear that there are different terms which can couple the same ideal space states. The factorization needed to construct an exact Hermitian equivalent operator is therefore impossible.

This is precisely the reason why the quadrupole operator given by Otsuka *et al.*[8] is different from ours and could only be constructed to reproduce fermion matrix elements approximately. It is namely necessary to resort to some approximation (as discussed in Ref. 8) to *enforce* a Hermitian structure on their quadrupole operator.

This is a particular example of a more general difficulty encountered with the OAI method. For general boson number it is namely not always possible to disentangle from the one equation which equates fermion and ideal space matrix elements the coefficients of terms which can couple the same ideal space states.

Although the approximation needed to get a Hermitian operator might still be able to yield acceptable numerical results (although this requires further investigation) the above exposition again serves to illustrate that one cannot always achieve a complete IBM-like structure equivalent to a given fermion structure. (See also Ref. 15.)

IV. CONCLUSION AND OUTLOOK

We have confirmed the result of Otsuka *et al.*[8] that the origin of the exchange term in the IBFM, when applied to spherical nuclei, is linked to the fermion quadrupole pairing interaction. From our analysis a comparison between the direct and exchange terms [Eqs. (29) and (31)] is immediately possible, since the formalism based on the Dyson mapping and subsequent similarity transformation yields the complete seniority boson image of any fermion operator.

We have found some support for the phenomenological choice of the IBFM exchange term in a single-j shell where only angular momentum coupling to the j value of the given shell is retained. It was however pointed out that even for the special case investigated, the direct term plays an important role in the complete description of that excited state which carries the j value of the shell. Furthermore the role of the direct term for multishell situations and for states with more than one boson requires further investigation, as well as the effect of the fermion interaction in determining this role.

We have also constructed the IBFM seniority image of the quadrupole operator U^2_μ using the similarity transformation as in Eq. (23). This seniority image differs from the analogous operator given by Otsuka *et al.*[8] As discussed, the difference stems from an unavoidable approximation made[8] in order to enforce a Hermitian structure on the quadrupole operator.

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APPENDIX A: SENIORITY TRANSFORMATION

We seek the solution of the commutator equation

$$[Z^{-1},H_0]=WZ^{-1} \ . \tag{A1}$$

To this end we make the ansatz

$$Z^{-1}\equiv \sum_{k=0}^{\infty}\left[\frac{1}{\hat{H}_0-H_0}W\right]^k \wedge \ , \tag{A2}$$

where $\hat{H}^0$ is defined[15] to be evaluated at the position indicated by the lone hat. Inserting this ansatz into the commutator equation (36) yields for the left-hand side

$$[(\hat{H}_0-H_0)^{-1}W\wedge,H_0] + [[(\hat{H}_0-H_0)^{-1}W]^2\wedge,H_0]+\cdots \ . \tag{A3}$$

The first commutator above simplifies as follows:

$$[(\hat{H}_0-H_0)^{-1}W\wedge,H_0]=(\hat{H}_0-H_0)^{-1}W\wedge H_0 - H_0(\hat{H}_0-H_0)^{-1}W\wedge \tag{A4}$$

$$=\hat{H}_0(\hat{H}_0-H_0)^{-1}W\wedge - H_0(\hat{H}_0-H_0)^{-2}W\wedge \tag{A5}$$

$$=W\wedge \ . \tag{A6}$$

Similarly the second commutator becomes $W(\hat{H}_0-H_0)^{-1}W\wedge$. Continuing in this manner, one obtains exactly the right-hand side of Eq. (36) with Z^{-1} given in Eq. (37), which proves that the ansatz is indeed a solution.

APPENDIX B: Z^{-1} COMMUTES WITH s

We show that s commutes with Z^{-1}. Define

$$M=n_s+n_B+n_a \tag{B1}$$

with

$$n_B={\sum}' n_{2J} \ . \tag{B2}$$

The structure of W in expression (14) implies that for any state $|M,n_s,n_B,n_a)$, the eigenvalue of M is left invariant under operation of W. We can therefore write

$$\hat{H}_0-H_0=-G_0[(\hat{n}_s-n_s)(\Omega+1-2M+\hat{n}_s+n_s+n_a)] \tag{B3}$$

Now, since $sf(n_s)=f(n_s+1)s$ and $sf(\hat{n}_s)\wedge =f(\hat{n}_s+1)\wedge s$, where f is any function of n_s and $\hat{n}_s$, and from the fact that

$$[s,B^{JM}]_{J\neq 0}=[s,B_{JM}]_{J\neq 0}=0 \ , \tag{B4}$$

it easily follows that Z^{-1} and $(A_{00})_D=s$ commute.

APPENDIX C: NUMBER-DEPENDENT COEFFICIENTS OF H^Q_s

The number-dependent coefficients of H^Q_s which are obtained after implementing the Dyson mapping and the subsequent similarity transformation Z, are as listed below:

$$\epsilon_s = -\frac{2G}{\Omega}\left[\frac{N-n_d-1}{\Omega-1-2n_d-n_a}\right], \tag{C1}$$

$$\epsilon_d = -G\left[\frac{\Omega-2N+2n_d}{\Omega}+\frac{2}{\Omega}\frac{(N-n_d-1)(N-n_d)}{\Omega-1-2n_d-n_a}-\left[180\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}^2+200\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2\right]\left[\frac{N-n_d}{\Omega-2n_d-n_a}\right]\right], \tag{C2}$$

$$v_0 = \frac{G}{\Omega}\left[\frac{\Omega-2N+4n_d-4}{\Omega+3-2n_d-n_a}+2\left[\frac{\Omega}{\Omega+3-2n_d-n_a}\right]\begin{Bmatrix}j & j & 2\\ j & j & 2\end{Bmatrix}\right], \tag{C3}$$

$$v_0' = \frac{G}{\Omega}, \tag{C4}$$

$$\begin{aligned} v_2 = -G\Bigg[& 5\sqrt{10\Omega}\left[\left[\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}-100\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\right]\begin{Bmatrix}j & j & 2\\ j & j & 2\\ 2 & 2 & 2\end{Bmatrix}-18\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}\begin{Bmatrix}j & j & 2\\ j & j & 2\\ 4 & 2 & 2\end{Bmatrix}\right] \\ & \times\left[\frac{1}{\Omega+2-2n_d-n_a}\right]-10\left[\frac{10}{\Omega}\right]^{1/2}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\left[\frac{N-2n_d+1}{\Omega+2-2n_d-n_a}\right] \\ & +\frac{100}{\Omega}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\left[\frac{\Omega+1-4N+2n_d-n_a}{\Omega+1-2n_d-n_a}\right]\Bigg], \end{aligned} \tag{C5}$$

$$v_2' = -10\left[\frac{10}{\Omega}\right]^{1/2} G\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\left[\frac{\Omega-2N-n_a}{\Omega-2-2n_d-n_a}\right], \tag{C6}$$

$$\begin{aligned} c_J = 5\hat{J}G\Bigg[& 10\begin{Bmatrix}j & j & 2\\ j & j & 2\\ 2 & 2 & J\end{Bmatrix}-20\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2\left[\frac{N-n_d+1}{\Omega+2-2n_d-n_a}\right]\delta_{J2} \\ & +\frac{1}{2\Omega}\left[\frac{(N-n_d+1)(N-n_d+2)}{\Omega+3-2n_d-n_a}-\frac{(N-n_d-1)(N-n_d)}{\Omega-1-2n_d-n_a}\right]\delta_{J0}+30\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}\left[\frac{N-n_d+1}{\Omega+2-2n_d-n_a}\right]\delta_{J4} \\ & +\left[400\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2\begin{Bmatrix}2 & 2 & 2\\ 2 & 2 & J\end{Bmatrix}+180\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}^2\begin{Bmatrix}2 & 2 & 4\\ 2 & 2 & J\end{Bmatrix}\right]\left[\frac{N-n_d}{\Omega-2n_d-n_a}\right]\Bigg], \end{aligned}$$

$$A_0 = \frac{10}{\sqrt{2\Omega}}G\left[\frac{1}{\Omega-2n_d-n_a}\right], \tag{C7}$$

$$A_2 = -120\sqrt{5}G\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\left[\frac{N-n_d}{\Omega-2n_d-n_a}\right], \tag{C8}$$

$$\Gamma = -\left[\frac{10}{\Omega}\right]^{1/2} G\left[\frac{\Omega-2N-n_a}{\Omega-2-2n_d-n_a}\right], \tag{C9}$$

$$\Gamma' = -\left[\frac{10}{\Omega}\right]^{1/2} G\left[\left[\frac{\Omega-2-2N+4n_d}{\Omega+2-2n_d-n_a}\right]+10\begin{Bmatrix}j & j & 2\\ j & j & 2\end{Bmatrix}\left[\frac{\Omega}{\Omega+2-2n_d-n_a}\right]-\left[\frac{2N-2n_d}{\Omega-2n_d-n_a}\right]\right]. \tag{C10}$$

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