Linear relation between the scattering length and the size of a loosely bound two-body system: One-dimensional model analysis

F. M. Toyama* and Y. Nogami
Department of Physics, McMaster University, Hamilton, Ontario, Canada L8S 4M1
(Received 29 April 1988)

Relativistic and nonrelativistic models in one dimension which simulate the deuteron and the $^3S_1$ nucleon-nucleon (NN) scattering are constructed and the relation between the scattering length a_t and the root-mean-square radius r_m of the simulated deuteron is examined for a variety of interactions. The linear relation between a_t and r_m, which was found by Klarsfeld *et al.* for realistic NN potentials, holds for a wide class of potentials, relativistic as well as nonrelativistic. Within the limitation of our one-dimensional models, this suggests that the discrepancy between theory and experiment which was pointed out by Klarsfeld *et al.* regarding a_t and r_m cannot be resolved by relativistic effects.

I. INTRODUCTION

In their extensive study of deuteron structure, Klarsfeld *et al.*[1] made an interesting observation regarding the triplet scattering length a_t and the root-mean-square (rms) radius r_m of the matter distribution in the deuteron. When a_t and r_m are calculated by using realistic nucleon-nucleon (NN) potentials and a_t is plotted against r_m, the points representing the potentials all cluster along a straight line. The experimental point, however, is clearly off the line; the realistic potentials which fit the experimental value of a_t overestimate r_m.

The purpose of this paper is to find an answer to the natural question: Can the discrepancy regarding a_t and r_m of the deuteron be resolved by considering relativistic effects? To this end we use the one-dimensional simulation of the relativistic two-body system which we proposed recently.[2] The answer that we find is negative. In the course of this study we found that the linear relation between the calculated values of a_t and r_m holds for a class of interactions much wider than that considered by Klarsfeld *et al.*[1] The potentials include relativistic ones and also those which lead to very unrealistic phase shifts at medium and higher energies.

The models that we use are briefly described in Sec. II, and the relation between a_t (which we denote by a in the following) and r_m are examined in Sec. III. The results are discussed in Sec. IV.

II. MODELS

We use the same relativistic and nonrelativistic models as those of Ref. 2. The relativistic model is based on the one-dimensional two-body Dirac equation, which, apart from spin-related effects, adequately illustrates relativistic aspects of the two-body problem. The equation reads,[3] in natural units

$$H\psi=E\psi, \quad H=\sum_i(\alpha_i p_i+\beta_i m)+U \ , \tag{2.1}$$

where $i(=1,2)$ refers to the two particles. The potential U consists of a Lorentz scalar and the zeroth component of a Lorentz vector, i.e.,

$$U=\beta_1\beta_2 S(x)+(1-\alpha_1\alpha_2)V(x) \ , \tag{2.2}$$

where $x=x_1-x_2$ is the relative coordinate. Note that the total momentum $P=p_1+p_2$ is a constant of the motion. Throughout this paper we confine ourselves to the center-of-mass system, i.e., $P=0$.

The wave function ψ has four components. Equation (2.1) can be reduced to an equation of one of the components of ψ, say ϕ_2 as defined in Ref. 2. If we further define χ by

$$\chi=(E+S-2V)^{-1/2}\phi_2 \ , \tag{2.3}$$

Eq. (2.1) can be reduced to a Schrödinger-like equation

$$\left[\frac{p^2}{m}+W\right]\chi=\left[\frac{E^2}{4m}-m\right]\chi \ , \tag{2.4}$$

where W is an energy-dependent potential, which is defined by Eq. (2.7) of Ref. 2. Some interesting consequences of the energy dependence of W were discussed therein.

We assume that S is attractive (negative) while V is repulsive (positive). This particular combination of S and V is suggested by the recent success of the Dirac phenomenology.[4] For the functional forms of S and V we choose

$$S(x)=-g_S e^{-(x/a_S)^2}, \quad V(x)=g_V e^{-(x/a_V)^2} \ . \tag{2.5}$$

The parameters in the potential are chosen such that there is a bound state which simulates the deuteron.

In one dimension there are two "partial waves", with even and odd parity. As discussed in Ref. 2, the odd-parity state is much more suitable for simulating a three-dimensional S state. We consider both of the even and odd parity states, however. The effective range expansion for the odd-parity state is in the usual form of

$$k\cot\delta=-a^{-1}+rk^2/2 \ ,$$

TABLE I. Parameters of the relativistic potential of Eq. (2.5), used for the odd-parity state, and the corresponding rms radius r_m and the scattering length a. The first letter R in the code of the potential means "relativistic," and the second letter O means "odd parity." Units are mass m for g, and m^{-1} for a and r_m.

Potential	a_S	$10g_S$	a_V	$10g_V$	r_m	a
*RO*1	5	3.7	3	4.6487	9.839	27.40
*RO*2	5	2.8	3	2.8132	9.660	26.96
*RO*3	5	2	5	0.4312	9.319	26.11
*RO*4	5	4	5	1.8982	9.474	26.51
*RO*5	5	8	5	3.8682	9.799	27.33
*RO*6	3	6	3	0.7444	8.783	24.72
*RO*7	3	8	3	1.3740	8.824	24.83

TABLE II. Parameters of the nonrelativistic core potential of Eq. (2.6), used for the odd-parity state and the corresponding rms radius r_m and the scattering length a. The first letter C in the code means "core potential." Units are mass m for h, and m^{-1} for a, b, and r_m.

Potential	b_A	$10h_A$	b_R	$10h_R$	r_m	a
*CO*1	5	2.78	2.5	21.186	9.768	27.23
*CO*2	5	1.6	0.5	4803.32	9.368	26.24
*CO*3	3	3.5	0.5	289.83	8.702	24.53

TABLE III. The same as for Table I except for the even-parity state. The value of a for the density-equivalent nonrelativistic model is also shown immediately below its relativistic counterpart.

Potential	a_S	$10g_S$	a_V	$10g_V$	r_m	a
*RE*1	5	0.35	3	0.36635	8.277	23.37
*DE*1						23.39
*RE*2	3	1	3	0.79623	8.021	22.67
*DE*2						22.68
*RE*3	5	0.5	5	0.37634	8.113	22.93
*DE*3						22.94
*RE*4	5	1	5	0.86321	8.161	23.06
*DE*4						23.07
*RE*5	5	3	5	2.7346	8.365	23.65
*DE*5						23.63
*RE*6	5	5	5	4.5049	8.564	24.20
*DE*6						24.16

TABLE IV. The same as for Table II except for the even-parity state.

Potential	b_A	$10h_A$	b_R	$10h_R$	r_m	a
*CE*1	5	0.252	0.15	3.6202	8.239	23.29

whereas it goes as

$$-k\tan\delta=-a^{-1}+rk^2/2$$

for the even-parity state. This difference is a reflection of the different boundary conditions on the wave function at the origin.[5]

We compare the results of our relativistic model with those of the nonrelativistic model with a potential of the form

$$U_{\rm NR}(x)=h_R\theta(b_R-|x|)-h_A e^{-(x/b_A)^2}\theta(|x|-b_R)\ , \tag{2.6}$$

where $\theta(x)=1$ (0) if $x>0$ (<0) and suffices A and R refer to the attractive and repulsive parts, respectively.

In addition to the model defined by Eq. (2.6), we consider the "density-equivalent" nonrelativistic model defined by

$$U_{\rm NR}(x)=\frac{1}{m}\rho_R{}^{-1/2}\frac{d^2}{dx^2}\rho_R^{1/2}+\left[\frac{E^2}{4m}-m\right]\ , \tag{2.7}$$

where ρ_R is the density in the bound state obtained in the relativistic model. The E in Eq. (2.7) is the energy of the relativistic bound state; $U_{\rm NR}$ is energy independent, and it has the same range as its relativistic counterpart. With this potential the Schrödinger equation yields a bound state with wave function $\psi_{\rm NR}=\rho_R{}^{1/2}$; hence $\psi^2_{\rm NR}=\rho_{\rm NR}=\rho_R$. The binding energy of this nonrelativistic system turns out to be $(2m-E)[(2m+E)/4m]$. In our model calculations, we consider a bound state with a very small binding energy; hence $(2m-E)[(2m+E)/4m]$ is practically indistinguishable from the relativistic binding energy $2m-E$ (Ref. 6). In this way we obtain a pair of models, relativistic and nonrelativistic, which has a bound state of identical structure and practically the same binding energy. As we shall see, however, this density-equivalent pair of potentials yields phase shifts significantly different from each other at medium and high energies. Actually this density-equivalent nonrelativistic model can be constructed only for the simulation by means of the even-parity state. In the odd-parity state, while $\rho_{\rm NR}(x=0)=0$, $\rho_R(0)$ is, in general, finite.[2]

III. SCATTERING LENGTH VERSUS ROOT-MEAN-SQUARE RADIUS

As we already mentioned, the odd-parity state simulates the S state of three dimensions much better than the even-parity state does. However, we try both of them. In choosing the values of the parameters in the potential we impose the constraint that there is a bound state with en-

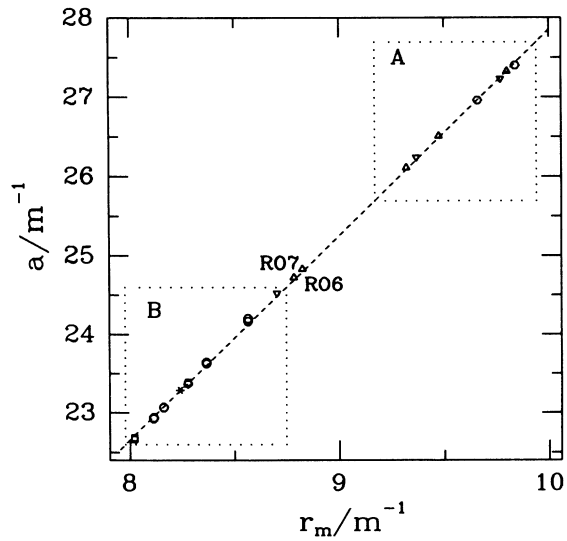

FIG. 1. The scattering length a vs the rms radius r_m of the bound state.

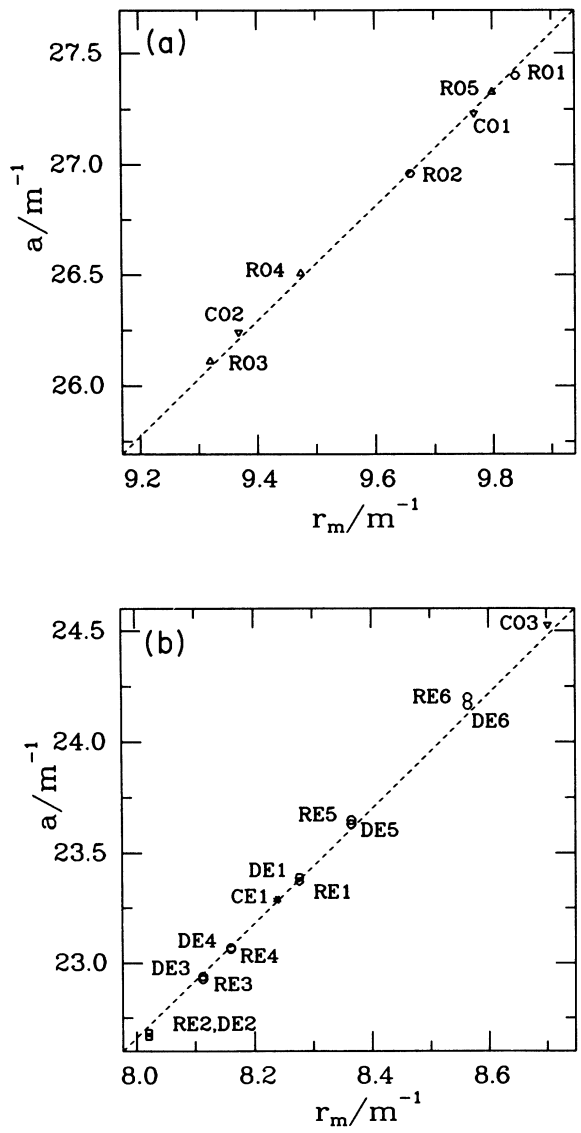

FIG. 2. (a) The details within box A of Fig. 1. (b) The details within box B of Fig. 1.

ergy $E=(2-0.002)m$, i.e., binding energy $B=0.002m$. If $m=1$ GeV, then $B=2$ MeV, and this bound state simulates the deuteron. Of course we could have fitted the exact deuteron binding energy 2.225 MeV in combination with the exact nucleon mass 939 MeV, but since we are doing a simulation we have chosen neat schematic figures. For the range of the attractive part of the potential we tried $5m^{-1}$ and $3m^{-1}$. Note that 1 GeV^{-1} $=0.1973$ fm. We consider a large variety of potentials. The phase shifts of some of the potentials simulate the $^3S_1$ NN phase shift very well, but others do not.

Tables I–IV list the potential parameters together with the rms radius r_m of the bound state and the scattering length a which result from the potentials. Tables I and II are for odd-parity state, while Tables III and IV are for the even-parity state. Tables I and III list the relativistic models and Tables II and IV the nonrelativistic models. Table III also lists the scattering length a calculated by using the density-equivalent nonrelativistic potential defined by Eq. (2.7). The code for the potentials is: The first letters R and C mean "relativistic potential" and "nonrelativistic core potential", respectively. The first letter D refers to the density-equivalent nonrelativistic model. The second letters E and O refer to the parity of the state used. In Table III, $RE1$ and $DE1$, e.g., are a density-equivalent pair, $RE1$ is relativistic and $DE1$ is nonrelativistic.

Figure 1 shows a vs r_m obtained from the potentials listed in Tables I–IV. Figures 2(a) and (b) show the details within two boxes, A and B, of Fig. 1. The points representing the various potentials all cluster closely along a straight line. This simulates the linear relation between a and r_m found by Klarsfeld *et al.*[1] We would like to emphasize the following, however. While the potentials considered in Ref. 1 are nonrelativistic, we have considered relativistic as well as nonrelativistic models. The potentials of Ref. 1 are all realistic,[7] while some of our potentials are unrealistic in the sense that they lead to phase shifts significantly different from the empirical $^3S_1$ NN phase shift at medium and high energies.

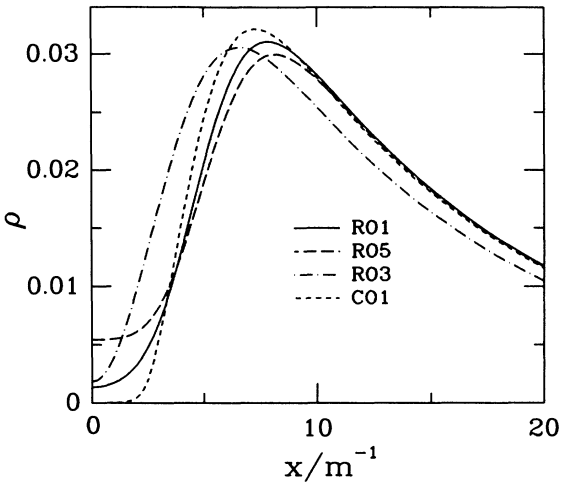

FIG. 4. The density distribution $\rho(x)$ in the bound state with odd parity, in units of mass m.

Figure 3 shows the phase shifts of some of the odd-parity models. The three of them, $RO1$, $RO5$, and $CO1$, are those examined in Ref. 2, and the phase shifts for them simulate the $^3S_1$ phase shift very well. They all change sign from positive to negative around $E=2.17m$. In contrast, the phase shift of potential $RO3$ is clearly unrealistic; it does not change sign at any reasonable energy.

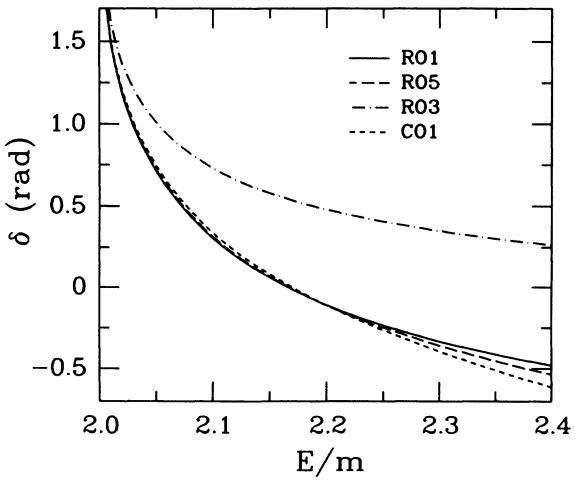

FIG. 3. The odd-parity phase shift $\delta(E)$.

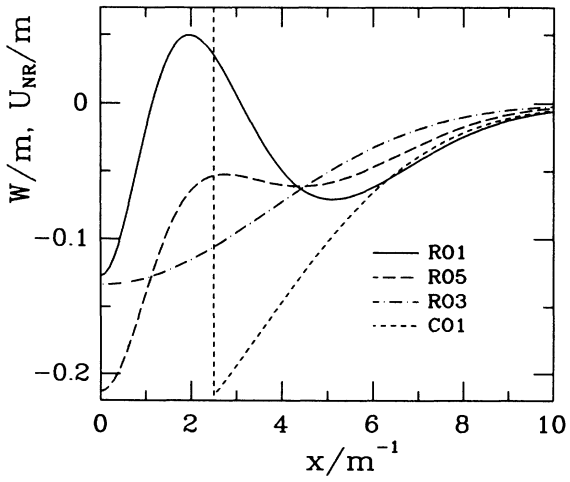

FIG. 5. The effective energy-dependent potential W for the bound state energy $E=1.998m$, compared with the nonrelativistic potential U_{NR} of Eq. (2.6). The odd-parity state is used.

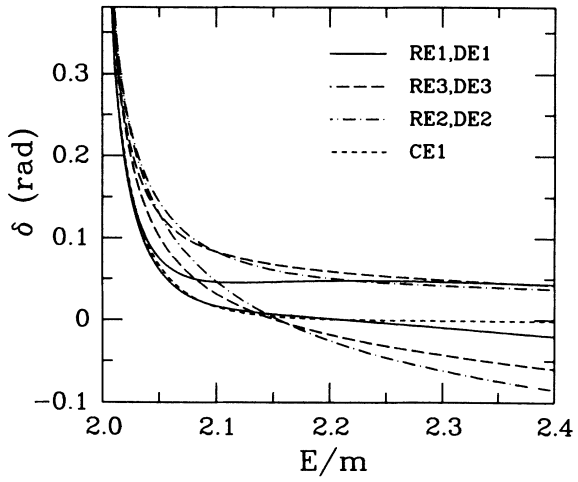

FIG. 6. The even-parity phase shift $\delta(E)$. For example, there are two solid lines. The upper one is for $DE1$ and the lower one for $RE1$. For all density-equivalent pairs, the nonrelativistic phase shift is greater than the relativistic counterpart.

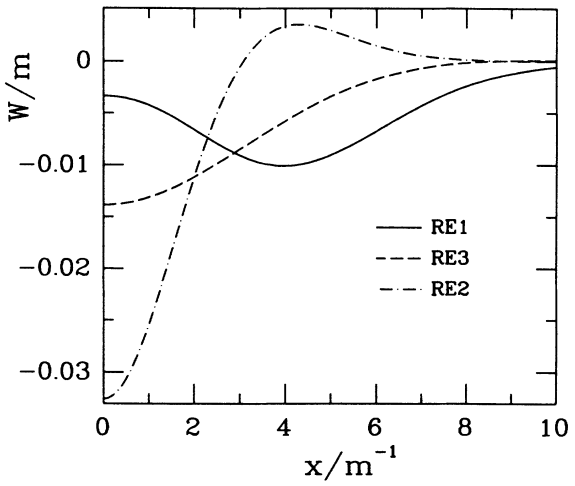

FIG. 8. The effective energy-dependent potential W for the bound-state energy $E=1.998m$, compared with the nonrelativistic potential U_{NR} of Eq. (2.6). The even-parity state is used.

In Fig. 2(a), the points for $RO1$, $RO5$, and $CO1$ are close to each other, while the $RO3$ point is isolated from them. Nevertheless they all lie along the same straight line. The r_m for $RO3$ is smaller than those for the other three; this is because potential $RO3$ is more attractive than the other three as can be surmised from the behavior of the phase shifts of Fig. 3.

Figure 4 shows the bound-state densities calculated with potentials $RO1$, $RO3$, $RO5$, and $CO1$. Figure 5 shows the effective energy-dependent potential W for the bound-state energy $E=1.998m$ for $RO1$, $RO3$, and $RO5$, and the nonrelativistic core potential $CO1$. The reason why we show these two figures is to emphasize that the structure of the bound state and its underlying potential are very different among those models. The linear relation between a and r_m is insensitive to such details.

Figures 6–9 are concerned with the simulation by means of the even-parity state. Potentials $RE1$, $RE3$, and $CE1$ were examined in Ref. 2. Figure 6 displays the phase shifts for three density-equivalent pairs (relativistic and nonrelativistic) and for one nonrelativistic model. The density-equivalent pair of potentials yields very different phase shifts. For example, there are two solid lines in Fig. 6; the upper one represents $DE1$ and the lower one $RE1$. For all such density-equivalent pairs, the phase shift of the nonrelativistic one is greater than the

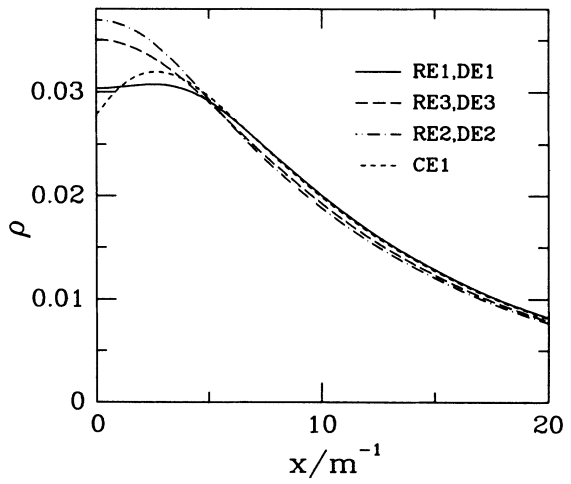

FIG. 7. The density distribution $\rho(x)$ in the bound state with even parity, in units of mass m.

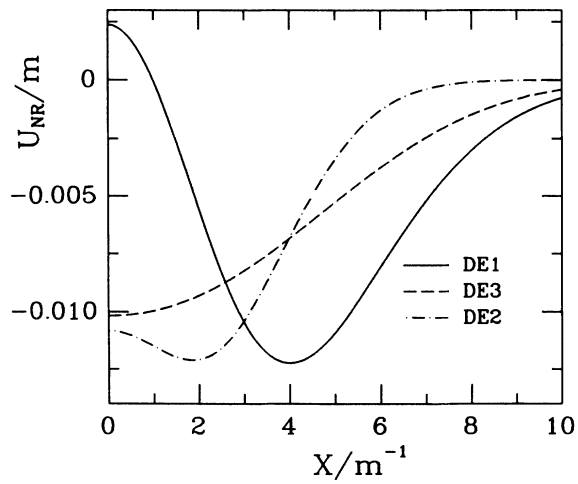

FIG. 9. The nonrelativistic potential of the density equivalent model defined by Eq. (2.7).

relativistic counterpart. Recall, however, in the plot of a vs r_m of Figs. 1, 2(a) and (b), a density-equivalent pair is very close together.[8] Figures 7 and 8 show the bound-state density and potential W, respectively, for the same set of potentials as for Fig. 6. Finally, Fig. 9 shows the nonrelativistic potentials for the density-equivalent models. Again the point to be noticed is that the structure of the bound state and the underlying potential differ significantly from one model to another. Nevertheless, the points representing a and r_m remain very close to the same line.

IV. DISCUSSIONS

We have constructed a number of one-dimensional models which simulate the deuteron. Some of them are relativistic while the others are nonrelativistic, but they all give the same binding energy $2m-E=0.002$ m of the simulated deuteron. We then found that the rms radius r_m and the scattering length a are linearly related to each other as seen in Figs. 1, 2(a) and (b). This is very similar to what Klarsfeld *et al.* found by using realistic nonrelativistic potentials.[1] Note however that we have included relativistic potentials and also unrealistic potentials (i.e., those giving phase shifts very different from the empirical $^3S_1$ NN phase shift).

As pointed out in Ref. 1, none of the realistic NN potential models reproduces the empirical triplet scattering length a_t and the deuteron rms radius r_m simultaneously. Naturally one wonders whether or not relativistic effects can rectify this situation. The answer to this question, suggested by our model analysis, is negative. We should add, however, that the models that we have used are one-dimensional ones, and the suggestion based on these models has to be substantiated by three-dimensional model analysis.

The potential W in the Schrödinger-like equation (2.4) of the relativistic model is energy dependent. At low energies, W is either entirely attractive or has only a very mild repulsive part; recall Fig. 5. As discussed in detail in Ref. 2, W changes from attraction to repulsion as the energy increases. This can cause the sign of the phase-shift change. Unlike the nonrelativistic case, the sign change of the phase shift from positive to negative does not require the presence of a short-range repulsion in the relativistic model. Therefore, we had thought that r_m would tend to be smaller in the relativistic case and the a-r_m relation would be different. We were somewhat surprised to find no distinction between relativistic and nonrelativistic models as far as the a-r_m relation is concerned. We were also surprised to find that unrealistic models (with unrealistic phase shifts) conform to the a-r_m relation very well.

The results of our analysis may give the impression that the linear relation between a and r_m is universal and entirely model independent. This is not quite so, however. If a tensor force (in three dimensions) is included, the a-r_m line shifts to the right. Here it is of course understood that the same binding energy is always fitted. The a-r_m plot of Ref. 1 should not be directly compared with ours because the potentials considered in Ref. 1 all have tensor forces. Since the potential representing the empirical a-r_m is situated to the left of the calculated a-r_m line (see, Fig. 6 of Ref. 1), we are interested in a mechanism which pushes the a-r_m line to the left. One such possibility is the nonlocality of the NN interaction. A preliminary calculation shows that the a-r_m relation is quite sensitive to the nonlocality of the interaction and it produces an effect in the expected direction. It is conceivable that the discrepancy regarding a_t and r_m pointed out by Klarsfeld *et al.*[1] is a signature of the nonlocality of the NN interaction.

ACKNOWLEDGMENTS

We would like to thank Dr. D. W. L. Sprung and Dr. W. van Dijk for interesting discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

---

*On leave from Institute of Computer Sciences, Kyoto Sangyo University, Kyoto 603, Japan.

[1] S. Klarsfeld, J. Martorell, J. A. Oteo, M. Nishimura, and D. W. L. Sprung, Nucl. Phys. **A456**, 373 (1986).

[2] Y. Nogami and F. M. Toyama, Phys. Rev. C **38**, 1578 (1988); see also W. Glöckle, Y. Nogami, and I. Fukui, Phys. Rev. D **35**, 584 (1987).

[3] We use the same notation as that of Ref. 2.

[4] J. A. McNeil, J. R. Shepard, and S. J. Wallace, Phys. Rev. Lett. **50**, 1439 (1983); J. R. Shepard, J. A. McNeil, and S. J. Wallace, *ibid.* **50**, 1443 (1983); B. C. Clark, S. Hama, R. L. Mercer, L. Ray, and B. D. Serot, *ibid.* **50**, 1644 (1983).

[5] For the definition of the phase shift in one dimension we follow J. H. Eberly, Am. J. Phys. **33**, 771 (1965).

[6] With the potential of Eq. (2.6) we fit exactly the same binding energy as the relativistic one.

[7] Among those potentials, older ones are less realistic than more recent ones.

[8] This feature of the density-equivalent pair is very similar to what was found for a density-equivalent pair of relativistic and nonrelativistic nonlocal-separable interactions in a one-body simulation of the NN interaction; Y. Nogami and W. van Dijk, Phys. Rev. C **34**, 1855 (1986); **36**, 1648(E) (1987).