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β-band moment of inertia anomaly

S. Kuyucak and I. Morrison
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia
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The moment of inertia of the $K=0^+$ bands in some of the rare-earth nuclei differ from other low-lying bands. Using the $1/N$ expansion technique, we show that the interacting boson model with arbitrary kinds of bosons and interactions leads to the same moment of inertia for all bands and hence does not allow such a variation to leading order. Since there are usually more 0^+ bands in this region than can be accommodated in collective models, this could provide a useful criterion for the selection of the collective β band.

The success of the interacting boson model[1] (IBM) in systematically "explaining" the structure of nuclei over extensive mass regions is well known. Particularly for even-even nuclei, extensive tabulations of model parameters have been derived to cover a wide range of nuclei. The properties of the low-lying collective states in even-even nuclei are adequately and uniformly treated in the model and generally this is true for states below the pairing gap as recently reviewed by Warner and Casten.[2] There is, however, a notable exception. In the rare-earth nuclei, where band structures are most pronounced, the β-band moment of inertia systematically differs from other bands. This problem was first raised by Bohr and Mottelson,[3] and the inability of the standard (sd) IBM to restore the deviation has been recently stressed.[2]

Commonly, the β band is taken as the lowest excited 0^+ band. However, especially in the rare-earth nuclei, there are usually several $K=0^+$ bands in close proximity not all of which can be accommodated in the collective model space. Thus care must be exercised in choosing the β band among several candidates. The purpose of this paper is to show that the 0^+ rotational bands with markedly different moment of inertia are outside of the IBM model space, and demonstrate with some examples that the above problem may be due to the wrong choice of the β band.

In a recent series of articles,[4-7] we have developed an angular momentum projected mean-field model which enables algebraic study of rotational systems and investigation of the role of higher spin bosons neglected in the standard IBM. Extension of the model from (sd) to (sdg) bosons offers explanations for (i) the extra bands (especially $K=1^+$ and 3^+ bands) observed in some rare-earth nuclei, (ii) absence of the predicted boson cutoff in $B(E2)$ values at spins $L\approx 2N$, (iii) $M1$ transitions among low-lying states, (iv) g-factor variations along the ground band. The possibility has been raised that the extra g-boson degree of freedom could also resolve the β-band moment of inertia anomaly.[2] In the previous work, we had used a simple quadrupole interaction, which is sufficient to demonstrate the above features, and to leading order, it leads to the same moment of inertia for all bands. Here we show that even allowing for arbitrarily high spin bosons and using a general Hamiltonian does not change this conclusion.

We consider a boson system with spins ($l=0,2,4,\ldots,p$) interacting via a general IBM-1 Hamiltonian with one- and two-body terms

$$H=\sum_l \varepsilon_l n_l+\sum_{k=0}^{2p}\kappa_k T^{(k)}\cdot T^{(k)}. \tag{1}$$

Here n_l is the boson number operator and $T^{(k)}$ is the boson multipole operator given by

$$T^{(k)}=\sum_{jl} t_{kjl}[b_j^\dagger \tilde{b}_l]^{(k)}. \tag{2}$$

The parameters of the Hamiltonian consist of the single boson energies ε_l, the multipole strengths and coefficients κ_k and t_{kjl}, respectively. The intrinsic boson operators $b_m^\dagger=\sum_l x_{lm} b_{lm}^\dagger$ provide a natural basis for low-lying collective bands which can be written as a condensate of N bosons and its simple excitations

$$|\phi_g\rangle=\frac{1}{\sqrt{N!}}(b^\dagger)^N\equiv|N\rangle,\quad |\phi_K\rangle=b_K^\dagger|N-1\rangle. \tag{3}$$

The structure coefficients x_{lm} are dynamically determined from $\langle H\rangle$

$$\langle H\rangle_{K,L}=\langle\phi_K|HP^L_{KK}|\phi_K\rangle/\langle\phi_K|P^L_{KK}|\phi_K\rangle, \tag{4}$$

by variation after projection (VAP) (Ref. 6).

Calculations for all multipole interactions are rather similar to the quadrupole interaction which is discussed at length elsewhere.[6] Therefore, we give here only the final results. The ground-band expectation value of the Hamiltonian (1), to first layer (i.e., to first order in excitation energies and moments of inertia) is given by

$$\langle H\rangle_{g,L}=NE+\left(1-\frac{\bar{L}}{2yN}\right)\left(E-\frac{E_1}{2y}\right)+\sum_k\kappa_k\left\{N(N-1)(A_k)^2+NC_k+N\left(1-\frac{\bar{L}}{2yN}\right)\right.$$
$$\left.\times\left[2(A_k)^2-\frac{1}{2y}\left(2A_{k1}-(1-\delta_{k0})\frac{\bar{k}}{2}A_k\right)A_k-\frac{1}{2y}(D_k)^2\delta_{k,\text{odd}}\right]\right\}, \tag{5}$$

where $\bar{L}=L(L+1)$ and the various quadratic forms are defined as

$$A_{kn}=\sum_{jl}\bar{j}^{n}\langle j0l0\,|\,k0\rangle t_{kjl}x_jx_l\,,$$
$$C_k=\sum_{jl}\frac{2k+1}{2l+1}(t_{kjl}x_l)^2\,,$$
$$D_k=\sum_{jl}\bar{j}^{1/2}\langle j1l0\,|\,k1\rangle t_{kjl}x_jx_l\,,\qquad(6)$$
$$E_n=\sum_{l}\bar{l}^{n}\varepsilon_l x_l^2,\quad y=\tfrac{1}{2}\sum_{l}\bar{l}x_l^2\,.$$

For convenience, we have suppressed the subscript n in A_{kn} when $n=0$ and also the quantum number $m=0$ in the ground-band mean fields in order to distinguish it from the β band. Special choices of parameters provide a check on Eq. (5), e.g., the one body part gives N for the number operator ($\varepsilon_l=1$), and choosing the dipole operator as the angular momentum operator $\{t_{1jl}=\delta_{jl}[l(l+1)(2l+1)/3]^{1/2}\}$ gives $\bar{L}$. Notice that $A_k=0$ for odd k and the odd multipoles are suppressed by $1/N$ compared to the even ones. Thus, assuming similar multipole strengths, the leading order mean-field amplitudes are not affected by the odd-multipole interactions. Calculations of this form give for the expectation value of the single-phonon K bands

$$\langle H\rangle_{K,L}=\langle H\rangle_{g,L}+\left[E(K)-E-\left(1-\frac{\bar{L}}{2yN}\right)\frac{b}{y}E'\delta_{K0}\right]$$
$$+\sum_k\kappa_k\left[2N\{[A_k(K)-A_k]A_k+[B_k(K)]^2\}-N\left(1-\frac{\bar{L}}{2yN}\right)\frac{2b}{y}A_kB_k(0)\delta_{K0}\right],\qquad(7)$$

with the quadratic forms defined as

$$A_k(K)=\sum_{jl}(-)^K\langle jKl-K\,|\,k0\rangle t_{kjl}x_{jK}x_{lK}\,,$$
$$B_k(K)=\sum_{jl}\langle jKl0\,|\,kK\rangle t_{kjl}x_{jK}x_l\,,\qquad(8)$$
$$E(K)=\sum_l\varepsilon_l x_lK^2,\quad E'=\sum_l\varepsilon_l x_{l0}x_l,\quad b=\sum_l\bar{l}x_{l0}x_l\,.$$

The moment of inertia of various bands is given by the inverse of the $\bar{L}$ terms in Eqs. (5) and (7). Continuing the $1/N$ expansion leads to band specific corrections of order $1/N$ which is in line with the *small* variation observed in the moments of inertia of most low-lying bands. From Eq. (7), it appears that, to leading order, all bands have the same moment of inertia except for the excited $K=0$ bands. In fact, as is shown below, the extra δ_{K0} terms in Eq. (7) disappear after variation, and the moment of inertia of the $K=0$ bands come in line with the others.

Applying the variational principle to the ground-band energy leads to the set of equations

$$\frac{\partial}{\partial x_l}(\langle H\rangle_{g,L}-\lambda\mathbf{x}\cdot\mathbf{x})=0,\quad l=0,2,\ldots,p\,,\qquad(9)$$

subject to the condition $\mathbf{x}\cdot\mathbf{x}=1$. These equations can be solved order by order using the following ansatz for the structure coefficients:

$$x_l=x_l^0+\frac{y_l}{N}+\frac{z_l}{N^2}\bar{L},\quad l=0,2,\ldots,p\,.\qquad(10)$$

Denoting the leading order terms in the energy formula (5) by H^0, the next order by H^1, etc., Eq. (9) can be cast into the form

$$\frac{\partial}{\partial x_l}[H^0-\lambda\mathbf{x}\cdot\mathbf{x}]_{\mathbf{x}^0}=0\,,\qquad(11a)$$
$$\frac{\partial}{\partial x_l}[H^0-\lambda\mathbf{x}\cdot\mathbf{x}]_{\mathbf{x}^0+\mathbf{y}/N}+\left[\frac{\partial H^1}{\partial x_l}\right]_{\mathbf{x}^0}=0\,,\qquad(11b)$$
$$\frac{\partial}{\partial x_l}[H^0-\lambda\mathbf{x}\cdot\mathbf{x}]_{\mathbf{x}^0+\mathbf{z}\bar{L}/N^2}+\left[\frac{\partial H^2}{\partial x_l}\right]_{\mathbf{x}^0}=0\,,\qquad(11c)$$

where we have used the fact that Eq. (9) should be satisfied independently for each order (1, $1/N$, $\bar{L}/N^2$) leading to Eqs. (11a), (11b), (11c), respectively. Similar sets of equations can be written down for the other bands where Eq. (11a) remains intact, but Eqs. (11b) and (11c) are different because of the changes in the higher order terms H^1 and H^2. Restoring the normalization factors $\mathbf{x}\cdot\mathbf{x}$ in H^0, we obtain from Eq. (11a)

$$4N^2\sum_k\kappa_kA_k\left(\sum_j\bar{t}_{kjl}x_j-A_kx_l\right)+2N(\varepsilon_l-E)x_l-2\lambda x_l=0\,,\qquad(12)$$

where $\bar{t}_{kjl}=\langle j0l0\,|\,k0\rangle t_{kjl}$ and we have suppressed the superscript 0 for convenience. Contrary to the odd multipoles, the one-body terms are included in Eq. (12) because the $1/N$ suppression factor is compensated by the much larger single boson energies. Multiplying Eq. (12) by x_l and summing over l gives $\lambda=0$. Thus Eq. (12) can be written in the more suggestive form

$$\sum_j\left(\sum_k\kappa_kA_k\bar{t}_{kjl}+\frac{\varepsilon_l}{2N}\delta_{jl}\right)x_j=\left(\sum_k\kappa_k(A_k)^2+\frac{E}{2N}\right)x_l\,,\qquad(13)$$

which forms a set of coupled nonlinear equations and has to be solved numerically by iteration. For our purpose, however, it is sufficient to note that the extra δ_{K0} terms in Eq. (7) (denoted as H_β) can be written as

$$H_\beta=-N\left(1-\frac{\bar{L}}{2yN}\right)\frac{2b}{y}\delta_{k0}$$
$$\times\sum_{jl}\left(\sum_k\kappa_kA_k\bar{t}_{kjl}+\frac{\varepsilon_l}{2N}\delta_{jl}\right)x_jx_{l0}\,.\qquad(14)$$

Substituting Eq. (13) in (14) gives

$$H_\beta=-N\left(1-\frac{\bar{L}}{2yN}\right)\frac{2b}{y}\delta_{k0}\left(\sum_k\kappa_k(A_k)^2+\frac{E}{2N}\right)\mathbf{x}\cdot\mathbf{x}_0\,,\qquad(15)$$

which vanishes due to the orthogonality of the ground and β bands.

To complete the proof, we next consider the effect of the variation on the energy formulas [Eqs. (5) and (7)]. Because of the extra δ_{K0} terms given in (14), Eq. (11c) for the $K=0^+$ bands is different than the others (derivative of H_β with respect to x_l does not vanish). Thus, the coefficients z are different for the $K=0^+$ bands, and in turn, this could lead to a variation in the moments of inertia.

We need to substitute the ansatz (10) only in the leading order term H^0, since the others (H^1,H^2) contribute to higher order than the first layer. To the same order, this can be written as

$$H^0\left(\mathbf{x}^0+\frac{\mathbf{y}}{N}+\frac{\mathbf{z}}{N^2}\bar{L}\right) = H^0(\mathbf{x}^0)+\sum_l\left(\frac{y_l}{N}+\frac{z_l}{N^2}\bar{L}\right)\left(\frac{\partial H^0}{\partial x_l}\right)_{\mathbf{x}^0}. \quad (16)$$

The correction terms in Eq. (16) vanish by virtue of the Hartree-Bose equations [(11a)–(13)], and hence, Eqs. (5) and (7) remain the same after variation. In other words, the leading order $1/N$ expansion results for the band excitation energies and the moments of inertia are not affected by variation. This is an important result in itself which greatly simplifies the application of the $1/N$ technique in complex situations.

We have tested the leading order $1/N$ results against exact numerical diagonalization using the computer code SDGBOSON.[8] Calculations, carried out for a quadrupole interaction plus single boson energies, indicate that the relative error is always less than $1/N$ and becomes much smaller for $N>10$. It appears that band mixing effects, which are still important for small boson numbers ($N\approx 6$–7) as indicated by staggering in the γ band, rapidly die off with increasing N and become negligible for $N>10$. Thus for deformed nuclei, the leading order $1/N$ results are accurate within a few percent.

In the rest of the paper, we examine the available rare-earth data in the light of the previous results. In Table I, we list the energies and the moment of inertia ratios $R_K=I_K/I_g$ of the γ and $K=0^+$ bands in the Dy, Er, Yb, and Hf isotopes. Included in the list are only the well formed rotors with energy ratio E_4/E_2 close to 3.3. Inspection of the table shows that the ratio R_γ mostly stays in the vicinity of 1, and only in a few cases the deviation is larger than $1/N$. Exceptions occur in the Hf isotopes (174–176) where $R_\gamma\approx 0.85$. In fact, in these nuclei, there is staggering in the γ band and the moment of inertia is not very well defined. Taking into account band mixing effects or simply using the even members of the γ band in extracting the moment of inertia rectifies R_γ here also. The situation is rather different for R_β which shows fluctuations much larger than $1/N$, sometimes as large as 1.5. Nevertheless, in the majority of nuclei in Table I, there is a $K=0$ band with R_β close to 1, though in some cases it is the second excited 0^+ band (e.g., Er isotopes).

The $E2$ transition rates could provide some further evidence as to the collectivity of the 0^+ band. In the consistent Q formalism (i.e., the same quadrupole operator is used in the Hamiltonian and the $E2$ operator), the leading order $\beta\rightarrow g$ to $\gamma\rightarrow g$ $B(E2)$ ratio is given by

$$R_{\beta\gamma}=\frac{B(E2;2_\beta\rightarrow 0_g)}{B(E2;2_\gamma\rightarrow 0_g)}=\frac{1}{2}\left[\frac{B_2(0)}{B_2(2)}\right]^2, \quad (17)$$

where $B_2(K)$ are defined in Eq. (8). For a simple quadrupole Hamiltonian, the eigenmode condition holds, and this

TABLE I. Energies (in keV) and moment of inertia ratios of γ and $K=0$ bands in rare-earth nuclei. Data from Ref. 9. The underlined 0^+ bands have $R_{\beta\gamma}\ll 0.1$.

	A	N	E_γ	R_γ	E_β	R_β	$E_{\beta'}$	$R_{\beta'}$
Dy	158	13	946	1.01	991	1.05		
	160	14	966	1.04	1275	1.16		
	162	15	888	1.07	1131	1.08	1400	1.50
	164	16	762	1.10	1655	1.21		
Er	162	13	901	1.01	1087	1.22	1420	1.27
	164	14	860	1.06	1246	1.33	1702	1.07
	166	15	786	1.10	1460	1.20	1703	0.99
	168	16	821	1.06	1217	1.34	1422	1.12
	170	17	934	1.03	891	1.14	1324	1.30
Yb	166	13	932	0.96	1043	1.01		
	168	14	984	1.06	1154	1.11	1197	1.10
	170	15	1146	1.05	1069	1.22	1229	1.09
	172	16	1466	0.95	1043	1.05	1405	1.10
	174	17	1634	1.01	1487	1.04	1886	1.05
Hf	170	13	987	1.00	880	1.24		
	172	14	1075	0.90	871	1.16	1336	1.50
	174	15	1227	0.84	828	1.26	1239	1.14
	176	16	1341	0.85	1150	1.15	1293	1.04
	178	15	1175	0.99	1199	1.20	1434	1.50
	180	14	1300	1.15	1107	1.01	1164	0.97

ratio vanishes.[6] However, in realistic applications, one should add the single boson energies in the Hamiltonian. We have performed some simple calculations in the (*sdg*) IBM framework including a *g*-boson energy of 0.5–1.0 MeV. Rough fits to the energies indicate that the ratio $R_{\beta\gamma}$ stays in the vicinity of 0.1. Experimentally,[2] $R_{\beta\gamma}$ changes between 0.1 and 0.5, except for the underlined 0^+ bands in Table I for which $R_{\beta\gamma} \ll 0.1$. Notice that all of these bands have anomalous moment of inertia. Conversely, in the Yb isotopes where $R_{\beta\gamma}$ is larger than 0.1, $R_\beta \approx 1$. Thus, the $E2$ data appear to support the idea that the 0^+ bands with anomalous moment of inertia are not collective β excitations but rather 2-*qp* excitations.

In summary, using the $1/N$ expansion formalism in a general IBM framework, we have shown that for well formed rotors, all collective bands have the same moment of inertia to leading order. This result is quite general and inclusion of higher spin bosons or arbitrary two-body interactions does not change it. Examination of the energy systematics and the $E2$ data in the rare-earth nuclei suggests that the choice of the first excited 0^+ band as the collective β band is not well founded and those with anomalous moment of inertia appear to be outside the collective model space.

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[1] A. Arima and F. Iachello, Adv. Nucl. Phys. **13**, 139 (1984).
[2] R. F. Casten and D. D. Warner, Rev. Mod. Phys. **60**, 389 (1988).
[3] A. Bohr and B. R. Mottelson, Phys. Scr. **25**, 28 (1982).
[4] S. Kuyucak and I. Morrison, Phys. Rev. Lett. **58**, 315 (1987).
[5] S. Kuyucak and I. Morrison, Phys. Rev. C **36**, 774 (1988).
[6] S. Kuyucak and I. Morrison, Ann. Phys. **181**, 79 (1988).
[7] S. Kuyucak and I. Morrison, Phys. Lett. B **202**, 169 (1988).
[8] I. Morrison, Computer code SDGBOSON, University of Melbourne (1986).
[9] M. Sakai and Y. Gono, At. Data Nucl. Data Tables **31**, 399 (1984).