Three-dimensional, spherically symmetric, saturating model of an N-boson condensate

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The variational Hartree method is applied to a system of N bosons interacting via Skyrme-type attractive and repulsive forces. When the system condenses into its ground state, the wave function for a single boson, ϕ, may be described by means of the nonlinear, time-independent Schrödinger equation, $\nabla^2\phi=\epsilon_1\phi-A\,|\phi|^2\phi+B\,|\phi|^4\phi$. In one dimension, this equation has a single, analytic, bound state solution and exhibits the property of saturation. In the physically more interesting case of three dimensions with spherical symmetry, no analytic solutions are known, so that a numerical solution must be resorted to, and this shows that saturation is again obtained. However, an analytic approximation to the three-dimensional wave function, which is very accurate for values of $\epsilon_1 B/A^2>0.1$, is deduced and studied. Approximate analytic expressions for the Hartree potential energy, kinetic energy, and mean square radius of the system are thereby derived, and applications to finite nuclei and infinite nuclear matter are considered.

I. INTRODUCTION

In a previous publication,[1] we considered the properties of the ground state condensate of a one-dimensional system of N bosons interacting via Skyrme-type attractive and repulsive forces. Using the variational Hartree method, we were able to formulate the problem in such a way that the single boson ground state wave function obeyed a nonlinear Schrödinger equation. Upon solving this equation, we found, as expected, that the interplay of the attractive and repulsive forces counterbalancing each other, led to the condensate achieving a state of saturation as the number of bosons in the system was increased without limit.

The reason for initially restricting our attention to one space dimension was that this enabled us to obtain analytic expressions for the single boson wave function, the single boson separation energy, the Hartree potential energy, the kinetic energy and the mean square length of the boson chain. We thus had an ideal, analytically solvable model exhibiting saturation, which is a fundamental feature necessary for the stability of any macroscopic system. We were able to see the behavior of the condensate evolving from a situation dominated by delta function attractive forces, when a very small number of bosons was present, to an equilibrated, uniform chain as the repulsive forces made their presence felt with increasing boson number and saturation was approached. The model thus provided an excellent testing ground to study this evolution in a readily controlled and easily understandable way.

As an additional bonus, by setting the strength of our repulsive force to zero, we were able to recover the mean field results of Calogero and Degasperis,[2] Nohl,[3] and Yoon and Negele[4] for an N-boson system interacting through purely attractive delta function forces only. These results, in fact, are identical to those obtained by McGuire,[5] who solved this problem exactly.

It is our intention here to extend our previous treatment so as to examine a spherically symmetric three-dimensional system, and thus deal with some physically more realistic examples. It is not immediately clear whether the properties of the one-dimensional solution will be retained upon effecting this change of dimensionality, but we do in fact find that saturation again occurs, so that real, physical systems in which this phenomenon is important can be studied. Unfortunately, no analytic solutions are known for the three-dimensional nonlinear Schrödinger equation which now describes the single boson wave function (to be discussed further in Sec. II) which we write as

$$i\frac{\partial\psi}{\partial t}+\nabla^2\psi=\epsilon_1\psi-A\,|\psi|^2\psi+B\,|\psi|^4\psi\ . \tag{1.1}$$

However, Anderson has studied this equation numerically[6] and has shown that it has stable solutions, provided that the parameters satisfy $0<\epsilon_1 B/A^2<\frac{3}{16}$. Furthermore, the changing forms of these numerical solutions for increasing values of $\epsilon_1 B/A^2$, together with our previous analytic one-dimensional solution, suggest an approximate analytic solution for this case also, which turns out to be excellent for $\epsilon_1 B/A^2>0.1$ and allows us to produce analytic approximations for the Hartree potential energy, the kinetic energy, and the mean square radius of the system. These approximations become better and better as saturation is approached more closely, enabling many of the attractive simplifications of the one-dimensional model to be carried over and applied to physically interesting examples in three dimensions.

The nonlinear Schrödinger equation of Eq. (1.1) has generated a great deal of recent interest in many apparently diverse areas of theoretical physics. With the parameter restrictions mentioned earlier, it supports stationary, stable soliton solutions, and has therefore been commonly used in investigations of classical soliton dynamics,[7] and in the closely related area of soliton models of hadrons.[8,9] In nuclear physics, it has been shown that the equations of nuclear hydrodynamics can be reformulated in terms of this equation, so that it has been applied to large nuclear systems to describe static and dynamic phenomena in heavy ion collisions.[10] Conversely, at the other extremity of the mass scale, it has been used to investigate the properties of light nuclei containing equal, even numbers of protons and neutrons as a Bose condensate of "alpha" particles[11] or, perhaps more precisely, spin-isospin quartets. The idea of Bose gas is, of course, more frequently encountered in low-temperature and condensed matter physics, where Eq. (1.1) has been applied to describe a linear system of bosons interacting via two-body attractive forces and three-body repulsive forces.[12] Various other novel applications have been suggested by Coleman[13] and Cohen *et al.*[14] in connection with Q balls. We therefore conclude that Eq. (1.1) is well worth further study in its own right because of this wealth of potential applications.

An additional collateral effect, or spinoff, is that wave functions and expectation values associated with the cubic nonlinear Schrödinger equation [having $B=0$ in Eq. (1.1)] can be deduced from the approximate analytic results associated with Eq. (1.1) by taking the limit as B approaches zero (as was done for the one-dimensional case in Ref. 1). This simplified form of Eq. (1.1) has also provoked a lot of interest in such varied areas as condensed matter physics, nonlinear optics, and quantum field theory.[15,16] It is also intimately related to the sigma model, where fermions are coupled to a single scalar field. The detailed theory and semiclassical approximation to this model may be found in Refs. 17 and 18.

In Sec. II we shall formulate the N-body problem in terms of the variational Hartree method to arrive at an eigenvalue equation for the ground state condensate. Exact solutions of this equation will be presented for one and three dimensions in Sec. III, while an approximate , analytic solution in the three-dimensional case (together with some analytic approximations to the expectation values of the system's observables) will be deduced in Sec. IV. In Sec. V we apply these results to the field of nuclear physics, and discuss our conclusions in Sec. VI.

II. FORMULATION OF THE PROBLEM

We wish to investigate the behavior of the ground state of a system of N bosons which interact with one another through a potential, $V(\mathbf{r}_1,\mathbf{r}_2)$, which consists of a delta function attraction and a density dependent delta function repulsion

$$V(\mathbf{r}_1,\mathbf{r}_2)=\{-\alpha+\beta\rho[\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)]\}\delta(\mathbf{r}_1-\mathbf{r}_2) . \tag{2.1}$$

This Skyrme-type effective interaction was suggested by Moszkowski[19] in his studies of infinite nuclear matter. It has the desirable feature of ensuring saturation because the density dependent repulsive term prevents two particles from getting too close together and so avoids the collapse associated with a simple delta function attractive potential.

We formulate the problem in exact analogy to Calogero and Degasperis[2] and write our (unnormalized) N-boson ground state trial wave function $\Phi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$ as

$$\Phi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=\prod_{i=1}^{N}\phi(\mathbf{r}_i) , \tag{2.2}$$

where we have placed all N bosons in the lowest single-particle orbit to obtain a many-body wave function for the Bose condensate of the system. We then use the Ritz variational principle to minimize the energy functional

$$E[\Phi]\equiv\langle\Phi\,|\,H-H_{\text{c.m.}}\,|\,\Phi\rangle .$$

This leads immediately to the Hartree eigenvalue equation [11] for the single-particle eigenfunction ϕ,

$$\frac{-\hbar^2}{2m}\nabla^2\phi=\epsilon_0\phi+(N-1)\alpha\,|\phi|^2\phi-N(N-1)\beta\,|\phi|^4\phi , \tag{2.3}$$

which has the form of a time-independent nonlinear Schrödinger equation.

We shall be primarily concerned with exact and approximate solutions of this equation so as to investigate the properties of the Bose condensate within the mean field approximation. We shall find it convenient to define the constants $\epsilon_1=-2m\epsilon_0/\hbar^2$, $A=2m(N-1)\alpha/\hbar^2$, and $B=2mN(N-1)\beta/\hbar^2$ which will allow us to rewrite Eq. (2.3) as

$$\nabla^2\phi=\epsilon_1\phi-A\,|\phi|^2\phi+B\,|\phi|^4\phi , \tag{2.4}$$

which is equivalent to Eq. (1.1) with the time dependence separated out. We shall now look for real, bound state ($\epsilon_0<0$) solutions in one and three dimensions.

III. EXACT SOLUTIONS IN ONE AND THREE DIMENSIONS

In one dimension the Laplacian is simply $d^2\phi/dx^2$ and Eq. (2.3) may be integrated analytically by standard techniques.[1] We apply the boundary conditions that ϕ (and, consequently, $d\phi/dx$) approaches zero as x approaches infinity, and that

$$|\phi(-x)|^2=|\phi(x)|^2 ,$$

which is equivalent to $d\phi/dx\,|_{x=0}=0$. The latter constraint indicates that we have no reason to prefer the positive over the negative x direction. This freedom to choose the position of the origin is related to the lack of translational invariance inherent in the Hartree method. We obtain the unnormalized single boson wave function

$$\phi(x)=\frac{2(\epsilon_1/A)^{1/2}}{\left\{1+\left[1-\frac{16\epsilon_1 B}{(3A^2)}\right]^{1/2}\cosh[2(\epsilon_1)^{1/2}x]\right\}^{1/2}} . \tag{3.1}$$

The value(s) of ϵ_1 may now be obtained by normalizing ϕ.

We find that there is a unique bound state energy eigenvalue, given by

$$\epsilon_1=\frac{3A^2}{16B}\tanh^2\left[\frac{B}{3}\right]^{1/2} . \tag{3.2}$$

Since $0\leq\tanh^2\sqrt{B/3}\leq 1$, this solution is clearly in line with Anderson's stability considerations[6] which require $0<\epsilon_1 B/A^2<\frac{3}{16}$. This one-dimensional analytic solution will serve to inspire a spherically symmetric three-dimensional approximation in Sec. IV.

In three dimensions, with spherical symmetry, (i.e., $L=0$) the Laplacian is $d^2\phi/dr^2+(2/r)(d\phi/dr)$ and no analytic solutions of Eq. (2.3) are known. A numerical approach is necessary to obtain exact solutions, and it is convenient to rescale both the wave function and the radius and write

$$\phi'=\sqrt{A/\epsilon_1}\phi,\quad r'=\sqrt{\epsilon_1}\, r ,$$

so that the differential equation becomes

$$\frac{d^2\phi'}{dr'^2}+\frac{2}{r'}\frac{d\phi'}{dr'}=\phi'-\phi'^3+\frac{\epsilon_1 B}{A^2}\phi'^5 . \tag{3.3}$$

This equation (and, in particular, the phase space trajectories of ϕ' for various values of $\epsilon_1 B/A^2$) has been numerically studied by Anderson.[6,20] To obtain an acceptable wave function we seek a solution which approaches zero as r' approaches infinity, and whose first derivative at the origin is zero [to avoid problems with $(2/r')(d\phi'/dr')$ in the Laplacian].

Anderson's basic conclusion is that stable solutions without any nodes, having a well-defined value at the origin, exist for $0<\epsilon_1 B/A^2<\frac{3}{16}$. Solutions with nodes are also possible, but they do not concern us here, since we are only interested in the ground state of the system. Our method of solution, based on his previous analysis, is therefore as follows: We replace the differential equation by a finite difference equation, choose a value for $\phi'(0)$, use a series solution of Eq. (3.3) valid for small r' to initiate our solution, and proceed to solve in the forward direction. We write the initiating series as

$$\phi'(r)=a_0+a_2r'^2+a_4r'^4+\cdots \tag{3.4}$$

where

$$a_2=a_0\left[1-a_0^2+\frac{\epsilon_1 B}{A^2}a_0^4\right]/6 \tag{3.5}$$

and

$$a_4=a_2\left[1-3a_0^2+5\frac{\epsilon_1 B}{A^2}a_0^4\right]/20 . \tag{3.6}$$

In general, the function obtained in this way for a given value of $\epsilon_1 B/A^2$, and an arbitrary value of a_0, will not have an acceptable asymptotic behavior at large r'. However, the nature of that unacceptability allows us to make an improved estimate of a_0, and after a few iterations a satisfactory numerical solution is obtained. It is particularly useful to appreciate that an upper bound on a_0 is provided by

$$a_0<\left[\frac{1+(1-4\epsilon_1 B/A^2)^{1/2}}{2\epsilon_1 B/A^2}\right]^{1/2} . \tag{3.7}$$

Having produced a single boson wave function, we now consider its normalization. Figure 1 shows the norm of ϕ', $\int_0^\infty|\phi'|^2r'^2dr'$ as a function of $\epsilon_1 B/A^2$, for values of the abscissa between 0 and 0.16. We see that the norm grows without limit as the value of $\frac{3}{16}$ is approached from below. For any value of $\epsilon_1 B/A^2$ in the prescribed range we are able to obtain an acceptable solution. However, it will not, in general, correspond to an integral number of bosons in the system. To see this, let us examine the norm more carefully. In one dimension the normalization condition provided us with a unique energy ϵ_0. A similar thing will happen here if we insist that N, which is contained in A and B, must be an integer. The normalization condition is $4\pi\int_0^\infty|\phi|^2r^2dr=1$, which in our rescaled variables becomes

$$\int_0^\infty|\phi'|^2r'^2dr'=\frac{A\sqrt{\epsilon_1}}{4\pi} . \tag{3.8}$$

Using this expression, we are able to convert our graph of norm vs $\epsilon_1 B/A^2$ into a graph of $\epsilon_1 B/A^2$ vs boson number N. This is shown in the upper panel of Fig. 2 for the case when the effective interaction parameters take the values $\alpha=1600$ MeV fm^3, $\beta=32\,000$ MeV fm^6, and

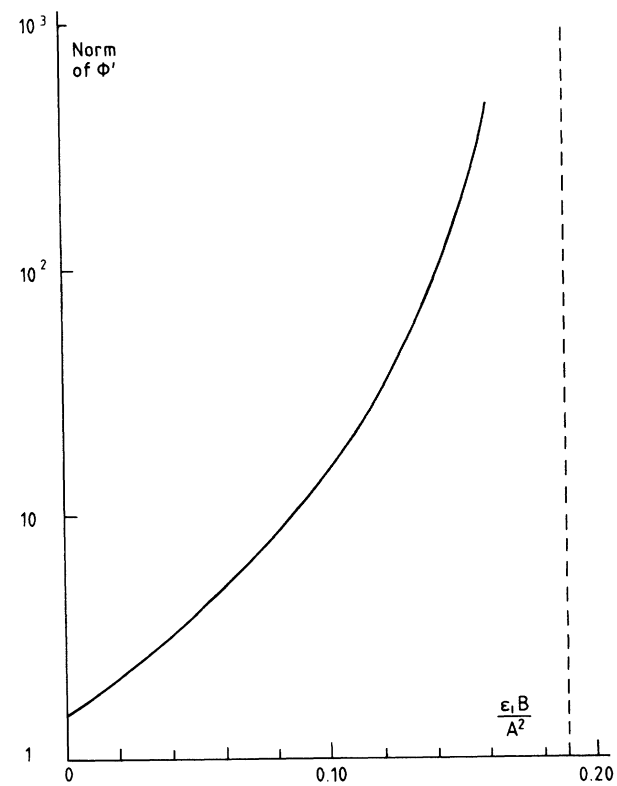

FIG. 1. The norm of the rescaled wave function, ϕ', of Eq. (3.3) as a function of the parameter $\epsilon_1 B/A^2$. Note that the norm grows without limit as the asymptotic value of $\epsilon_1 B/A^2=\frac{3}{16}$ is approached from below. These results are in agreement with those of Ref. 20.

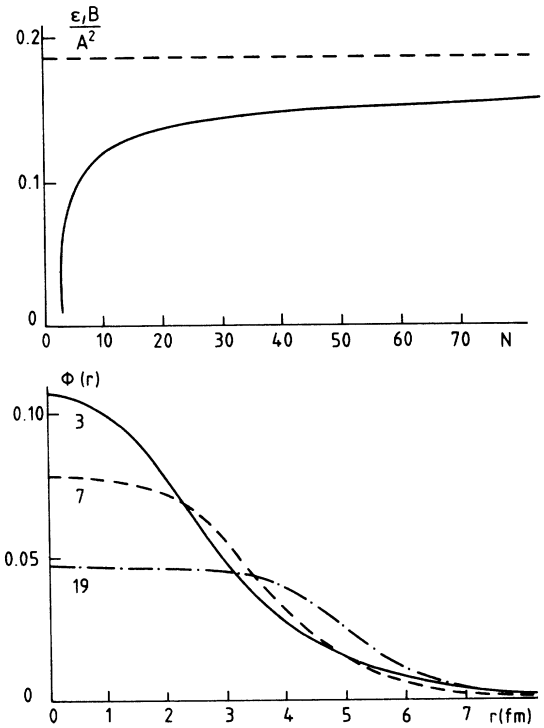

FIG. 2. The upper panel shows the parameter $\epsilon_1 B/A^2$ as function of boson number, deduced from the normalization condition of Eq. (3.8), using the interboson potential parameters $\alpha = 1600$ MeV fm^3 and $\beta = 32\,000$ MeV fm^6. The lower panel shows some specimen unscaled single boson wave functions, ϕ, as a function of r, corresponding to $N=3$, 7, and 19 bosons. As N increases, $\phi(r)$ takes on the appearance of a Fermi function.

$m = 939$ MeV/c^2 (the nucleon mass).

We see that only certain discrete values of $\epsilon_1 B/A^2$ (or, equivalently, ϵ_1) correspond to integral values of N. A possible alternative strategy is therefore for us to specify the particle number and perform a twofold iteration on the energy and a_0 until we satisfy the normalization condition and the boundary condition at large r', so as to obtain a nodeless wave function for that desired value of N. Some examples of the unscaled quantities ϕ vs r are shown in the lower panel of Fig. 2 corresponding to $N=3$, 7, and 19 using the parameter values mentioned earlier. As a further check on the validity of the method outlined here, it has been applied (with suitable modifications) to the one-dimensional case, and found to give complete agreement with the known analytic wave functions and eigenvalues.

There are two points to notice about the curve in the upper panel of Fig. 2. Firstly, as N increases, the value of $\epsilon_1 B/A^2$ approaches the asymptotic value of $\frac{3}{16}$, which implies that saturation is being attained, and that when it is reached, the energy to remove a single boson, ϵ_0, will be given by

$$\lim_{N\to\infty} \epsilon_0 = \frac{-3\alpha^2}{16\beta} , \tag{3.9}$$

which has a value of 15 MeV with the parameters being used here. Secondly, we see that the curve has a minimum value of N, and a small region where two values of ϵ_1 correspond to the same value of N. With our chosen parameters we see that this implies that no solution is possible for two bosons, and hence that three bosons provide the minimum number needed to achieve a bound state, while the double valued section of the curve corresponds to a nonintegral boson number. It is interesting to note that this means that if we identified our bosons as alpha particles, and the ground states of the various N-alpha nuclei as Bose condensates of alpha particles, we would say that ^{8}Be is unstable against breakup into two alpha particles, whereas ^{12}C is bound against alpha emission.

However, we do not wish to take this point too seriously, since our mean field approach is not expected to be valid for such small values of N. Apart from this, the boson pair interaction of equation (2.1) is appropriate for neutral alpha particles. Now, in the absence of Coulomb repulsion, it is an empirical fact that the corresponding scattering length is positive, indicating that such a system could indeed be bound. It is therefore probable that the nonbinding of two particles in our model is small-N quirk of our theory, and not necessarily related to the physical nonexistence of an alpha-alpha bound state.

The lower panel of Fig. 2 shows how the single boson wave function evolves with increasing particle number. As expected, for a few particles, the individual bosons are localized near to the origin. As N increases, and saturation is approached, the wave function takes approximately the form of a Fermi function, being nearly constant for some distance out from the origin before falling to zero over a relatively short distance which we can identify at the surface region. In fact, we can even get an expression for the saturation density without solving the differential equation. We identify the density of the condensate as $\rho = N|\phi|^2$. Then, when saturation is achieved in an infinite system, we shall have $\nabla^2\phi = 0$, and Eq. (2.4) may be written as

$$\epsilon_0^{\text{sat}} = \alpha\rho_{\text{sat}} - \beta\rho_{\text{sat}}^2 , \tag{3.10}$$

where we have used the largeness of N to ignore the difference between N and $(N-1)$. Using Eq. (3.9) for ϵ_0^{sat}, we obtain

$$\rho_{\text{sat}} = \lim_{N\to\infty} N|\phi^2| = \frac{3\alpha}{4\beta} \tag{3.11}$$

for the number of bosons per unit volume. Our parameters yield a value of 0.0375 bosons/fm^3, [so that if each boson consisted of four nucleons we should have 0.15 nucleons/fm^3, which lies within the expected range of 0.16 ± 0.015 nucleons/fm^3 (Ref. 21)].

IV. ANALYTIC APPROXIMATIONS IN THREE DIMENSIONS

We are unable to analytically integrate Eq. (2.4) in three dimensions (with $L=0$) because of the presence of the term $(2/r)(d\phi/dr)$ in the Laplacian. However, inspection of Fig. 2 shows that, as N becomes larger, the region where ϕ varies significantly with r gets pushed out further and further from the origin. In these cases we see that at very large or very small distances, $d\phi/dr$ is small. It only takes substantial values at the surface, where $2/r$ will, in its turn, be a small quantity. For N "large enough" it may therefore be a good approximation to throw away $(2r)(d\phi/dr)$ and solve what amounts to the one-dimensional nonlinear Schrödinger equation, subject to the same boundary conditions (namely, $d\phi/dr\,|_{r=0}=0$ and $\phi\rightarrow 0$ as $r\rightarrow\infty$). In this approximation we therefore obtain an unnormalized, single boson wave function identical to that given in Eq. (3.1). However, the normalization condition is now

$$4\pi\int_0^\infty |\phi|^2 r^2 dr=1 ,$$

instead of the linear condition

$$\int_{-\infty}^{\infty}|\phi|^2 dx=1 ,$$

so our eigenenergies and other expectation values are different.

The expressions for the norm, Hartree potential energy, kinetic energy, and mean square radius may all be evaluated by integrating appropriate integrands around the rectangular contour in the Argand plane shown in Fig. 3. As we take R towards infinity, the contributions from the vertical sections vanish, and we obtain the desired quantities in terms of residues at the two enclosed poles.

The normalization integral gives us a transcendental equation for the separation energy, ϵ_1, of the form

$$\frac{\pi}{2\epsilon_1}\sqrt{\frac{1}{3B}}\,\theta(\pi^2+\theta^2)=1 , \tag{4.1}$$

where

$$\theta=\tanh^{-1}\left[\frac{16\epsilon_1 B}{3A^2}\right]^{1/2} . \tag{4.2}$$

We have already seen the limiting saturation value of $\epsilon_1=-2m\epsilon_0/\hbar^2$ as N becomes large and $\tanh\theta$ approaches 1 in Eq. (3.9). In addition, we can obtain the value of ϵ_1 in the absence of a repulsive force by letting B become vanishingly small. We find

$$\lim_{B\rightarrow 0}\epsilon_1=\left[\frac{2\pi^3}{3A}\right]^2 . \tag{4.3}$$

The expectation value of r^2 can be calculated along the same lines to give

$$\langle r^2\rangle=\frac{9A^2\tanh^2\theta(3\theta^2+7\pi^2)}{80\pi^2\theta^2(\theta^2+\pi^2)^2} . \tag{4.4}$$

In the absence of repulsion, we obtain

$$\lim_{B\rightarrow 0}\langle r^2\rangle=\frac{63}{80}\frac{A^2}{\pi^4} . \tag{4.5}$$

When the system increases in size and saturates, the single boson wave function remains almost constant out to some large radius where it falls rapidly to zero. We may then relate the mean square radius to $\langle r^2\rangle$ by a factor of $\frac{5}{3}$. Therefore, as the particle number becomes large, we predict that the root mean square radius of our condensate will be given by

$$\lim_{N\rightarrow\infty}\left[\frac{5\langle r^2\rangle}{3}\right]^{1/2}=\left[\frac{N\beta}{\pi\alpha}\right]^{1/3} , \tag{4.6}$$

which increases in proportion to $N^{1/3}$ as we would expect for a saturated, spherical system. We could have obtained this value for the rms radius of the condensate without taking the large N limit of Eq. (4.4) by simply considering the saturation density, $\rho_{\rm sat}=3\alpha/(4\beta)$, [obtained in Eq. (3.11) without even solving the differential equation] and taking the N particles in volume $4\pi R^3/3$ to have this density.

The Hartree potential energy for a single particle is found by evaluating

$$\langle V_H\rangle=4\pi\int_0^\infty (N-1)(-\alpha+N\beta\phi^2)\phi^4 r^2 dr , \tag{4.7}$$

and yields

$$\langle V_H\rangle=\frac{\pi\sqrt{\epsilon_1}}{6A\tanh^3\theta}[(18\theta-3\pi^2\theta-3\theta^3)\tanh^2\theta -(\pi^2+3\theta^2)\tanh\theta+\theta(\pi^2+\theta^2)] . \tag{4.8}$$

The total Hartree potential energy is obtained on multiplying this result by $N/2$ (to avoid double counting). The no-repulsion limit of this expression is

$$\lim_{B\rightarrow 0}\langle V_H\rangle=-\frac{16\pi^4}{9A^2}\left[\frac{\pi^2}{6}-1\right] , \tag{4.9}$$

while the large N limit is obviously given by the saturation value

$$\lim_{N\rightarrow\infty}\langle V_H\rangle=-\frac{3\alpha^2}{16\beta} . \tag{4.10}$$

The single-particle kinetic energy is found from

$$\langle T\rangle=-\frac{\hbar^2}{2m}4\pi\int_0^\infty \phi\nabla^2\phi r^2 dr , \tag{4.11}$$

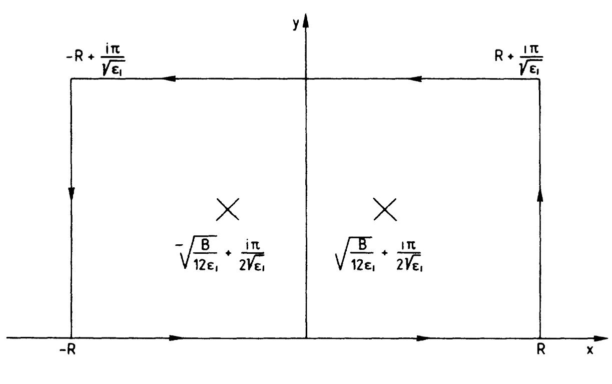

FIG. 3. The contour used to obtain the approximate analytic expressions of Sec. IV. Two poles are enclosed, and as R approaches infinity the contributions from the vertical sections to all the quantities considered, vanish.

and is (setting $\hbar=m=1$)

$$\langle T\rangle=\frac{\pi\sqrt{\epsilon_1}}{6A\tanh^3\theta}[(6\theta+\pi^2\theta+\theta^3)\tanh^2\theta+(\pi^2+3\theta^2)\tanh\theta-\theta(\pi^2+\theta^2)]\ . \tag{4.12}$$

The small B limit of this is given by

$$\lim_{B\to 0}\langle T\rangle=\frac{8\pi^4}{9A^2}\left[1+\frac{\pi^2}{12}\right], \tag{4.13}$$

while the large N limit varies like $N^{-1/3}$,

$$\lim_{N\to\infty}\langle T\rangle=\left[\frac{9\pi\epsilon_1^{5/2}}{64N\alpha}\right]^{1/3}. \tag{4.14}$$

This means that as the particle number increases, the total energy becomes purely potential with the single-particle kinetic energy approaching zero. This is to be expected since at saturation we effectively have a ball of hard spheres which have lost their freedom of individual movement due to the presence of the (infinitely) large number of other spheres around them. It is also a consequence of the Heisenberg uncertainty principle. As the system increases in size, we lose all knowledge of an individual particle's position, and the uncertainty in its momentum may therefore approach zero, leading to a vanishing single-particle kinetic energy.

These analytic expressions are expected to become valid as the particle number becomes large, and they certainly have the correct qualitative behavior to describe a saturating system. In Sec. V we shall test them out in a concrete, physical example.

V. SOME APPLICATIONS IN NUCLEAR PHYSICS

Since alpha decay was observed experimentally before the discovery of the neutron, it motivated Gamow to make some early attempts to describe nuclei as conglomerates of alpha particles. Although this model subsequently fell from favor with the introduction of proton-neutron models of the nucleus, it has been persistently revived, in one form or another, over the (almost) 60 years which have elapsed since then.

Wheeler[22] used the resonating group method to construct a ^{8}Be wave function consisting of two alpha clusters in which the individual nucleons were properly antisymmetrized. This approach has also been vigorously pursued by Edwards,[23] Wildermuth,[24] and Neudachin *et al.*[25] An alternative approach, in which ^{8}Be was described as a pair of alpha clusters with fixed centers (and whose component nucleons were again properly antisymmetrized), was proposed by Margenau.[26] Similar work was done by Biel,[27] and some unpublished work by Bloch was elaborated upon by Brink[28] to develop an ingenious model which has recently been revived because it lends itself to cranking.[29–31]

A philosphically very different approach, in which ^{8}Be was taken to be composed of two structureless alpha particles obeying Bose-Einstein statistics, was introduced by Wefelmeier[32] and further developed by Teller and Wheeler,[33] Dennison,[34] and Kameny.[35] Our own boson model is clearly most compatible with these latter ideas, and we shall endeavor to apply it to nuclear physics by identifying our bosons as spatially correlated combinations of two protons and two neutrons having total spin and isospin of zero and loosely referred to as alpha particles. We shall fit our two free parameters, α and β, to the established properties of nonrelativistic nuclear matter, and see to what extent we can reproduce the observed properties of the ground states of the N-alpha nuclei (by which we mean those light nuclei containing equal even numbers of protons and neutrons).

For completeness we should point out that the study of four nucleon correlations in nuclei has enjoyed something of a renaissance recently. There has been a suggestion of the existence of alphalike condensates within the framework of the interacting boson model,[36] and also some BCS-like calculations, including four-particle correlations, of the structure of the superfluid and low-lying excited states of atomic nuclei.[37]

In earlier sections of this paper we have already suggested the parameter values $\alpha=1600$ MeV fm^3 and $\beta=32\,000$ MeV fm^6. In the limit of a very large, saturating system, the analytic expression given in Sec. IV become exact, and we can see that this choice is consistent with the known properties of nonrelativistic nuclear matter. The saturation density was given in Eq. (3.11) as $3\alpha/4\beta$, implying a value of 0.15 nucleons/fm^3, in agreement with conventional wisdom [see for example the recent review by Negele (Ref. 21)]. In addition, if we write the number of nucleons, A_0, as four times the numbers of bosons, N, we can rewrite Eq. (4.6) as

$$(R_0^2)^{1/2}=\lim_{A_0\to\infty}\left[\frac{5\langle r^2\rangle}{3}\right]^{1/2}=\frac{1}{2}\left[\frac{2\beta}{\pi\alpha}\right]^{1/3}A_0^{1/3}\ . \tag{5.1}$$

Hence, we see that this same ratio of parameters leads to the relation $(R_0^2)^{1/2}\approx 1.17A_0^{1/3}$ fm. This result is compatible with measurements across the entire periodic table of the matter radii of atomic nuclei obtained from neutron scattering experiments.

The energy to separate an alpha particle from the medium was given in Eq. (3.9) as $3\alpha^2/16\beta$, which leads to a value of 15 MeV. The compressibility of nuclear matter can be related to this same combination of parameters by differentiating the total energy of the saturated system, E, twice with respect to the density, ρ, [see Eq. (3.10), (4.10), and (4.14), and write $N=4A_0$] as follows:

$$K=\frac{1}{\rho^2}\frac{d^2}{d\rho^2}\left[\frac{E}{A_0}\right]=-\frac{9\alpha^2}{4\beta}\ . \tag{5.2}$$

This gives a value of 180 MeV, which lies within the range of accepted values and which are usually quoted as lying between 180 and 360 MeV (Ref. 38).

In our model, only two of the four nuclear matter quantities considered are independent, and they give us values for the ratios α/β and α^2/β (thus serving to fix the two unknown parameters). It is nevertheless gratifying that we are able to achieve acceptable agreement with the results of more sophisticated calculations in which these four quantities are not so trivially related. We shall now investigate to what extent our model, with these parameters, can be applied to the ground state properties of the light N-alpha nuclei.

We begin by using the numerical solutions of Eq. (2.4), described in Sec. III, to calculate the charge density, charge radius, alpha particle separation energy, and binding energy for the N-alpha nuclei ranging from ^{12}C to ^{44}Ti. As mentioned in Sec. III, the ^{8}Be nucleus has no bound states with our chosen parameters, which happens to be in qualitative agreement with experiment. However, we feel that this success is somewhat fortuitous, since we hardly expect a mean field approach to provide an adequate description of two boson system, and is probably a small-N quirk of our theory.

To calculate the charge density, ρ_{ch}, we assume that the neutron and proton distributions coincide, and associate two protons with each boson so that $\rho_{ch}=2N|\phi|^2$, and we are treating the individual particles as pointlike. There are no great qualitative differences between our calculated distributions for the various N-α nuclei, and so we present two typical examples from the light (^{16}O) and heavy (^{40}Ca) extremities of the mass range in Fig. 4. The charge densities of both these nuclei have been measured by high-energy elastic electron scattering and we compare our calculations with some simple parametrizations of the data suggested by Hofstadter.[39]

In the upper panel of Fig. 4, we compare our calculation with the analytic form

$$\rho_{ch}(^{16}\mathrm{O})=\rho_0\left[1+\frac{\alpha r^2}{a_0^2}\right]\exp(-r^2/a_0^2)\ , \qquad (5.3)$$

where $\alpha=2$, $a_0=1.77$ fm and $\rho_0=0.064\,77$ protons/fm^{-3} (which is normalized to eight protons). This formula is motivated by considerations of the single-particle orbitals in a spherical harmonic oscillator shell model, and gives an excellent reproduction of the data. Our calculated distribution does not show a central minimum and has too large a tail, but since we did not expect a mean field description of only four bosons to be particularly accurate, we consider the outcome to be surprisingly good despite these obvious shortcomings.

In the lower panel of Fig. 4 we compare our calculation for the proton density in ^{40}Ca with a Fermi function which describes the experimental data very accurately.[39]

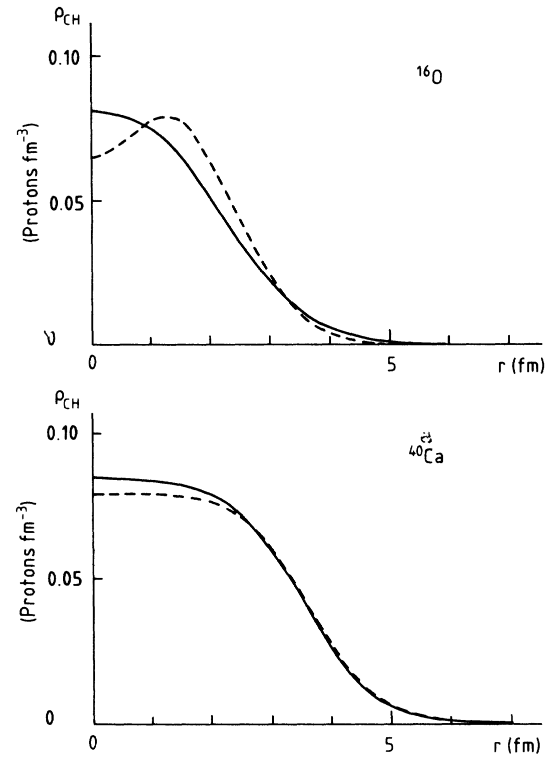

FIG. 4. Calculated (solid lines) and experimental (broken lines) charge distributions for ^{16}O and ^{40}Ca. Interboson potential parameters of $\alpha=1600$ MeV fm^3 and $\beta=32\,000$ MeV fm^6 are used, and the experimental distributions are from Ref. 39.

$$\rho_{ch}(^{40}\mathrm{Ca})=\frac{\rho_0}{1+\exp[(r-c)/z]}\ , \qquad (5.4)$$

where $c=3.659$ fm, $z=0.5455$ fm, and $\rho_0=0.079\,94$ protons/fm^{-3} (and is normalized to 20 protons). The agreement between calculation and experiment in this case is considerably better, reflecting the improved adequacy of the mean field approximation for the description of ten bosons. In particular, the qualitative features of a

TABLE I. Calculated and experimental values of the alpha particle separation energy, root mean square radius, and binding energy (BE) for N-alpha nuclei. We use $\alpha=1600$ MeV fm^3 and $\beta=32\,000$ MeV fm^6 for the boson potential parameters. Experimental data are from Refs. 40–42.

	$-\epsilon_0$ (MeV)		$(\langle r^2\rangle)^{1/2}$ (fm)		BE (MeV)	
Nuclei	calc.	expt.	calc.	expt.	calc.	expt.
^{12}C	3.04	7.37	3.42	2.46	82.8	92.2
^{16}O	4.92	7.16	3.15	2.73	113.2	127.6
^{20}Ne	6.09	4.73	3.15	2.91	144.5	160.6
^{24}Mg	6.91	9.31	3.21	3.03	176.3	198.3
^{28}Si	7.53	9.98	3.28	3.13	208.4	236.5
^{32}S	8.02	6.95	3.36	3.25	240.7	271.8
^{36}Ar	8.42	6.64	3.44	3.33	273.3	307.1
^{40}Ca	8.75	7.04	3.51	3.49	306.0	342.0
^{44}Ti	9.03	5.13	3.59	3.59	338.8	375.5

central plateau and a less diffuse surface region where the density falls quite rapidly to zero are clearly discernible. The calculations for the intermediate nuclei show a gradual evolution from the curve of the upper panel of Fig. 4 to that of the lower panel, and are accompanied by a corresponding improvement in agreement with the data as N increases and the validity of the mean field approximation improves. This pattern of improved agreement is also noticeable in the charge radii (see Table I) which are calculated to be larger than their experimental counterparts[40,41] in the lighter nuclei, but are closely comparable for ^{40}Ca and ^{44}Ti.

Table I also shows a comparison between the calculated and experimental alpha particle separation energies, $-\epsilon_0$, and binding energies for the N-alpha nuclei. The separation energy is calculated from the normalization condition on the single boson wave function, and, in our model, rises monotonically towards the saturation value of -15 MeV. The experimental values[42] are much more irregular and do not exhibit any kind of systematic pattern. In particular, we note that the energies to remove an alpha particle from ^{20}Ne and ^{44}Ti are significantly lower than from the other nuclei. This is a feature which can be attributed to the presence of double shell closures at $N=Z=8$ in ^{16}O and $N=Z=20$ in ^{40}Ca facilitating the removal of the alpha particle, and is obviously not present in our model. Although we are not sensitive to details which depend on the internal fermionic structure of alpha particles, the order of magnitude of our separation energies (calculated with parameter values fitted to nuclear matter properties) is correct, and might be expected to give better agreement in heavier nuclei, far from shell closures. Unfortunately, heavier N-alpha nuclei are unstable because of the influence of the Coulomb force which is ultimately responsible for favoring an excess of neutrons over protons, and so our conjecture cannot be tested in this context.

The binding energies in Table I are calculated by summing the Hartree potential and kinetic energies, giving the energy to separate the system into N individual bosons, and then adding N times the binding energy of a free alpha particle, to separate the system into its constituent nucleons. We thus write the binding energy in MeV as

$$-E=\frac{N}{2}\langle V_H\rangle+N\langle T\rangle-28.295N \ . \qquad (5.5)$$

Overall, our results for the binding energies of the 4-N nuclei are generally below the experimental values [obtained from a consideration of the proton and neutron mass defects, and the mass defect of the nucleus in question (Ref. 42)], but can be considered satisfactory for a simple two parameter model which was not fitted *a priori* to these data.

These results actually open up an interesting connection with neutral alpha matter. Equation (3.11) gave a saturation density of 0.0375 bosons/fm^3, and Eq. (5.5) implies a saturation binding energy per boson of -35.795 MeV (since $\langle T\rangle$ vanishes and $\langle V_H\rangle\rightarrow -15$ MeV in the large N limit. These values are within the ranges of 0.036–0.086 alphas/fm^3 and -40 to -10 MeV/alpha, respectively, obtained by Johnson and Clark using realistic alpha-alpha potentials in a Jastrow variational calculation.[43]

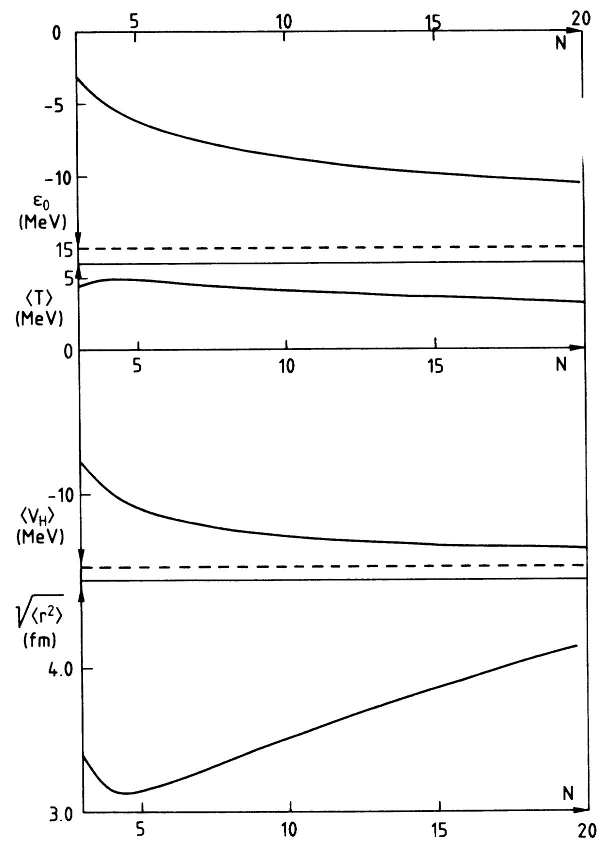

FIG. 5. The upper panel shows the calculated alpha particle separation energy as a function of boson number, N. As N increases, the saturation value of -15 MeV is approached. The middle panel shows the expectation values of the single boson kinetic and Hartree potential energies as functions of N. The former tends to zero and the lattice approaches -15 MeV as saturation sets in. The lower panel shows the root mean square radius as a function of N. In all cases parameter values of $\alpha=1600$ MeV fm^3 and $\beta=32\,000$ MeV fm^6 are taken.

Figure 5 shows the behavior of ϵ_0, $\langle T\rangle$, $\langle V_H\rangle$, and $(\langle r^2\rangle)^{1/2}$ as functions of the boson number N obtained from our numerical solutions. The upper panel shows the monotonic increase in the magnitude of the alpha particle separation energy, and its slow approach to the saturation value of 15 MeV as N is increased. The central panel shows the expectation values of the single-particle kinetic, $\langle T\rangle$, and Hartree potential, $\langle V_H\rangle$, energies as functions of N. The former reaches a maximum at about $N=4$ and then declines (eventually like $N^{-1/3}$) towards its asymptotic value of zero, while the latter approaches the saturation value of -15 MeV. Finally, the lower panel shows $(\langle r^2\rangle)^{1/2}$, which has a minimum value between $N=4$ and $N=5$, but then rises without limit as N is increased further (ultimately like $N^{1/3}$).

TABLE II. Correlations of root mean square radius and binding energies (BE) with the alpha particle separation energy in the solid sphere approximation for N-alpha nuclei. We use $\alpha = 1600$ MeV fm^3 and $\beta = 32\,000$ MeV fm^6 for the boson potential parameters. Experimental data are from Refs. 40–42.

Nuclei	$-\epsilon_0$ (MeV) expt.	$(\langle r^2 \rangle)$ fm calc.	$(\langle r^2 \rangle)$ fm expt.	BE (MeV) calc.	BE (MeV) expt.
^{12}C	7.37	2.60	2.46	95.9	92.2
^{16}O	7.16	2.81	2.73	127.5	127.6
^{20}Ne	4.73	2.96	2.91	153.3	160.6
^{24}Mg	9.31	3.25	3.03	197.7	198.3
^{28}Si	9.98	3.44	3.13	233.0	236.5
^{32}S	6.95	3.50	3.25	254.2	271.8
^{36}Ar	6.64	3.63	3.33	284.5	307.1
^{40}Ca	7.04	3.77	3.49	318.2	342.0
^{44}Ti	5.13	3.85	3.59	339.5	375.5

We could certainly improve our agreement with the experimental data in Table I by a different choice of α and β. However, we do not feel that this would be a physically reasonable procedure since with so few bosons in the system the mean field approximation is only just beginning to be validated in the heavier N-alpha nuclei, and so we prefer to retain the parameter values which fit the characteristics of nuclear matter. Even with these caveats, our calculations are acceptably close to the experimental results and exhibit a reasonable global behavior.

The approximate analytic expressions derived in Sec. IV are expected to be highly accurate for large boson number when the contribution of the nuclear surface [which was effectively ignored by throwing away $(2/r')(d\phi'/dr')$] is less important. Nevertheless, by applying the analytic expressions for $\langle V_H \rangle$, $\langle T \rangle$, and $\langle r^2 \rangle$ to the light N-alpha nuclei, we have found that they give values within 10% or less of those obtained from the numerically calculated wave functions for $6 \le N \le 11$, which corresponds to $\epsilon_1 B / A^2 > 0.1$ with our choice of parameters. We have checked that this agreement does indeed become progressively better as N is increased still further, and saturation is approached more closely.

In view of the proximity of these analytic approximations to the exact numerical results, we were motivated to examine an even more extreme approximation which we might call the solid sphere model. In this case, we avoided solution of the differential equation (2.4) completely by assuming that the single boson wave function ϕ was constant out to some cutoff radius R and zero beyond. We then normalized ϕ within a sphere of radius R and so obtained relations between R, the alpha particle separation energy ϵ_0 and the binding energy, E. Employing the same values of α and β as before, and taking ϵ_0 from experiment, we present the resulting values of $(\langle r^2 \rangle)^{1/2}$ and E in Table II. They are inferior to those of Table I (especially when we note that they involve a greater experimental input), but the fact that they are at all comparable with the data is a reflection of the leptodermous nature of atomic nuclei (i.e., their density is more or less constant throughout their interior and falls rapidly to zero in the surface region).

VI. CONCLUSIONS

We have presented a spherically symmetric, three-dimensional model of a system of N bosons interacting through attractive and repulsive Skyrme-type forces. The mean field description of the ground state condensate of this system has been formulated as a nonlinear Schrödinger equation for the single boson wave function, ϕ, [Eq. (2.4)]. We have solved this differential equation numerically, subject to the boundary conditions that ϕ vanishes at large distances and that its derivative at the origin is zero. We have also proposed an approximation which allows a modified version of the equation to be solved analytically (subject to the same boundary conditions) and which further allows analytic evaluation of the expectation values of the kinetic energy, Hartree potential energy, and the mean square radius of the Bose condensate. Equation (2.4) is known to have stable solutions for $0 < \epsilon_1 B / A^2 < \frac{3}{16}$, and our approximate analytic expectation values were found to be within 10% (or better) of the exact numerical ones for $\epsilon_1 B / A^2 > 0.1$.

The interplay between the attractive and repulsive Skyrme-type forces leads to saturation as the boson number increases. In this limit we have shown that the single boson separation energy and Hartree potential energy both attain a constant value, while the single boson kinetic energy vanishes and the radius grows in proportion to $N^{1/3}$.

We have applied our results to the field of nuclear physics by interpreting our bosons as spatially correlated conglomerates of two protons and two neutrons whose spins and isospins are coupled to zero (and loosely called alpha particles). By fitting our two free parameters to the properties of nonrelativistic nuclear matter, we have been able to obtain a surprisingly good description of the ground state properties of the light N-alpha nuclei, although the finer details remain beyond our reach.

In view of the widespread occurrence of the basic nonlinear differential equation (2.4) in so many disparate fields of physics, we anticipate a variety of future applications of our model and, in particular, of the approximate analytic results presented in Sec. IV.

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[1]A. C. Merchant and M. P. Isidro Filho, Phys. Rev. A **37**, 1371 (1988).
[2]F. Calogero and A. Degasperis, Phys. Rev. A **11**, 265 (1975).
[3]C. R. Nohl, Ann. Phys. (N.Y.) **96**, 234 (1976).
[4]B. Yoon and J. W. Negele, Phys. Rev. A **16**, 1451 (1977).
[5]J. B. McGuire, J. Math. Phys. **5**, 622 (1964).
[6]D. L. T. Anderson, J. Math. Phys. **12**, 945 (1971).
[7]V. G. Makhan'kov, Phys. Rep. **35**, 1 (1978).
[8]R. Friedberg, T. D. Lee, and A. Sirlin, Nucl. Phys. **B115**, 32 (1976).
[9]T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, London, 1981), p. 128 ff.
[10]V. G. Kartavenko, Yad. Fiz. **40**, 377 (1984) [Sov. J. Nucl. Phys. **40**, 240 (1984)].
[11]L. Dohnert and H. Harris, in *Proceedings of the Ninth Workshop in Nuclear Physics, Buenos Aires, Argentina, 1986,* edited by A. O. Machiavelli, H. M. Sofia, and E. Ventura (World-Scientific, Singapore, 1987), p. 460.
[12]A. S. Kovalev and A. M. Kosevich, Fiz. Nik. Temp. **2**, 913 (1976) [Sov. J. Low Temp. Phys. **2**, 449 (1976)].
[13]S. Coleman, Nucl. Phys. **B262**, 263 (1985).
[14]A. Cohen, S. Coleman, H. Georgi, and A. Manohar, Nucl. Phys. **B272**, 301 (1986).
[15]A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica (Amsterdam) **1D**, 1 (1980); and for a variety of reviews see *Solitons in Condensed Matter Physics,* edited by A. R. Bishop and T. Schneider (Springer, Berlin, 1978).
[16]R. Rajaraman, in *Solitons and Instantons, an Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982), p. 9 ff.
[17]D. Campbell and Y. T. Liao, Phys. Rev. D **14**, 2093 (1976).
[18]D. Campbell, in *Nuclear Physics with Heavy Ions and Mesons,* Proceedings of the Les Houches Summer School of Theoretical Physics, Les Houches, 1977, edited by R. Balian, M. Rho, and G. Ripka (North-Holland, Amsterdam, 1978), Vol. 2, p. 673.
[19]S. A. Moszkowski, Phys. Rev. C **2**, 402 (1970).
[20]D. L. T. Anderson, Ph.D. thesis, University of St. Andrews, Scotland, 1969.
[21]J. W. Negele, Rev. Mod. Phys. **54**, 913 (1982).
[22]J. A. Wheeler, Phys. Rev. **52**, 1083 (1937).
[23]S. Edwards, Proc. Cambridge Philos. Soc. **48**, 652 (1952).
[24]K. Wildermuth and Th. Kanellopoulos, Nucl. Phys. **9**, 449 (1958).
[25]Z. Matthies, V. G. Neudachin, and Yu. F. Smirnov, Nucl. Phys. **49**, 97 (1963).
[26]H. Margenau, Phys. Rev. **59**, 37 (1941).
[27]S. J. Biel, Proc. Phys. Soc. **A70**, 866 (1957).
[28]D. M. Brink, in Proceedings of the International School of Physics "Enrico Fermi," Course XXXVI, Varenna, 1966, edited by C. Bloch (Academic, New York, 1966).
[29]E. Caurier and B. Grammaticos, J. Phys. G **9**, L125 (1983).
[30]W. D. M. Rae and S. Marsh, Phys. Lett. **161B**, 251 (1985).
[31]S. Marsh and W. D. M. Rae, Phys. Lett. B **180**, 185 (1986).
[32]W. Wefelmeier, Naturwissenschaften **25**, 525 (1937).
[33]E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938).
[34]D. Dennison, Phys. Rev. **57**, 454 (1940); **96**, 378 (1954).
[35]S. L. Kameny, Phys. Rev. **103**, 358 (1956).
[36]Y. Gambhir, P. Ring, and P. Schuck, Phys. Rev. Lett. **51**, 1235 (1983).
[37]M. Apostol, I. Bulboacă, F. Cârstoiu, O. Dumitrescu, and M. Horoi, Nucl. Phys. **A470**, 64 (1987).
[38]M. I. Sobel, P. J. Siemens, J. P. Bondorf, and H. A. Bethe, Nucl. Phys. **A251**, 502 (1975).
[39]R. Hofstadter, Annu. Rev. Nucl. Sci. **7**, 231 (1957).
[40]C. W. de Jager, H. de Vries, and C. de Vries, At. Nucl. Data Tables **14**, 479 (1974).
[41]J. M. Finn, H. Crannell, P. L. Hallowell, J. T. O'Brien, and S. Penner, Nucl. Phys. **A274**, 28 (1976).
[42]A. H. Wapstra and G. Audi, Nucl. Phys. **A432**, 1 (1985).
[43]M. T. Johnson and J. W. Clark, KINAM **2**, 3 (1980).