PHYSICAL REVIEW C VOLUME 38, NUMBER 3 SEPTEMBER 1988

Competing coexistence models in medium-mass nuclei

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(Received 13 October 1987)

Simple phenomenological two-state coexistence models have previously been developed to describe the ground state and some low-lying $0^{+'}$ excited state in isotopic chains of even-even medium-mass nuclei. Specifically, these models allow for mixing between a basis ground state and a basis 0^+ intruder state to form the wave functions for the physical ground state and $0^{+'}$ excited state. The amount of data accounted for by these models is amazing, considering their simplicity. However, these two-state phenomenological models contain inconsistencies that have previously been overlooked. We point out some of these inconsistencies and emphasize that they can be removed by a slight generalization of the two-state coexistence model, without destroying agreement between data and model predictions.

I. INTRODUCTION

It is well known that complete *unrestricted* shell-model calculations for nuclei beyond the *sd* shell are impractical because of the large dimensions of the model spaces. Only recently,[1] the early stages of such large scale shell-model calculations have been successfully developed in that the coefficients of fractional parentage and the single-shell matrix elements for *fp*-shell nuclei in isospin formalism have been calculated using a newly formulated Gramian method.

Girod and Grammaticos[2] performed Hartree-Fock and Hartree-Fock-Bogolyubov calculations of the static ground-state properties of a number of nuclei ranging from ^{12}C to ^{240}Pu, and including the germanium isotopes $^{68-76}$Ge. Those calculations contained triaxial self-consistent symmetries, but no statically triaxial shapes were found for germanium. Rather, the results suggested a transition from an oblate shape in ^{68}Ge to prolate in ^{76}Ge, with ^{72}Ge being spherical.

Similar self-consistent calculations have been performed[3] for nuclei around zirconium. In both cases, no information about transition strengths or particle-transfer amplitudes is available. Such calculations are of little use in attempts to explain such data.

Kumar[4] performed dynamical-deformation calculations for the germanium isotopes, and obtained some kind of a transition from light mass to heavy mass, but the magnitude of the effect was underestimated. While, in principle, "two-nucleon transfer amplitudes can be computed in that model, they are not currently available."[5]

It is frequently necessary to develop simple phenomenological models when describing medium-mass nuclei. An example of such a model as applied to medium-mass even-even nuclei is a two-state coexistence model developed to describe the ground state of the nucleus and some low-lying $0^{+'}$ excited state. These models assume dominant mixing between a basis 0^+ ground state and a basis 0^+ intruder state to form the wave functions for the physical ground state and some $0^{+'}$ excited state. Specifically, if $\Psi_A[{}^AX(\text{g.s.})]$ and $\Psi_A[{}^AX(0^{+'})]$ represent the orthonormal wave functions for the *physical* ground state and excited $0^{+'}$ state in nucleus ^AX and if ϕ_g^A and ϕ_e^A represent an orthonormal pair of basis ground and excited states, one writes

$$\begin{aligned}\Psi_A[{}^AX(\text{g.s.})]&=\alpha_A\phi_g^A+\beta_A\phi_e^A\ ,\\ \Psi_A[{}^AX(0^{+'})]&=\beta_A\phi_g^A-\alpha_A\phi_e^A\ ,\end{aligned}\tag{1}$$

with $\alpha_A^2+\beta_A^2=1$. Applications of these coexistence two-state wave functions to the germanium,[6-10] zirconium,[11-25] and molybdenum[23,26] isotopes show some very nice agreement with experiment. However, in many cases, there exist serious disagreements that have been overlooked. The purpose of this report is to point out these inconsistencies and to emphasize that the conflict between model and experiment does not lie in the two-state model assumption, but rather in the choice of basis states.

II. THE SUCCESSES AND FAILURES OF EARLY TWO-STATE COEXISTENCE MODELS FOR THE GERMANIUM ISOTOPES

Early attempts to use a two-state coexistence model for describing the ground state and 0_2^+ state in the germanium isotopes were those of Fournier *et al.*[6] and Monahan and Arns.[27] The successes and failures of these attempts are described in a review article of Vergnes.[9] Vergnes showed in that article that the neutron-configuration-mixed wave functions of Fournier, although successful in describing the weak population of the ^{72}Ge(0_2^+) state in one-neutron pickup, are not consistent (at least in a simple way) with one-proton stripping to the same state. In addition, there exists conflicting evidence between these simple wave functions and measured $1p_{3/2}$ proton occupancies for the ground states of ^{72}Ge and ^{71}Ga and $1p_{1/2}, 0g_{9/2}$ neutron occupancies for the ground state of ^{71}Ga.

Attempts[7–10] to resurrect the two-state model as applied to ^{72}Ge(g.s.) and ^{72}Ge(0_2^+) were made by Vergnes, van den Berg, Ardouin, and others. They turned around the argument of Fournier and assumed that the 40 neutrons in ^{72}Ge form a reasonable closed core with configuration mixing in the *protons*. A similar idea[11–19] applied to ^{90}Zr many years earlier had proven to be relatively successful. Specifically in ^{71}Ga and ^{72}Ge, Vergnes *et al.* write for the physical wave functions:

$$\Psi_{71}[^{71}\mathrm{Ga(g.s.)}] = a_{71}\pi(1p_{3/2})^3_{3/2} + b_{71}\pi(1p_{3/2})^1_{3/2}\pi(0f_{5/2})^2_0 \,, \quad (2a)$$

$$\Psi_{72}[^{72}\mathrm{Ge(g.s.)}] = \alpha_{72}\pi(1p_{3/2})^4_0 + \beta_{72}\pi(1p_{3/2})^2_0\pi(0f_{5/2})^2_0 \,, \quad (2b)$$

$$\Psi_{72}[^{72}\mathrm{Ge}(0_2^+)] = \beta_{72}\pi(1p_{3/2})^4_0 - \alpha_{72}\pi(1p_{3/2})^2_0\pi(0f_{5/2})^2_0 \,. \quad (2c)$$

These still satisfy (by orthogonality and weak coupling) the fact that the ^{73}Ge(p,d)^{72}Ge reactions does not populate the 0_2^+ state in ^{72}Ge. The value of a_{71} is calculated from knowledge of the $1p_{3/2}$ proton-spectroscopic-strength data measured in the ^{71}Ga(d,^{3}He) reaction[28] yielding $3a_{71}^2 + b_{71}^2 = 2.76$. This equation, along with the normalization $a_{71}^2 + b_{71}^2 = 1$ gives $a_{71}^2 = 0.88$ and $b_{71}^2 = 0.12$.

To make these proton wave functions consistent with the large $\sigma(0_2^+)/\sigma$(g.s.) ratio in the ^{71}Ga(^{3}He,d)^{72}Ge reaction, the values of mixing probabilities for the ^{72}Ge wave functions were required to be given by $\alpha_{72}^2 = 0.37$ and $\beta_{72}^2 = 0.63$. As a further test of these parameters, one can calculate the $1p_{3/2}$ proton occupation number in the ground state of ^{72}Ge. From Eq. (2b) one obtains $4\alpha_{72}^2 + 2\beta_{72}^2$ which becomes 2.74 with the α_{72}^2 and β_{72}^2 values above. This is in close agreement with the experimental[29] value of 2.87.

Motivated in part by the observed rapid A dependence of $0f_{5/2}$ proton occupation numbers in the ground states of the even-mass germanium isotopes, the forms of Eqs. (2) were extended to include *all* even-mass germanium nuclei by writing

$$\begin{aligned} \Psi_A[^A\mathrm{Ge(g.s.)}] &= \propto_A \pi(1p_{3/2})^4_0 + \beta_A \pi(1p_{3/2})^2_0\pi(0f_{5/2})^2_0 \,, \\ \Psi_A[^A\mathrm{Ge}(0_2^+)] &= \beta_A \pi(1p_{3/2})^4_0 - \alpha_A \pi(1p_{3/2})^2_0\pi(0f_{5/2})^2_0 \,. \end{aligned} \quad (3)$$

In addition, it was assumed that the neutron configuration of the ground state is the same as that of the 0_2^+ excited state for a given neutron number N. We shall refer to this model for the germanium ground state and 0_2^+ state wave functions as the proton particle-hole (πph) model. The "best-fit" mixing probabilities for the four even-mass germanium and two odd-mass gallium isotopes obtained in Refs. 9 and 10 are summarized in Table I. As shown in Table I of Ref. 9, this simple two-state coexistence model is consistent with several direct-reaction results.

TABLE I. Parameters used in the simple two-state proton excitation model (taken from Refs. 9 and 10).

	Germanium		Gallium	
A	α_A^2	β_A^2	a_{A-1}^2	b_{A-1}^2
70	0.37	0.63	0.80	0.20
72	0.37	0.63	0.89	0.11
74	0.03	0.97		
76	0.03	0.97		
78	0.03	0.97		

Unfortunately, the success of that model is limited because there exist much data that are in conflict with the proton wave functions of Eq. (3) with the extra assumption that $\Psi_A^\nu[^A\mathrm{Ge(g.s.)}] = \Psi_A^\nu[^A\mathrm{Ge}(0_2^+)]$ for a given value of A. In a recent article by Rotbard *et al.*,[30] it is *still reported* that the πph wave functions of Eq. (3) are compatible with the $0f_{5/2}$ proton occupation numbers in the ground states of the stable even-mass germanium isotopes even though *it is obviously not so.* In fact, it is an interesting dichotomy to note that one of the serious flaws in Eq. (3) lies in these proton occupation numbers which were one of the initial motivations behind their development. The $0f_{5/2}$ proton occupation numbers in the ground state of ^{A}Ge (denoted by $\langle 0f_{5/2}\rangle_A$) have been measured by Rotbard *et al.*[29] and are given in Table II (normalized to give a sum of 4 for $\langle 0f_{5/2}\rangle$ and $\langle 1p_{(1/2+3/2)}\rangle$ combined) and plotted in Fig. 1 with the predictions of the wave functions of Eq. (3) based on the α_A^2 and β_A^2 value given in Table I (the Rergl predictions will be discussed later). The mass-dependent trend predicted by the πph wave functions of Eq. (3) is reproduced qualitatively but not quantitatively for the heavier-mass ^{74}Ge and ^{76}Ge. In fact, it is clear from the wave functions of Eq. (3) that $\langle 0f_{5/2}\rangle_A$ has the value $2\beta_A^2$ which is less than or equal to 2 for any value over the full range in β_A and must therefore always disagree with the measurements of the heavier-mass ^{74}Ge and ^{76}Ge.

A more serious conflict between the πph model and ex-

TABLE II. The experimental $0f_{5/2}$ proton occupation numbers $\langle 0f_{5/2}\rangle_A$ in the ground states of the even-mass germanium isotopes normalized so that the sum $\langle 0f_{5/2}\rangle_A + \langle 1p\rangle_A = 4$.

Experimental[a]	Vergnes' model[b]	Rergl model[c]
1.17±0.10	1.26	1.13±0.08
1.30±0.10	1.26	1.33±0.06
2.16±0.10	1.94	2.16±0.05
2.37±0.10	1.94	2.33±0.06
χ^2 value[d] →	$\chi^2 \approx 4.93$	$\chi^2 \approx 0.64$

[a] Taken from Ref. 29.

[b] Using the mixing probabilities of Table I and $\langle 0f_{5/2}\rangle_A = 2\beta_A^2$.

[c] Taken from Ref. 43.

[d] $\chi^2 = (1/N)\sum[(\text{exp.} - (\text{calc.})/(\text{uncer.})]^2$.

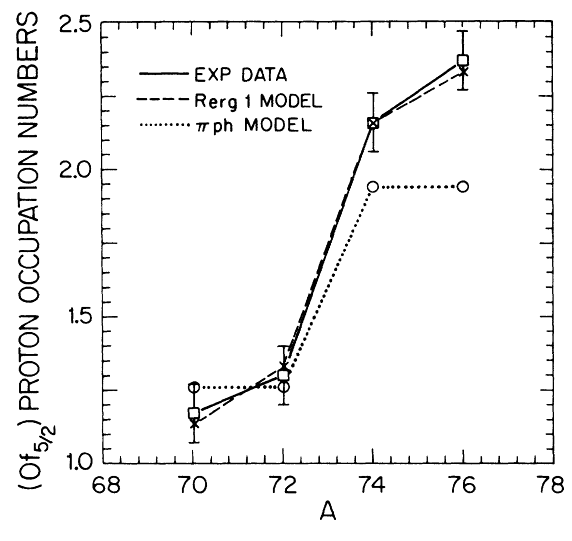

FIG. 1. Plot of the $0f_{5/2}$ proton occupation numbers in the ground states of the even-even stable germanium isotopes (normalized so that $\langle 0f_{5/2}\rangle_A+\langle 1p\rangle_A=4$) vs A, along with the predictions of the Vergnes two-state coexistence model and the improved Rergl coexistence model. The experiment is the solid line, the SML is the dotted line, and the Rergl model is the dashed line.

periment can be found in the two-neutron cross-section ratio data in the $^{A+2}\mathrm{Ge}(p,t)^{A}\mathrm{Ge}$ and $^{A}\mathrm{Ge}(t,p)^{A+2}\mathrm{Ge}$ reactions. To calculate these ratios, we form the $2n$-overlap integral from the wave functions in Eq. (3) with the (t,p) and (p,t) operator sandwiched in between. Since two-neutron transfer operates only on the neutron part of the wave function and it is assumed that $\Psi_A^{\nu}[^{A}\mathrm{Ge(g.s.)}]=\Psi_A^{\nu}[^{A}\mathrm{Ge}(0_2^+)]$, these would divide out leaving only the overlaps of the proton parts. Because these wave functions assume very simple shell-model basis states for the proton part of the wave function, we must necessarily obtain *basis-state* $2n$-transfer overlap ratios of unity for ground→ground and excited→excited $2n$ transitions in the basis states and zero for ground→excited and excited→ground $2n$ transitions in the basis states. With this, the cross-section ratio $\sigma[^{A}\mathrm{Ge}(t,p)^{A+2}\mathrm{Ge}(0_2^+)]/\sigma[^{A}\mathrm{Ge}(t,p)^{A+2}\mathrm{Ge(g.s.)}]$, denoted by T_A^2, becomes (in the πph model)

$$T_A^2=\left[\frac{\alpha_A\beta_{A+2}-\beta_A\alpha_{A+2}}{\alpha_A\alpha_{A+2}+\beta_A\beta_{A+2}}\right]^2 \tag{4}$$

or

$$T_A^2=\left[\frac{x_A-x_{A+2}}{1+x_Ax_{A+2}}\right]^2, \tag{5}$$

with

$$x_A=\alpha_A/\beta_A\ . \tag{6}$$

A similar calculation for the (p,t) cross-section ratio $\sigma[^{A+2}\mathrm{Ge}(p,t)^{A}\mathrm{Ge}(0_2^+)]$ divided by $\sigma[^{A+2}\mathrm{Ge}(p,t)^{A}\mathrm{Ge(g.s.)}]$, which we denote by P_A^2, yields the same results as in Eq. (4) so that

$$P_A^2=T_A^2\ \ (\text{all } A)\ . \tag{7}$$

It is interesting to note that this condition on the cross-section ratios is recognized in Ref. 10 [Eq. (2)] and believed to produce a "reasonable" fit to the experimental data. In a "least-squared" sense, the "best-fit" calculated values (in this πph limit) of the cross-section ratios [denoted by T_A^2 and P_A^2 in (t,p) and (p,t), respectively] as functions of the *experimental* cross-section ratios with their uncertainties [denoted by $T_{A0}^2\pm\Delta T_{A0}^2$ and $P_{A0}^2\pm\Delta P_{A0}^2$ in (t,p) and (p,t) respectively] are obtained by minimizing the "least-squares" function

$$\chi^2=(1/N)\sum_{A=A'}^{A''}\{[(T_A^2-T_{A0}^2)/\Delta T_{A0}^2]^2+[(P_A^2-P_{A0}^2)/\Delta P_{A0}^2]^2\}\ , \tag{8}$$

subject to the condition $T_A^2-P_A^2=0$ for all A. The result of this minimization leads to the best-fit values given by

$$T_A^2=P_A^2=\frac{T_{A0}^2/(\Delta T_{A0}^2)^2+P_{A0}^2/(\Delta P_{A0}^2)^2}{1/(\Delta T_{A0}^2)^2+1/(\Delta P_{A0}^2)^2}\ , \tag{9}$$

which is just the weighted average of the experimental (t,p) and (p,t) measurements. In Table III we present the experimental[31–37] (corrected for Q-value effects) and model-calculated values of the $\mathrm{Ge}(p,t)$ and $\mathrm{Ge}(t,p)$ cross-section ratios using the mixing probability parameters quoted in Refs. 9 and 10 (i.e., Table I) along with the best-fit calculations deduced from Eq. (9) above. We also plot these ratios versus A in Fig. 2. It is clear from Table III that these experimental ratios do not satisfy the condition of Eq. (7), with the most notable exception occur-

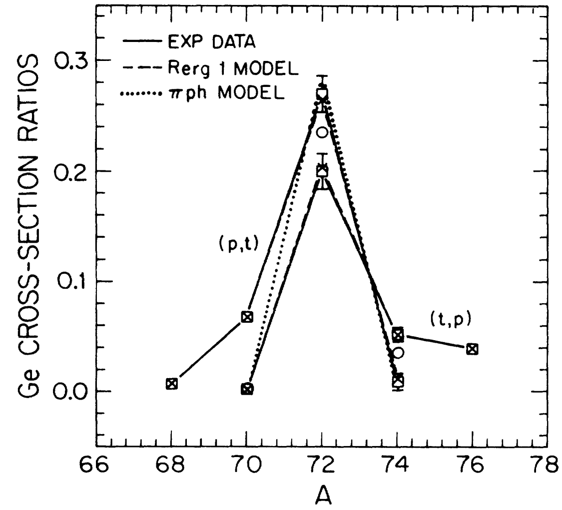

FIG. 2. Plot of the cross-section ratios T_A^2 and P_A^2 vs A for germanium along with the predictions of the Vergnes two-state coexistence model and the improved Rergl coexistence model. Also plotted are the best-fit SML results. The experiment is the solid line, the Vergnes πph limit is the dotted line, the Rergl model is the dashed line and the best-fit SML results are the open circles with the point at $A=70$ overlapping the experimental (t,p) data.

TABLE III. Experimental and calculated two-neutron 0_2^+/g.s. cross-section ratios in germanium, T_A^2 and P_A^2, using both the Vergnes' mixing probabilities in Table I and the best-fit SML results.

Ratio	Experimental	Vergnes' πph model	Best-fit SML
P_{68}^2	0.0071 ± 0.0007		
P_{70}^2	0.0680 ± 0.0040	0.0	0.0030
P_{72}^2	0.270 ± 0.016	0.2847	0.2350
P_{74}^2	0.0090 ± 0.0072	0.0	0.0354
T_{70}^2	0.0020 ± 0.0005	0.0	0.0030
T_{72}^2	0.200 ± 0.016	0.2847	0.2350
T_{74}^2	0.0520 ± 0.0057	0.0	0.0354
T_{76}^2	0.0391 ± 0.0040	0.0	
	χ^2 value →	$\chi^2\approx68.0$	$\chi^2\approx49.9$

ring at $A=70$. Specifically, of the ratios of experimental cross-section ratios T_{A0}^2/P_{A0}^2, given by 0.029 ± 0.008, 0.741 ± 0.074 and 2.78 ± 2.3 for $A=70$, 72, and 74, respectively, only the $A=74$ result is equal to unity within the experimental uncertainties. These are plotted versus neutron number N in Fig. 3 along with the simple prediction [Eq. (7)] of unity. The large uncertainty in T_{740}^2/P_{740}^2 is due to the large uncertainty in P_{740}^2 (see Table III). Also shown in that figure are the same observables for zirconium and molybdenum which will be discussed in a later section.

If we consider the cross-section ratios directly, we see that according to Vergnes' numbers in Table , $x_{70}=x_{72}$ and $x_{74}=x_{76}=x_{78}$ and so all T_A^2 (and hence P_A^2) values are predicted to be identically zero except for $T_{72}^2=P_{72}^2$, for which Vergnes gets the value 0.28. Only T_{72}^2 and P_{72}^2 are in approximate agreement with experiment. Even the best-fit values (open circles in Fig. 2) are outside the experimental uncertainty for $A=72$ and 74. It is clear then that there are serious conflicts between the πph wave functions of Eq. (3) and the data consisting of $0f_{5/2}$ proton occupation numbers in the ground states of ^{A}Ge and two-neutron cross-section ratios in the even germanium isotopes.

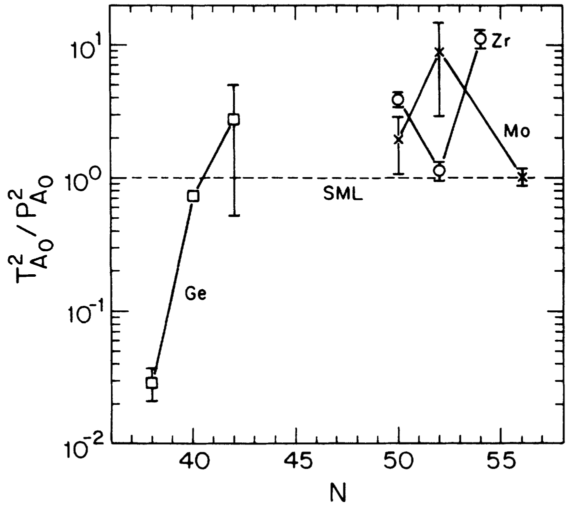

FIG. 3. Plot of the experimental cross-section ratio *ratios* T_{A0}^2/P_{A0}^2 vs N for the germanium, zirconium, and molybdenum isotopes as compared to unity required by the SML [Eq. (7) in the text].

As a further demonstration of the ineptness of the two-proton excitation wave functions of Eq. (3), we focus on the ^{70}Zn(^{16}O,^{14}C)^{72}Ge reaction that was measured recently.[38] As shown in that reference, the wave functions of Eq. (3) are not capable of reproducing the extreme weakness (≈0.01) of the measured 0_2^+/g.s. cross-section ratio in the above two-proton stripping reaction. Specifically, the calculated results are at least more than an order of magnitude too large.

It is clear then that *both* the $2n$ excitation wave functions of Fournier *and* the $2p$ excitation wave functions of Eq. (3) are in conflict with certain experimental data. As suggested in Refs. 38 and 39, perhaps an α-particle excitation is the correct way of viewing the wave functions at least for the low-lying 0^+ states in ^{72}Ge.

III. THE GENERALIZED TWO-STATE COEXISTENCE MODEL (RERG1)

In each of the above refinements of the two-state analysis of the even germanium isotopes, existing conflicts with certain data were resolved, but at the same time new conflicts arose. These considerations force one to investigate whether the two-state coexistence model is capable of describing the physical ground state and $0^{+'}$ excited state in the even germanium isotopes. Perhaps a third basis state with comparable strength is mixing also with the basis ground state and excited state. Then a two-state coexistence model would be inadequate for describing the physical ground state and 0^+ excited state in the even germanium isotopes. This is surely a possibility. However, it does complicate the original simplicity of the two-state coexistence model. As an alternate possibility, perhaps the choice of the form for the basis states ϕ_g^A and ϕ_e^A has so far been two restrictive.

In an attempt to keep only two-state mixing, we have left[40–42] Eq. (1) in its original form but made no attempt to present an explicit form for ϕ_g^A and ϕ_e^A. Our goal merely was to investigate whether a two-state coexistence model with some reasonable assumptions is capable of describing both the $0f_{5/2}$ proton-occupation-number data in ^{A}Ge(g.s.) and the $0^{+'}$/g.s. two-neutron cross-section ratio data in a comprehensive and quantitative way.

In Eq. (1) we make no assumptions about proton-neutron factorization in the basis-state wave function and, as mentioned, we offer no form for the basis-state wave functions. The parameter α_A^2 serves as a measure of the mixing between ϕ_g^A and ϕ_e^A and we assume only that the $2n$-transfer overlap ratios between the basis states ϕ_g^A and ϕ_e^A (denoted by r_A, s_A, and R_A and shown schematically in Fig. 1 of Ref. 40) are A independent and given by r, s, and R, respectively, and $r=s$. We denote (as explained in Ref. 41) the two-state coexistence model described by Eq. (1) and the above two assumptions as the *Rergl* model. Using the Rergl model, we have been able to describe the $0^{+'}$/g.s. cross-section ratios in the Ge(p,t) and Ge(t,p) reactions[40,42] along with the (p,t) and (t,p) reactions[41] on a variety of other nuclei for which enough data exist for a comprehensive comparison between model and experiment. The results of the Rergl calculation for (p,t) and (t,p) ratios in germanium[40–42] are given in Table IV and plotted in Fig. 2 where we obtain almost an exact fit to the T_A^2 and P_A^2 ratio data. In addition, further application[43] of the Rergl two-state coexistence wave functions to the $0f_{5/2}$ proton occupation numbers in the ground states of the even-mass germanium isotopes shows (Table II and Fig. 1) also a nearly exact fit to the data. In both Figs. 1 and 2, there exists only one free parameter R which is restricted to a finite range of values. Specifying a value of R is equivalent to making a specific choice of basis states.

The detailed mathematical development of the Rergl model is presented in Refs. 40 and 41 and as mentioned, the detailed application of the Rergl model to the germanium isotopes is given in Refs. 41 and 42. One major result of those developments is that the two-state Rergl coexistence model is consistent with the two-neutron cross-section ratio data *if and only if* the experimental data satisfy

$$L_A \equiv \frac{(T_{A+2}-P_A)(T_{A-2}-P_A)(1+T_A^2)}{(P_{A+2}-T_A)(P_{A-2}-T_A)(1+P_A^2)} = 1 , \qquad (10)$$

for all A. This follows only from the two Rergl assumptions about the $2n$-transfer overlap ratios in the basis state and the requirement that the model be consistent with the data. Equation (10) serves as an *immediate* test as to whether or not the two-state coexistence model with the above Rergl assumptions can be made consistent with the $0^{+'}$/g.s. two-neutron cross-section ratio data. By minimizing the "least-squared" function of Eq. (8) (with P^2 and T^2 replaced by P and T, respectively) subject to the conditions $L_A = 1$ for $A = A'+2, A'+4, \ldots, A''-2$, one obtains the best-fit Rergl calculations (Table IV) along with the best choice of signs in the square roots T_A and P_A. Note that the restrictions on the data implied by Eq. (10) are less restrictive than those of Eq. (7) (i.e., $P_A^2 = T_A^2$).

When Eq. (10) is satisfied for all A, we then obtain a *constraint* equation between the basis-state $2n$-transfer overlap ratios r and R, given by

$$r^2 = R + K_A(R+1)^2 , \qquad (11)$$

where K_A is defined by Eq. (24) of Ref. 40, is independent of A, is completely determined by the experimental data, and can be chosen to be negative. In other words, assuming that the two-state Rergl coexistence model is sufficient for quantitatively describing the cross-section ratio data in (t,p) and (p,t) [i.e., Eq. (10) is satisfied for all A], we allow the data [via Eq. (11)] to dictate the relationship between the $2n$-transfer overlap ratio between the basis states. In this sense Eq. (11) becomes a very simple test as to whether a particular choice of basis states is consistent with the two-neutron transfer data.

TABLE IV. Experimental and calculated two-neutron 0_2^+/g.s. cross-section ratios in germanium, T_A^2 and P_A^2, using both the Vergnes' πph and Rergl models.

Ratio	Experimental	πph model[a]	Rergl model[b] ($0^{+'}=0_2^+$ in ^{76}Ge)	($0^{+'}=0_3^+$ in ^{76}Ge)
P_{68}^2	0.0071±0.0007		0.0071	0.0071
P_{70}^2	0.0680±0.0040	0.0	0.0682	0.0680
P_{72}^2	0.270±0.016	0.2847	0.2675	0.2740
P_{74}^2	0.0090±0.0072	0.0	0.0114	0.0060
T_{70}^2	0.0020±0.0005	0.0	0.0020	0.0020
T_{72}^2	0.200±0.016	0.2847	0.2032	0.1950
T_{74}^2 [c]	0.0520±0.0057	0.0	0.0516	
T_{74}^2 [d]	0.0250±0.0034			0.0250
T_{76}^2 [e]	0.0391±0.0040	0.0	0.0391	0.0391
T_{76}^2 [f]	0.0190±0.0020		0.0190	0.0190
χ^2 value →		$\chi^2 \approx 68.0$	$\chi^2 \approx 0.023$	$\chi^2 \approx 0.042$

[a] Using the mixing probabilities in Table I.

[b] References 40–42.

[c] Using $0^{+'}=0_2^+$ in ^{76}Ge.

[d] Using $0^{+'}=0_3^+$ in ^{76}Ge.

[e] Using $0^{+'}=0_2^+$ in ^{78}Ge.

[f] Using $0^{+'}=0_3^+$ in ^{78}Ge.

TABLE V. The w_A and v_A Rergl parameters. Note that in the SML one must necessarily have $v_A \equiv 0$ and $w_A \equiv -\frac{1}{2}$.

	$0^{+'}=0_2^+$ in ^{76}Ge		$0^{+'}=0_3^+$ in ^{76}Ge	
A	v_A	w_A	v_A	w_A
70	+0.0651	−0.5352	+0.0629	−0.5292
72	+0.0501	−0.5545	+0.0502	−0.5478
74	−0.0207	−0.5710	−0.0141	−0.5679
76	−0.0452	−0.5586	−0.0314	−0.5618

For instance, Eq. (11) clearly shows that the πph wave functions of Eq. (3), which require $R=1$ and $r=0$, can hold *if and only if* $K_A=-\frac{1}{4}$ which[40] is possible only when $P_A+T_A=0$, or in terms of the measurable quantities P_A^2 and T_A^2, $P_A^2=T_A^2$. This is precisely the result obtained earlier in Eq. (7).

Since r and R are related via Eq. (11), the mixing probability α_A^2 becomes a function of the *one* parameter R and is given by Eq. (33) of Ref. 40. In the Rergl analysis of the even germanium isotopes, we had assumed that the physical mixed $0^{+'}$ excited state was the 0_2^+ in ^{70,72,74}Ge, while for ^{76}Ge we considered two calculations—one with $0^{+'}=0_2^+$ and one with $0^{+'}=0_3^+$. The parameters w_A and v_A (which are $-\frac{1}{2}$ and 0, respectively, in the simple limit of $R=1$ and $r=0$) are defined by Eqs. (27) and (28) of Ref. 40 and depend only on the values of T_A and P_A. The mixing probabilities α_A^2 (which are functions of R) and the values of w_A and v_A that result from that analysis and that produce the fits in Tables II and IV are given in Table V and Figs. 3 and 7 of Ref. 42.

IV. THE SIMILARITIES AND DIFFERENCES BETWEEN THE RERG1 MODEL AND EARLIER TWO-STATE COEXISTENCE MODELS

In discussing the similarities and differences between the Rergl model and earlier πph models of Eq. (3), we must examine the expressions for T_A and P_A since they represent the starting (and ending) points in both calculations. In the limit of Eq. (3), these become

$$T_A=\frac{x_A-x_{A+2}}{1+x_Ax_{A+2}} \quad\text{and}\quad P_A=\frac{x_{A+2}-x_A}{1+x_Ax_{A+2}}\ , \tag{12}$$

with $x_A=\alpha_A/\beta_A$. In the more general case in which Eq. (1) applies with the Rergl assumptions about the $2n$-transfer overlap ratios in the basis states, we have

$$T_A=\frac{x_A+(1-x_Ax_{A+2})r-x_{A+2}R}{R+(x_{A+2}+x_A)r+x_{A+2}x_A}\ , \tag{13a}$$

and

$$P_A=\frac{x_{A+2}+(1-x_Ax_{A+2})r-x_AR}{R+(x_{A+2}+x_A)r+x_{A+2}x_A}\ , \tag{13b}$$

Thus the πph model as applied earlier to germanium is one that assumes *both* $R=1$ *and* $r=0$. In general, we refer to the limit with $R\equiv1$ *and* $r\equiv0$ as the simple-model limit (SML) of which the πph model is an example. As mentioned, in the more general Rergl model, this is possible only when $K_A=-\frac{1}{4}$. In the standard interacting-boson model (IBM), in which[44] the basis state ϕ_g^A is vibrational and ϕ_e^A is rotational, the quantity r is zero and R is much smaller than unity.

As noted in Refs. 40 and 45 and proven in Appendix D of Ref. 41, it is possible to transform via a unitary transformation from the basis set $\{\phi_g^A,\phi_e^A\}$ to an appropriate basis set $\{\phi_g'^A,\phi_e'^A\}$, such that

$$\begin{aligned}\Psi_A[{}^AX(\text{g.s.})]&=\alpha'_A\phi_g'^A+\beta'_A\phi_e'^A\ ,\\ \Psi_A[{}^AX(0^{+'})]&=\beta'_A\phi_g'^A-\alpha'_A\phi_e'^A\ ,\end{aligned} \tag{14}$$

which will lead to calculated two-neutron cross-section ratios

$$T_A=\frac{x'_A-x'_{A+2}\Omega}{\Omega+x'_Ax'_{A+2}} \quad\text{and}\quad P_A=\frac{x'_{A+2}-x'_A\Omega}{\Omega+x'_Ax'_{A+2}}\ , \tag{15}$$

where $x'_A=\alpha'_A/\beta'_A$,

$$\Omega=\frac{1-\Delta}{1+\Delta}\ , \tag{16}$$

and $\Delta=\pm(1+4K_A)^{1/2}$ (remember that K_A is independent of A).

If one compares Eq. (15) with Eq. (12), one sees that in this "prime" basis, we almost have the SML. In fact, it is clear that the deviation of Ω from unity (or the deviation of Δ from zero) is a measure of the validity of the SML. So we see that the Rergl generalization proposed in Ref. 40 is "almost" just a unitary transformation away from the simpler πph models. However, there exist (see Appendix D in Ref. 41) an infinite number of these unitary transformation matrices $\overline{U}(R)$ all described in terms of the one parameter R. In addition, it is important to note that in the prime basis, one is requiring that the *vertices* of the ellipse represented by Eq. (16) in the R-r plane determine the values of Ω, Δ, and x'_A (i.e., when $r=0$ in Eq. (16), then $R=\Omega$). This, however, may not always be possible for a given assumed phase convention.

If in Eq. (11) we try requiring that $R=1$, but then one must insist that $r=\Delta$ and Eqs. (13a) and (13b) become (with $x''_A=x_A$ *evaluated* at $r=\Delta$ and $R=1$)

$$T_A=\frac{x''_A+(1-x''_Ax''_{A+2})\Delta-x''_{A+2}}{1+(x''_{A+2}+x''_A)\Delta+x''_Ax''_{A+2}}\ , \tag{17a}$$

and

$$P_A=\frac{x''_{A+2}+(1-x''_Ax''_{A+2})\Delta-x''_A}{1+(x''_{A+2}+x''_A)\Delta+x''_Ax''_{A+2}}\ . \tag{17b}$$

In this sense, the deviation of Δ from zero is again shown to be a measure of how well the SML will reproduce the $0^{+'}$/g.s. cross-section ratios in the (p,t) and (t,p) reactions.

To summarize, one can assume a simple two-state coexistence analysis of these cross-section ratios with $r=0$, provided we let $R=\Omega$, or with $R=1$ provided we let $r=\Delta$, but we may not allow for *both* $r=0$ *and* $R=1$

unless the data is very symmetric in that $P_A^2=T_A^2$ for all A. In this sense either the deviation of Δ from zero or the deviation of Ω from unity become very sensitive tests as to whether SML wave functions will consistently describe the data. If the experimental ratios T_{A0}^2 and P_{A0}^2 are equal to within their uncertainties, then the best the SML can do in a least-squared sense is given by Eq. (7). For the germanium data using $0^{+'}=0_3^+$ in ^{76}Ge, we have [40–42] $K_A(\mathrm{Ge})=-0.2452\pm0.0009$, giving $\Delta(\mathrm{Ge})=0.139\pm0.013$ and $\Omega(\mathrm{Ge})=0.756\pm0.020$, results clearly not equal to zero and one, respectively.

In Refs. 40 and 41 it is shown that the quantity $\epsilon_{A,A'}$, defined by

$$\epsilon_{A,A'}=\frac{x_A-x_{A'}}{1+x_Ax_{A'}}\ , \tag{18}$$

depends only on T_A and P_A and is given by

$$\epsilon_{A,A'}=\frac{w_A-w_{A'}}{v_A+v_{A'}}=\frac{v_{A'}-v_A}{1+w_A+w_{A'}}\ . \tag{19}$$

We note that $\epsilon_{A,A'}$ are elements of a skew-symmetric matrix that satisfy the equations

$$\epsilon_{A,A'}+\epsilon_{A',A}=0 \tag{20}$$

and

$$\epsilon_{A,A''}=\frac{\epsilon_{A,A'}+\epsilon_{A',A''}}{1-\epsilon_{A,A'}\epsilon_{A',A''}}\ . \tag{21}$$

In terms of $\epsilon_{A,A'}$ one can show that in the Rergl model, T_A and P_A become

$$T_A=+\epsilon_{A,A+2}+\frac{v_A(\epsilon_{A,A+2}^2+1)}{w_A-v_A\epsilon_{A,A+2}} \tag{22a}$$

and

$$P_A=-\epsilon_{A,A+2}+\frac{v_{A+2}(\epsilon_{A,A+2}^2+1)}{w_A+v_{A+2}\epsilon_{A,A+2}}\ , \tag{22b}$$

while in the SML [i.e., Eq. (12)], one has

$$T_A=+\epsilon_{A,A+2}\quad\text{and}\quad P_A=-\epsilon_{A,A+2}\ . \tag{23}$$

In this sense, Eqs. (22a) and (22b) represent higher-order corrections to Eq. (23). In Table VI we present the $\epsilon_{A,A'}$ values deduced from Table I and Eq. (18) (the πph model) and those deduced from Table V and Eq. (19) (the Rergl model) for the case in which the physical excited $0^{+'}$ state mixed with the ground state is the 0_2^+ state in all even Ge isotopes. Note that in the SML, one has $v_A=0$ and $w_A=-\frac{1}{2}$ for all A and so Eq. (19) does not produce $\epsilon_{A,A'}$. Instead, the defining equation [Eq. (18)] must be used. In addition, one notes that under the SML, the best-fit values of $\epsilon_{A,A+2}$ are obtained from Eq. (23) with the additional $\epsilon_{A,A'}$ values obtained via Eq. (21).

TABLE VI. The $\epsilon_{A,A'}$ parameters in the Rerg 1 model (assuming that $0^{+'}=0_2^+$ in all ^{A}Ge) and in Vergnes' πph model.

A / A'	70	72	74	76
	Vergnes' πph results[a] for $\epsilon_{A,A'}$			
70	0.0	0.0	+0.5335	+0.5335
72	0.0	0.0	+0.5335	+0.5335
74	−0.5335	−0.5335	0.0	0.0
76	−0.5335	−0.5335	0.0	0.0
	Rergl results[b] from $\epsilon_{A,A'}$			
70	0.0	+0.1675	+0.8063	+1.1759
72	−0.1675	0.0	+0.5612	+0.8367
74	−0.8063	−0.5612	0.0	+0.1882
76	−1.1759	−0.8367	−0.1882	0.0

[a] Using the mixing probabilities in Table I and $x_A=\alpha_A/\beta_A\geq0$ and Eq. (18) in the text.

[b] Using the w_A and v_A parameters in Table V and Eq. (19) in the text.

V. COEXISTENCE MODELS AS APPLIED TO ZIRCONIUM AND MOLYBDENUM

A. Zirconium

Early two-state mixing models for the zirconium isotopes[11–25] were first applied to the low-lying 0^+ states of ^{90}Zr by Bayman *et al.*[11] These existed many years before similar two-state mixing models were postulated in the germanium isotopes.[6–10] Early in the life of the shell model it was discovered that 50 neutrons form a very stable closed core. It was, of course, then reasonable to assume that only the protons are active in the ground state and first excited 0_2^+ of ^{90}Zr and thus allow for only two valence protons that are distributed in the $1p_{1/2}$ and $0g_{9/2}$ subshells. Since then many others[12–20] have found this simple idea on ^{90}Zr consistent with a variety of experimental data.

The same idea was later extended[18,20–25] to include all the zirconium isotopes. In particular, Saha *et al.*[22] and Tickle *et al.*[24] used a two-state coexistence model[18] for describing the ground state *and* 0_2^+ excited state in ^{A}Zr in an attempt to account for cross-section ratio data in the $^{A+2}$Mo(^{6}Li,^{8}Be)^{A}Zr and ^{100}Mo(d,^{6}Li)^{96}Zr reactions. Their results show that proton configurations given by

$$\begin{aligned}\Psi_A^\pi[\,^A\mathrm{Zr(g.s.)}]&=\gamma_A\pi(1p_{1/2})_0^2+\delta_A\pi(0g_{9/2})_0^2\ ,\\ \Psi_A^\pi[\,^A\mathrm{Zr}(0_2^+)]&=\delta_A\pi(1p_{1/2})_0^2-\gamma_A\pi(0g_{9/2})_0^2\ ,\end{aligned} \tag{24}$$

for zirconium and a *pure* $[(1p_{1/2})^2(0g_{9/2})^2]_0$ proton configuration for the molybdenum ground state lead to 0_2^+/g.s. two-proton and α-transfer cross-section ratios in reasonable agreement with experiment.

Following Preedom *et al.*[18] and Cates *et al.*,[20] Saha *et al.*[22] extended the zirconium wave functions to include the 0_2^+ excited state in ^{A}Zr and made the additional assumption that the *neutron* configurations of the zirconium excited states for *each* even zirconium isotopes be the same as those of the ground states (i.e., $\Psi^{\nu}_{A}[{}^{A}\mathrm{Zr(g.s.)}]=\Psi^{\nu}_{A}[{}^{A}\mathrm{Zr}(0_2^+)]$). Van den Berg *et al.*[25] used these same zirconium proton wave functions of Eq. (24) and *pure* $[(1p_{1/2})^2(0g_{9/2})^2]_0$ proton molybdenum ground-state wave functions to describe the $^{A+4}\mathrm{Mo}(d,{}^{6}\mathrm{Li})^{A}\mathrm{Zr}$ reactions. However, it is this *oversimplification* of the neutron part of the zirconium wave functions that is in conflict with experiment, for then the SML above applies so that [via Eq. (7)] the predicted 0_2^+/g.s. cross-section ratios in the $^{A+2}\mathrm{Zr}(p,t)$ and $^{A}\mathrm{Zr}(p,t)$ must be identical. The (p,t) and (t,p) reactions on zirconium have been measured[21,46–49] and the $\sigma(0_2^+)/\sigma(\mathrm{g.s.})$ ratios are summarized in Table VII and Fig. 4 along with the predictions of the SML wave functions using the mixing probabilities taken from Refs. 18, 20, 22, and 24 and summarized in Ref. 25, and the Rergl mixing probabilities deduced in Ref. 41 and reproduced here in Fig. 5. As is evident in Table VII and Fig. 4, the condition $P_A^2=T_A^2$ (required by the proton wave functions of Eq. (24) and the *additional* assumption that $\Psi^{\nu}_{A}[{}^{A}\mathrm{Zr(g.s.)}]=\Psi^{\nu}_{A}[{}^{A}\mathrm{Zr}(0_2^+)]$ for given A) is not satisfied within the experimental uncertainties for any value of A. In particular, the cross-section ratio ratios, T_{A0}^2/P_{A0}^2 (plotted versus neutron number N in Fig. 3), are 3.88 ± 0.49, 1.14 ± 0.17, and 11.1 ± 1.7 for $N=50$, 52, and 54, respectively. None of these ratios overlaps unity with the largest deviations occurring at $N=50$ and 54. The best-fit results using the SML in the Zr isotopes are [via Eq. (9)] given by $T_A^2=P_A^2=0.014$, 0.036, and 0.042 for $A=90$, 92, and 94, respectively. In terms of the Rergl parameters K_A, Ω, and Δ, we find for the zirconium data[41] $K_A(\mathrm{Zr})=-0.2385\pm0.0030$ giving $\Delta(\mathrm{Zr})=0.215\pm0.028$ and $\Omega(\mathrm{Zr})=0.647\pm0.038$, so that $\Delta\neq0$ and $\Omega\neq1$ within the uncertainty of the data.

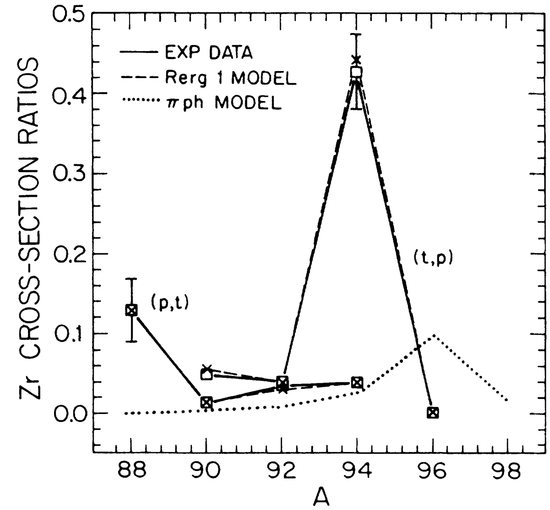

FIG. 4. Plot of the cross-section ratios T_A^2 and P_A^2 vs A for zirconium along with the predictions of the van den Berg πph two-state coexistence model and the improved Rergl coexistence model. The experiment is the solid line, the πph limit is the dotted line, and the Rergl model is the dashed line.

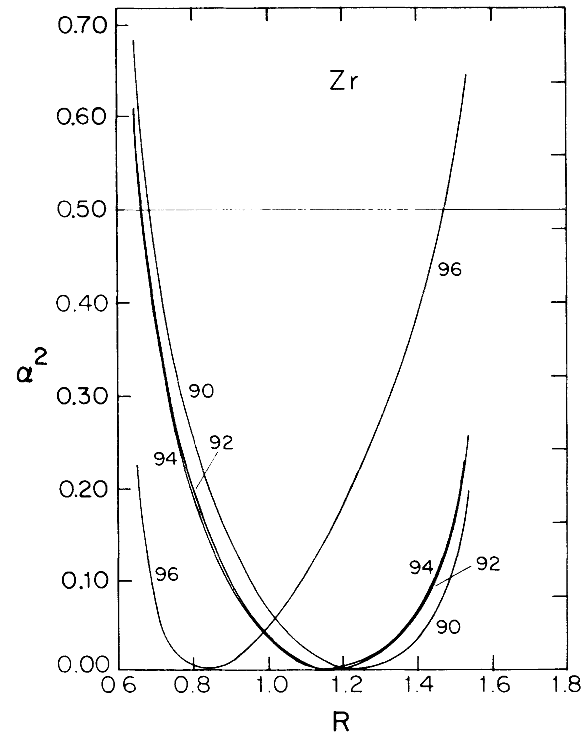

FIG. 5. Plot of the mixing probabilities α_A^2 as a function of R in the Rergl model as applied to the zirconium isotopes assuming $0^{+\prime}=0_2^+$ in all $^{A}\mathrm{Zr}$.

The fact that the zirconium wave functions of Eq. (24) are not consistent with other data was recognized earlier,[50] where it was claimed that an inclusion of $1p_{3/2}$ core excitation in $^{90}\mathrm{Zr}$ was necessary to remove a factor-of-six discrepancy between measured and calculated lifetime for the decay of the first excited 0^+ state. Specifically, they wrote for the ground state of $^{90}\mathrm{Zr}$

$$\Psi^{\pi}_{90}[{}^{90}\mathrm{Zr(g.s.)}]=\gamma_{90}\pi(1p_{1/2})^2_0+\delta_{90}\pi(0g_{9/2})^2_0 +\kappa_{90}\pi[(1p_{3/2})^{-2}(1p_{1/2})^2(0g_{9/2})^2]_0 . \tag{25}$$

Our Rergl results (Table VII) indicate that a two-state coexistence model is adequate for describing the two-neutron 0_2^+/g.s. cross-section ratios in the zirconium isotopes. If both Eq. (25) and the Rergl results are to be consistent, one must require that the proton part of the basis states ϕ_g^{90} and ϕ_e^{90} for $^{90}\mathrm{Zr}$ contain the three configurations $\pi(1p_{1/2})^2_0$, $\pi(0g_{9/2})^2_0$, and $\pi[(1p_{3/2})^{-2}(1p_{1/2})^2(0g_{9/2})^2]_0$.

B. Molybdenum

Early work investigating the possibility of using a two-state mixing model for describing the low-lying 0^+ states in the molybdenum isotopes was performed by Catara *et al.*[23] In that reference they used a two-state mixing model for molybdenum and zirconium to describe the cross sections leading to zirconium via various two-

TABLE VII. Mixing probabilities for the πph zirconium and molybdenum wave functions (taken from Refs. 18, 22, 24, and 25) and the two-neutron cross-section ratio predictions using both these mixing probabilities and the Rergl model results.

			Zirconium				
	πph model		Experimental		πph model	Rergl	
A	γ_A^2	δ_A^2	P_A^2	T_A^2	$P_A^2=T_A^2$	P_A^2	T_A^2
88	0.59	0.41	0.129±0.039		0.0	0.129	
90	0.59	0.41	0.013±0.001	0.050±0.005	0.008	0.013	0.057
92	0.50	0.50	0.035±0.003	0.040±0.005	0.025	0.031	0.037
94	0.66	0.34	0.039±0.004	0.427±0.047	0.099	0.038	0.442
96	0.90	0.10		0.001±0.001	0.015		0.001
98	0.96	0.04					
					$\chi^2\approx 60.59$	$\chi^2\approx 0.533$	
			Molybdenum				
	πph model		Experimental		πph model	Rergl	
A	ζ_A^2	η_A^2	P_A^2	T_A^2	$P_A^2=T_A^2$	P_A^2	T_A^2
90			0.074±0.030			0.074	
92			0.048±0.021	0.094±0.012		0.042	0.095
94	0.44	0.56	0.001±0.001	0.013±0.001	0.152	0.002	0.013
96	0.12	0.88		0.017±0.003	0.076		0.017
98	0.34	0.66	0.158±0.014	0.162±0.018	0.052	0.158	0.163
100	0.15	0.85		0.48 ±0.13			0.479
					$\chi^2\approx 8521$	$\chi^2\approx 0.121$	

nucleon and four-nucleon transfer reactions. These wave functions were later simplified by van den Berg *et al.*[26] and written as

$$\Psi_A^{\pi}[{}^{A}\mathrm{Mo}(\mathrm{g.s.})]=\zeta_A\pi(0g_{9/2})_0^2+\eta_A\pi(0g_{9/2})_0^4(\mathrm{hole})_0^2 , \tag{26}$$

$$\Psi_A^{\pi}[{}^{A}\mathrm{Mo}(0_2^+)]=\eta_A\pi(0g_{9/2})_0^2-\zeta_A\pi(0g_{9/2})_0^4(\mathrm{hole})_0^2 ,$$

where $(\mathrm{hole})_0^2$ is a sum over two-proton hole states in a $Z=40$ core, i.e.,

$$(\mathrm{hole})_0^2= a_A\pi(1p_{1/2})_0^{-2}+b_A\pi(1p_{3/2})_0^{-2} +c_A\pi(0f_{5/2})_0^{-2} . \tag{27}$$

This *is* consistent with experiment, but van den Berg *et al.*[26] also assumed the neutron wave functions for the ground state and first-excited 0_2^+ state in the even molybdenum isotopes were the same. This implies that the SML of the Rergl analysis above should describe the 0_2^+/g.s. cross-section ratio data in Mo(p,t) and Mo(t,p), i.e., $P_A^2=T_A^2$. The Mo(p,t) and Mo(t,p) reactions[47,51–61] have been measured and the results are summarized in Table VII and plotted versus A in Fig. 6. It is again clear that the condition $P_A^2=T_A^2$ (required by the πph wave functions van den Berg used) is not satisfied for all A. The value of the cross-section ratio ratios, T_{A0}^2/P_{A0}^2 (plotted versus neutron number N in Fig. 3), are 1.97±0.90, 8.88±5.9, and 1.02±0.15 for $N=50$, 52, and 56, respectively, with the largest deviation from unity occurring at $N=52$. The best-fit results using the SML in the Mo isotopes are [via Eq. (9)] given by $T_A^2=P_A^2=0.083$, 0.007, and 0.160 for $A=92$, 94, and 98, respectively. Out of the three sets of nuclei considered in this report, the molybdenum two-neutron cross-section ratios follow the SML best. However, the mixing probabilities used in Ref. 26 (taken from Ref. 23) and reproduced here in Table VII result in an enormous deviation

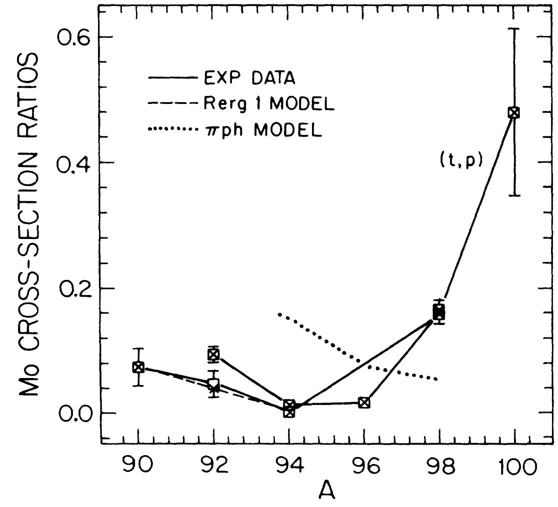

FIG. 6. Plot of the cross-section ratios T_A^2 and P_A^2 vs A for molybdenum along with the predictions of the van den Berg πph two-state coexistence model and the improved Rergl coexistence model. The experiment is the solid line, the πph limit is the dotted line, and the Rergl model is the dashed line.

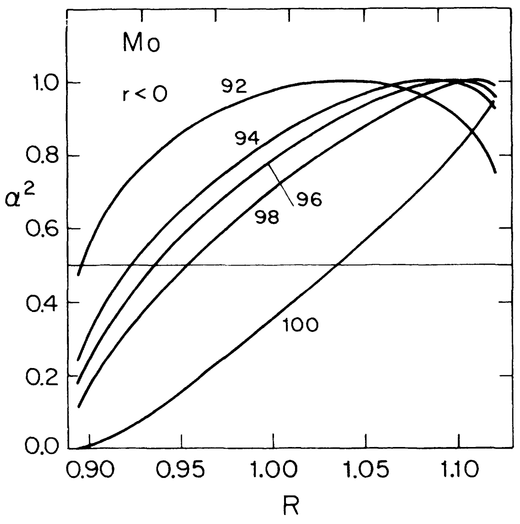

FIG. 7. Plot of the mixing probabilities α_A^2 as a function of R in the Rergl model as applied to the molybdenum isotopes assuming $0^{+'}=0_2^+$ in all ^{A}Mo.

between model and experiment. In the more general Rergl analysis, we obtain from the molybdenum data[41] the mixing probabilities given in Fig. 7 resulting in the fits shown in Table VII and Fig. 6. In addition, we obtain the value $K_A(\mathrm{Mo})=-0.2492\pm0.0003$ giving $\Delta(\mathrm{Mo})=0.057\pm0.011$ and $\Omega(\mathrm{Mo})=0.893\pm0.019$. Again, $\Delta\neq0$ and $\Omega\neq1$ within the uncertainties of the data.

We can conclude then that the wave functions of Eqs. (26) and (27) *along* with the *extra* assumption that the neutron part of the wave function is the same for the ground state and the first-excited 0_2^+ state are not consistent with the two-neutron cross-section ratio data in zirconium and molybdenum, respectively. In most cases, it is this extra assumption about the neutron configurations that leads to inconsistencies with the two-neutron transfer cross-section ratio data.

VI. CONCLUSIONS

We have shown that the simple πph coexistence models of Vergnes and van den Berg as applied to germanium, zirconium, and molybdenum—although very successful in describing some direct-transfer data in those nuclei—suffer from some serious inconsistencies. Without losing much of the original simplicity in the two-state model analysis, we have generalized the two-state coexistence model wave functions by using Eq. (1) in its original general form and requiring only the two Rergl assumptions about the $2n$-transfer overlap ratios among the basis states and thus removing many inconsistencies between the experimental data and model as far as two-neutron data is concerned.

We have presented a simple test in Eq. (10) as to whether the two-state model wave functions are capable of describing the two-neutron transfer ratio data. In addition, instead of assuming both $R=1$ *and* $r=0$ in the basis state $2n$-transfer overlap ratios, we have allowed the $\sigma(0^{+'})/\sigma(\mathrm{g.s.})$ ratio data in the (p,t) and (t,p) reactions to dictate the nature of the relationship between r and R via Eq. (11).

We have shown that the earlier simpler models will work provided $K_A=-\frac{1}{4}$ or alternatively, $\Delta=0$, or alternatively, $\Omega=1$, all possible only when $P_A^2=T_A^2$. The requirements $R=1$ and $r=0$, taken separately, could, in principle, mathematically account for the $\sigma(0^{+'})/\sigma(\mathrm{g.s.})$ ratio data in (p,t) and (t,p), but taken together, they do not (unless $P_A^2=T_A^2$).

The two-state model basis wave functions of the Rergl analysis are infinite in number in that the parameter R is a variable and not determined from (p,t) and (t,p) data alone. In addition, these Rergl mixing probabilities are found to be consistent with other experimental data in the germanium isotopes, particularly the $0f_{5/2}$ proton occupancies in the ground states of the even-mass Ge isotopes.

Finally, of course, our model suffers from the fact that we offer no form for the basis-state wave functions. However, the model is developed far enough so that any choice for the basis-state wave functions can be easily tested against Eq. (11). In this sense, the Rergl model allows the two-neutron cross-section ratio data to dictate the relationship between the $2n$-transfer overlap ratios in the basis states and thus becomes the first step in their (ϕ_g^A and ϕ_e^A) explicit construction.

We acknowledge financial support from the National Science Foundation.

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