Nonlocal separable potential in the one-dimensional Dirac equation

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The one-dimensional Dirac equation is solved for a separable potential of the form of Lorentz scalar plus vector, $(\beta g+h)v(x)v(x')$. Exact analytic solutions are obtained for bound and scattering states for arbitrary $v(x)$. For a particular combination of the values of g and h, degeneracy of the bound state occurs, and total reflection also takes place for a certain incident energy. The limiting case, in which $v(x)$ becomes a delta function, is discussed in detail.

Recently, McKellar and Stephenson examined the Dirac equation with a local delta-function potential in the context of quarks in one-dimensional periodic structure.[1] They showed that, when the delta function acts at $x=0$, the use of the usual formula

$$\int_{-\infty}^{\infty}\delta(x)\psi(x)dx=\psi(0)=\tfrac{1}{2}[\psi(0^+)+\psi(0^-)] \tag{1}$$

is incorrect. In particular, they pointed out that when the potential is a Lorentz scalar, Eq. (1) leads to an apparently unphysical result. The connection formula in this case reads

$$\psi(0^+)=\exp[-2i\alpha\beta\tanh^{-1}(g/2)]\psi(0^-) \tag{2a}$$

or

$$\psi(0^+)=-\exp[-2i\alpha\beta\coth^{-1}(g/2)]\psi(0^-)\ , \tag{2b}$$

for $|g|<2$ and $|g|>2$, respectively. This suggests that, when $g=\pm 2$, an infinite phase factor occurs in the wave function from $x=0^-$ to $x=0^+$.

While we have also proved that the use of Eq. (1) for a local delta-function potential is improper, we have pointed out that Eq. (1) *is* correct for the delta-function limit of a *nonlocal separable* potential.[2] The purpose of this paper is to solve the Dirac equation[3,4]

$$(\alpha p+\beta m)\psi(x)+(\beta g+h)v(x)\int_{-\infty}^{\infty}dx'v(x')\psi(x')=E\psi(x) \tag{3}$$

for any reasonable form of $v(x)$, with $v(x)=v(-x)$, and then examine the limit $v(x)\to\delta(x)$. We shall confirm the validity of Eq. (1) in this case and point out that the difficulty related to Eqs. (2a) and (2b) with $g=\pm 2$ is only an apparent one.

The Green's function for Eq. (3) satisfies

$$(\alpha p+\beta m-E)G(x,x')=\delta(x-x')\ . \tag{4}$$

For $E^2<m^2$, we have

$$G(x,x')=\frac{1}{2\kappa}e^{-\kappa|x-x'|}[i\alpha\kappa\,\mathrm{sgn}(x-x')+\beta m+E]\ , \tag{5}$$

where $\kappa=(m^2-E^2)^{1/2}$. For $E^2>m^2$, we replace $i\kappa$ by $k=(E^2-m^2)^{1/2}$. For the bound state (if any), $\psi(x)$ is given by

$$\psi(x)=-\int_{-\infty}^{\infty}dx'G(x,x')v(x')(\beta g+h)\chi\ , \tag{6}$$

where

$$\chi=\int_{-\infty}^{\infty}dx\,v(x)\psi(x)\ . \tag{7}$$

Equations (6) and (7) require

$$\mathrm{Det}\left[1+\int dx\int dx'G(x,x')v(x)v(x')(\beta g+h)\right]=0\ . \tag{8}$$

Using Eq. (5), we can reduce Eq. (8) to

$$[2\kappa+(m+E)(g+h)\mathcal{J}][2\kappa+(m-E)(g-h)\mathcal{J}]=0\ , \tag{9}$$

where

$$\mathcal{J}=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'e^{-\kappa|x-x'|}v(x)v(x')\ . \tag{10}$$

If $(g+h)<0$, there is a bound state with energy

$$E=\frac{4-(g+h)^2\mathcal{J}^2}{4+(g+h)^2\mathcal{J}^2}m\ . \tag{11a}$$

Similarly, if $(g-h)<0$, there is a bound state with energy

$$E=-\frac{4-(g-h)^2\mathcal{J}^2}{4+(g-h)^2\mathcal{J}^2}m\ . \tag{11b}$$

If

$$(g^2-h^2)\mathcal{J}^2=4\ , \tag{12}$$

the E's of Eq. (11a) and (11b) coincide; the two states become degenerate. This is a rare occurrence of degeneracy for a one-dimensional problem.[5,6]

For $E^2>m^2$, the transmission coefficient T and the reflection coefficient R can be explicitly determined. The results are

$$T=\{k[4-(g^2-h^2)|J|^2]-4(gm+hE)\,\mathrm{Im}J\}^2/D \tag{13}$$

and

$$R=[4(gE+hm)\,\mathrm{Re}J]^2/D\ , \tag{14}$$

where

$$J=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'e^{ik|x-x'|}v(x)v(x') \tag{15}$$

and

$$D=|\,k[4+(g^2-h^2)J^2]+4i(gm+hE)J\,|^2\ . \tag{16}$$

In the limit $v(x)\to\delta(x)$, we find $\mathcal{J}\to 1$ and $J\to 1$. On the other hand, if one sets $v(x)=\delta(x)$ and uses Eq. (1) from the outset, one obtains exactly the same results: The bound state energy, the transmission and reflection coefficients are those of Eqs. (11), (13), and (14), with $\mathcal{J}=J=1$.

Let us now examine the relation between $\psi(0^+)$ and $\psi(0^-)$ in the limit $v(x)\to\delta(x)$. Equations (3) and (1) lead to

$$-i\alpha[\psi(0^+)-\psi(0^-)]+\tfrac{1}{2}(\beta g+h)[\psi(0^+)+\psi(0^-)]=0\ , \tag{17}$$

which can be rearranged as

$$(4-g^2+h^2)\psi(0^+)=[4+g^2-h^2-4i\alpha(\beta g+h)]\psi(0^-)\ , \tag{18}$$

or as

$$[4+g^2-h^2+4i\alpha(\beta g+h)]\psi(0^+)=(4-g^2+h^2)\psi(0^-)\ . \tag{19}$$

If $g^2\neq 4$ and $h=0$, either Eq. (18) or Eq. (19) can be rewritten in the form of Eq. (2). On the other hand, if $g^2=4$ and $h=0$, or more generally, if $g^2-h^2=4$, $\psi(0^+)$ and $\psi(0^-)$ become unrelated. Instead, they satisfy

$$[2+i\alpha(\beta g+h)]\psi(0^+)=0 \tag{20}$$

and

$$[2-i\alpha(\beta g+h)]\psi(0^-)=0\ . \tag{21}$$

That is, the upper and lower components on a given side of $x=0$ are related. When $g^2-h^2=4$ and $(g+h)<0$, there are two degenerate bound states. One of them has even parity and the other odd parity. Since an arbitrary linear combination of the two degenerate solutions is also an eigenfunction with the same eigenvalue, the relative phase between the wave functions for $x>0$ and $x<0$ is arbitrary. When $g^2-h^2=4$, there is no transmission. When a wave is incident from the left of $x=0$, there is no wave transmitted to the right of $x=0$. In other words, the two regions $x>0$ and $x<0$ are completely separated.

As we noted earlier, the degeneracy of the bound states with E occurs for any $v(x)$ if $(g^2-h^2)\mathcal{J}^2=4$. In this situation, the phases of the asymptotic wave functions, $\psi(+\infty)$ and $\psi(-\infty)$, become unrelated. However, the complete disappearance of transmission for any incident energy E is peculiar to the delta-function case. For an arbitrary $v(x)$, J of Eq. (15) depends on k, and hence, T of Eq. (13) can vanish only for a certain value of k.

In summary, unlike the case of a local delta-function potential, Eq. (1) is correct for a separable delta-function potential. When $g^2-h^2=4$ and $(g+h)<0$, the bound state becomes degenerate, and when $g^2-h^2=4$ the transmission vanishes. This is related to the indeterminacy of the relative phase between $\psi(0^+)$ and $\psi(0^-)$, but there is nothing unphysical about this. A similar situation also arises for an arbitrary form factor $v(x)$ of the potential.

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[1]B. H. McKellar and G. J. Stephenson, Jr., Phys. Rev. C **35**, 2262 (1987). We use slightly different notations. Their α_x, S_x, and S_e correspond to our α, g, and H, respectively.

[2]M. G. Calkin, D. Kiang, and Y. Nogami, Am. J. Phys. **55**, 737 (1987). See also B. Sutherland and D. C. Mattis, Phys. Rev. A **24**, 1194 (1981).

[3]A separable Lorentz vector potential in one dimension was used by M. L. Glasser, Am. J. Phys. **51**, 936 (1983). The interesting features found for the Lorentz scalar potential discussed in the present paper are not present in the pure vector case.

[4]Nonlocal separable potentials have been extensively used as a model for nonrelativistic nucleon-nucleon interaction. See, for example, Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); T. R. Mongan, Phys. Rev. **178**, 1597 (1969). In a Dirac equation, nonlocal separable potentials in three dimensions have been considered by Y. Nogami and W. van Dijk, Phys. Rev. C **34**, 1855 (1986) to study relativistic effects in nucleon-nucleon interactions. They took different form factors for the Lorentz scalar and vector parts of the potential.

[5]This does not violate the no-degeneracy theorem (see Ref. 6) as the theorem is not applicable to nonlocal potentials.

[6]F. A. B. Coutinno, Y. Nogami, and F. M. Toyama, private communication.