Brief Reports

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Semiclassical analysis of the SO(3,1) S matrix

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We perform a semiclassical analysis of the SO(3,1) S matrix. In particular, we discuss the near-far components of the amplitude and describe the nuclear rainbow scattering. We use the semiclassical inverse scattering theory to obtain the underlying potential, which is found to behave as $r^{-1/2}e^{-r/a}$ at large separations.

During the last few years an algebraic approach to quantal scattering problems has been developed by one of us together with Alhassid, Gürsey, and others.[1] In this approach, S matrices are derived from general group-theoretic considerations. One of these, based on the algebra of SO(3,1), has the general form

$$S_l(k)=\frac{\Gamma[l+1+iv(l,k)]}{\Gamma[l+1-iv(l,k)]}\ , \tag{1}$$

and promises to provide a useful tool in the analysis of experimental data.[2] For each particular problem, one chooses a function $v(l,k)$, computes $S_l(k)$ and from it the differential cross section, $d\sigma/d\Omega$. The function $v(l,k)$, called the "algebraic potential" is taken to be real for nonabsorptive scattering and complex for absorptive scattering. Since the algebraic approach is formulated at the level of S matrices, it is not clear what is the underlying physics in a Schrödinger picture. In order to find a potential that, inserted in the Schrodinger equation, produces the same S matrix, one needs to solve the inverse scattering problem. This is, in general, rather complicated. The solution is much simpler when a semiclassical approximation is taken. In this paper we perform a semiclassical analysis of the scattering matrix, Eq. (1).

We consider the case of an absorption-free S matrix. The key ingredient is the classical deflection function $\Theta(l)\equiv 2d\delta_l/dl$, where $S_l\equiv\exp(2i\delta_l)$. Since, for $v(l,k)$ real, an explicit form exists for δ_l in terms of l,[2] the classical deflection function can be calculated exactly in numerical form. However, since we are interested in general properties, we consider an approximate form (very accurate for large l) obtained by using the following asymptotic form of the Γ functions[3] appearing in (1),

$$\Gamma(z)\approx(2\pi)^{1/2}e^{-z}(z)^{z-1/2}\ ,\qquad z=l+1\pm iv(l,k)\ . \tag{2}$$

This gives

$$\begin{aligned}2i\delta_l=&\frac{1}{i}\arctan\left[\frac{v(l,k)}{l+1}\right]\\&+[l+1+iv(l,k)]\ln[l+1+iv(l,k)]\\&-[l+1-iv(l,k)]\ln[l+1-iv(l,k)]-2iv(l,k)\ .\end{aligned} \tag{3}$$

The classical deflection function is obtained immediately from (3) by taking the derivative with respect to l, neglecting $\frac{1}{2}$ compared with l, and introducing $\lambda=l+1/2$:

$$\Theta(\lambda)=2\arctan\left[\frac{v(\lambda,k)}{\lambda}\right]+\frac{dv(\lambda,k)}{d\lambda}\ln[\lambda^2+v(\lambda,k)^2]\ . \tag{4}$$

One can see from (4) that a parametrization of $v(l,k)$ provides a simple parametrization of the classical deflection function $\Theta(\lambda)$ which can, in turn, be used to analyze the data. We consider now in particular the form of $v(l,k)$ used to analyze heavy-ion scattering:[2]

$$v(l,k)=v_c(l,k)+v_s(l,k)=\frac{Z_1Z_2e^2\mu}{\hbar^2k}+\frac{v_0(k)}{1+\exp\{[l-l_0(k)]/\Delta(k)\}}\ . \tag{5}$$

For pure Coulomb scattering, $v(l,k)=\eta=Z_1Z_2e^2\mu/\hbar^2k$ and the second term in (4) vanishes identically, leaving the famous Rutherford deflection function. The presence of nuclear scattering changes the first term in (4) slightly, but gives rise to a sharply localized second term, which is negative for positive v_0. We show in Fig. 1 a typical deflection function. This deflection function exhibits several interesting features. The nuclear rainbow [minimum in $\Theta(\lambda)$] comes directly from the second term $(dv/d\lambda)$. The first term in $\Theta(\lambda)$ is a modified Coulomb deflection function, which, at small impact parameters $(\lambda\to 0)$ behaves as $\pi-[2\lambda/(\eta+v_0)]$. This behavior, however, is appropriate to the approximate deflection function and is different from the exact result.

Another interesting feature is that the parametrization (5) does not exhibit exact orbiting, i.e., the deflection function never goes to $-\infty$. However, for large v_0 or small Δ, the deflection function becomes more and more negative. Exact orbiting can be obtained only by using a parametrization of $v(l,k)$ which is a discontinuous function of l.

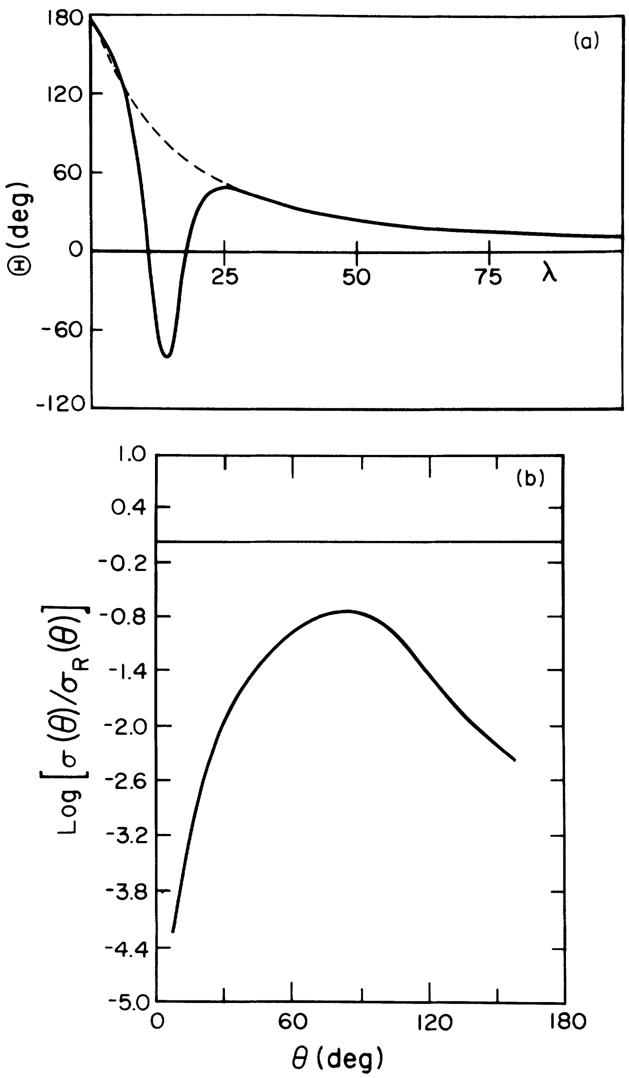

FIG. 1. (a) The classical deflection function and (b) the far-side contribution to the elastic-scattering cross section for the case $\eta=12.58$, $E=20.0$ MeV, $k=3.12\ \text{fm}^{-1}$, $v_0=4$, $\lambda_0=14$, and $\Delta=2$.

We now turn to a discussion of the scattering amplitude. This can be obtained from[4]

$$f(\theta)=f^{(+)}(\theta)+f^{(-)}(\theta) =\frac{1}{\sqrt{\sin\theta}}[I(\theta)-iI(-\theta)]\ , \quad (6)$$

$$I(\theta)=-\frac{e^{i\pi/4}}{k\sqrt{2\pi}}\int_0^{+\infty}d\lambda\,\lambda^{1/2}\frac{\Gamma[\lambda+1/2+iv(\lambda,k)]}{\Gamma[\lambda+1/2-iv(\lambda,k)]}e^{i\lambda\theta}\ ,$$

where $f^{(+)}(\theta)$ and $f^{(-)}\theta$ represent the near- and far-side components of $f(\theta)$. It is clear from the structure of the deflection function (Fig. 1) that $f^{(+)}(\theta)$ is purely Coulomb at very small angles and very close to Coulomb at angles close to π. At intermediate angles it is affected strongly by the nuclear interaction. The interesting feature of $f^{(+)}(\theta)$ is its back-angle behavior. With the aid of the stationary-phase method, we are able to evaluate the amplitude in this angle region. If we assume that $l_0(k)/\Delta(k)\gg 1$ in (5) (a condition usually met in heavy-ion scattering), then

$$v_s(\lambda,k)\underset{\lambda\to 0}{\approx}v_0(k)\ ,$$
$$\frac{dv_s(\lambda,k)}{d\lambda}\underset{\lambda\to 0}{\approx}0\ , \quad (7)$$
$$\Theta(\lambda)\underset{\lambda\to 0}{\approx}2\arctan\left[\frac{\eta+v_0(k)}{\lambda}\right].$$

Use of Eq. (6) gives $f^{(+)}(\theta)$ and the cross section

$$\sigma^{(+)}(\theta)=\frac{[\eta+v_0(k)]^2}{4k^2\sin^4(\theta/2)}\ , \quad (8)$$

which reduces to the Rutherford formula when $v_0=0$.

As far as the far-side amplitude is concerned, the structure of $\Theta(\lambda)$ clearly indicates that it will be predominantly composed of nuclear rainbow scattering. In fact, one can see that the second term in $\Theta(\lambda)$, Eq. (4), dominates $f^{(-)}(\theta)$. As an example of the behavior of $|f^{(-)}(\theta)|^2$ we show, in the lower part of Fig. 1, results obtained which the parametrization of the upper part. The nuclear rainbow here is structureless due to the narrow nature of $\Theta(\lambda)$. We have repeated the calculation for different values of the parameters, as shown in Fig. 2. The far-side cross section shows clearly in this case well-developed Airy's oscillations, owing to the broad nature of $\Theta(\lambda)$.

Having discussed briefly the semiclassical limit of the SO(3,1) S matrix, we turn now to the question of determining the underlying potential in r space. This can be done by using the semiclassical inverse scattering method.[5] Knowing the deflection function $\Theta(\lambda)$, one can obtain the potential $V(r)$ by computing the integral

$$T(s,k)=\frac{1}{\pi}\int_{ks}^{\infty}(\lambda^2-k^2s^2)^{-1/2}\Theta(\lambda)d\lambda\ . \quad (9)$$

The potential $V(r)$ is given by

$$V(r)=E\{1-\exp[-2T(s,k)]\} \quad (10)$$

with s obtained by inverting

$$r(s)=s\exp[T(s,k)] \tag{11}$$

and $E=k^2\hbar^2/2\mu$. From the above equations one can obtain numerically the potential $V(r)$. For large r, the equations can be solved approximately. Large r corresponds semiclassically to large λ. In this region,

$$\begin{aligned} v_s(\lambda,k) &\xrightarrow[\lambda\to\infty]{} v_0(k)e^{-\lambda/\Delta(k)} ,\\ \frac{dv(\lambda,k)}{d\lambda} &\xrightarrow[\lambda\to\infty]{} -\frac{v_0(k)}{\Delta(k)}e^{\lambda_0(k)/\Delta(k)}e^{-\lambda/\Delta(k)} ,\\ \Theta(\lambda) &\xrightarrow[\lambda\to\infty]{} 2\frac{\eta}{\lambda}-\frac{v_0(k)}{\Delta(k)}e^{\lambda_0(k)/\Delta(k)} \\ &\qquad\times e^{-\lambda/\Delta(k)}\ln(\lambda^2+\eta^2) , \end{aligned} \tag{12}$$

where $\lambda_0=l_0+\frac{1}{2}$. One then obtains

$$T(s,k)\approx\frac{\eta}{ks}-\frac{v_0(k)}{\pi\Delta(k)}\ln(k^2s^2+\eta^2) \times e^{\lambda_0/\Delta(k)}K_0\left[\frac{ks}{\Delta(k)}\right] , \tag{13}$$

where K_0 is the modified Bessel function. Using the asymptotic form of K_0 and $\exp T\approx 1+T$, yields

$$V(r)\approx\frac{2\eta E}{kr}-\left[\frac{2}{\pi\Delta(k)}\right]^{1/2}Ev_0(k)\ln(k^2s^2+\eta^2) \times\frac{e^{-(ks-\lambda_0)/\Delta(k)}}{\sqrt{ks}} \tag{14}$$

with $s\approx r$. From Eq. (14) one can see that the potential $V(r)$ consists of the Coulomb potential, $2\eta E/kr$, and a nuclear potential which varies with r, for large r, as $\ln(kr)\exp(-kr/\Delta)/\sqrt{kr}$. A Woods-Saxon well in r,

$$V_{\rm WS}=-V_0/\{1+\exp[(r-r_0)/\delta]\} , \tag{15}$$

would produce a behavior as $\exp(-r/\delta)$, very similar to that obtained in Eq. (14). One notes that if the surface diffuseness δ is constant in r space, Eq. (14) implies that the surface diffuseness Δ in l space should increase with k. We have also checked the accuracy of Eq. (14) by direct numerical evaluation of the integrals. We note that, if we neglect the slowly varying logarithm term, the SO(3,1) S matrix with the parametrization (5) produces a potential, $r^{-1/2}e^{-kr/\Delta}$, which is different from that obtained using the eikonal approximation for an SO(3,2) S matrix with the same parametrization $r^{-5/2}e^{-kr/\Delta}$.[6] This is due to the different form of the S matrix. The eikonal analysis of Amado and Sparrow would produce the same result as our analysis if applied to the SO(3,1) amplitude.

In conclusion, we have performed a semiclassical analysis of the SO(3,1) amplitude with the parametrization (5) which exposes the correct physical behavior of the model when applied to heavy-ion scattering. Our analysis has been limited to absorption-free scattering and should be repeated for absorptive scattering. In the latter case one could use the method recently developed by two of us in order to relate the absorption-modified amplitude to the absorption-free amplitude.[7] Conversely, one could directly construct absorption-modified amplitudes by making the "algebraic potential" $v(l,k)$ complex as discussed in Ref. 2.

Apart from elucidating the physical nature of the SO(3,1) amplitude, the analysis presented here could be of practical interest in those situations where the semiclassical approximation is a good one. In particular we have in mind here molecular scattering data. In this case the potential $v(l,k)$ can be taken to be real and an appropriate parametrization of it would produce a parametrization of the deflection function $\Theta(\lambda)$, which is of practical importance. Results of the analysis of atom-atom, atom-molecule, and molecule-molecule collisions using this method will be presented elsewhere.

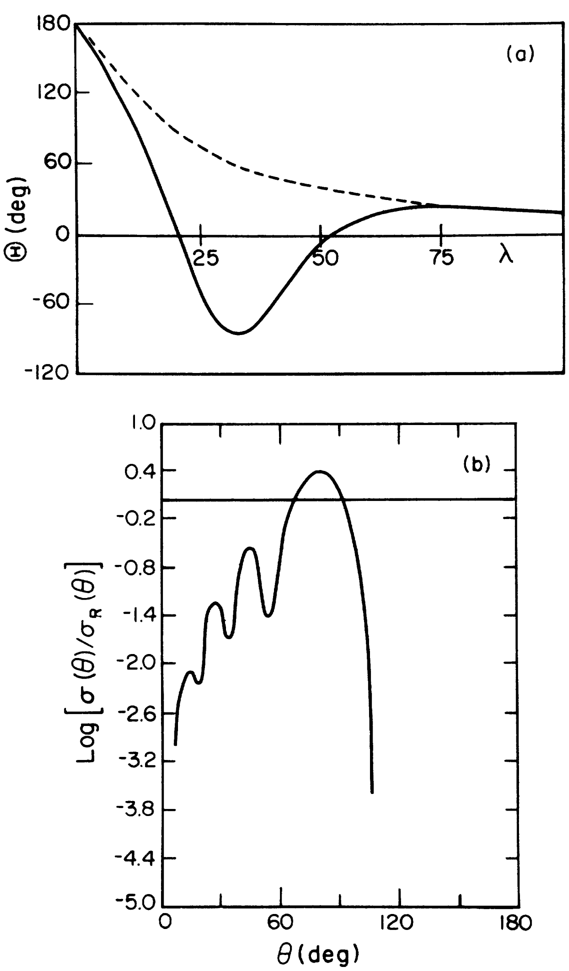

FIG. 2. Same as Fig. 1 with $\eta=18.22$, $E=100.0$ MeV, $k=7.63$ fm^{-1}, $v_0=12$, $\lambda_0=30$, and $\Delta=8$.

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