Comment on "Algebraic analysis of physical and spurious states in Dyson boson mapping"

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By stressing the application of boson mapping techniques to the truncated collective branch of nuclear excitations, the usefulness of two recently developed methods for the treatment of spurious states are compared. It is shown why the identification proposed by Geyer *et al.* is superior to the use of the "Majorana" operator S as proposed by Park.

A recent paper by Park,[1] on which we comment here, begins with the following statement: "The Dyson boson mapping method has been introduced to study low-lying collective states." This important qualification, namely that one anticipates profitable use of boson methods only for collective excitations, has been stressed previously by several authors. (See, e.g., Refs. 2 and 3.) Nothing is gained through boson methods if one is not able to identify a collective subspace which decouples from the "noncollective" states albeit only in some approximate way. The required truncation to a collective subspace at some stage is the central theme of the present Comment.

We feel obliged to react to the paper by Park,[1] because in his conclusion he states that he has found a method to deal with the spurious eigenstates of boson-mapped fermion Hamiltonians by introducing his operator S, and he furthermore maintains that the method of Geyer *et al.*[4] dealing with the same problem is not practical. In this Comment we demonstrate that this view actually applies the other way around: Park's method is difficult to employ in conjunction with a truncation to the collective branch, whereas his criticism that in the method of Geyer *et al.*[4] one "requires calculating matrix elements for many different physical boson operators between all eigenstates of a given physical boson Hamiltonian" does not apply at all.

The origin of the spurious states can be understood by examining the properties and nature of the boson Fock space. Consider first the Dyson mapping of bifermion operators

$$\begin{aligned} c_\alpha^\dagger c_\beta^\dagger &\equiv b^{\alpha\beta} \rightarrow R^{\alpha\beta} \equiv B^{\alpha\beta} - B^{\alpha\theta} B^{\beta\rho} B_{\theta\rho} \ , \\ c_\beta c_\alpha &\equiv b_{\alpha\beta} \rightarrow R_{\alpha\beta} \equiv B_{\alpha\beta} \ , \\ c_\alpha^\dagger c_\beta &\equiv b_\beta^\alpha \rightarrow A_\beta^\alpha \equiv B^{\alpha\theta} B_{\beta\theta} \ . \end{aligned} \tag{1}$$

The boson operators $R^{\alpha\beta}$, $R_{\alpha\beta}$, and A^α_β are constructed in this way to satisfy the bifermion algebra. (See Ref. 4 for the notation used. When indices are not important we use $R^\dagger$, R, A, $B^\dagger$, and B as abbreviations for the corresponding indexed operators.)

With the repeated application of $R^\dagger$'s on the boson vacuum state one builds up the physical subspace of the complete boson Fock space, the latter being obtained by the repeated application of $B^\dagger$'s. The physical subspace is characterized by a one-to-one correspondence with fermion states, whereas the states of its complement are termed unphysical. In general a diagonalization in the complete boson Fock space will produce some eigenstates which have components in the unphysical subspace and are therefore termed spurious. (See Ref. 4 for a complete discussion.)

In order to introduce a collective subspace, new bosons

$$B^n = \tfrac{1}{2} \sum \chi^n_{\alpha\beta} B^{\alpha\beta}$$

are defined, where $\chi^n_{\alpha\beta}$ is a complete set of transformation coefficients. One now endeavors to find a set $\chi^n_{\alpha\beta}$ which contains a subset characterized by only a small number of the values n and the property that the operators R^n, R_n, and $A^n_{n'} = B^n B_{n'}$ form a closed algebra under commutation. This is then a subalgebra of the original algebra. It is important for the usefulness of the boson method that this new set of boson operators contains a much smaller number of elements than the original set which was indexed by $\alpha\beta$, etc. Furthermore, it is necessary for the application of the method that the Hamiltonian and other physically relevant operators can be expressed in terms of this subset of operators R^n, etc. This may only be possible in some approximation and it is the objective of microscopic theory to find such approximations and to determine their validity. Progress along this road has been limited. One has therefore had to fall back on carefully constructed models in which the decoupling of a collective subspace has been built in.[5,6] One of the simplest models in which various facets of the occurrence of spurious states have been studied[4] has an Sp(4) symmetry, and its properties in terms of Dyson boson mappings have been given in some detail.[7-9]

However, to comment on Park's method it is already sufficient to consider the SU(2) or quasispin model. Some discussion centering around the adequacy of the boson basis in this model was already given in Ref. 2, but as also pointed out by Gambhir *et al.*,[10] the model is too limited to expose effects which are obtained from linear dependencies in the physical basis as found in the Sp(4) case and in other more realistic applications.

To appreciate various aspects of the problem it is useful first to consider the "Hermitian" and "non-Hermitian" Dyson images which play a role in Park's construction of the operator S. Park unfortunately does not differentiate between the relative merits of the two possibilities, and we therefore first illustrate this aspect in the case of a pairing Hamiltonian

$$H=\lambda \sum_{\alpha\beta>0} b^{\alpha-\alpha} b_{\beta-\beta} \ . \tag{2}$$

The "non-Hermitian" Dyson image, which here emerges as the proper one in the context of an anticipated truncation, is obtained by the mapping

$$\begin{aligned} b^{\alpha-\alpha} &\rightarrow R^{\alpha-\alpha}=B^{\alpha-\alpha}-B^{\alpha\mu}B^{\nu-\alpha}B_{\nu\mu} \ , \\ b_{\beta-\beta} &\rightarrow R_{\beta-\beta}=B_{\beta-\beta} \ , \end{aligned} \tag{3}$$

and a subsequent introduction of collective bosons. In the present example it is possible to find a subset of transformation coefficients, as discussed above, which contains only one distinct element,

$$\chi^c_{\alpha\beta}=\text{sgn}(\alpha)\delta_{\alpha-\beta}/\sqrt{\Omega} \ ,$$

where Ω is the number of positive α values. [This is just the collective Tamm-Dancoff approximation (TDA) wave function.] We therefore obtain one collective boson

$$B^c=\sum_{\alpha>0} \chi^c_{\alpha-\alpha} B^{\alpha-\alpha}$$

and the corresponding collective operator

$$R^c=\sqrt{\Omega}\left[B^c-B^c\left[B^cB_c+2\sum_n \beta^n\beta_n\right]\Big/\Omega\right] \ . \tag{4}$$

Here the β^n create noncollective bosons which are obtained from the set of all $\chi^n_{\mu\nu}$ orthogonal to the collective $\chi^c_{\alpha-\alpha}$. Furthermore, $R_c=\sqrt{\Omega}B_c$. One also introduces the one-body operators

$$b^\mu_\nu \rightarrow A^\mu_\nu=\sum_{mn\gamma} \chi^{\mu\gamma}_m \chi^n_{\nu\gamma} B^m B_n \tag{5}$$

and a corresponding boson number operator

$$A^c_c=2\left[B^cB_c+\sum \beta^n\beta_n\right] \ . \tag{6}$$

The collective operators R^c, R_c, and A^c_c form a closed algebra which is essentially SU(2).

Using the expressions above one now maps the pairing Hamiltonian (2) onto

$$H_D=\lambda R^cR_c=\lambda\left[\Omega B^cB_c-B^c\left[B^cB_c+2\sum \beta^n\beta_n\right]B_c\right] \ . \tag{7}$$

(Actually R^c contains additional terms of the type $\beta^{n_1}\beta^{n_2}B_c$ and $\beta^{n_1}\beta^{n_2}\beta_{n_3}$. In H_D, however, these terms have no influence on the spectrum; neither do they affect the closure of the collective algebra, and have therefore been omitted.) The complete spectrum of the pairing Hamiltonian can be obtained from the diagonal boson Hamiltonian H_D above. One can now truncate to the collective subspace, by simply dropping all β_n-boson terms in R^c and A^c_c. The algebra still closes and the results obtained are still exact; namely, for the state with n collective bosons one has

$$E_n=\lambda\Omega n[1-(n-1)/\Omega] \ . \tag{8}$$

If one were now to repeat the above procedure with what Park terms the Hermitian image of the pairing Hamiltonian, namely first map, then transform to collective bosons, and finally simply truncate to the (single) collective boson, one finds that the first step leads to

$$\begin{aligned} H&=\lambda \sum_{\alpha\beta>0} b^\alpha_\beta b^{-\alpha}_{-\beta} \\ &\rightarrow \lambda \sum_{\alpha\beta>0,\mu\nu} B^{\alpha\mu}B_{\beta\mu}B^{\nu-\alpha}B_{\nu-\beta} \ . \end{aligned} \tag{9}$$

Depending on whether one now first rewrites the boson Hamiltonian in normal ordered form before truncating to the collective branch, or first truncates followed by normal ordering, one obtains the following two truncated boson Hamiltonians, respectively (see also Ref. 2 for a further discussion of this aspect),

$$H_{\text{trunc}}=\lambda\Omega(B^cB_c+B^cB^cB_cB_c/\Omega^2) \ , \tag{10a}$$

$$H'_{\text{trunc}}=\lambda(B^cB_c+B^cB^cB_cB_c)/\Omega \ . \tag{10b}$$

The corresponding n boson collective energies are given by

$$E_n=\lambda\Omega n[1+(n-1)/\Omega^2] \tag{11a}$$

and

$$E'_n=\lambda n^2/\Omega \ . \tag{11b}$$

It is clear that the procedure fails in both cases. In the second instance not even the leading term comes out correctly, while in the first both the sign and magnitude of the interactionlike term are wrong.

This same feature was pointed out in the propagator approach to elementary excitations[11] some time ago where a graphical explanation was indicated. In the boson approach, it can be understood by observing that the first mapping was in terms of generators of SU(2) while the second was not. Truncation in the first case amounts to selecting the collective realization of the algebra and there is no approximation involved, while in the second case it amounts to discarding terms about the importance of which one has neither *a priori* knowledge nor control. It will probably always fail as in this example.

Turning now to the operator S in Park's paper, we realize that it is defined in terms of the difference between two expressions which in the fermion space are rearrangements of one another. Park makes use of the fact that S is zero in the physical subspace of the boson space because there is a one-to-one relation with the fermion space. On the other hand, it is different from zero in the rest of the boson space where the relation to the fermion space is absent. By way of the SU(2) example, we now show that S is also nonzero if one truncates to the collective subspace, the reason being that S does not leave the collective subspace invariant, or, stated differently, it cannot be written in terms of the generators which form the collective subalgebra only.

Following Park S is written as

$$S=N^2-N-K \ , \tag{12}$$

where

$$N=B^{\alpha\beta}B_{\alpha\beta} \ ,$$
$$K=R^{\alpha\beta}B_{\alpha\beta} \ . \tag{13}$$

Transforming these expressions to the collective representation and truncating to the truly collective boson, one finds for our SU(2) case (independently of whether one first truncates or first imposes normal ordering)

$$S=4n^2-4n+2n(n-1)/\Omega \ . \tag{14}$$

This expression is generally different from zero, except for $n=1$. This example therefore illustrates that in the collective subspace S cannot be used as proposed by Park.

The same problems occur if one tries to apply the method to the Sp(4) model or other more complicated cases. One has difficulties in obtaining a unique prescription for calculating S in the collective subspace in those cases.

We do not imply that Park has in fact proposed that S should be calculated by the methods which are shown here to lead to erroneous results. We also agree with Park that it would be very valuable to have an operator available which could be used to remove the spurious states from the physically interesting part of a spectrum, as is common procedure for other kinds of spurious states. What we do expose here are the typical difficulties encountered with Park's operator S within the framework in which one would want to work.

Turning to Park's criticism of our method—namely that it is not practical because one has to calculate too many matrix elements for many states—we wish to refer back to our explicit application to the Sp(4) model.[4] There we constructed an operator Q from the collective generators and found that our results hold independently of our choice of x and y. We also stated that individual parts of the Hamiltonian may serve as a test operator Q. Whatever one does, there are very few operators which one has to calculate to obtain confidence in the result. On the second point made by Park that one would have to do the calculation for all eigenstates, we again refer back to the Sp(4) example where we showed that one obtains zeros which are needed for the identification of spurious states, for any pair of physical and unphysical states. Normally, when most states are physical, one simply continues with confidence by obtaining one matrix element which is nonzero. When one finds a zero one only has to check whether it is accidental, as discussed in our paper, or whether it is due to some symmetry. In the latter case $Q_{\alpha\beta}$ and $Q_{\beta\alpha}$ will both be zero. When one finds that most states obtained are unphysical, one is clearly not working with the correct formalism.

In summary, we state that the method proposed by Park is not applicable to calculations which are truncated to collective subspace because the operator S does not leave this subspace invariant, and, secondly, that the method of Geyer *et al.*[4] is particularly useful in the collective subspace where the calculation of very few matrix elements—which could even be parts of the Hamiltonian—have to be calculated to obtain confidence in the physical nature of the eigenstates. The question as to what extent this method can be used when one only has an approximate decoupling of a collective subspace, remains open.

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