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Microscopic approach to enforced SU(6) symmetry in random phase approximation

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Explicit construction of Dyson, Holstein-Primakoff, and Schwinger representations for random-phase approximation phonon operators is given within Lie algebraic framework. It is shown that these representations emerge as exact boson realizations of quadrupole collective algebra, but the enforcement of SU(6) symmetry involves important constraints embodied in definite nonlinear conditions imposed on random-phase approximation phonon amplitudes. The constructed Schwinger representation could be employed to provide alternative approach to the interacting boson model parameters, avoiding standard procedures of mapping the shell-model *SD* subspace into the *sd* boson space.

I. INTRODUCTION

An important role in the microscopic approach to nuclear collectivity is played by the concept of the random-phase approximation (RPA) phonon, which is defined in the fermion space.[1] It is often useful to map RPA phonon operators into the boson space.[1] In particular, in connection with the success of the interacting-boson model (IBM) (Refs. 2 and 3) as "new phenomenology," and with attempts to give a microscopic foundation of the IBM,[3-6] it is of interest to study the mapping of RPA quadrupole phonon operators onto the boson space spanned by the s and d bosons of IBM.

The Holstein-Primakoff quadrupole boson representation, which presents a boson representation in a closed form, has been obtained for Tamm-Dancoff phonon operators in the so-called SU(6) approximation.[7,8] For the generalized coordinates and momenta q,p, the Holstein-Primakoff representation (HPR) has been introduced heuristically.[9,10] In fact, no complete analysis of the consequences from the enforced SU(6) symmetry has been done and not all SU(6) enforcing conditions were stated explicitly; thus some constraints on the amplitudes have been overlooked. The point is in the difficulty of guessing the SU(6) enforcing conditions in the case of RPA phonon operators. On the other hand, there are well known advantages of the RPA over the Tamm-Dancoff approximation (TDA): (i) The RPA treats ground and excited states on an equal footing, and (ii) the RPA properly treats the inherent symmetries (spurious states are separated from the physical ones).[1,11]

These reasons provide motivation for a detailed study of a microscopic derivation of HPR and a Schwinger representation (SR) of the quadrupole collective RPA phonon operator. We will also consider the Dyson representation (DR). It should be noted that our derivation differs from the traditional method of boson mappings.[1,12]

Our approach differs from the procedure utilized in Ref. 13 in the fact that we are using exact realizations. We introduce the quadrupole RPA phonon operator in a standard way, and following the approach of Jolos, Janssen, and Dönau,[9,10] we enforce the closure of the Lie algebra (cf. Secs. II and III), referred to as quadrupole collective algebra (QCA). We show that this algebra is isomorphic to the SU(6) Cartan-Weyl canonical algebra. The apparatus of classical Lie algebras seems to be ideally suited for our purpose. We construct explicitly the microscopic Dyson, Holstein-Primakoff, and Schwinger representations and show that they occur as exact boson realizations of the QCA, provided definite constraints are fulfilled [SU(6) enforcing conditions]. In addition to the SU(6) enforcing conditions modified with respect to the previous TDA case due to the presence of backward going amplitudes, we deduce a new set of restrictions which should be fulfilled by the amplitudes of RPA phonon operators. If we neglect the backward going amplitudes, i.e., if we consider the TDA, these additional constraints convert into trivial identities. These new restrictions should be verified in comparison to data, as it has been done previously for Tamm-Dancoff phonon operators.[14] The microscopic derivation of the DR, HPR, and SR, presented in this paper, is by no means a straightforward generalization of the TDA case.

At the same time, advanced microscopic nuclear models (such as nuclear field theory,[15] the quasiparticle phonon model,[16] and boson expansion theories[17,18]) are based instead on RPA phonon operators, not on TD ones. Hence the obtained microscopic realizations can be (and were) used[19] to construct the "SU(6) limit" of microscopic nuclear models based on pairing plus quadrupole-quadrupole interactions.

Keeping only the quadrupole collective degree of freedom, it is possible to separate out a part of the Hamiltonian which is expressed in terms of the QCA, which consists of RPA phonon operators and their commutators. In this way we explicitly construct microscopic SU(6) Hamiltonians in the DR, HPR, and SR. In particular, substituting the SR of QCA generators into a microscopic Hamiltonian with a separated quadrupole degree of freedom, we can obtain the IBM Hamiltonian, with the microscopic expressions of the parameters that appear with different boson structures.[19] This method of obtaining the IBM from the microscopic point of view is an alternative to the standard approach.[3–6,20–22]

II. OUTLINE OF THE PROCEDURE

It is well known that for the SU(6) Cartan-Weyl canonical algebra[23] there exist the DR and HPR. They have been explicitly constructed in Ref. 24. For the quadrupole collective algebra (QCA), associated with SU(6) boson symmetry, the DR, HPR, and SR have been investigated,[25,26] but not on the microscopic basis.

To obtain the microscopic DR, HPR, and SR we perform the following steps.

First we introduce the RPA phonon creation and annihilation operators[1,16]

$$Q^\dagger_{2\mu 1}=\tfrac{1}{2}\sum_{jj'}[\Psi^{21}_{jj'}A^\dagger(jj';2\mu)-(-1)^{2-\mu}\phi^{21}_{jj'}A(jj';2-\mu)] , \quad (1)$$

$$Q_{2\mu 1}=\tfrac{1}{2}\sum_{jj'}[\Psi^{21}_{jj'}A(jj';2\mu)-(-1)^{2-\mu}\phi^{21}_{jj'}A^\dagger(jj';2-\mu)] . \quad (2)$$

Generally, the RPA operator is denoted $Q_{\lambda\mu i}$, where λ denotes multipolarity, μ denotes z projection in the laboratory system, and i is the label of the solution of RPA dynamical equation ($i=1$ corresponds to the collective solution). $A^\dagger$ and A denote two-quasiparticle creation and annihilation operators,

$$A^\dagger(jj';2\mu)=\sum_{mm'}\langle jmj'm'\,|\,2\mu\rangle a^\dagger_{jm}a^\dagger_{j'm'} , \quad (3)$$

$$A(jj';2-\mu)=\sum_{mm'}\langle jmj'm'\,|\,2-\mu\rangle a_{j'm'}a_{jm} , \quad (4)$$

where $a^\dagger_{jm}$ and a_{jm} are quasiparticle creation and annihilation operators in the spherical basis, respectively.

In the second step we introduce the generating state[1] with the ansatz[27]

$$|\alpha\rangle=\exp\left[\sum_{\mu=-2}^{2}\alpha^*_{2\mu}Q^\dagger_{2\mu 1}\right]|0\rangle , \quad (5)$$

where $|0\rangle$ is the vacuum for RPA phonons, not for TD, as in Ref. 27,

$$Q_{2\mu 1}|0\rangle=0 . \quad (6)$$

Now we study the matrix elements of the operators $Q_{2\mu 1}$, $Q^\dagger_{2\mu 1}$, and $[Q_{2\mu 1},Q^\dagger_{\nu 1}]$ between the generating states (5). They read[24]

$$\langle\alpha|Q_{2\mu 1}|\alpha'\rangle=\left\langle 0\left|\left[Q_\mu+\sum_{m=1}^{\infty}\sum_{\tau_1,\ldots,\tau_m}\frac{(-1)^m}{m!}\{[\cdots[[Q_\mu,Q_{\tau_1}],Q_{\tau_2}],\ldots,Q_{\tau_m}]\alpha_{\tau_1}\alpha_{\tau_2}\cdots\alpha_{\tau_m}\}\right]\right.\right.$$
$$\left.\left.\times\exp\left[\sum_{\tau=-2}^{2}\alpha_\tau Q_\tau\right]\right|\alpha'\right\rangle , \quad (7)$$

$$\langle\alpha|Q^\dagger_{2\mu 1}|\alpha'\rangle=\left\langle 0\left|\left[Q^\dagger_\mu+\sum_{m=1}^{\infty}\sum_{\tau_1,\ldots,\tau_m}\frac{(-1)^m}{m!}[\cdots[Q^\dagger_\mu,Q_{\tau_1}],\ldots,Q_{\tau_m}]\alpha_{\tau_1}\cdots\alpha_{\tau_m}\right]\exp\left[\sum_{\tau=-2}^{2}\alpha_\tau Q_\tau\right]\right|\alpha'\right\rangle , \quad (8)$$

$$\langle\alpha|[Q_{2\mu 1},Q^\dagger_{2\nu 1}]|\alpha'\rangle=\left\langle 0\left|\left[[Q_\mu,Q^\dagger_\nu]+\sum_{m=1}^{\infty}\sum_{\tau_1,\ldots,\tau_m}\frac{(-1)^m}{m!}[\cdots[[Q_\mu,Q^\dagger_\nu],Q_{\tau_1}],Q_{\tau_2}],\ldots,Q_{\tau_m}]\alpha_{\tau_1}\cdots\alpha_{\tau_m}\right]\right.\right.$$
$$\left.\left.\times\exp\left[\sum_{\tau=-2}^{2}\alpha_\tau Q_\tau\right]\right|\alpha'\right\rangle . \quad (9)$$

On the right-hand side (rhs) of Eqs. (7)–(9) the notation Q_τ and $Q^\dagger_\tau$ stands for $Q_{2\tau 1}$ and $Q^\dagger_{2\tau 1}$, respectively.

As it turns out, the multipole commutators of the RPA-phonon operators contain not only the operators $Q_{2\mu 1}$ and $Q^\dagger_{2\mu 1}$ but the noncollective roots $Q_{\lambda\mu i}$ and $Q^\dagger_{\lambda\nu i}$ also [cf. Eqs. (16)–(18)].

The general RPA solution for creation and annihilation operator has the standard form[16]

$$Q^\dagger_{\lambda\mu i}=\tfrac{1}{2}\sum_{jj'}[\Psi^{\lambda i}_{jj'}A^\dagger(jj';\lambda\mu)-(-1)^{\lambda-\mu}\phi^{\lambda i}_{jj'}A(jj';\lambda-\mu)] , \quad (10)$$

$$Q_{\lambda\mu i}=\tfrac{1}{2}\sum_{jj'}[\Psi^{\lambda i}_{jj'}A(jj';\lambda\mu)-(-1)^{\lambda-\mu}\phi^{\lambda i}_{jj'}A^\dagger(jj';\lambda-\mu)] , \quad (11)$$

with

$$A^\dagger(jj';\lambda\mu)=\sum_{mm'}\langle jmj'm'\,|\,\lambda\mu\rangle a^\dagger_{jm}a^\dagger_{j'm'} , \quad (12)$$

$$A(jj';\lambda-\mu)=\sum_{mm'}\langle jmj'm'\,|\,\lambda-\mu\rangle a_{j'm'}a_{jm} , \quad (13)$$

with the closure relations

$$\sum_{jj'}[\Psi^{\lambda i}_{jj'}\Psi^{\lambda i'}_{jj'}-\Phi^{\lambda i}_{jj'}\Phi^{\lambda i'}_{jj'}]=2\delta_{ii'}\ ,$$

$$\sum_{jj'}[\Psi^{\lambda i}_{jj'}\Phi^{\lambda i'}_{jj'}-\Psi^{\lambda i'}_{jj'}\Phi^{\lambda i}_{jj'}]=0\ ,$$

$$\sum_{i}(\Psi^{\lambda i}_{j_1j_1'}\Psi^{\lambda i}_{j_2j_2'}-\Phi^{\lambda i}_{j_1j_1'}\Phi^{\lambda i}_{j_2j_2'})=\delta_{j_1j_2}\delta_{j_1'j_2'}-(-1)^{j_1+j_2-\lambda}\delta_{j_1j_2'}\delta_{j_2j_1'}\ .$$

The inverse transformation reads

$$A^{\dagger}(jj';\lambda\mu)=\sum_{i}\Psi^{\lambda i}_{jj'}Q^{\dagger}_{\lambda\mu i}+(-1)^{\lambda-\mu}\Phi^{\lambda i}_{jj'}Q_{\lambda-\mu i}\ , \tag{14}$$

$$A(jj';\lambda-\mu)=\sum_{i}\Psi^{\lambda i}_{jj'}Q_{\lambda-\mu i}+(-1)^{\lambda+\mu}\Phi^{\lambda i}_{jj'}Q^{\dagger}_{\lambda\mu i}\ . \tag{15}$$

Using Eqs. (10)–(13) and the anticommutation relations for quasiparticle operators $a^{\dagger}_{jm},a_{jm}$, we obtain the expressions for commutators appearing in Eqs. (7)–(9).

The single commutators are

$$[Q_{\lambda\mu i},Q^{\dagger}_{\lambda_0\mu_0 i_0}]=\delta_{\lambda\lambda_0}\delta_{\mu\mu_0}\delta_{ii_0}-\sum_{j_1j_2j_1'}\sum_{\lambda'\mu'}B(j_1'j_2;\lambda'\mu')(-1)^{j_1'+j_2}(2\lambda+1)^{1/2}(2\lambda'+1)^{1/2}\langle\lambda\mu\lambda'\mu'\mid\lambda_0\mu_0\rangle$$
$$\times\left[\Psi^{\lambda i}_{j_1j_2}\Psi^{\lambda_0 i_0}_{j_1'j_1}(-1)^{\lambda'}\begin{Bmatrix}\lambda_0&\lambda&\lambda'\\ j_2&j_1'&j_1\end{Bmatrix}-(-1)^{\lambda-\lambda_0}\Phi^{\lambda_0 i_0}_{j_1j_2}\Phi^{\lambda i}_{j_1'j_1}\begin{Bmatrix}\lambda&\lambda_0&\lambda'\\ j_2&j_1'&j_1\end{Bmatrix}\right]\ , \tag{16}$$

$$[Q^{\dagger}_{\lambda\mu i},Q^{\dagger}_{\lambda_0\mu_0 i_0}]=\sum_{j_1j_2j_1'}\sum_{\lambda'\mu'}B(j_1j_1';\lambda'\mu')(-1)^{j_1+j_1'}(2\lambda+1)^{1/2}(2\lambda_0+1)^{1/2}\langle\lambda\mu\lambda_0\mu_0\mid\lambda'\mu'\rangle$$
$$\times\left[\Psi^{\lambda_0 i_0}_{j_1j_2}\Phi^{\lambda i}_{j_2j_1'}\begin{Bmatrix}\lambda_0&\lambda&\lambda'\\ j_1'&j_1&j_2\end{Bmatrix}(-1)^{\lambda+\lambda_0}-\Psi^{\lambda i}_{j_1j_2}\Phi^{\lambda_0 i_0}_{j_2j_1'}\begin{Bmatrix}\lambda&\lambda_0&\lambda'\\ j_1'&j_1&j_2\end{Bmatrix}(-1)^{\lambda'}\right]\ , \tag{17}$$

where

$$B(jj';\lambda\mu)=\sum_{mm'}\langle jmj'm'\mid\lambda\mu\rangle a^{\dagger}_{jm}(-1)^{j'+m'}a_{j'-m'}\ . \tag{17a}$$

We also need the commutator

$$[B(j_1j_2;\lambda\mu),Q^{\dagger}_{2\nu 1}]=\sqrt{5}(2\lambda+1)^{1/2}\sum_{\lambda'\mu'i'}\langle 2\nu\lambda\mu\mid\lambda'\mu'\rangle\left\{Q^{\dagger}_{\lambda'\mu'i'}\sum_{j_1'}(-1)^{j_1+j_1'}\left[\begin{Bmatrix}\lambda'&\lambda&2\\ j_2&j_1'&j_1\end{Bmatrix}(-1)^{\lambda}\Psi^{\lambda'i'}_{j_1j_1'}\Psi^{21}_{j_2j_1'}\right.\right.$$
$$\left.+\begin{Bmatrix}2&\lambda&\lambda'\\ j_2&j_1'&j_1\end{Bmatrix}\Phi^{21}_{j_1j_1'}\Phi^{\lambda'i'}_{j_2j_1'}\right]$$
$$+(-1)^{\lambda'-\mu'}Q_{\lambda'-\mu'i'}\sum_{j_1'}(-1)^{j_1+j_1'}\left[(-1)^{\lambda}\begin{Bmatrix}\lambda'&\lambda&2\\ j_2&j_1'&j_1\end{Bmatrix}\Phi^{\lambda'i'}_{j_1j_1'}\Psi^{21}_{j_2j_1'}\right.$$
$$\left.\left.+\begin{Bmatrix}2&\lambda&\lambda'\\ j_2&j_1'&j_1\end{Bmatrix}\Phi^{21}_{j_1j_1'}\Psi^{\lambda'i'}_{j_2j_1'}\right]\right\}\ . \tag{18}$$

Using expressions (10)–(18), one obtains, in a straightforward way, double commutators which appear in (7)–(9). Let us write only one double commutator in order to illustrate terms which are included and left out in the SU(6) microscopic boson model of Ref. 9,

$$
\begin{aligned}
&[Q^{\dagger}_{\lambda'\mu' i'},[Q_{\lambda\mu i},Q^{\dagger}_{\lambda_0\mu_0 i_0}]] \\
&\quad=\sum_{j_1 j_2 j_1' j_2'}\ \sum_{k\kappa\lambda''\mu'' i''}[(2\lambda'+1)(2\lambda+1)(2\lambda_0+1)(2\lambda''+1)]^{1/2} \\
&\qquad\times\Bigg\{(-1)^{\lambda+\lambda_0+\lambda''+k}\Bigg[\langle\lambda_0\mu_0\lambda'\mu'\mid k\kappa\rangle\langle\lambda''\mu''\lambda\mu\mid k\kappa\rangle(-1)^{j_1'-j_1}\begin{Bmatrix} j_2' & j_1' & \lambda'' \\ j_2 & j_1 & \lambda \\ \lambda' & \lambda_0 & k\end{Bmatrix} \\
&\qquad\qquad\times\left(Q^{\dagger}_{\lambda''\mu'' i''}\Psi^{\lambda i}_{j_1 j_2}\Psi^{\lambda_0 i_0}_{j_1' j_1}\Psi^{\lambda' i'}_{j_2 j_2'}\Psi^{\lambda'' i''}_{j_1' j_2'}+(-1)^{\lambda''-\mu''}Q_{\lambda''-\mu'' i''}\Psi^{\lambda i}_{j_1 j_2}\Psi^{\lambda_0 i_0}_{j_1' j_1}\Psi^{\lambda' i'}_{j_2 j_2'}\Phi^{\lambda'' i''}_{j_1' j_2'}\right) \\
&\qquad\qquad-\langle\lambda-\mu\lambda'\mu'\mid k\kappa\rangle\langle\lambda''\mu''\lambda_0-\mu_0\mid k\kappa\rangle(-1)^{\lambda-\mu+\lambda_0-\mu_0}(-1)^{j_1'-j_1}\begin{Bmatrix} j_2' & j_1' & \lambda'' \\ j_2 & j_1 & \lambda_0 \\ \lambda' & \lambda & k\end{Bmatrix} \\
&\qquad\qquad\times\left(Q^{\dagger}_{\lambda''\mu'' i''}\Phi^{\lambda_0 i_0}_{j_1 j_2}\Phi^{\lambda i}_{j_1' j_1}\Psi^{\lambda' i'}_{j_2 j_2'}\Psi^{\lambda'' i''}_{j_1' j_2'}+(-1)^{\lambda''-\mu''}Q_{\lambda''-\mu'' i''}\Phi^{\lambda_0 i_0}_{j_1 j_2}\Phi^{\lambda i}_{j_1' j_1}\Psi^{\lambda' i'}_{j_2 j_2'}\Phi^{\lambda'' i''}_{j_1' j_2'}\right)\Bigg] \\
&\qquad+(-1)^{\lambda'+\lambda+\lambda_0+k}\Bigg[\langle\lambda_0\mu_0\lambda''\mu''\mid k\kappa\rangle(-1)^{\lambda'-\mu'}\langle\lambda'-\mu'\lambda\mu\mid k\kappa\rangle \\
&\qquad\qquad\times(-1)^{j_1'-j_1}\begin{Bmatrix} j_2' & j_1' & \lambda' \\ j_2 & j_1 & \lambda \\ \lambda'' & \lambda_0 & k\end{Bmatrix}\Big(Q_{\lambda''\mu'' i''}\Psi^{\lambda i}_{j_1 j_2}\Psi^{\lambda_0 i_0}_{j_1' j_1}\Psi^{\lambda'' i''}_{j_2 j_2'}\Phi^{\lambda' i'}_{j_1' j_2'} \\
&\qquad\qquad+(-1)^{\lambda''-\mu''}Q^{\dagger}_{\lambda''-\mu'' i''}\Psi^{\lambda i}_{j_1 j_2}\Psi^{\lambda_0 i_0}_{j_1' j_1}\Phi^{\lambda'' i''}_{j_2 j_2'}\Phi^{\lambda' i'}_{j_1' j_2'}\Big) \\
&\qquad\qquad-(-1)^{\lambda'-\mu'}\langle\lambda'-\mu'\lambda_0-\mu_0\mid k\kappa\rangle(-1)^{\lambda_0-\mu_0}\langle\lambda-\mu\lambda''\mu''\mid k\kappa\rangle \\
&\qquad\qquad\times(-1)^{\lambda-\mu}(-1)^{j_1'-j_1}\begin{Bmatrix} j_2' & j_1' & \lambda \\ j_2 & j_1 & \lambda_0 \\ \lambda'' & \lambda & k\end{Bmatrix}\Big(Q_{\lambda''\mu'' i''}\Phi^{\lambda_0 i_0}_{j_1 j_2}\Phi^{\lambda i}_{j_1' j_1}\Phi^{\lambda' i'}_{j_1' j_2'}\Psi^{\lambda'' i''}_{j_2 j_2'} \\
&\qquad\qquad+(-1)^{\lambda''-\mu''}Q^{\dagger}_{\lambda''-\mu'' i''}\Phi^{\lambda_0 i_0}_{j_1 j_2}\Phi^{\lambda i}_{j_1' j_1}\Phi^{\lambda' i'}_{j_1' j_2'}\Phi^{\lambda'' i''}_{j_2 j_2'}\Big)\Bigg]\Bigg\}\ . \qquad (19)
\end{aligned}
$$

It is seen that even if we set $\lambda_0=\lambda=\lambda'=2$ and $i_0=i=i'=1$, i.e., restrict our consideration to the commutators among the RPA quadrupole collective operators, there are the terms with $\lambda''\neq 2$ and/or $i''\neq 1$ which "scatter" out of quadrupole collective subspace due to summation on the rhs of Eq. (19). These terms are referred to as "scattering terms."

However, if we enforce λ'' to $\lambda''=2$ only, and i'' to $i''=1$, i.e., if we neglect the "scattering terms," the expression significantly simplifies. In this case the double commutators are approximated by the terms which do not lead out of the RPA quadrupole collective subspace. This approximation will be referred to as "scattering" approximation. In this way the original, untractable algebra SO(2Ω), closed by two-fermion operators $\{aa,a^{\dagger}a^{\dagger},a^{\dagger}a\}$, will be restricted to a tractable but enforced SU(6) Lie algebra, i.e., SO(2Ω)$\supset$SU(6).

If the scattering approximation is imposed, the double commutators appearing in expressions (7)–(9) can be (cf Sec. III) expressed as linear combinations of $Q_{2\tau 1}$ and $Q^{\dagger}_{2\tau 1}$. Therefore, the set of operators

$$\{Q_{\mu},Q^{\dagger}_{\mu},[Q_{\mu},Q^{\dagger}_{\mu_0}],[Q_{\mu},Q_{\mu_0}],[Q^{\dagger}_{\mu},Q^{\dagger}_{\mu_0}]\} \qquad (20)$$

constitutes a closed algebra. It should be pointed out that the operators (20) generally do not close an algebra—the closure was enforced by neglecting scattering terms. Here we have introduced the notation

$Q_\mu, Q^\dagger_\mu$, labeling quadrupole collective operators associated with the "scattering" approximation.

In the following step we shall impose the condition that the operators (20) form a Lie algebra. This will be referred to as Lie algebra enforcement. We will show that in this case the set of 35 operators

$$\{Q_\mu, Q^\dagger_\mu, [Q_\mu, Q^\dagger_{\mu_0}]\} \quad (\mu, \mu_0 = 0, \pm 1, \pm 2)$$

forms an algebra isomorphic to the SU(6) Cartan-Weyl canonical algebra.[23,24,28] This algebra is referred to as quadrupole collective algebra (QCA).

The two requirements together, the "scattering" approximation and Lie algebra enforcement, are referred to as the SU(6) approximation. We will show that in this case the commutators $[Q,Q]$ and $[Q^\dagger, Q^\dagger]$ vanish.

Employing SU(6) approximation the infinite series in (7)–(9) terminate at the double commutators and we shall obtain closed expressions for the matrix elements $\langle \alpha | Q_\mu | \alpha' \rangle$, $\langle \alpha | Q^\dagger_\mu | \alpha' \rangle$, and $\langle \alpha | [Q_\mu, Q^\dagger_\nu] | \alpha' \rangle$ in terms of α_μ and their derivatives $\partial/\partial\alpha_\mu$.

In the next step α_μ and $\partial/\partial\alpha_\mu$ will be mapped into quadrupole boson creation and annihilation operators

$$\alpha_\mu \to b^\dagger_\mu, \quad \frac{\partial}{\partial\alpha_\mu} \to b_\mu .$$

Upon this replacement we shall obtain microscopic Dyson boson realization for $Q_\mu, Q^\dagger_\mu$ and $[Q_\mu, Q^\dagger_\nu]$.

In the following step, using the microscopic DR we shall derive the microscopic HPR and microscopic SR. Alternatively, we shall derive the microscopic HPR directly, by performing a complete analysis of the enforced QCA closed by $\{Q_\mu, Q^\dagger_\mu, [Q_\mu, Q^\dagger_\nu]\}$. Having explicitly constructed the microscopic HPR of the QCA in a straightforward way, we construct the microscopic SR of this algebra.

III. QUADRUPOLE COLLECTIVE ALGEBRA IN TERMS OF RPA QUADRUPOLE PHONON OPERATORS AND RPA-SU(6)-ENFORCING CONDITIONS

Imposing the SU(6) scattering approximation, we can present the double commutator (19) in the compact form

$$[Q^\dagger_{\mu'}, [Q_\mu, Q^\dagger_{\mu_0}]] = 2 \sum_{\mu''=-2}^{2} (Q^\dagger_{\mu''} C_{\mu'\mu\mu_0\mu''} + Q_{\mu''} D_{\mu'\mu\mu_0\mu''}) , \tag{21}$$

with

$$C_{\mu'\mu\mu_0\mu''} = \sum_{K=0}^{4} C_K \sum_{\kappa=-K}^{K} \langle 2\mu' 2\mu_0 | K\kappa \rangle \langle 2\mu'' 2\mu | K\kappa \rangle , \tag{22}$$

$$D_{\mu'\mu\mu_0\mu''} = \sum_{K=0}^{4} D_K \sum_{\kappa=-K}^{K} [(-1)^{\mu'} \langle 2\mu 2-\mu' | K\kappa \rangle \langle 2\mu_0 2\mu'' | K\kappa \rangle + (-1)^{\mu''} \langle 2\mu' 2\mu_0 | K\kappa \rangle \langle 2-\mu'' 2\mu | K\kappa \rangle] , \tag{23}$$

$$C_K = \tfrac{25}{2} \sum_{j_1 j_2 j'_1 j'_2} (-1)^{j'_1 - j_1} \begin{Bmatrix} j'_2 & j'_1 & 2 \\ j_2 & j_1 & 2 \\ 2 & 2 & K \end{Bmatrix} (\Psi_{j_1 j_2} \Psi_{j'_1 j_1} \Psi_{j_2 j'_2} \Psi_{j'_1 j'_2} - \Phi_{j_1 j_2} \Phi_{j'_1 j_1} \Phi_{j_2 j'_2} \Phi_{j'_1 j'_2}) , \tag{24}$$

$$D_K = \tfrac{25}{2} \sum_{j_1 j_2 j'_1 j'_2} (-1)^{j'_1 - j_1} \begin{Bmatrix} j'_2 & j'_1 & 2 \\ j_2 & j_1 & 2 \\ 2 & 2 & K \end{Bmatrix} (\Psi_{j_1 j_2} \Psi_{j'_1 j_1} \Psi_{j_2 j'_2} \Phi_{j'_1 j'_2} - \Phi_{j_1 j_2} \Phi_{j'_1 j_1} \Psi_{j_2 j'_2} \Phi_{j'_1 j'_2}) . \tag{25}$$

Here, Ψ and Φ stand for Ψ^{21} and Φ^{21}, respectively. In obtaining Eqs. (21)–(25) we have used the relations[16]

$$\Psi^{\lambda i}_{jj'} = (-1)^{j-j'-\lambda} \Psi^{\lambda i}_{j'j} \quad \Phi^{\lambda i}_{jj'} = (-1)^{j-j'-\lambda} \Phi^{\lambda i}_{j'j} . \tag{26}$$

Let us write the expression for another double commutator,

$$[Q_{\mu'}, [Q_\mu, Q^\dagger_{\mu_0}]] = -2 \sum_{\mu''} (Q_{\mu''} C_{\mu'\mu_0\mu\mu''} + Q^\dagger_{\mu''} D_{\mu'\mu_0\mu\mu''}) . \tag{27}$$

If we consider the Tamm-Dancoff phonon operators, i.e., by neglecting the backward going amplitudes Φ, from Eqs. (17), (21)–(25), and (27) there follows

$$[Q^\dagger_\mu, Q^\dagger_{\mu_0}] \equiv [Q_\mu, Q_{\mu_0}] \equiv 0 ,$$

$$D_{\mu'\mu\mu_0\mu''} \equiv 0 ,$$

$$[Q^\dagger, [Q, Q^\dagger]] \sim Q^\dagger ,$$

$$[Q, [Q, Q^\dagger]] \sim Q ,$$

and in Eqs. (7)–(9) the infinite series terminate at double commutators. In this case we easily obtain exact boson realization of $\{Q^{\mathrm{TD}}_\mu, Q^{\dagger\mathrm{TD}}_\mu, [Q_\mu, Q^\dagger_\nu]^{\mathrm{TD}}\}$ in the form of the DR. As for C^{TD}_K, it exactly reproduces the corresponding quantity C_λ given in Ref. 14.

The case of RPA phonon operators is more complicated because RPA phonon operators do not automatically commute [see Eq. (17)], and the double commutators contain both Q and $Q^\dagger$ terms [cf. Eqs. (21) and (27)]. There is no apparent termination of the infinite series in (7)–(9) and, therefore, the exact boson realization does not emerge in a simple way as it does in the Tamm-Dancoff case.

On the other hand, we can directly check that the DR and HPR should exist as particular solutions for the commutation relations satisfied by $\{Q_\mu, Q_\mu^\dagger, [Q_\mu, Q_\nu^\dagger]\}$. The main point is to derive conditions under which the DR and HPR occur as exact realizations of QCA. This will also result in the microscopic expression for N, the quantum number associated with the linear Casimir operator of the SU(6) group.[2,9,10]

We have started with the set of operators

$$\{Q_{\lambda\mu i}, Q^\dagger_{\lambda\mu i}, [Q_{\lambda\mu i}, Q^\dagger_{\lambda_0\mu_0 i_0}], [Q_{\lambda\mu i}, Q_{\lambda_0\mu_0 i_0}], [Q^\dagger_{\lambda\mu i}, Q^\dagger_{\lambda_0\mu_0 i_0}]\}$$

corresponding to the algebra SO($2n$). Applying the "scattering" approximation, this set of operators was restricted to a smaller set,

$$\{Q_\mu, Q_\mu^\dagger [Q_\mu, Q^\dagger_{\mu_0}], [Q_\mu, Q_{\mu_0}], [Q^\dagger_\mu, Q^\dagger_{\mu_0}]\} \ . \tag{28}$$

Now we apply the Lie algebra enforcement: We require that the operators of the restricted algebra (28) satisfy Jacobi identities. In this way we obtain a number of constraints imposed on the RPA amplitudes Ψ and Φ (cf. the Appendix):

$$W(j_1 j_2;K) = \sum_{j'} (-1)^{j_1+j'} \begin{Bmatrix} 2 & K & 2 \\ j_1 & j' & j_2 \end{Bmatrix} \Phi_{j_1 j'} \Psi_{j_2 j'} = 0 \ , \tag{29}$$

for $K=1,3$ and any $\{j_1 j_2\}$;

$$W(j_1 j_2;K) = 0 \tag{30}$$

for $K=0,2,4$ and any $\{j_1 j_2\}$;

$$D_K = 0 \quad (K=0,1,2,3,4) \ , \tag{31}$$

$$C_1 = C_3 = 0 \ , \tag{32}$$

$$C_0 = C_2 = C_4 = C \quad (K \text{ independence of } C_K) \ , \tag{33}$$

where D_K and C_K are defined by Eqs. (25) and (24), respectively.

The constraints (29)–(33) will be referred to as RPA-SU(6)–enforcing conditions. In the Tamm-Dancoff case ($\Phi_{jj'}=0$) the relations (29)–(31) are automatically satisfied. In the RPA case these conditions are nontrivial. The most intricate point is to establish $D_K=0$. This is a crucial step in the microscopic derivation of the DR, HPR, and SR.

We note that up until now Ψ and Φ are assumed to be the solutions of standard RPA equations. However, the task of determining Ψ and Φ could be reformulated, taking into account the constraints (29)–(33) which lead to a system of dynamical linear equations for Ψ, Φ and frequency ω, coupled with the nonlinear RPA-SU(6)–enforcing conditions.

In the Appendix we show explicitly that, due to (29) and (30), there holds

$$[Q_\mu^\dagger, Q_\nu^\dagger] = 0, \quad [Q_\mu, Q_\nu] = 0 \ . \tag{34}$$

We stress that all these relations stem unambiguously from the corresponding Jacobi identities (cf. the Appendix). The enforcing RPA-SU(6) conditions (29)–(33) are essential for our microscopic derivation of the DR and HPR for RPA phonon operators. On the other hand, these relations resolve some of the problems appearing in Ref. 9: The difficulties of Ref. 9 were that the relation $K_l = L_l$ was not shown, and that the boson number N included terms which should exactly vanish as required by Jacobi identities. Now, by substituting the expressions

$$p_{jj'} = \tfrac{1}{2}(\Psi_{jj'} - \Phi_{jj'}), \quad q_{jj'} = \tfrac{1}{2}(\Psi_{jj'} + \Phi_{jj'})$$

into the expressions for the quantities K_l and L_l of Ref. 9, we obtain

$$K_l = C_l - 2D_l, \quad L_l = C_l + 2D_l \ .$$

From $D_l = 0$ [Eq. (30)], there follows $K_l = L_l$ and

$$N = \mathrm{Int}\left[\frac{2}{K}\right] = \mathrm{Int}\left[\frac{1}{L}\right] = \mathrm{Int}\left[\frac{1}{C}\right] \ . \tag{35}$$

It should be noted that relations (34) and (35) have been proved to hold exactly if the SU(6) enforcing condition is assumed; we do not use the RPA assumption that the number of quasiparticles in the ground state is small.

We can easily show that the quantities C_0, C_2, and C_4 are related to the norm of the two-phonon state,

$$\frac{1}{\sqrt{2}}[Q_2^\dagger, Q_2^\dagger]_{JM} |\tilde{0}\rangle \ ,$$

where $|\tilde{0}\rangle$ denotes the vacuum of Q_μ, $Q_\mu |\tilde{0}\rangle = 0$. By a straightforward derivation, we obtain, for the two-phonon norm,

$$\left|\left| \frac{1}{\sqrt{2}}[Q_2^\dagger, Q_2^\dagger]_{JM} |\tilde{0}\rangle \right|\right| = \tfrac{1}{2} \sum_{\mu_0' \mu' \mu_0 \mu} \langle 2\mu' 2\mu_0' | JM \rangle \langle 2\mu 2\mu_0 | JM \rangle \langle \tilde{0} | Q_{\mu_0'} [[Q_{\mu'}, Q_\mu^\dagger], Q^\dagger_{\mu_0}] | \tilde{0} \rangle + 1$$

$$= 1 - \tfrac{1}{2} \sum_{\mu_0' \mu' \mu_0 \mu} \langle 2\mu' 2\mu_0' | JM \rangle \langle 2\mu 2\mu_0 | JM \rangle C (\delta_{\mu'\mu} \delta_{\mu_0 \mu_0'} + \delta_{\mu' \mu_0} \delta_{\mu \mu_0'}) = 1 - C \ ,$$

i.e., C takes into account, on the average, the Pauli principle.

We note that in view of Eq. (34) the enforced RPA phonon operators commute and therefore odd momenta do not contribute to the two-phonon norm; thus C_1 and C_3 should not contribute, in agreement with Eq. (32).

RPA-SU(6)-enforcing conditions lead to essential simplification of the algebra (28). By virtue of Eqs. (34) and (35), the commutators $[Q_\mu, Q_{\mu_0}]$ and $[Q_\mu^\dagger, Q_{\mu_0}^\dagger]$ drop out, and we are left with the set of 35 linearly independent operators,

$$\{Q_\mu, Q_\mu^\dagger, [Q_\mu, Q_{\mu_0}^\dagger]\} ,$$

which is just the number of operators corresponding to the SU(6) algebra.

Using Eqs. (31)–(33) the double commutators (21) and (27) can be brought into a simple form:

$$[Q_{\mu'}^\dagger, [Q_\mu, Q_{\mu_0}^\dagger]] = C\delta_{\mu'\mu} Q_{\mu_0}^\dagger + C\delta_{\mu\mu_0} Q_{\mu'}^\dagger , \tag{36}$$

$$[Q_{\mu'}, [Q_\mu, Q_{\mu_0}^\dagger]] = -C\delta_{\mu'\mu_0} Q_\mu - C\delta_{\mu\mu_0} Q_{\mu'} . \tag{37}$$

Therefrom, we have

$$[[Q_\mu, Q_\nu^\dagger], [Q_\rho, Q_\tau^\dagger]] = C\delta_{\rho\nu}[Q_\mu, Q_\tau^\dagger] - C\delta_{\mu\tau}[Q_\rho, Q_\nu^\dagger] . \tag{38}$$

Here we have used the relation

$$\sum_{K=0,2,4,} \sum_{\kappa=-K}^{K} \langle 2\mu 2\nu \mid K\kappa \rangle \langle 2\lambda 2\rho \mid K\kappa \rangle = \tfrac{1}{2}(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\lambda\nu}\delta_{\rho\mu}) .$$

Now we turn to the microscopic derivation of the DR of the QCA.

IV. MICROSCOPIC DYSON REALIZATION FOR THE QCA

Let us now insert the commutation relations (35)–(37) into the rhs of Eqs. (7)–(9).

From Eq. (7) we get

$$\langle \tilde{\alpha} \mid Q_\mu \mid \tilde{\alpha}' \rangle = \left\langle \tilde{0} \left| Q_\mu \exp\left[\sum_{\tau=-2}^{2} \alpha_\tau Q_\tau \right] \right| \tilde{\alpha}' \right\rangle = \left\langle \tilde{0} \left| \exp\left[\sum_{\tau=-2}^{2} \alpha_\tau Q_\tau \right] Q_\mu \right| \tilde{\alpha}' \right\rangle , \tag{39}$$

which can be written in the form

$$\langle \tilde{\alpha} \mid Q_\mu \mid \tilde{\alpha}' \rangle = \frac{\partial}{\partial \alpha_\mu} \langle \tilde{\alpha} \mid \tilde{\alpha}' \rangle . \tag{40}$$

Similarly, from Eqs. (8) and (9) we get

$$\langle \tilde{\alpha} \mid Q_\mu^\dagger \mid \tilde{\alpha}' \rangle = \left\langle \tilde{0} \left| \left[Q_\mu^\dagger + \sum_{\tau_1} \frac{(-1)^1}{1!}[Q_\mu^\dagger, Q_{\tau_1}]\alpha_{\tau_1} + \sum_{\tau_1\tau_2} \frac{(-1)^2}{2!}[[Q_\mu^\dagger, Q_{\tau_1}], Q_{\tau_2}]\alpha_{\tau_1}\alpha_{\tau_2} \right] \exp\left[\sum_{\tau=-2}^{2} \alpha_\tau Q_\tau \right] \right| \tilde{\alpha}' \right\rangle , \tag{41}$$

$$\langle \tilde{\alpha} \mid [Q_\mu, Q_\nu^\dagger] \mid \tilde{\alpha}' \rangle = \left\langle \tilde{0} \left| \left[[Q_\mu, Q_\nu^\dagger] + \sum_{\tau_1} \frac{(-1)^1}{1!}[[Q_\mu, Q_\nu^\dagger], Q_{\tau_1}]\alpha_{\tau_1} \right] \exp\left[\sum_{\tau=-2}^{2} \alpha_\tau Q_\tau \right] \right| \tilde{\alpha}' \right\rangle . \tag{42}$$

In order to simplify (41) and (42), let us determine the effect of Q_μ, $Q_\mu^\dagger$, and $[Q_\mu, Q_\nu^\dagger]$ on $|\tilde{0}\rangle$. In Sec. VII, we will show that the QCA is isomorphic to the SU(6) Cartan-Weyl canonical algebra; Q_μ are raising generators, $Q_\mu^\dagger$ are lowering generators, and $[Q_\mu, Q_\nu^\dagger]$ are lowering generators for $\nu > \mu$ and belong to a Cartan Abelian subalgebra for $\nu = \mu$.

Previously, we have considered the RPA vacuum (6): $Q_\mu |0\rangle = \langle 0 | Q_\mu^\dagger = 0$. When we replace the RPA quadrupole phonon operators $\{Q_{2\mu 1}, Q_{2\mu 1}^\dagger\}$ by $\{Q_\mu, Q_\mu^\dagger\}$, which belong to the SU(6) Lie algebra, as it is natural [because of QCA=SU(6) isomorphism] to replace the original phonon vacuum $|0\rangle$ by the highest weight state of the totally symmetric irreducible representation of SU(6). This is a straightforward group-theoretical identification of the state $|\tilde{0}\rangle$. As a consequence, we have

$$Q_\mu |\tilde{0}\rangle = \langle \tilde{0} | Q_\mu^\dagger = 0 , \tag{43}$$

$$\langle \tilde{0} | [Q_\mu, Q_\nu^\dagger] = \delta_{\mu\nu} \langle \tilde{0} | . \tag{44}$$

Using (37), (43), and (44) we transform the relations (41) and (42) as follows:

$$\langle \tilde{\alpha} \mid Q_\mu^\dagger \mid \tilde{\alpha}' \rangle = \left[\alpha_\mu - \alpha_\mu C \sum_{\tau=-2}^{2} \alpha_\tau \frac{\partial}{\partial \alpha_\tau} \right] \langle \tilde{\alpha} \mid \tilde{\alpha}' \rangle ,$$

$$\langle \tilde{\alpha} \mid [Q_\mu, Q_\nu^\dagger] \mid \tilde{\alpha}' \rangle = \left[\delta_{\mu\nu} \left[1 - C \sum_\tau \alpha_\tau \frac{\partial}{\partial \alpha_\tau} \right] - C\alpha_\nu \frac{\partial}{\partial \alpha_\mu} \right] \langle \tilde{\alpha} \mid \tilde{\alpha}' \rangle .$$

In this way we have obtained closed expressions for the operators of the QCA in terms of α and $\partial/\partial\alpha$:

$$Q_\mu \rightarrow \frac{\partial}{\partial \alpha_\mu},$$

$$Q_\mu^\dagger \rightarrow \alpha_\mu - C\alpha_\mu \sum_{\tau=-2}^{2} \alpha_\tau \frac{\partial}{\partial \alpha_\tau},$$

$$[Q_\mu, Q_\nu^\dagger] \rightarrow \delta_{\mu\nu}\left[1 - C\sum_{\tau=-2}^{2} \alpha_\tau \frac{\partial}{\partial\alpha_\tau}\right] - C\alpha_\nu \frac{\partial}{\partial \alpha_\mu}.$$

Making the replacement

$$\alpha_\mu \rightarrow b_\mu^\dagger, \quad \frac{\partial}{\partial\alpha_\mu} \rightarrow b_\mu,$$

we obtain Dyson boson realization for the RPA phonon operators

$$Q_\mu^{\mathrm{DR}} = b_\mu,$$

$$Q_\mu^{\dagger\mathrm{DR}} = b_\mu^\dagger\left[1 - C\sum_\tau b_\tau^\dagger b_\tau\right],$$

$$[Q_\mu, Q_\nu^\dagger]^{\mathrm{DR}} = \delta_{\mu\nu}\left[1 - C\sum_\tau b_\tau^\dagger b_\tau\right] - Cb_\nu^\dagger b_\mu.$$

Let us introduce

$$N = \mathrm{Int}\left[\frac{1}{C}\right]$$

as the integer nearest to $1/C$.

Now we present the DR in the standard form[1]

$$Q_\mu^{\mathrm{DR}} = b_\mu, \tag{45}$$

$$Q_\mu^{\dagger\mathrm{DR}} = b_\mu^\dagger\left[1 - \frac{1}{N}\sum_{\tau=-2}^{2} b_\tau^\dagger b_\tau\right], \tag{46}$$

$$[Q_\mu, Q_\nu^\dagger]^{\mathrm{DR}} = \delta_{\mu\nu}\left[1 - \frac{1}{N}\sum_{\tau=-2}^{2} b_\tau^\dagger b_\tau\right] - \frac{1}{N} b_\nu^\dagger b_\mu. \tag{47}$$

V. MICROSCOPIC HOLSTEIN-PRIMAKOFF BOSON REALIZATION FOR THE QCA

Using the microscopic DR (45)–(47), it is easy to derive the HPR. To this end we employ an orthogonalization procedure with the transformation operator[24,26]

$$\hat{O} = \left[\left[N - \sum_\nu b_\nu^\dagger b_\nu\right]!/N!\right]^{1/2}.$$

In this way we obtain

$$Q_\mu^{\dagger\mathrm{HPR}} = \hat{O}Q_\mu^{\dagger\mathrm{DR}}\hat{O}^{-1} = b_\mu^\dagger\left[1 - \frac{\hat{N}}{N}\right]^{1/2}, \tag{48}$$

$$Q_\mu^{\mathrm{HPR}} = \hat{O}Q_\mu^{\mathrm{DR}}\hat{O}^{-1} = \left[1 - \frac{\hat{N}}{N}\right]^{1/2} b_\mu, \tag{49}$$

$$[Q_\mu, Q_\nu^\dagger]^{\mathrm{HPR}} = \delta_{\mu\nu}\frac{N - \hat{N}}{N} - \frac{1}{N}b_\nu^\dagger b_\mu, \tag{50}$$

with

$$\hat{N} = \sum_\nu b_\nu^\dagger b_\nu.$$

This presents the desired microscopic HPR for RPA phonon operators and their commutators.

It should be noted that there is also a direct method[12,29] of deriving the microscopic HPR for the simplified QCA given by commutation relations (36)–(38). Following the method of Refs. 12 and 29, we introduce the ansatz

$$Q_\mu^\dagger = b_\mu^\dagger + \sum_{\tau\rho\sigma} X_{\tau\rho\sigma}^{(3)} b_\tau^\dagger b_\rho b_\sigma + \sum_{\tau\rho_1\sigma_1\rho_2\sigma_2} X_{\tau\rho_1\sigma_1\rho_2\sigma_2}^{(5)} b_\tau^\dagger b_{\rho_1}^\dagger b_{\sigma_1} b_{\rho_2}^\dagger b_{\sigma_2} + \cdots,$$

$$Q_\mu = b_\mu + \sum_{\tau\rho\sigma} X_{\tau\rho\sigma}^{(3)*} b_\sigma^\dagger b_\rho b_\tau^+ \cdots,$$

and calculate the coefficients order by order. In this way we obtain

$$Q_\mu^\dagger = b_\mu^\dagger - b_\mu^\dagger \frac{1}{2}\frac{\hat{N}}{N} - b_\mu^\dagger \frac{1}{8}\left[\frac{\hat{N}}{N}\right]^2 - \cdots$$

$$= b_\mu^\dagger \sum_{n=0}^{\infty} \frac{(2n-3)!!}{n!2^n}\left[\frac{\hat{N}}{N}\right]^n = b_\mu^\dagger\left[1 - \frac{\hat{N}}{N}\right]^{1/2},$$

and, analogously,

$$Q_\mu = \left[1 - \frac{\hat{N}}{N}\right]^{1/2} b_\mu.$$

In this way the HPR was obtained in the Tamm-Dancoff case, i.e., for the simplified QCA ($\phi = 0$).[1]

VI. MICROSCOPIC SCHWINGER REALIZATION FOR THE QCA

Schwinger boson realization for RPA phonon operators is directly obtained from the Holstein-Primakoff realization (48)–(50) by replacing

$$N - \sum_\lambda b_\lambda^\dagger b_\lambda \rightarrow s^\dagger s,$$

$$b_\mu^\dagger\left[1 - \frac{1}{N}\sum_\lambda b_\lambda^\dagger b_\lambda\right]^{1/2} \rightarrow N^{-1/2} d_\mu^\dagger s,$$

where $s^\dagger$ and $d_\mu^\dagger$ denotes creation operators for s and d bosons, respectively.

In this way, we obtain

$$Q_\mu^{\dagger\mathrm{SR}} = d_\mu^\dagger s N^{-1/2}, \tag{51}$$

$$Q_\mu^{\mathrm{SR}} = s^\dagger d_\mu N^{-1/2}, \tag{52}$$

$$[Q_\mu, Q_\nu^\dagger]^{\mathrm{SR}} = \frac{1}{N}\delta_{\mu\nu} s^\dagger s - \frac{1}{N} d_\nu^\dagger d_\mu. \tag{53}$$

Schwinger boson realization is associated[26,28] with the interacting boson model (IBM).[2–4] In the present framework this representation is particularly convenient for establishing the one-to-one correspondence between the QCA and Cartan-Weyl canonical algebra,

$$\{Q_\mu, Q_\mu^\dagger, [Q_\mu, Q_\nu^\dagger]\} \leftrightarrow \{H_k, E_\alpha\} .$$

VII. ISOMORPHISM BETWEEN THE QCA AND CARTAN-WEYL CANONICAL ALGEBRA

In order to complete our derivation of boson realizations, we have to establish isomorphism between the QCA and Cartan-Weyl canonical algebra (CWCA).[23,24,26] This isomorphism was used in obtaining relations (43) and (44). Furthermore, establishing this isomorphism enables us to clarify the Lie-algebraic meaning of $Q_\mu, Q_\mu^\dagger$. This is also important for understanding why the SU(6) algebra appears as the dynamical algebra associated with the quadrupole degree of freedom.

In order to express one-to-one correspondence between the QCA and CWCA, it is necessary to introduce a correspondence between indices utilized in both algebras. In the QCA index μ denotes the projection of $L=2$ angular momentum in the laboratory system. In the QCA we also need the combined index $\{\mu\nu\}$ related to $[Q_\mu, Q_\nu^\dagger]$. Taking $\nu \geq \mu$ we have two types of commutators, $[Q_\mu, Q_\nu^\dagger]$ and $[Q_\nu, Q_\mu^\dagger]$. In the CWCA there are $k=\beta$ indices related to the Cartan Abelian subalgebra ($k=1,\ldots,5$) and five simple roots ($\beta=1,\ldots,5$), and γ indices which specify the raising operators connected with nonsimple roots ($\gamma=6,\ldots,15$). We take the following unambiguous convention for the correspondence $\{\mu\} \leftrightarrow \{k=\beta\}$, $\{\mu\nu\} \leftrightarrow \{\gamma\}$:

(a)

μ	-2	-1	0	1	2
$k=\beta$	1	2	3	4	5

and

(b)

γ	6	7	8	9	10	11	12	13	14	15
$10i+j$	23	24	25	26	34	35	36	45	46	56

For a given γ we first determine the corresponding i and j using Table (b), and then we assign $i-1 \leftrightarrow \mu$, $j-1 \leftrightarrow \nu$, using Table (a).

First, we establish the role of Q_μ and $Q_\mu^\dagger$ as raising and lowering operators, respectively. To this end we are going to use a standard commutation relation for the SU(6) CWCA,[23,26,28]

$$[E_\alpha, E_{-\alpha}] = \mathbf{r}(\alpha)\cdot\mathbf{H} .$$

Using (51) and (52), or directly (53), we have

$$[Q_\mu, Q_\mu^\dagger] = \frac{1}{N}(s^\dagger s - d_\mu^\dagger d_\mu) .$$

Using the root system of the SU(6) CWCA from Ref. 28, we obtain

$$s^\dagger s - d_\mu^\dagger d_\mu = 12\mathbf{r}(\beta)\cdot\mathbf{H} ,$$

with $\{\mu\} \leftrightarrow \{\beta\}$ [cf. Table (a)]. Here, $\mathbf{r}(\beta)$ are five simple roots of the SU(6) CWCA and $\mathbf{H}=\{H_k\}$ are the elements of the Cartan Abelian subalgebra:

$$\begin{aligned}
H_1 &= \frac{1}{2\sqrt{6}}(s^\dagger s - d_{-2}^\dagger d_{-2}) , \\
H_2 &= \frac{1}{6\sqrt{2}}(s^\dagger s + d_{-2}^\dagger d_{-2} - 2d_{-1}^\dagger d_{-1}) , \\
H_3 &= \tfrac{1}{12}(s^\dagger s + d_{-2}^\dagger d_{-2} + d_{-1}^\dagger d_{-1} - 3d_0^\dagger d_0) , \\
H_4 &= \frac{1}{4\sqrt{15}}(s^\dagger s + d_{-2}^\dagger d_{-2} + d_{-1}^\dagger d_{-1} + d_0^\dagger d_0 - 4d_1^\dagger d_1) , \\
H_5 &= \frac{1}{6\sqrt{10}}(s^\dagger s + d_{-2}^\dagger d_{-2} + d_{-1}^\dagger d_{-1} + d_0^\dagger d_0 \\
&\quad + d_1^\dagger d_1 - 5d_2^\dagger d_2) .
\end{aligned} \tag{54}$$

Thus,

$$[Q_\beta, Q_\beta^\dagger] = \frac{12}{N}\mathbf{r}(\beta)\cdot\mathbf{H} . \tag{55}$$

From Eq. (55) it follows that Q_μ and $Q_\mu^\dagger$ can be identified with the raising and lowering generators of the SU(6) CWCA:[28]

$$\frac{1}{\sqrt{12}}s^\dagger d_\mu = E_\beta \leftrightarrow Q_\beta = \left[1 - \frac{\sum_\lambda b_\lambda^\dagger b_\lambda}{N}\right]^{1/2} b_\mu$$
$$= s^\dagger d_\mu N^{-1/2} , \tag{56}$$

$$\frac{1}{\sqrt{12}}d_\mu^\dagger s = E_{-\beta} \leftrightarrow Q_\beta^\dagger = b_\mu^\dagger \left[1 - \frac{\sum_\lambda b_\lambda^\dagger b_\lambda}{N}\right]^{1/2}$$
$$= d_\mu^\dagger s N^{-1/2} . \tag{57}$$

Using relation (38) we see that diagonal commutators $[Q_\mu, Q_\mu^\dagger]$ vanish. Therefore, they must be linearly related to the elements of the Cartan Abelian subalgebra of the SU(6) CWCA,

$$\mathbf{H} = M\mathbf{H}' , \tag{58}$$

where

$$\mathbf{H}=\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_5 \end{bmatrix}$$

is given by Eqs. (54), and elements of the Cartan Abelian subalgebra of the QCA are

$$\mathbf{H}'=N\begin{bmatrix} [Q_{-2},Q^{\dagger}_{-2}] \\ \vdots \\ [Q_2,Q^{\dagger}_2] \end{bmatrix}. \tag{59}$$

The explicit form of M matrix can be easily found with the help of Eqs. (54), (58), and (59):

$$M=\begin{bmatrix} \frac{1}{2\sqrt{6}} & 0 & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{12} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{4\sqrt{15}} & -\frac{1}{4\sqrt{15}} & -\frac{1}{4\sqrt{15}} & \frac{1}{\sqrt{15}} & 0 \\ -\frac{1}{6\sqrt{10}} & -\frac{1}{6\sqrt{10}} & -\frac{1}{6\sqrt{10}} & -\frac{1}{6\sqrt{10}} & \frac{5}{6\sqrt{10}} \end{bmatrix} \tag{60}$$

Thus, there is also a one-to-one correspondence,

$$\{H_k\}\leftrightarrow\{N[Q_\mu,Q^{\dagger}_\mu]\},\quad k=\beta\ . \tag{61}$$

Using Eq. (53) for $\nu>\mu$, we have

$$[N[Q_\mu,Q^{\dagger}_\nu],(N[Q_\mu,Q^{\dagger}_\nu])^{\dagger}]=\tfrac{1}{12}(d^{\dagger}_\nu d_\nu-d^{\dagger}_\mu d_\mu)\ .$$

Using the root system of the SU(6) CWCA, we get, for any values $\{\mu\nu\}$ (Refs. 26 and 28),

$$\tfrac{1}{12}(d^{\dagger}_\nu d_\nu-d^{\dagger}_\mu d_\mu)=-\mathbf{r}(\gamma)\cdot\mathbf{H}\ ,$$

where $\{\gamma\}\leftrightarrow\{\mu\nu\}$.

Therefrom it follows that $N[Q_\mu,Q^{\dagger}_\nu]^{\dagger}=N[Q_\nu,Q^{\dagger}_\mu]$ and $N[Q_\mu,Q^{\dagger}_\nu]$ play the roles of raising and lowering generators in the QCA, respectively, corresponding to the nonsimple roots

$$\frac{1}{\sqrt{12}}d^{\dagger}_\mu d_\nu=E_\gamma\leftrightarrow N[Q_\nu,Q^{\dagger}_\mu]=b^{\dagger}_\mu b_\nu=d^{\dagger}_\mu d_\nu\ , \tag{62}$$

$$\frac{1}{\sqrt{12}}d^{\dagger}_\nu d_\mu=E_{-\gamma}\leftrightarrow N[Q_\mu,Q^{\dagger}_\nu]=b^{\dagger}_\nu b_\mu=d^{\dagger}_\nu d_\mu\ . \tag{63}$$

Relations (56), (57), (62), and (63) are of the type

$$\text{SR}=\text{CWCA}\leftrightarrow\text{QCA}=\text{HPR}=\text{SR}\ .$$

Thus, by use of relations (56), (57), and (61)–(63), we have established the isomorphism $\{\text{QCA}\}\leftrightarrow\{\text{CWCA}\}$. It is easy to check that other commutation relations of the SU(6) CWCA are satisfied too.

We have shown that the operators $Q^{\dagger}_\mu$ and $[Q_\mu,Q^{\dagger}_\nu]$ for $\nu>\mu$ are the lowering generators. Therefore, the result of these operators acting on the highest weight state $\langle\tilde{0}|$ is zero [Eqs. (43) and (44)]. On the other hand, the operators $|Q_\mu,Q^{\dagger}_\mu|$ belong to the Cartan Abelian subalgebra and thus the highest weight state $|\tilde{0}\rangle$ is their eigenstate.

In fact, the highest weight state for the totally symmetric irreducible representation of SU(6) is a vacuum of b bosons, $|\tilde{0}\rangle=|0\rangle_b$. From Eqs. (50), (57), and (63) it then follows that

$$\langle\tilde{0}|Q^{\dagger}_\beta={}_b\langle 0|b^{\dagger}_\mu\left[1-\frac{\hat{N}}{N}\right]^{1/2}=0\ ,$$

$$\langle\tilde{0}|\ [Q_\mu,Q^{\dagger}_\nu]=\begin{cases} {}_b\langle 0|b^{\dagger}_\nu b_\mu=0, & \mu\neq\nu \\ {}_b\langle 0|\delta_{\mu\nu}, & \mu=\nu\ . \end{cases}$$

Thus, $|\tilde{0}\rangle$ is the zero-energy state of the free Hamiltonian.

The consequence of our proof that Q_μ and $Q^{\dagger}_\mu$ are the raising and lowering operators, respectively, corresponding to the simple roots, is a group-theoretical identification of the generating state (5): this state is, in fact, the SU(6) coherent state.[23,24,30,31]

Let us also mention a simple interpretation of relations (31), (34), and (35) in analogy with the CWCA. Relations (34) and (35) are associated with the vanishing commutators between the raising and lowering generators corresponding to simple roots ($[E_\beta,E_{\beta'}]=0$). The vanishing of $D_K,D_{\mu'\mu\mu_0\mu''}$, Eq. (31), is associated with the fact that raising and lowering generators do not mix on the rhs's of commutators of the CWCA ($[H_k,E_{\pm|\alpha|}]=r_k(\alpha)E_{\pm|\alpha|}$).

VIII. BOSON IMAGES OF OPERATORS $B(jj;2\mu)$ AND $B(jj;00)$

Finally, we have to determine the boson images of the operators $B(jj';\lambda\mu)$, defined by Eq. (17a). These operators enter, on an equal footing with RPA phonon operators, into model Hamiltonians and in other physical operators. Therefore, we have to construct their boson images too.

Our starting point is the ansatz (A6). We have to determine explicitly the coefficients $X_{\rho\tau}(j_1j_2;LM)$. To this end we calculate the commutators $[B(j_1j_2;LM),Q_\mu^\dagger]$ by using Eqs. (A15), (A16), and (36). We obtain the relation

$$X_{\mu\nu}(j_1j_2;LM)+\delta_{\mu\nu}\sum_{\mu'}X_{\mu'\mu'}(j_1j_2;LM)=N\sqrt{5}(2L+1)^{1/2}\langle 2\mu LM\mid 2\nu\rangle S^{(+)}(j_1j_2;L)\ ,$$

where we have used $\mathrm{Int}[1/C]=N$. The quantity $S^+(j_1j_2;LM)$ is defined by Eq. (A4).

Taking diagonal matrix elements, $\mu=\nu$ and summing over μ, we obtain $\sum_\mu X_{\mu\mu}(j_1j_2;LM)$. Thus, finally we obtain

$$X_{\mu\nu}(j_1j_2;LM)=N\sqrt{5}(2L+1)^{1/2}\langle 2\mu LM\mid 2\nu\rangle S^{(+)}(j_1j_2;LM)-\delta_{\mu\nu}\tfrac{5}{6}N\frac{\delta_{j_1j_2}}{(2j_1+1)^{1/2}}\sum_{j'}(\Psi^2_{j_1j'}+\Phi^2_{j_1j'})\ . \tag{64}$$

In particular, for $L=0$ we obtain

$$X_{\mu\nu}(j_1j_2;00)=N\delta_{\mu\nu}\delta_{j_1j_2}(2j_1+1)^{-1/2}\times\tfrac{1}{6}\sum_{j'}(\Psi^2_{j_1j'}+\Phi^2_{j_1j'})\ . \tag{65}$$

For physical applications of direct interest, we have the operators $B(j_1j_2;2M)$ and $B(jj;00)$. The latter is proportional to the quasiparticle number operator [cf. (17a)]

$$B(jj;00)=(2j+1)^{-1/2}\sum_m a^\dagger_{jm}a_{jm}\ . \tag{66}$$

Using (A6), (50), and (65), we obtain

$$B(jj;00)=(2j+1)^{-1/2}\sum_{j'}(\Psi^2_{jj'}+\Phi^2_{jj'})\sum_{\tau=-2}^{2}b^\dagger_\tau b_\tau\ ,$$

and comparing with (66) we have

$$\sum_j a^\dagger_{jm}a_{jm}=\sum_{jj'}(\Psi^2_{jj'}+\Phi^2_{jj'})\sum_{\tau=-2}^{2}b^\dagger_\tau b_\tau\ . \tag{67}$$

Thus, in the case of SU(6) enforced symmetry the quasiparticle number operator is proportional to the number operator of quadrupole bosons.

The physical significance of the operators $B(j_1j_2;2M)$ stems from the fact that they enter the quadrupole moment operator and generate quasiparticle-quadrupole phonon interactions in microscopic nuclear models based on pairing plus quadrupole-quadrupole force.[15,16]

From (64) we have

$$X_{\mu\nu}(j_1j_2;2M)=5N\langle 2\mu 2M\mid 2\nu\rangle\sum_{j'}(-1)^{j_1+j'}\begin{Bmatrix} j_1 & j' & 2\\ 2 & 2 & j_2\end{Bmatrix}(\Psi_{j_1j'}\Psi_{j_2j'}+\Phi_{j_1j'}\Phi_{j_2j'})=5N\langle 2\mu 2M\mid 2\nu\rangle S^{(+)}(j_1j_2;2)\ . \tag{68}$$

It can be easily checked that the diagonal part of $X_{\mu\nu}(j_1j_2;LM)$ for $L=2,4$ does not contribute to $B(j_1j_2;LM)$. Then, we can write Eq. (A6) in the factorized form

$$B(j_1j_2;LM)=5S^{(+)}(j_1j_2;L)(b_2^\dagger\times\tilde{b}_2)_{LM},\quad L=2,4 \tag{69}$$

where the tensor product is defined as

$$(b_2^\dagger\times\tilde{b}_2)_{LM}=\sum_{\rho\tau}\langle 2\rho 2\tau\mid LM\rangle b^\dagger_{2\tau}(-1)^\rho b_{2-\rho}\ ,$$

and we have used the HPR of $[Q,Q^\dagger]$ given by Eq. (50). We note that the expression (69) is a generalization of the boson image of the operator $B(jj;LM)$ in the case of a single-j shell.[14]

IX. CONCLUSIONS

We have explicitly constructed microscopic Dyson, Holstein-Primakoff, and Schwinger realizations for the RPA quadrupole phonon operators. Using as a starting point the standard RPA quadrupole phonon operators, we enforce these operators and their commutators to close the Lie algebra—the QCA. Due to the presence of backward going amplitudes Φ, the derivation becomes rather complex. The analysis performed on the corollaries steming from Jacobi identities showed that simplifications appear in a nontrivial way and at a price of a set of constraints on the RPA quadrupole phonon amplitudes. In terms of these operators and their commutators, the QCA comes out directly in the canonical form. Thus, for our purposes of explicitly constructing the SR, HPR, and DR, the RPA quadrupole phonon operators turned out to be a natural and suitable representation.

In conclusion, we have constructed—in the framework of the RPA—three types of boson images of all elements $\{Q_\mu,Q^\dagger_\nu,[Q_\mu,Q^\dagger_\nu]\leftrightarrow B\}$ needed for an SU(6) Hamiltonian expressed in terms of bosons. Using the

Schwinger realization of the above elements of the QCA, we shall obtain the interacting boson model (IBM),[2-4] while using Holstein-Primakoff realization we shall obtain the truncated quadrupole phonon model (TQM).[10,19] Thus, the obtained microscopic realizations provide a new microscopic foundation of the IBM, alternative to the traditionally employed approach.[4] An advantage of this approach is that the total boson number N appears as a microscopic quantity (35). The efficiency of this approach is under investigation and preliminary results are encouraging.[19]

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APPENDIX: DERIVATION OF SU(6)-RPA-ENFORCING CONDITIONS FROM JACOBI IDENTITIES

If we assume that the SU(6) "scattering" approximation is valid, the commutators (16)–(18) can be written in the following compact form:

$$[Q_\mu,Q^\dagger_{\mu_0}]=\delta_{\mu\mu_0}-\sum_{j_1j_2LM}B(j_1j_2;LM)\sqrt{5}(2L+1)^{1/2}\langle 2\mu LM\,|\,2\mu_0\rangle S^{(-)}(j_1j_2;L)\;, \tag{A1}$$

$$[Q^\dagger_\mu,Q^\dagger_{\mu_0}]=5\sum_{j_1j_2LM}B(j_1j_2;LM)\langle 2\mu 2\mu_0\,|\,LM\rangle[1-(-1)^L]W(j_2j_1;L)(-1)^{j_1+j_2}\;, \tag{A2}$$

$$[B(j_1j_2;LM),\;Q^\dagger_{\mu'}]=\sqrt{5}(2L+1)^{1/2}\sum_{\mu''}\langle 2\mu' LM\,|\,2\mu''\rangle\{Q^\dagger_{\mu''}S^{(+)}(j_1j_2;L)+(-1)^{\mu''}Q_{-\mu''}[1+(-1)^L]W(j_1j_2;L)\}\;, \tag{A3}$$

with

$$S^{(\pm)}(j_1j_2;L)=\sum_{j'}(-1)^{j_1+j'}\begin{Bmatrix} j_1 & j' & 2\\ 2 & L & j_2\end{Bmatrix}[(-1)^L\Psi_{j_1j'}\Psi_{j_2j'}\pm\Phi_{j_1j'}\Phi_{j_2j'}]\;, \tag{A4}$$

$$W(j_1j_2;L)=\sum_{j'}(-1)^{j_1+j'}\begin{Bmatrix} j_1 & j' & 2\\ 2 & L & j_2\end{Bmatrix}\Phi_{j_1j'}\Psi_{j_2j'}\;. \tag{A5}$$

By comparing commutation relations (21) and (A3), we observe that the B-tensor operators commute with $Q^\dagger$ operators in the same way as the $[Q,Q^\dagger]$ commutators, i.e, B's should be linear combinations of $[Q,Q^\dagger]$'s. Also taking into account (A1), we introduce the following ansatz for B operators:

$$B(j_1j_2;LM)=\sum_{\rho\tau}X_{\rho\tau}(j_1j_2;LM)(\delta_{\rho\tau}-[Q_\rho,Q^\dagger_\tau])\;. \tag{A6}$$

Jacobi identity for operators $\hat{a},\hat{b},\hat{c}$ will be denoted $[\hat{a},\hat{b},\hat{c}]=0$. Let us now consider the Jacobi identities (i)–(iii) as follows.

(i) $$[B(j_1j_2;LM),Q^\dagger_\mu,Q^\dagger_\nu]=[[B(j_1j_2;LM),Q^\dagger_\mu],Q^\dagger_\nu]+[[Q^\dagger_\mu,Q^\dagger_\nu],B(j_1j_2;LM)]+[[Q^\dagger_\nu,B(j_1,j_2;LM)],Q^\dagger_\mu]=0\;.$$

Utilizing (A1)–(A3) and the commutator

$$[B(j'_1j'_2;L'M'),B(j_1j_2;LM)]=(-1)^{L+L'}\sqrt{(2L+1)(2L'+1)}\times\sum_{L''M''}\left[\delta_{j_1j'_2}(-1)^{j'_1+j_2}B(j'_1j_2;L''M'')(-1)^{L+L'-L''}\begin{Bmatrix} j_1 & j_2 & L\\ L'' & L' & j'_1\end{Bmatrix}-\delta_{j'_1j_2}(-1)^{j_1+j'_2}B(j_2j'_2;L''M'')\begin{Bmatrix} j'_1 & j'_2 & L'\\ L'' & L & j_1\end{Bmatrix}\right]C^{L''M''}_{L'M'LM}\;,$$

we can represent this Jacobi identity in the form

$$\sum_{j'_1j'_2L'M'}Y(j_1j_2;LM,\mu,\nu\,|\,j'_1j'_2;L'M')B(j'_1j'_2;L'M')=0\;, \tag{A7}$$

where

$$Y(j_1j_2;LM,\mu,\nu\,|\,j_1'j_2';L'M')\equiv 5\sqrt{5}(2L+1)^{1/2}S^{(+)}(j_1j_2;L)W(j_2'j_1';L')(-1)^{j_1'+j_2'}[(-1)^{L'}-1]$$
$$\times\sum_{\mu''}\{[\langle 2\mu LM\,|\,2\mu''\rangle\langle 2-\nu L'M'\,|\,2\mu''\rangle(-1)^{\nu}]-[\mu\rightleftarrows\nu]\}$$
$$+\sqrt{5}(2L+1)^{1/2}S^{(-)}(j_1'j_2';L')W(j_1j_2;L)[1+(-1)^{L}]$$
$$\times\sum_{\mu''}\{[\langle 2\nu LM\,|\,2\mu''\rangle\langle 2-\mu''L'M'\,|\,2\mu\rangle(-1)^{\mu''}]-[\mu\rightleftarrows\nu]\}$$
$$+5(-1)^{L}(2L+1)^{1/2}\sum_{L''M''}(2L''+1)^{1/2}[(-1)^{L''}-1]\langle 2\mu 2\nu\,|\,L''M''\rangle$$
$$\times\langle L''M''LM\,|\,L'M'\rangle\left[(-1)^{L+L''-L'}\begin{Bmatrix} j_1 & j_2' & L\\ L' & L'' & j_1'\end{Bmatrix}(-1)^{j_1+j_2'}W(j_1j_1';L'')\delta_{j_2j_2'}\right.$$
$$\left.-\begin{Bmatrix} j_2 & j_2' & L''\\ L' & L & j_1'\end{Bmatrix}(-1)^{j_1'+j_2}W(j_2'j_2;L'')\delta_{j_1j_1'}\right]. \qquad \text{(A8)}$$

The symbol $[\mu\rightleftarrows\nu]$ means that in the expression in parentheses to the left the index μ should be replaced by ν and vice versa.

$$[[B(j_1j_2;LM),Q_\mu^\dagger],Q_\nu^\dagger]+[[Q_\nu^\dagger B(j_1j_2;LM)],Q_\mu^\dagger]=[[B,Q_\mu^\dagger],Q_\nu^\dagger]-[[B,Q_\nu^\dagger],Q_\mu^\dagger],\quad \mu\rightleftarrows\nu$$

and the third group of terms originates from the double commutator

$$[[Q_\mu^\dagger,Q_\nu^\dagger],B(j_1j_2;LM)]$$

when we use commutator

$$[B(j_1'j_2';L'M'),B(j_1j_2;LM)]$$

written above.

In order that the identity (A7) be valid, it is necessary that

$$Y(j_1j_2;LM,\mu,\nu\,|\,j_1'j_2';L'M')=0 \qquad \text{(A9)}$$

for any set of indices

$$\{j_1j_2;LM,\mu,\nu\,|\,j_1'j_2';L'M'\}\ .$$

We see from (A8) that if we take L' to be even and L to be odd, then only the third group of terms in (A8) is different from zero. If, moreover, we take L' to be equal to zero, we obtain (A9) in the form

$$10\langle 2\mu 2\nu\,|\,L\,-M\rangle(-1)^{L-M}W(j_1j_2;L)\times\delta_{j_1'j_2'}\left[\frac{\delta_{j_1j_1'}}{\sqrt{2j_1+1}}-\frac{\delta_{j_2j_2'}}{\sqrt{2j_2+1}}\right]=0$$

for odd L values ($L=1,3$) and any set $\{j_1j_2;\mu,\nu\,|\,j_1'j_2'\}$. We see that if $j_1'=j_2'=j_1$, but $j_2\neq j_2'$, then we deduce

$$10\langle 2\mu 2\nu\,|\,L\,-M\rangle(-1)^{L-M}W(j_1j_2;L)\frac{1}{\sqrt{2j_1+1}}=0$$

for odd L values and any $\{j_1j_2;\mu,\nu\}$. We can always take such values satisfying $\mu+\nu=-M$ that the Clebsch-Gordan coefficients are different from zero. Hence, we infer that $W(j_1j_2;L)$ must vanish for odd momenta $L=1,3$ and any $\{j_1j_2\}$ pair of a single particle set of quantum numbers

$$W(j_1j_2;L)=0\ , \qquad \text{(A10)}$$

and hence relation (29) in the text has been proved.

Let us now take L to be even and L' to be odd. In this case the first group of terms in (A8) gives zero, and the third group of terms does not contribute because, due to the factors $[(-1)^{L'}-1]$ and $[(-1)^{L''}-1]$, there appear $W(j_1j_2;L)$ for odd L, which, in accordance with (A10), is zero. Only the second group of terms contributes to the lhs of (A9). In the case $L=$ even, $L'=$ odd, (A9) reads

$$\sqrt{5}(2L+1)^{1/2}S^{(-)}(j_1'j_2';L')W(j_1j_2;L)2\sum_{\mu''}[\langle 2\nu LM\,|\,2\mu''\rangle\langle 2-\mu''L'M'\,|\,2\mu\rangle(-1)^{\mu''}$$
$$-\langle 2\mu LM\,|\,2\mu''\rangle\langle 2-\mu''L'M'\,|\,2\nu\rangle(-1)^{\mu''}]=0\ . \qquad \text{(A11)}$$

Let us set $L=0$; then (A11) gives

$$2\sqrt{5}S^{(-)}(j_1'j_2';L')W(j_1j_2;0)[1-(-1)^{L'}]\langle 2-\nu L'M'\,|\,2\mu\rangle=0\ .$$

There are two possibilities:

(1) $S^{(-)}(j_1'j_2';L')=0$,

(2) $W(j_1j_2;0)=0$.

If we assume $S^{(-)}(j_1'j_2';L')=0$, we come to the conclusion that $[Q_\mu,Q_\nu^\dagger]=\delta_{\mu\nu}$, i.e., the single commutators (A1) reduce to a c number. We prefer to go beyond the RPA so that we have to choose the second possibility, i.e., $W(j_1j_2;0)=0$.

Let us now take the set of indices

$$L=2,\ \ M=1,\ \ \mu=-2,\ \ \nu=1,\ \ L'=3,\ \ M'=-3\ .$$

In this case (A11) reduces to

$$-10S^{(-)}(j_1'j_2';3)W(j_1j_2;2)(-1)\tfrac{1}{14}\sqrt{30}=0\ .$$

Using the same reasoning as above, we infer

$$W(j_1j_2;2)=0\ . \tag{A12}$$

Let us now take the set of indices

$$L=4,\ \ M=0,\ \ \mu=-1,\ \ \nu=2,\ \ L'=1,\ \ M'=0\ .$$

In this case (A11) reads

$$2\sqrt{5}\sqrt{9}S^{(-)}(j_1'j_2';0)W(j_1j_2;4)(-1)\frac{1}{2}\frac{1}{\sqrt{21}}=0\ .$$

From here we conclude

$$W(j_1j_2;4)=0\ . \tag{A13}$$

Therefore we have shown the enforced vanishing of $W(j_1j_2;L)$ for even values of L and thus the validity of relation (30) has been established:

$$W(j_1j_2;L)=0,\ \ L=0,2,4\ . \tag{A14}$$

From (A10) and the explicit form (A2) of the commutator $[Q_\mu^\dagger,Q_\nu^\dagger]$, it follows immediately that $[Q_\mu^\dagger,Q_\nu^\dagger]$, and also $[Q_\mu,Q_\nu]$, vanish if the Lie algebra enforcement is imposed. Thus (34) has been proved.

The consequence of relations (A3) and (A14) is that the commutators $[B,Q^\dagger]$ do not contain Q terms, but only $Q^\dagger$ terms:

$$[B(j_1j_2;LM),Q_{\mu'}^\dagger]=\sqrt{5}(2L+1)^{1/2}\sum_{\mu''}\langle 2\mu'LM\mid 2\mu''\rangle S^{(+)}(j_1j_2;L)Q_{\mu''}^\dagger\ . \tag{A15}$$

As a result of (A1) and (A15), we find that the double commutators $[Q^\dagger,[Q,Q^\dagger]]$ contain only $Q^\dagger$ terms,

$$[Q_{\mu'}^\dagger,[Q_\mu,Q_{\mu_0}^\dagger]]=2\sum_{\mu''}C_{\mu'\mu\mu_0\mu''}Q_{\mu''}^\dagger\ , \tag{A16}$$

where $C_{\mu'\mu\mu_0\mu''}$ is given by Eq. (22).

On the other hand, if we use (A3) instead of (A15) for the commutator $[Q_\mu,Q_{\mu_0}^\dagger]$, a straightforward calculation gives

$$[Q_{\mu'}^\dagger,[Q_\mu,Q_{\mu_0}^\dagger]]=2\sum_{\mu''}[C_{\mu'\mu\mu_0\mu''}Q_{\mu''}^\dagger+D_{\mu'\mu\mu_0\mu''}Q_{\mu''}]\ , \tag{A17}$$

where $D_{\mu'\mu\mu_0\mu''}$ is given by Eq. (23). The comparison of (A16) and (A17) leads to the result

$$D_{\mu'\mu\mu_0\mu''}=0\ \ \text{for any}\ \mu'\mu,\mu_0,\mu''\ .$$

From the definition for the D matrix (23), we then have

$$D_K=0,\ \ K=0,1,2,3,4\ . \tag{A18}$$

In this way the relation (31) is proved. Thus, the relation (A18) is a direct corollary of relation (A14),

$$\text{(ii)}\ \ [Q_{\mu'},Q_\mu,Q_{\mu_0}^\dagger]=0\ .$$

Using the relations (27), (34), and (A18), we obtain

$$\sum_{\mu''}Q_{\mu''}(C_{\mu'\mu_0\mu\mu''}-C_{\mu\mu_0\mu'\mu''})=0\ ,$$

i.e.,

$$C_{\mu'\mu_0\mu\mu''}=C_{\mu\mu_0\mu'\mu''}\ \ \text{for any}\ \mu',\mu_0,\mu,\mu''\ .$$

Using (22) this gives

$$\sum_K C_K[(-1)^K-1]\sum_\kappa\langle 2\mu'2\mu\mid K\kappa\rangle\langle 2\mu_0 2\mu''\mid K\kappa\rangle=0. \tag{A19}$$

Taking two sets of μ values—$\{\mu',\mu,\mu_0,\mu''\}=\{1,0,2,-1\}$ and $\{-1,1,-1,1\}$—we obtain

$$C_1-C_3=0,\ \ C_1+4C_3=0\ .$$

Thus, we have obtained the relation (32),

$$C_1=C_3=0\ .$$

We now turn our attention to the third Jacobi identity,

$$\text{(iii)}\ \ [[Q_\mu,Q_\nu^\dagger],[Q_\tau,Q_\rho^\dagger],Q_\xi^\dagger]=0\ .$$

Using relations (21), (27), and (A18), we obtain

$$\sum_{\epsilon'}Q_{\epsilon'}^\dagger\sum_\epsilon(C_{\tau\rho\mu\epsilon}C_{\nu\epsilon\xi\epsilon'}-C_{\rho\tau\nu\epsilon}C_{\xi\mu\epsilon\epsilon'}+C_{\nu\mu\xi\epsilon}C_{\rho\tau\epsilon\epsilon'}-C_{\rho\tau\xi\epsilon}C_{\nu\mu\epsilon\epsilon'})=0\ ,$$

i.e.,

$$\sum_\epsilon(C_{\tau\rho\mu\epsilon}C_{\nu\epsilon\xi\epsilon'}-C_{\rho\tau\nu\epsilon}C_{\xi\mu\epsilon\epsilon'}+C_{\nu\mu\xi\epsilon}C_{\rho\tau\epsilon\epsilon'}-C_{\rho\tau\xi\epsilon}C_{\nu\mu\epsilon\epsilon'})=0 \tag{A20}$$

for any set $\mu,\nu,\tau,\rho,\xi,\epsilon'$.

Using definition (22), we get a simple selection rule for indices in each term appearing in (A20): the term $C_{\alpha\beta\gamma\delta}C_{\alpha'\beta'\gamma'\delta'}$ can be different from zero only if

$$\alpha+\gamma=\alpha'+\gamma' \ , \quad \beta'+\delta'=\beta+\delta \ . \tag{A21}$$

For $\alpha+\gamma=\beta+\delta=0$, from (22) it follows that $C_{\alpha\beta\gamma\delta}$ has the form

$$C_{\alpha\beta\gamma\delta}=(m_0C_0+m_2C_2+m_4C_4)R \ , \tag{A22}$$

with the integers m_0,m_2,m_4 satisfying the conditions

$$m_0+m_2+m_4=\begin{cases}0 & \text{if } \alpha\neq\beta,\\ 1/R & \text{if } \alpha=\beta \ ,\end{cases} \tag{A23}$$

and R is a real constant. The condition follows from the orthogonality relation for Clebsch-Gordan coefficients in definition (22) [cf. the relation below Eq. (38)].

For $\alpha+\gamma=\beta+\delta\neq 0$, the expression for $C_{\alpha\beta\gamma\delta}$ takes the form

$$C_{\alpha\beta\gamma\delta}=(m_2'C_2+m_4'C_4)R' \ .$$

The integers m_2',m_4' satisfy the condition

$$m_2'+m_4'=\begin{cases}0 & \text{if } \alpha\neq\beta,\\ 1/R & \text{if } \alpha=\beta \ .\end{cases}$$

On the basis of definition (22), we can represent identities (A20) in a more transparent form by taking different concrete $\{\mu,\nu,\tau,\rho,\xi,\epsilon'\}$ sets. For the series of five sets $\{0,-1,1,2,0,0\}$, $\{2,2,0,1,-1,0\}$, $\{2,0,2,2,1,-1\}$, $\{2,2,0,-1,1,0\}$, and $\{-1,0,1,0,0,0\}$, we get, respectively,

$$(C_4-C_2)(14C_0+10C_2-24C_4)=0 \ , \tag{A24}$$

$$(C_4-C_2)(7C_0-10C_2+3C_4)=0 \ , \tag{A25}$$

$$(C_4-C_2)(2C_2+5C_4)=0 \ , \tag{A26}$$

$$(C_4-C_2)(14C_0+25C_2-39C_4)=0 \ , \tag{A27}$$

$$(14C_0+15C_2+6C_4)(14C_0+10C_2-24C_4)=0 \ . \tag{A28}$$

Physical meanings have positive real C values. Hence, (A26) implies $C_4=C_2$. On the other hand, from (A28) stems $14C_0+10C_2-24C_4=0$. Therefrom, we obtain $C_0=C_2=C_4=C$, i.e., Eqs. (33) have been shown. All other relations—(A24), (A25), and (A27)—are automatically satisfied.

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