Nuclear structure effects in double-beta decay

J. Engel and P. Vogel
Physics Department, California Institute of Technology, Pasadena, California 91125

M. R. Zirnbauer
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125
(Received 8 June 1987)

Using the quasiparticle random phase approximation, we calculate the nuclear matrix elements governing two-neutrino and neutrinoless double-beta decay. We show that a consistent treatment, including both particle-hole and particle-particle interactions, helps to resolve the longstanding discrepancy between experimental and calculated two-neutrino decay rates. The particle-particle force, which allows us to bring calculated EC/β^+ decay rates in semimagic nuclei into closer agreement with experiment, is in large part responsible for suppressing calculated two-neutrino decay rates that are otherwise too fast. We test the validity of our procedure by comparing quasiparticle random phase approximation results with exact solutions for a solvable model, in which the suppression of two-neutrino decay by the particle-particle interaction is confirmed. We then extend our approach to the neutrinoless decay associated with a finite Majorana neutrino mass and, conceivably, with majoron emission, and demonstrate that the nuclear matrix elements governing these processes are also suppressed. We present predicted half-lives for both two-neutrino and neutrinoless double-beta decay in several candidate nuclei, and discuss the difficulties associated with the calculation of such highly suppressed quantities.

I. INTRODUCTION

Double-beta decay is the process by which a nucleus with neutron and proton number (N,Z) undergoes a transition to the nucleus $(N-2,Z+2)$, when the transition to the intermediate nucleus $(N-1,Z+1)$ is energetically forbidden or highly retarded. There are two modes of double-beta decay, one involving the emission of two antineutrinos and two electrons (2ν mode), and another in which only electrons are emitted (0ν mode). The 2ν mode of double-beta decay occurs in second order in the standard theory of weak interactions and its rate is independent of a possible small neutrino mass. On the other hand, the 0ν mode violates the law of lepton number conservation and requires the existence of massive Majorana neutrinos. Observation of 0ν double-beta decay would therefore signal physics beyond the standard minimal electroweak model. For this reason, considerable experimental effort is being devoted to the study of this mode of nuclear decay. In order to plan and correctly interpret the results of experiments on double-beta decay, one has to know the various nuclear matrix elements associated with the decay process. The calculation of these matrix elements is the subject of our paper.

The theory and the experimental status of the two modes of double-beta decay have been reviewed by Doi *et al.*[1] and by Haxton and Stephenson,[2] and we refer to these papers for details. Even though a careful analysis in the framework of general gauge theories[3] shows that the 0ν mode indeed always requires the existence of a massive Majorana neutrino, one should distinguish between two mechanisms for 0ν decay: one in which the decay rate involves the effective neutrino Majorana mass $\langle m_\nu \rangle$ (mass mechanism), and a second one in which the rate contains the strength of a weak interaction involving right-handed leptonic currents (RHC mechanism). In the present work, we consider the nuclear structure aspects of the 2ν mode and of the mass mechanism for the 0ν mode. We hope to treat the RHC mechanism for the 0ν mode elsewhere.

A third, somewhat exotic mechanism for 0ν double-beta decay involves the emission of a hypothetical scalar particle, the majoron, along with the two electrons. This decay is believed to be governed by the same nuclear matrix elements that describe the mass mechanism for the 0ν mode.[1] Hence our calculations are applicable to majoron emission as well. On the other hand, we do not consider the decay mediated by very heavy neutrinos.[5]

Half-lives for the 2ν double-beta decay have recently been calculated by Vogel and Zirnbauer[6] (this paper will hereafter be referred to as I), who used the quasiparticle random phase approximation (QRPA) to deal with the nuclear structure aspects of the process. It was shown in I that the inclusion of the particle-particle component of the residual nucleon-nucleon interaction results in a considerable suppression of the decay rate. This suppression, in turn, provides a possible resolution of the longstanding discrepancy between calculated and measured 2ν double-beta decay rates.[1,2] In the present work, we elaborate on this result and extend the treatment to the nuclear matrix elements of the 0ν mode, showing that they are also strongly affected by the residual nucleon-nucleon force.

 ©1988 The American Physical Society

II. RATE FORMULAS AND SUM RULES

To obtain the formula for the 2ν decay rate, we use the standard allowed approximation and assume that Gamow-Teller transitions dominate over Fermi transitions in medium-mass and heavy nuclei. The inverse half-life of the 2ν decay connecting the 0^+ ground states of two even-even nuclei is then given by the formula

$$[T_{1/2}^{2\nu}(0_i^+ \to 0_f^+)]^{-1} = G^{2\nu}(E_{\max}, Z)\,|\,M_{\rm GT}^{2\nu}\,|^2 \,, \tag{2.1}$$

where

$$M_{\rm GT}^{2\nu} = \sum_m \frac{\left\langle 0_f^+ \left| \sum_l \boldsymbol{\sigma}(l)\tau^+(l) \right| 1_m^+ \right\rangle \cdot \left\langle 1_m^+ \left| \sum_k \boldsymbol{\sigma}(k)\tau^+(k) \right| 0_i^+ \right\rangle}{E_m - (M_i + M_f)/2} \,. \tag{2.2}$$

Here, $|\,0_i^+\rangle$ ($|\,0_f^+\rangle$) is the 0^+ ground state of the initial (final) nucleus with mass M_i (M_f); $|\,1_m^+\rangle$ are the 1^+ states in the intermediate odd-odd nucleus with energies E_m; $\boldsymbol{\sigma}(l)$ are the usual Pauli spin operators for the lth nucleon; and $\tau^+(l)$ is the isospin raising operator changing a neutron into a proton (we follow Refs. 1 and 2 and use the normalization $\langle \mathrm{p}\,|\,\tau^+\,|\,\mathrm{n}\rangle = 1$; more standard notation would use t^+ instead of τ^+). The function $G^{2\nu}(E_{\max}, Z)$ results from integration over the lepton phase space. This function has been calculated by Doi *et al.*,[1] and we give the values needed for the cases treated here in Table I (see Sec. VI). We note that in our calculation the sum over the intermediate states in Eq. (2.2) is performed explicitly. While our procedure allows us to use the customary closure approximation, the explicit summation is more accurate and therefore preferable.

In the 0ν case, one is justified in neglecting the variation of the energy denominator with nuclear excitation and in performing the summation over the intermediate nuclear states by closure. The neutrino propagation is then characterized by the function $H(\bar{E}, r)$ ("neutrino potential"),

$$H(\bar{E}, r) = \frac{1}{4\pi^2} \int d^3k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k(k+\bar{E})} = \Phi(\bar{E}r)/r \,, \tag{2.3}$$

where $\bar{E} = \langle E \rangle - (M_i + M_f)/2$ and $\langle E \rangle$ is the "typical" excitation energy of the intermediate nucleus. The function $\Phi(x)$ is given by

$$\Phi(x) = \frac{2}{\pi}[\sin(x)\mathrm{ci}(x) - \cos(x)\mathrm{si}(x)] \simeq e^{-1.5x} \,, \tag{2.4}$$

the last expression being valid with an accuracy better than 15% for $x \lesssim 0.5$, which is the range of the variable $x = \bar{E}r$ of interest here.

With these approximations, we can write the inverse half-life for 0ν double-beta decay in the form

$$[T_{1/2}^{0\nu}(0_i^+ \to 0_f^+)]^{-1} = G^{0\nu}(E_{\max}, Z) \left| M_{\rm GT}^{0\nu} - \frac{g_V^2}{g_A^2} M_F^{0\nu} \right|^2 \langle m_\nu \rangle^2 \,, \tag{2.5}$$

where the 0ν Gamow-Teller and Fermi matrix elements are given by

$$M_{\rm GT}^{0\nu} = \left\langle 0_f^+ \left| R \sum_{k,l} K(k,l)\boldsymbol{\sigma}(k)\cdot\boldsymbol{\sigma}(l)\tau^+(k)\tau^+(l) \right| 0_i^+ \right\rangle \,, \tag{2.6a}$$

$$M_F^{0\nu} = \left\langle 0_f^+ \left| R \sum_{k,l} K(k,l)\tau^+(k)\tau^+(l) \right| 0_i^+ \right\rangle \,, \tag{2.6b}$$

$$K(k,l) = H(\bar{E}, |\,\mathbf{r}(k) - \mathbf{r}(l)\,|) \,. \tag{2.6c}$$

The summations in Eqs. (2.6a) and (2.6b) are over all pairs of nucleons k,l, with relative vector $\mathbf{r}(k) - \mathbf{r}(l)$. The function $G^{0\nu}$ again results from integration over lepton phase space and is given for the decays of interest in Table IV of Sec. VIII. [The nuclear radius R in Eqs. (2.6a) and (2.6b) was included to make $M_{\rm GT}^{0\nu}$ and $M_F^{0\nu}$ dimensionless; a corresponding compensating factor is contained in $G^{0\nu}$.] As was mentioned earlier, the 0ν double-beta decay associated with majoron emission is governed by the same nuclear matrix elements as is the 0ν mode, but has a different electron spectrum and therefore also a different phase space factor. The ratio of half-lives for these two 0ν modes is

$$\frac{T_{1/2}^{0\nu}}{T_{1/2}^{\rm maj}} = \frac{1}{\langle v_H \rangle^2} \frac{G^{\rm maj}(E_{\max}, Z)}{G^{0\nu}(E_{\max}, Z)} \,, \quad \frac{1}{\langle v_H \rangle} = \frac{\langle g \rangle}{\langle m_\nu \rangle} \,, \tag{2.7}$$

where $\langle v_H \rangle$ is the vacuum expectation value of the additional Higgs scalar,[1] and $\langle g \rangle$ is the majoron-neutrino coupling constant. A table for the function $G^{\rm maj}$ can be found in Ref. 1.

The neutrino propagator, Eq. (2.3), favors small internucleon separations. It is therefore important to include explicitly the effect of short-range nucleon-nucleon correlations created by the "hard-core," which prevents two nucleons from approaching one another too closely. This can be achieved by substituting for K in (2.6) the modified function $\rho K \rho$, with

$$\rho = 1 - e^{-\gamma_1 r^2}(1 - \gamma_2 r^2) \,, \tag{2.8}$$

thereby suppressing contributions to the radial integral from small nucleon-nucleon separations r. The parameter values we use are $\gamma_1 = 1.1\ \mathrm{fm}^{-2}$ and $\gamma_2 = 0.68\ \mathrm{fm}^{-2}$.[7] We find that the inclusion of short-range correlations affects the final results appreciably, contrary to the result obtained by Tomoda *et al.*[4] and Grotz and Klapdor.[8]

However, the finite size of the nucleon does not significantly affect the matrix elements (2.6).

Our task, in short then, is the evaluation of the matrix elements $M_{\rm GT}^{2\nu}$, $M_{\rm GT}^{0\nu}$, and $M_F^{0\nu}$, Eqs. (2.2) and (2.6), with K given by

$$K(k,l)=\rho(r_{k,l})\frac{e^{-1.5\bar{E}r_{k,l}}}{r_{k,l}}\rho(r_{k,l}) , \qquad (2.9a)$$

$$r_{k,l}=|\mathbf{r}(k)-\mathbf{r}(l)| . \qquad (2.9b)$$

The first matrix element involves just one-body operators but an explicit summation over the 1^+ states of the intermediate nucleus. The latter two matrix elements involve two-body operators and the initial and final nuclei in their respective ground states. In Sec. III, we present formulae for these matrix elements obtained within the quasiparticle random phase approximation (QRPA).

To conclude this section, we mention a certain class of sum rules that are of relevance for double-beta decay. We consider the quantities

$$S_{J\lambda}^{-}(m)=4\pi|\langle m\,|(Y_\lambda\sigma)^{(J)}\tau^+|0^+\rangle|^2 , \qquad (2.10a)$$

$$S_{J\lambda}^{+}(n)=4\pi|\langle n\,|(Y_\lambda\sigma)^{(J)}\tau^-|0^+\rangle|^2 . \qquad (2.10b)$$

These are generalized Gamow-Teller strengths corresponding to charge-raising and charge-lowering transitions, respectively. They are connected by the sum rule

$$\sum_m S_{J\lambda}^{-}(m)-\sum_n S_{J\lambda}^{+}(n)=(2J+1)(N-Z) , \qquad (2.11)$$

which is exact and model independent as long as the set of states $|m\rangle$ in the nucleus $(N-1,Z+1)$ and $|n\rangle$ in the nucleus $(N+1,Z-1)$ is complete, and as long as only nucleon degrees of freedom participate. [Coupling to the Δ isobar modifies (2.11).] A similar sum rule holds for the generalized Fermi operators $Y_\lambda\tau^{\pm}$, which excite only states with natural parity and angular momentum $\lambda=J$. The right-hand side (rhs) of Eq. (2.11) is simply given by $(N-Z)$ for these operators.

The usual Gamow-Teller strength corresponds to $J=1$, $\lambda=0$. In nuclei with a sizable neutron excess, the total strength S_{10}^{+}, which we denote by β^+ to indicate that it determines the rate of nuclear positron decay, is suppressed due to Pauli blocking of the accessible neutron orbits. The total strength S_{10}^{-} (denoted by β^-) is therefore expected to be given essentially by the sum rule (2.11). It is important that calculations of the type performed here reproduce this feature. Moreover, from the (p,n) reaction at forward angles and appropriate energies,[9,10] information on the β^- strength is available for a number of nuclei, including several double-beta decay candidates. From the analysis of such data, it follows that the giant Gamow-Teller resonance and the lower-lying 1^+ states *actually do not* exhaust the sum rule (2.11) entirely, but yield only $\sim 60\%$ of it. To account for this in a crude way, we choose $g_A=1.0$ (instead of $g_A=1.26$) throughout. (See, however, Ref. 11 for a treatment of Δ-nucleon-hole admixtures.)

III. QUASIPARTICLE RANDOM PHASE APPROXIMATION

As the theoretical tool for our calculations, we use the random phase approximation based on the quasiparticle formalism (QRPA). (The generalization of the usual QRPA to charge-changing modes is due to Halbleib and Sorensen.[12] Particle-particle interactions were first included in the QRPA by Cha.[13]) The use of quasiparticles enables us to include pairing correlations in the nuclear ground state in a simple fashion. Particle and quasiparticle creation and annihilation operators for spherical shell-model states labeled by (jm) are related to each other by the Bogoliubov transformation

$$a_{jm}^{\dagger}=u_j c_{jm}^{\dagger}+v_j\tilde{c}_{jm} , \qquad (3.1a)$$

$$\tilde{a}_{jm}=-v_j c_{jm}^{\dagger}+u_j\tilde{c}_{jm} , \qquad (3.1b)$$

where $\tilde{a}_{jm}=(-)^{j-m}a_{j-m}$ and $u_j^2+v_j^2=1$. The vacuum of the quasiparticle operators c and $c^{\dagger}$ (the BCS ground state) will be denoted by $|O\rangle$.

Our goal is to calculate transition amplitudes associated with charge changing but otherwise as yet unspecified one-body transition operators T^{JM}, connecting the 0^+ ground state of an even-even nucleus with any of the J^{π} excited states of the neighboring odd-odd nuclei. We also require the excitation energy of the states in the odd-odd nuclei. Assuming the nuclear motion to be harmonic, we describe such states by the solutions of the QRPA eigenvalue equations

$$\begin{bmatrix} A & B \\ -B & -A \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}=\omega\begin{bmatrix} X \\ Y \end{bmatrix} , \qquad (3.2)$$

where the matrices A and B are defined as

$$\begin{aligned} A_{pn,p'n'}^{J}&=\langle O\,|(c_p^{\dagger}c_n^{\dagger})^{(JM)^{\dagger}}\hat{H}(c_{p'}^{\dagger}c_{n'}^{\dagger})^{(JM)}|O\rangle \\ &=(\tilde{\varepsilon}_p+\tilde{\varepsilon}_n)\delta_{pn,p'n'} \\ &\quad+\tilde{V}_{pn,p'n'}^{J}(u_pv_nu_{p'}v_{n'}+v_pu_nv_{p'}u_{n'}) \\ &\quad+V_{pn,p'n'}^{J}(u_pu_nu_{p'}u_{n'}+v_pv_nv_{p'}v_{n'}) , \end{aligned} \qquad (3.3)$$

and

$$\begin{aligned} B_{pn,p'n'}^{J}&=\langle O\,|\hat{H}(c_p^{\dagger}c_n^{\dagger})^{(J-M)}(-)^M(c_{p'}^{\dagger}c_{n'}^{\dagger})^{(JM)}|O\rangle \\ &=(-)^J[\tilde{V}_{pn,p'n'}^{J}(v_pu_nu_{p'}v_{n'}+u_pv_nv_{p'}u_{n'})-V_{pn,p'n'}^{J}(u_pu_nv_{p'}v_{n'}+v_pv_nu_{p'}u_{n'})] . \end{aligned} \qquad (3.4)$$

Here $\tilde{\varepsilon}$ are the quasiparticle energies obtained by solving the BCS equations for the nuclear Hamiltonian, $\hat{H}$;

$$(c_p^\dagger c_n^\dagger)^{(JM)} = \sum_{m_p, m_n} \langle j_p m_p j_n m_n \mid JM \rangle c^\dagger_{j_p m_p} c^\dagger_{j_n m_n}$$

creates a proton and a neutron quasiparticle coupled to total angular momentum J and projection M [note that M is not summed over in Eqs. (3.3) and (3.4)]; $X = X^J_{pn}(m)$ and $Y = Y^J_{pn}(m)$ are the forward and backward QRPA amplitudes, respectively; and $\omega = \omega^J(m)$ are the QRPA frequencies. [The label m is used to distinguish between different solutions of Eqs. (3.2).]

The quantities $\tilde{V}^J$ and V^J are two-body matrix elements of the nucleon-nucleon interaction $\hat{V}$; the former corresponds to the particle-hole part of the neutron-proton interaction, and the latter to the particle-particle part. One can visualize this assignment by considering the case of a doubly magic nucleus (N,Z) with no pairing. The system of Eqs. (3.2) then decomposes into four mutually decoupled matrices. Two of these matrices contain $\tilde{V}^J$ and describe states in the particle-hole nuclei $(N-1,Z+1)$ and $(N+1,Z-1)$, while the other two contain V^J and describe the states in the particle-particle and hole-hole nuclei $(N+1,Z+1)$ and $(N-1,Z-1)$. In the general case of open-shell nuclei, all four submatrices interact and form the large system (3.2). We wish to emphasize that this feature is not an artifact of particle-number nonconservation due to the use of the Bogoliubov transformation (3.1). In a number-conserving theory, both $\tilde{V}^J$ and V^J appear too, admixing higher-seniority components into paired states.

The next issue we address is the choice of these interaction matrix elements. In a first-principles calculation they would be related to each other by angular momentum recoupling. However, because of our use of a QRPA treatment within an effective shell-model space, we are forced to take these two parts of the interaction as independent. The approach we have chosen is to parametrize $\tilde{V}^J$ and V^J by the respective matrix elements of a δ-function interaction. (While there is some evidence[26] that a δ function is not appropriate for $f_{7/2}$-shell calculations in lighter nuclei, it does not really bear on this work. We will discuss lighter nuclei in a subsequent publication.[28]) The detailed expressions for these are listed in the Appendix for completeness. Because a δ-function interaction acts only in orbitally symmetric states, the total antisymmetry of the wave function leads to a nonvanishing two-body interaction matrix element only if either $S=0$, $T=1$ or $S=1$, $T=0$. We thus have four basic interaction constants at our disposal: two in the particle-hole channel, $\alpha_0 = g_{\rm ph}^{S=0,T=1}$ and $\alpha_1 = g_{\rm ph}^{S=1,T=0}$, and two in the particle-particle channel, $\alpha_0' = g_{\rm pp}^{S=0,T=1}$ and $\alpha_1' = g_{\rm pp}^{S=1,T=0}$. In the following section we explain in detail how we determine these constants.

We now turn to the evaluation of transition matrix elements for open-shell nuclei, using the general QRPA equations (3.2). Let T^{JM} be a one-body operator that transforms a neutron into a proton, thereby raising the total charge by one unit. The matrix element of this operator between the (correlated) ground state $|0\rangle$ of the even-even nucleus (N,Z) and an excited state $|m;JM\rangle$ ("one-phonon state") of the neighboring odd-odd nucleus $(N-1,Z+1)$, as described within the QRPA, is given by

$$\langle m;JM \mid T^{JM} \mid 0 \rangle_{\rm QRPA} = \sum_{pn} [t^{-,J}_{pn} X^J_{pn}(m) + t^{+,J}_{pn} Y^J_{pn}(m)] , \tag{3.5}$$

where

$$t^{-,J}_{pn} = \langle O \mid (c_p^\dagger c_n^\dagger)^{(JM)\dagger} \hat{T}^{JM} \mid O \rangle = \frac{u_p v_n}{\sqrt{2J+1}} \langle p \| T^J \| n \rangle , \tag{3.6a}$$

$$\begin{aligned} t^{+,J}_{pn} &= (-)^M \langle O \mid \hat{T}^{J-M} (c_p^\dagger c_n^\dagger)^{(JM)} \mid O \rangle \\ &= (-)^J \frac{v_p u_n}{\sqrt{2J+1}} \langle p \| T^J \| n \rangle . \end{aligned} \tag{3.6b}$$

$\langle p \| T^J \| n \rangle$ is the reduced matrix element of de Shalit and Talmi.[14]

Next, we consider the Hermitian conjugate of the operator $T^{JM}, T^{JM^\dagger}$, which induces transitions from (N,Z) to $(N+1,Z-1)$. The corresponding transition matrix element is also approximated by (3.5) but with X and Y (or t^- and t^+) interchanged:

$$\begin{aligned} &(-)^M \langle n;J-M \mid T^{JM^\dagger} \mid 0 \rangle_{\rm QRPA} \\ &\qquad = \sum_{pn} [t^{-,J}_{pn} Y^J_{pn}(n) + t^{+,J}_{pn} X^J_{pn}(n)] . \end{aligned} \tag{3.7}$$

We denote the total transition strengths by

$$\begin{aligned} S_J^- &= \sum_{m,M} |\langle m;JM \mid T^{JM} \mid 0 \rangle_{\rm QRPA}|^2 , \\ S_J^+ &= \sum_{n,M} |(-)^M \langle n;J-M \mid T^{JM^\dagger} \mid 0 \rangle_{\rm QRPA}|^2 . \end{aligned} \tag{3.8}$$

The difference between these strengths is conserved since from (3.5) and (3.7)

$$\begin{aligned} &(S_J^- - S_J^+)/(2J+1) \\ &\quad = [(t^-)^T (t^+)^T] \begin{bmatrix} XX^T - YY^T & XY^T - YX^T \\ XX^T - YX^T & -XX^T + YY^T \end{bmatrix} \begin{bmatrix} t^- \\ t^+ \end{bmatrix} \\ &\quad = \sum_{pn} [(t^{-,J}_{pn})^2 - (t^{+,J}_{pn})^2] = \sum_{pn} \frac{(v_n^2 - v_p^2)}{(2J+1)} \langle p \| T^J \| n \rangle^2 , \end{aligned} \tag{3.9}$$

where the second equality follows from the orthogonality relations obeyed by the solutions of Eqs. (3.2). For the operators $T^{JM} = (Y_\lambda \sigma)^{(JM)} \tau^+$, further evaluation of (3.9) yields the exact sum rules (2.11). The preservation of these sum rules is an attractive feature of the QRPA scheme.

Although the difference between total strengths is conserved, one might be concerned that the individual strengths are spuriously enhanced (or suppressed) by the nonconservation of particle number in the QRPA, as reflected, for example, by the admixture of $(N+1,Z+1)$, $(N-1,Z-1)$ components to the wave function. This, however, is not a serious problem. For the case of a closed-shell nucleus, the only nonvanishing matrix elements t^- of T^J are those with particle states p and hole states n, so that the sum in (3.5) is in fact restricted to the amplitudes connecting the nucleus (N,Z) with states in the nucleus $(N-1,Z+1)$, as required. In the more general case of open-shell nuclei, we find by numerical calculation that the Gamow-Teller β^+ strength, which is a particularly sensitive quantity, changes by no more than $\sim 20\%$ when the BCS state $|O\rangle$ is projected onto good particle number.

To calculate the various nuclear matrix elements associated with double-beta decay, we make one further approximation beyond the use of the QRPA. We either replace the amplitudes of the charge-lowering transition operator for the final nucleus by those for the initial nucleus, or we replace the amplitudes of the charge-raising transition operator for the initial nucleus by those for the final nucleus. In either case, we obtain a generic matrix element of the form

$$M \equiv \sum_{pn,p'n';Jm} Z^J_{pn,p'n'}(m)[u_p v_n Y^J_{pn}(m) + v_p u_n (-)^J X^J_{pn}(m)] \times [u_{p'} v_{n'} X^J_{p'n'}(m) + v_{p'} u_{n'} (-)^J Y^J_{p'n'}(m)] \ , \tag{3.10}$$

where u, v, X, and Y are the BCS occupation amplitudes and QRPA solutions for the initial nucleus, or for the final nucleus. The different decay modes are described by different forms of the coefficient Z, which is given by

$$Z^J_{pn,p'n'}(m) = \delta_{J,1} \frac{\langle p \| \sigma\tau^+ \| n \rangle \langle p' \| \sigma\tau^+ \| n' \rangle}{\omega(m) - (M_i + M_f)/2} \tag{3.11}$$

for the 2ν Gamow-Teller mode ($M^{2\nu}_{\mathrm{GT}}$), and by

$$Z^J_{pn,p'n'} = (2J+1) \sum_{J'} \sqrt{2J'+1} (-)^{j_{\mathrm{p}}+j_{\mathrm{n'}}+J'} W(j_{\mathrm{p}} j_{\mathrm{n}} j_{\mathrm{p'}} j_{\mathrm{n'}}; JJ') \times \langle (j_{\mathrm{p}}(1) j_{\mathrm{p'}}(2))^{(J')} \| RK(1,2) \sigma(1)\cdot\sigma(2) \tau^+(1)\tau^+(2) \| (j_{\mathrm{n}}(1) j_{\mathrm{n'}}(2))^{(J')} \rangle \tag{3.12}$$

for the 0ν Gamow-Teller mode ($M^{0\nu}_{\mathrm{GT}}$). The Fermi analog of the latter mode is obtained by omitting the term $\sigma(1)\cdot\sigma(2)$ from the reduced matrix element in (3.12). We observe that, owing to the presence of the operator K for the 0ν matrix elements, the corresponding summation over states in the intermediate nucleus involves *all* spins and parities. (The sum over parities has not been written explicitly to avoid a further complication of the notation. The Fermi matrix element involves only natural parity states.) This means that we have to solve the QRPA equations for all spins and parities, not just for 1^+ as in I. To evaluate the reduced two-body matrix element in (3.12), we make the usual assumption that the radial single-particle wave functions are well described by harmonic-oscillator eigenstates and use Moshinsky brackets to transform the wave functions to relative and center-of-mass coordinates.

The additional approximation made in deriving Eq. (3.10) is expected to be reasonable for nuclei away from closed shells. An improved treatment would consist in evaluating $\langle m;JM \mid T^{JM} \mid 0^+_i \rangle$ from a QRPA calculation for the initial nucleus, and $(-)^M \langle m;J-M \mid T^{JM\dagger} \mid 0^+_f \rangle$ from a QRPA calculation for the final nucleus. However, such an approach requires some way of identifying the intermediate one-phonon states, which emerge as *different* from the two calculations, due to the approximate nature of the QRPA. Prescriptions for getting around this problem have been offered by Grotz and Klapdor,[8] and by Civitarese *et al.*[15] We do not follow these references here (see, however, Sec. VII) but stick to our previous scheme, which is to perform two separate calculations for the initial and final nucleus and average the resulting matrix elements. This procedure works at least as well as the prescriptions of Refs. 8 and 15 for the exactly solvable model presented in I, as we shall see later on.

To conclude this section, we examine the role of ground-state correlations in a simple one-mode approximation for which analytical results can be obtained. These results are of a rather qualitative nature, but they exhibit the essential features of the more realistic situation. Although the following considerations can be applied to any charge-lowering operator, what we particularly have in mind is the Gamow-Teller operator $\sigma\tau^-$.

We imagine a toy nucleus with a *single* charge-changing mode pn. In such a situation, the QRPA matrices A and B become single numbers, a and b, which we parametrize as

$$\begin{aligned} a &= \omega_0 + g^{\mathrm{eff}}_{\mathrm{ph}}(u_p^2 v_n^2 + v_p^2 u_n^2) + g^{\mathrm{eff}}_{\mathrm{pp}}(u_p^2 u_n^2 + v_p^2 v_n^2) \\ &\equiv \omega_0' + a' \ , \\ \omega_0' &= \omega_0 + g^{\mathrm{eff}}_{\mathrm{pp}}, \quad a' = (g^{\mathrm{eff}}_{\mathrm{ph}} - g^{\mathrm{eff}}_{\mathrm{pp}})(u_p^2 v_n^2 + v_p^2 v_n^2) \ , \\ b &= -2(g^{\mathrm{eff}}_{\mathrm{ph}} - g^{\mathrm{eff}}_{\mathrm{pp}}) v_p u_p v_n u_n \ . \end{aligned} \tag{3.13}$$

In order for this to correspond to a real nucleus, we take $\omega_0 = \epsilon_p + \epsilon_n$ to be roughly the spin-orbit splitting plus twice the pairing gap, and the effective interaction constant $g^{\mathrm{eff}}_{\mathrm{ph}}$ ($g^{\mathrm{eff}}_{\mathrm{pp}}$) to be an average particle-hole (particle-particle) interaction matrix element multiplied by the effective number of two-quasiparticle configurations participating in the collective vibrational motion.

With the restriction to a single mode, Eqs. (3.2) become a system of two linear equations that is easily solved,

$$\omega=(a^2-b^2)^{1/2},\quad y=-x\frac{b}{a+(a^2-b^2)^{1/2}}\ , \tag{3.14}$$

x and y being the forward and backward amplitudes of the single collective mode with frequency ω. According to (3.7), the amplitude for the transition connecting the ground state of the neighboring even-even nucleus with this mode is given by

$$\begin{aligned} t&=t^+x+t^-y\\ &=t^+x\left[1-\frac{t^-}{t^+}\frac{b}{a+(a^2-b^2)^{1/2}}\right]\\ &=\frac{t^+x}{a+(a^2-b^2)^{1/2}}\{(a-a')+[(a^2-b^2)^{1/2}\\ &\qquad -(a'^2-b^2)^{1/2}]\} \end{aligned} \tag{3.15}$$

where, to obtain the second line, we have used the identity

$$-\frac{t^-}{t^+}=\frac{u_pv_n}{v_pu_n}=\frac{a'+(a'^2-b^2)^{1/2}}{b}\ . \tag{3.16}$$

The formula (3.15) is still exact within the single-mode approximation. To express the rhs of Eq. (3.15) in terms of known physical quantities, we observe that $t^+x\simeq t^+\equiv t_{\rm BCS}$ is roughly the β^+ amplitude calculated within the independent quasiparticle model, and we approximate the denominator $a+(a^2-b^2)^{1/2}$ by $\sim 2(a^2-b^2)^{1/2}=2\omega$. We then obtain

$$\begin{aligned} t&\simeq\tfrac{1}{2}t_{\rm BCS}\left\{\frac{\omega_0'}{\omega}+1-\left[1-2\frac{a\omega_0'}{\omega^2}+\left[\frac{\omega_0'}{\omega}\right]^2\right]^{1/2}\right\}\\ &\simeq t_{\rm BCS}\frac{\omega_0'}{\omega}\ . \end{aligned} \tag{3.17}$$

In the last step, we have replaced a by ω also under the square root, which is rather a good approximation in practice. (We find that the Tamm-Dancoff and RPA energies do not differ very much from each other.)

Formula (3.17) is quite instructive. It shows that, since $|\omega_0'/\omega|<1$, ground-state correlations of the type considered here act to *decrease* the β^+ amplitude. In the absence of the particle-particle interaction, the reduction factor is always positive. For the case of the giant Gamow-Teller resonance we estimate $\omega_0\sim 7-10$ MeV and $\omega\approx 15-16$ MeV, so that the strength is reduced by a factor on the order of 2 from its independent quasiparticle value, in good agreement with what we find in realistic calculations. On the other hand, formula (3.17) shows that much more substantial reductions may occur when the attractive particle-particle interaction is turned on; the β^+ amplitude actually passes through zero as the (unobservable) energy $\omega_0'=\omega_0+g_{\rm pp}^{\rm eff}=0$ changes sign. At the same time, the β^- amplitude

$$t^-x+t^+y=t^-x\left[1+\frac{a'-(a'^2-b^2)^{1/2}}{a+(a^2-b^2)^{1/2}}\right]\simeq t^-x \tag{3.18}$$

is largely independent of the interaction.

IV. SINGLE-PARTICLE LEVELS AND DETERMINATION OF PARAMETERS

As is always the case in calculations of the present type, the results depend to a certain extent on the single-particle level scheme adopted. Here we use the same number of proton and neutron levels, and choose a sufficiently large number of states so as to satisfy the Gamow-Teller sum rule (2.11). In practice, this means that we include all subshells within 10 MeV of the Fermi energy and their spin-orbit partners. We find that the numerical results depend in a rather important way on details of the single-particle energies, in particular on the spin-orbit splitting. To illustrate this unfortunate dependence, we have performed the calculation with two different level schemes. Scheme (A) uses the single-particle energies of the Coulomb-corrected Woods-Saxon potential advanced by Bertsch,[16] while scheme (B) uses the same parameters for the central part of the potential, but employs the spin-orbit force of Ref. 17 with the proton strength adjusted to $V_{\rm SO}=5.2$ MeV in order to reproduce better the sequence of states in doubly magic nuclei with one proton added or removed. While most of the results quoted in what follows refer to scheme (A), we will periodically mention results from scheme (B) to illustrate the kinds of differences that can arise.

To obtain the quasiparticle energies and the pairing amplitudes u and v we solve the BCS equations, employing for consistency a δ-function interaction rather than the usual schematic interaction. The experimental pairing energies, determined from the binding energy differences of the odd- and even-A nuclei, are reproduced by choosing the corresponding interaction strength $g_{\rm pair}\sim-(260-290)$ MeV fm^3, with rather little dependence on mass number. We are able to reproduce essentially all pairing energies of nonmagic nuclei without varying $g_{\rm pair}$ by more than 10%. We also note that the amplitudes u and v obtained in this way differ by as much as 30% from those obtained from the standard pairing force with a constant G.

In Sec. III, we described the four adjustable parameters that enter into the RPA portion of our calculations. We turn now to the issue of how they are fixed. The two particle-hole parameters α_0 and α_1 largely determine the energies of the isobaric analog and Gamow-Teller giant resonances. We obtain their values by fitting these energies over a wide range of nuclei, and are able to reproduce all of them to within 1 MeV. In extracting the excitation energies of the giant resonance states, we take the ground-state energies of the odd-odd nuclei to be the sum of the lowest neutron and proton quasiparticle energies. (The particle-hole piece of the interaction does not significantly affect low-lying states.) With this prescription, the best values for the two particle-hole parameters are, in scheme (A), $\alpha_0=-890$ MeV fm^3 and $\alpha_1=-1010$ MeV fm^3. [With our choice of sign convention for the two-body interaction matrix elements (A1)–(A5) and (A8), $\alpha_0,\alpha_1<0$ corresponds to a *repulsive* particle-hole interaction, while $\alpha_0',\alpha_1'<0$ corresponds to an *attractive* particle-particle interaction.] The numbers do not change significantly when level scheme (B) is used.

The Gamow-Teller giant resonance typically contains $\sim 75\%$ of the total S_{10}^{-} strength, when contributions within 2 MeV of the strongest state are added. In the double-beta decay candidates, the 1^+ states below the giant resonance contain 15–25 % of the total strength, in agreement with the data.[10] The isobaric analog state contains essentially all of the total strength $(N-Z)$.

To fix the two particle-particle parameters α_0' and α_1', we require physical data that are sensitive to particle-particle forces. Energies of low-lying states in nuclei such as ^{210}Bi or ^{42}Sc that contain one proton and one neutron outside doubly closed shells immediately suggest themselves. However, the zero range of the force and the RPA correlations together contribute more binding to the lowest-lying states than seems to be called for by the data. This difficulty has been encountered previously[18] and is not entirely surprising. Therefore, to determine the parameter α_1' we turn, as in I, to β^+ strengths in neutron-deficient nuclei. We discuss the procedure used, and the physics involved, in the next section; here we consider the other particle-particle parameter α_0'. Since the 2ν double-beta decay proceeds exclusively through 1^+ intermediate states, this parameter is not needed to calculate the matrix element, Eq. (2.2); see Eqs. (A1) and (A8a). When we extend our calculations to the 0ν mode, however, all multipoles appear in the intermediate nucleus; those with natural parity are affected by the value of α_0' as well as by that of α_1'.

The parameter α_0' is the strength of the $S=0$, $T=1$ component of the effective nucleon-nucleon interaction, while α_1' is the strength of the corresponding $S=1$, $T=0$ component. Analysis of the low-energy two-nucleon data suggest that the spin-singlet force is $\sim 60\%$ of the spin-triplet one, which leads us to adopt $\alpha_0'=0.6\alpha_1'$. We are encouraged that the value of α_0' that emerges after α_1' is fixed from positron decay (see next section) is close to the value of the parameter g_{pair} discussed above that multiplies the delta-function interaction in the neutron-neutron and proton-proton channels. Since α_0' is a measure of the neutron-proton pairing, this result means that we are using an approximately isoscalar pairing interaction (even though we treat the neutron-neutron and proton-proton pairing within BCS theory, while the neutron-proton components is treated within the RPA). We have followed the prescription $\alpha_0'=0.6\alpha_1'$ consistently in all our calculations.

V. β^+/EC DECAY OF SEMIMAGIC NUCLEI

Neutron-deficient nuclei with a magic number of neutrons, such as ^{148}Dy, ^{150}Er, and ^{152}Yb with $N=82$, and ^{94}Ru and ^{96}Pd with $N=50$, are expected to undergo very fast positron decay (or electron capture). In the extreme single-particle model, the rates are determined by the $ph_{11/2}\rightarrow nh_{9/2}$ transition in the $N=82$ case, and by the $pg_{9/2}\rightarrow ng_{7/2}$ transition in the $N=50$ case. The Gamow-Teller β^+ strength is obtained from the experimental ft values through the relation

$$B(\mathrm{GT})=\frac{6160}{g_A^2\times ft} . \tag{5.1}$$

The single-particle value of B(GT) for even-mass systems is simply

$$B(\mathrm{GT})_{\mathrm{s.p.}}=N_{\mathrm{p}}\frac{4l}{2l+1} , \tag{5.2}$$

where l is the orbital angular momentum of the corresponding subshell and N_{p} is the number of protons in the partially filled $j=l+\frac{1}{2}$ shell. The measured B(GT) values[19] are ~ 7 times smaller than the single-particle values for the above nuclei. Despite the rather large Q values for these decays, only one (or two, in the case of ^{96}Pd) strong final state is observed. Our calculations also indicate that essentially all the strength is concentrated in a few low-lying states. We thus conclude that the total β^+ strength in these nuclei is strongly quenched.

In calculating the β^+ strength one must consider several corrections to the single-particle model.[20] Pairing and particle-hole interactions together reduce the strength by approximately a factor of 2 with respect to the single-particle value in the above semimagic nuclei. The effect of distant states (Δ isobars, two-particle–two-hole states), responsible for the "missing strength" in the giant Gamow-Teller resonance, can be crudely included by choosing $g_A=1.0$. After taking these effects into account, we are still left with a calculated β^+ strength a factor of 2–3 larger than the experimental one. We argued in paper I that the inclusion of particle-particle interactions leads to an increase in ground-state correlations (governed by the amplitudes Y) and, consequently, to a further decrease of the β^+ decay rate. Our numerical calculations show that this is indeed the case.

In Fig. 1 we show the dependence of the calculated strength B(GT) on the coupling constant α_1' of the particle-particle interaction, Eq. (A8a). We see that a single value, $\alpha_1'=-405$ MeV fm^3, explains the data for both nuclei shown. For the other three nuclei, ^{96}Pd, ^{150}Er, and ^{152}Yb, the same value of α_1' gives the B(GT) values 3.00, 1.41, and 2.28, respectively, while the experimental values (using $g_A=1.0$) are 2.06, 1.56, and 2.06. These data allow us to restrict α_1' with some confidence to a window between about -390 and -432 MeV fm^3, with an optimum value of -405 MeV fm^3. The calculations were performed with the single-particle level scheme (A). When they are repeated with the level scheme (B), the optimum value for α_1' is -430 MeV fm^3. The lower and upper limits of our window, quoted above, are deliberately chosen far enough apart to include the optimal values of α_1' in both schemes.

The conclusions of this section are the following: (a) It is possible to explain the experimental B(GT) values of semimagic nuclei by choosing a *single* value—or a small range of values—of the particle-particle interaction constant. We have used values within this "window" in all our calculations of double-beta decay. (b) In nuclei with a sizable neutron excess, where the β^+ strength represents a relatively small part of the sum rule (2.12), the particle-particle interaction is essential in evaluating

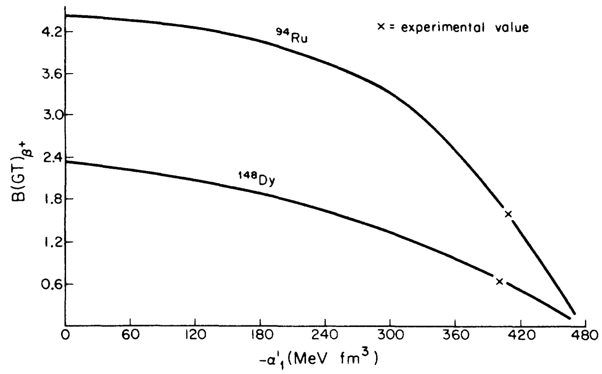

FIG. 1. Predicted B(GT) values for the EC/β^+ decay of two semimagic nuclei as a function of α'_1. The crosses denote the experimental values.

this strength. Cha[13] was apparently the first one to include this component of the residual interaction in his QRPA calculation of β^+ decay. However, he determined the coupling constant by fitting to energies of low-lying states, a procedure which, as we suggested in Sec. IV, may not give reliable results for a zero-range force. We believe that this is the reason his calculated β^+ strengths are larger than the experimental ones.

VI. 2ν MODE OF DOUBLE-BETA DECAY

The behavior of $M_{\rm GT}^{2\nu}$, Eq. (2.2), is largely governed by the matrix elements

$$\langle 0_f^+ | \sigma\tau^+ | 1_m^+ \rangle = \langle 1_m^+ | \sigma\tau^- | 0_f^+ \rangle$$

of the Gamow-Teller operator $\sigma\tau^-$. The latter are strongly affected by Pauli blocking, which prevents the transformation of a proton in the ground state of the final nucleus into a neutron in an unoccupied single-particle state of the intermediate nucleus. In fact, the β^+ strength, and hence $M_{\rm GT}^{2\nu}$, vanish in the extreme single-particle model for most of the double-beta decay candidates considered here. However, the pairing interaction, by smearing the occupation probabilities of states near the Fermi energy, allows the 2ν decay to proceed even in nuclei with a large neutron excess. Within the independent quasiparticle approximation, $M_{\rm GT}^{2\nu}$ is given by

$$M_{\rm GT}^{2\nu} = -\frac{1}{\Delta\bar{E}} \sum_{pn} \langle p \| \sigma\tau^+ \| n \rangle^2 u_p v_p u_n v_n \ . \tag{6.1}$$

All partially occupied proton and neutron states contribute coherently, there can be no cancellation, and the result depends mainly on the size of the pairing gap. (In practice this dependence is almost linear.) The same expression for $M_{\rm GT}^{2\nu}$ is obtained within the Tamm-Dancoff approximation ($B=0$) if we perform the summation over intermediate states by assuming closure. The expression remains unchanged because the X amplitudes disappear upon using the orthonormality relation $\sum_m X_{pn}(m) X_{p'n'}(m) = \delta_{pp'}\delta_{nn'}$.

Ground-state correlations, characterized by the amplitudes Y_{pn}^1, decrease the β^+ strength and thus the 2ν decay rate. We write the second equation in (3.2) in the form

$$-BX = (A + \omega) Y \ .$$

Since $(A+\omega)$ is positive definite, the relative signs of X and Y are determined by the signs of B. For $J=1$, B is largely negative, because the particle-hole interaction is repulsive and the particle-particle interaction is attractive. (Notice that there is a slight difference of phase convention compared to I.) The two terms $v_p u_n X_{pn}^1$ and $u_p v_n Y_{pn}^1$ in Eq. (3.10) therefore *interfere destructively.* Moreover, the two terms are of similar magnitude because the small size of the amplitude Y is compensated for by the larger occupation factor. In Eq. (3.4), these factors are such that the particle-particle matrix elements V contribute about as much to the matrix B as do the particle-hole matrix elements $\tilde{V}$.

Ground-state correlations induced by the particle-hole interaction reduce $M_{\rm GT}^{2\nu}$ by a factor of 2–3 when compared to the independent quasiparticle case. The β^+ strength is reduced correspondingly, whereas the β^- strength remains large and essentially constant since it cannot be reduced beyond the lower bound imposed by the sum rule (2.11). Eventually, the β^+ strength goes through a minimum with increasing particle-particle coupling strength, while $M_{\rm GT}^{2\nu}$ passes through zero, as shown in I. Ultimately, at a certain value of the particle-particle coupling constant, the QRPA breaks down and leads to unphysical complex frequencies. The matrix element passes through zero before this point is reached.

In Fig. 2 we show the dependence of β^+ and $M_{\rm GT}^{2\nu}$ in ^{76}Ge on the particle-particle interaction constant α'_1. This is a typical situation. However, the point at which $M_{\rm GT}^{2\nu}$ vanishes varies by $\sim 10\%$ from one nucleus to another. In Table I we give the resulting 2ν half-lives for different values of α'_1, including the upper and lower limits of the "window" fixed from positron decay. Let

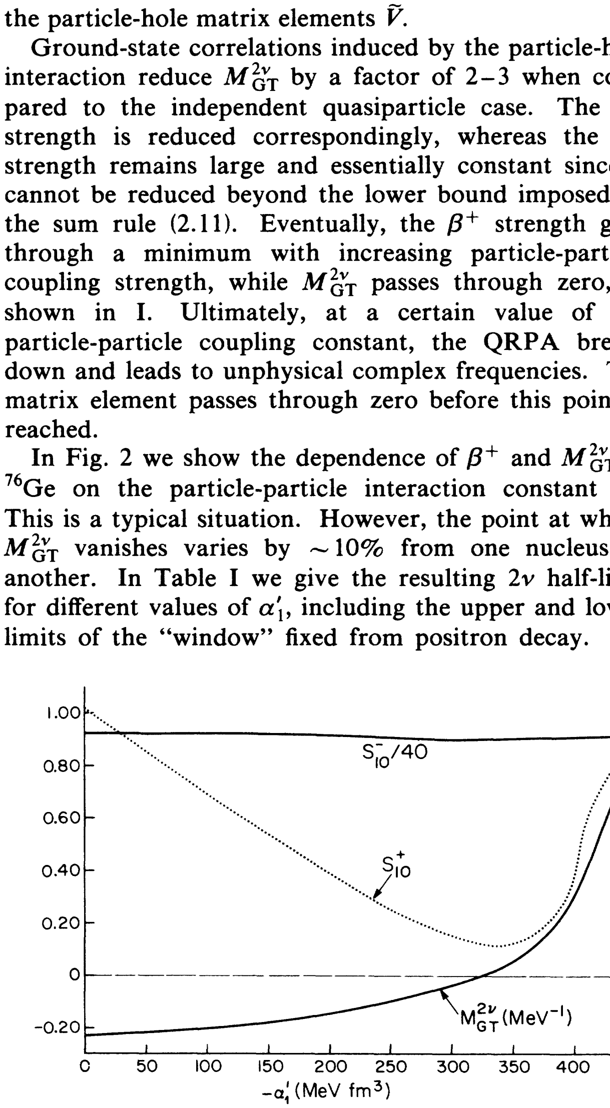

FIG. 2. The β^+ strength, the β^- strength (scaled by $\frac{1}{40}$), and $M_{\rm GT}^{2\nu}$ for the decay ^{76}Ge$\to$^{76}Se as a function of α'_1. The β^- strength is calculated for the initial nucleus, ^{76}Ge, the β^- strength for the final nucleus, ^{76}Se, and $M_{\rm GT}^{2\nu}$ is obtained by taking the average of the matrix elements calculated for the initial and final nucleus, as described in the text.

TABLE I. The phase space function $G^{2\nu}(E_{max},Z)$, Eq. (2.1), and 2ν double-beta decay half-lives for different values of α_1'. Half-lives are given in years (yr), and $G^{2\nu}$ in yr^{-1} if the energy denominator in Eq. (2.2) is represented in units of the electron mass. The column labeled "pairing" is calculated using the independent quasiparticle approximation, i.e., both the particle-hole and particle-particle interactions are set to zero. The last three columns correspond to three different values of α_1', which is quoted in units of MeV fm^3. A blank entry means that the QRPA gave unstable solutions in the final nucleus. The entry marked with an asterisk was calculated using the initial-nucleus solution only.

	$G^{2\nu}$	Pairing	0	-390	-432
^{76}Ge	5.20×10^{-20}	2.1×10^{20}	1.3×10^{21}	1.3×10^{21}	1.5×10^{20}
^{82}Se	1.73×10^{-18}	1.1×10^{19}	1.0×10^{20}	1.2×10^{20}	2.7×10^{19}
^{96}Zr	7.66×10^{-18}	3.7×10^{17}	3.6×10^{18}	8.5×10^{18}*	
^{100}Mo	3.74×10^{-18}	3.2×10^{17}	2.9×10^{18}	6.0×10^{18}	
^{128}Te	3.36×10^{-22}	9.6×10^{21}	1.1×10^{23}	5.5×10^{23}	9.2×10^{22}
^{130}Te	1.91×10^{-18}	2.4×10^{18}	3.9×10^{19}	2.2×10^{20}	4.6×10^{19}
^{136}Xe	1.92×10^{-18}	1.1×10^{19}	2.3×10^{20}	8.2×10^{20}	2.0×10^{20}

us stress once more that in the vicinity of the zero of $M_{GT}^{2\nu}$ the half-lives vary dramatically and could be affected in a major way by other nuclear structure phenomena, e.g., quadrupole ground-state correlations. The results presented here differ slightly from those in I. This reflects a minor change in the parameters of the Wood-Saxon potential used and demonstrates how sensitive the actual half-lives are to details of the calculation.

To illustrate the extreme sensitivity of the 2ν half-lives to α_1', we note that the experimental half-life in ^{130}Te ($T_{1/2}=2.6\times10^{21}$ yr) (Ref. 21) is obtained with the value $\alpha_1'=-354$ MeV fm^3, just 12% smaller than our optimum value of -405 MeV fm^3. Similarly, in ^{82}Se, we obtain the experimental value $T_{1/2}=1.3\times10^{20}$ yr given in Ref. 21 with $\alpha_1'=-380$ MeV fm^3, a mere 6% smaller than our optimum value. We thus confirm the main conclusion of I: inclusion of the particle-particle component of the nucleon-nucleon interaction, required on general grounds, with coupling constants determined from the experimental β^+ strength of semimagic nuclei, leads to a strong suppression of the 2ν double-beta decay rate and to calculated half-lives reasonably close to the experimental ones.

VII. EXACTLY SOLVABLE MODEL

The random phase approximation is a reliable and well-tested tool for studying properties of collective vibrational excitations in nuclei. However, the double-beta decay matrix element M_{GT} is a strongly suppressed, noncollective quantity, and it is therefore proper to ask whether the QRPA can be meaningfully used in its calculation. Although we are unable to provide a general answer to this question, there does exist the possibility of testing the QRPA within the exactly solvable model introduced in I. We present here the details of this model and its solution.

We begin with some motivation for the particular model we introduce. First of all, since what we are seeking is a description of transitions mediated by the Gamow-Teller operator, which acts only on the spin and isospin quantum numbers, a *minimal* schematic model should treat the nuclear spin and isospin degrees of freedom as "active," and the orbital degrees of freedom as "frozen." Second, the model must be sufficiently general to include pairing, the particle-hole interaction in the $J^\pi=1^+$ channel, and the corresponding particle-particle interaction, which we find to be of such great importance. Third, in order for the model to be exactly solvable, the Hamiltonian should be built solely from operators forming a closed algebra. These considerations lead us to the following construction.

We consider a set of (degenerate) single-particle orbitals, characterized by their orbital angular momentum l, spin $s=\frac{1}{2}$, isospin $t=\frac{1}{2}$, and the respective projections l_z, s_z, and t_z. Particle creation operators are denoted by $a^\dagger$, and (time-reversed) annihilation operators by

$$\tilde{a}_{l,l_z;s_z;t_z}=(-)^{l_z}(-)^{1/2-s_z}(-)^{1/2-t_z}a_{l,-l_z;-s_z;-t_z} . \quad (7.1)$$

Using LS coupling, we introduce the operators

$$D_{S,S_z;T,T_z}=\frac{1}{2}\sum_{ll_z}\sum_{s_zs_z't_zt_z'}\langle s_zs_z'\mid SS_z\rangle\langle t_zt_z'\mid TT_z\rangle[a^\dagger_{l,l_z;s_z;t_z},\tilde{a}_{l,-l_z;s_z';t_z'}] ,$$

$$C_{S,S_z;T,T_z}=\frac{1}{\sqrt{2}}\sum_{ll_z}\sum_{s_zs_z't_zt_z'}\langle s_zs_z'\mid SS_z\rangle\langle t_zt_z'\mid TT_z\rangle a^\dagger_{l,l_z;s_z;t_z}a^\dagger_{l,-l_z;s_z';t_z'} , \quad (7.2)$$

where $\langle s_z s_z' \mid SS_z \rangle \equiv \langle \frac{1}{2}, s_z; \frac{1}{2}, s_z' \mid SS_z \rangle$ are the Clebsch-Gordan coefficients coupling two angular momenta $\frac{1}{2}$ to total angular momentum S and projection S_z. The operators (7.2) represent a *complete* set of pair and particle-hole operators with orbital angular momentum $L=0$. The individual operators are denoted by $J_\mu = D_{1,\mu;0,0}$ (spin generators), $T^\nu = D_{0,0;1,\nu}$ (isospin generators), $F_\mu^\nu = D_{1,\mu;1,\nu}$ (Gamow-Teller operators), $N = D_{0,0;0,0}$ (half the particle number minus the number of orbital states), $S_{\rm nn}^+ = C_{0,0;1,-1}$ (neutron pair), $S_{\rm pn}^+ = C_{0,0;1,0}$ (neutron-proton pair), $S_{\rm pp}^+ = C_{0,0;1,+1}$ (proton pair), and $P_\mu^+ = C_{1,\mu;0,0}$ (spin-aligned neutron-proton pair). The adjoint pair operators are $P_\mu^- = (-)^\mu (P_{-\mu}^+)^\dagger$, $S_{\rm nn}^- = (S_{\rm nn}^+)^\dagger$, etc. This set of operators does indeed form an algebra, because the entire set of operators $a_\alpha^\dagger a_\beta^\dagger$, $a_\alpha a_\beta$, and $[a_\alpha^\dagger, a_\beta]$ closes under commutation, and the commutator of two operators with $L=0$ must again be an operator with $L=0$.

Counting the number of operators, we find that there are six pair creation operators, six pair annihilation operators, and 16 particle-hole operators including the (shifted) number operator N. The algebra formed by these operators is that of SO(8).[22] The particle-hole operators J_μ, T^ν, and F_μ^ν yield an SU(4) subalgebra, generating Wigner's supermultiplet symmetry.[23] We may therefore regard the present schematic model as an extension of Wigner's SU(4) to include pair operators.

Within SO(8), the most general two-body Hamiltonian that is invariant under rotations in both spin and isospin space, is given by

$$H = g_{\rm pair}(S_{\rm pp}^+ S_{\rm pp}^- + S_{\rm pn}^+ S_{\rm pn}^- + S_{\rm nn}^+ S_{\rm nn}^-) + g_{\rm ph} \sum_{\mu\nu} (-)^{\mu+\nu} F_\mu^\nu F_{-\mu}^{-\nu} + g_{\rm pp} P^+ \cdot P^- \;, \tag{7.3}$$

where $g_{\rm pair}$, $g_{\rm ph}$, and $g_{\rm pp}$ are the pairing, particle-hole, and particle-particle interaction constants, respectively. Although the Hamiltonian (7.3) contains all components of the nuclear residual interaction that are of direct relevance for double-beta decay, important features of real nuclei are missing from it, namely the nondegeneracy of the single-particle orbitals, and particularly the splitting of spin-orbit partners by the spin-orbit force. Nonetheless, we restrict ourselves here to the degenerate case in order to keep the model exactly solvable.

A convenient basis in which to diagonalize the Hamiltonian (7.3) is generated by repeatedly acting on the shell-model vacuum $|0\rangle$ (closed-shell state) with the pair operators S^+ and P^+:

$$|\mathbf{m}\rangle \equiv (S_{\rm pp}^+)^{m_{\rm pp}} (S_{\rm pn}^+)^{m_{\rm pn}} (S_{\rm nn}^+)^{m_{\rm nn}} \left[\prod_{\mu=-1}^{+1} (P_\mu^+)^{m_\mu} \right] |0\rangle \;. \tag{7.4}$$

To obtain the matrix elements of the Hamiltonian, the Gamow-Teller operator, and the unit operator between these states, we proceed as follows. Introducing the pair operator

$$\zeta^\dagger = \sum_\nu \alpha_\nu C_{0,0;1,\nu} + \sum_\mu \beta_\mu C_{1,\mu;0,0} \;, \tag{7.5}$$

we define a coherent state by

$$|\gamma\rangle = \exp(\zeta^\dagger)|0\rangle \;. \tag{7.6}$$

The overlap of this coherent state with itself is given by

$$\langle \gamma | \gamma \rangle = Z^\Omega, \quad Z = 1 + \gamma^* \cdot \tilde{\gamma} + |\tfrac{1}{2}\gamma\cdot\gamma|^2 \;, \tag{7.7}$$

where $\Omega = \sum_l (2l+1)$ is the orbital degeneracy, and

$$\gamma^*\cdot\tilde{\gamma} = \alpha^*\cdot\tilde{\alpha} + \beta^*\cdot\tilde{\beta}, \quad \alpha^*\cdot\tilde{\alpha} = \sum_\nu \alpha_\nu^* \alpha_\nu \;,$$

$$\beta^*\cdot\tilde{\beta} = \sum_\mu \beta_\mu^* \beta_\mu, \quad \gamma\cdot\gamma = -\alpha\cdot\alpha + \beta\cdot\beta \;, \tag{7.8}$$

$$\alpha\cdot\alpha = \sum_\nu (-)^\nu \alpha_\nu \alpha_{-\nu}, \quad \beta\cdot\beta = \sum_\mu (-)^\mu \beta_\mu \beta_{-\mu} \;.$$

Matrix elements of H between two coherent states can be evaluated by using

$$\langle \gamma | H | \gamma \rangle = \langle 0 | e^{\zeta} e^{\zeta^\dagger} ([H,\zeta^\dagger] + \tfrac{1}{2}[[H,\zeta^\dagger],\zeta^\dagger]) | 0 \rangle \;. \tag{7.9}$$

The expression on the rhs of Eq. (7.9) can be written as a second-order differential operator in the parameters γ acting on the overlap function $\langle \gamma | \gamma \rangle$. Using this fact, we find for the Hamiltonian (7.3),

$$\begin{aligned}\langle \gamma | H | \gamma \rangle = {} & \Omega Z^{\Omega-1}[g_{\rm pair}(\alpha^*\cdot\tilde{\alpha} + 3|\tfrac{1}{2}\gamma\cdot\gamma|^2) + g_{\rm ph}\gamma^*\cdot\tilde{\gamma} + g_{\rm pp}(\beta^*\cdot\tilde{\beta} + 3|\tfrac{1}{2}\gamma\cdot\gamma|^2)] \\ & + \Omega(\Omega-1)Z^{\Omega-2}\{g_{\rm pair}[(\alpha^*\cdot\tilde{\alpha})(1+|\tfrac{1}{2}\gamma\cdot\gamma|^2) - (\alpha^*\cdot\alpha^*)(\tfrac{1}{2}\gamma\cdot\gamma) - (\tfrac{1}{2}\gamma^*\cdot\gamma^*)(\alpha\cdot\alpha)] \\ & + g_{\rm ph}[2(\alpha^*\cdot\tilde{\alpha})(\beta^*\cdot\tilde{\beta}) + (\alpha^*\cdot\alpha^*)(\beta\cdot\beta) + (\beta^*\cdot\beta^*)(\alpha\cdot\alpha)] \\ & + g_{\rm pp}[(\beta^*\cdot\tilde{\beta})(1+|\tfrac{1}{2}\gamma\cdot\gamma|^2) + (\beta^*\cdot\beta^*)(\tfrac{1}{2}\gamma\cdot\gamma) + (\tfrac{1}{2}\gamma^*\cdot\gamma^*)(\beta\cdot\beta)]\} \;. \end{aligned} \tag{7.10}$$

Similarly,

$$\langle \gamma | F_\mu^\nu | \gamma \rangle = \langle 0 | e^{\zeta} e^{\zeta^\dagger} [F_\mu^\nu, \zeta^\dagger] | 0 \rangle$$
$$= \Omega Z^{\Omega-1} [\beta_\mu^* (-)^\nu \alpha_{-\nu} + \alpha_\nu^* (-)^\mu \beta_{-\mu}] . \quad (7.11)$$

From Eqs. (7.7), (7.10), and (7.11), we derive all matrix elements $\langle \mathbf{m}' | H | \mathbf{m} \rangle$, $\langle \mathbf{m}' | F_\mu^\nu | \mathbf{m} \rangle$, and $\langle \mathbf{m}' | \mathbf{m} \rangle$ for the states (7.4) by Taylor expansion.

These matrix elements having been determined, it is straightforward to obtain $M_{\rm GT}$, Eq. (2.2), by exact numerical calculation. In the first step, we diagonalize the overlap matrix $\langle \mathbf{m}' | \mathbf{m} \rangle$ to generate a basis of orthogonal normalized states $|\tilde{\mathbf{m}}\rangle$. Next, we diagonalize the Hamiltonian matrix $\langle \tilde{\mathbf{m}}' | H | \tilde{\mathbf{m}} \rangle$ in this orthonormal basis. Both steps are performed for the initial, intermediate, and final nucleus of the double-beta decay. Finally, we use the expression (7.11) to calculate the transition matrix elements in (2.2) and obtain $M_{\rm GT}$. In the actual calculation, we use a basis $|\mathbf{m}\rangle$ with good quantum numbers $J_z = m_{+1} - m_{-1}$ and $T_z = \frac{1}{2}(m_{\rm pp} - m_{\rm nn})$ but we do not explicitly project onto good total spin and isospin. The resulting matrices have dimension 10–100 for valence neutron and proton numbers that correspond to an actual nucleus.

To obtain the QRPA approximation to $M_{\rm GT}$ for the present model, we observe that because of the orbital degeneracy we need only consider a single two-quasiparticle configuration. The QRPA matrices A and B are therefore single numbers. Following a standard prescription, we calculate the particle-hole matrix element $\tilde{V}$ from the particle-hole interaction $g_{\rm ph} \sum_{\mu\nu} F_\mu^\nu (-)^{\mu+\nu} F_{-\mu}^{-\nu}$, and the particle-particle matrix element V from the particle-particle interaction $g_{\rm pp} P^+ \cdot P^-$, thereby neglecting a small contribution to $\tilde{V}$ from $g_{\rm pp} P^+ \cdot P^-$, and a small contribution to V from $g_{\rm ph} \sum_{\mu\nu} F_\mu^\nu (-)^{\mu+\nu} F_{-\mu}^{-\nu}$. The quantities a and b are then given by Eqs. (3.13), with $\omega_0 = -\Omega g_{\rm pair}$, $g_{\rm ph}^{\rm eff} = \Omega g_{\rm ph}$, and $g_{\rm pp}^{\rm eff} = \Omega g_{\rm pp}$. The rest of our procedure is defined in Sec. III.

The resulting approximation to the double-beta decay matrix element is compared to the exact result in Fig. 3. We display the matrix element as a function of $g_{\rm pp}/g_{\rm pair}$ and use the same parameter set as in I, namely $g_{\rm ph} = +1.5$, $g_{\rm pair} = -0.3$, with two proton particles and four neutron holes in a shell with $\Omega = 15$. [Equation (5) of I contains a spurious factor $\frac{1}{2}$.] We observe that the matrix element passes through zero at $g_{\rm pp}/g_{\rm pair} = 1$.

Also shown in Fig. 3 is the result obtained from the formula of Civitarese, Faessler, and Tomoda (CFT),[15]

$$M_{\rm GT} = -\frac{1}{\Delta E} \langle p \| \sigma \tau^+ \| n \rangle^2$$
$$\times (u_p v_n x - v_p u_n y)(\bar{v}_p \bar{u}_n \bar{x} - \bar{u}_p \bar{v}_n \bar{y})(x\bar{x} - y\bar{y}) . \quad (7.12)$$

Here, barred (unbarred) quantities refer to the final (initial) nucleus of the double-beta decay. As compared to Eq. (6.1), the CFT formula has the attractive feature of being *exact* for a model with degenerate single-particle energies and pairing between like nucleons but no interactions between neutrons and protons. ($x = \bar{x} = 1$ and $y = \bar{y} = 0$ in this case.) However, in the presence of a neutron-proton interaction its performance is less good. The figure shows that the CFT value always exceeds in magnitude both the initial and final nucleus solution, while the exact result lies somewhere in between. This can be understood by noting that the β^- strength (β^+ strength) decreases (increases) as Z and N approach each other. The CFT formula takes the β^- amplitude from the initial nucleus (which is larger than the corresponding amplitude for the final nucleus), and the β^+ amplitude from the final nucleus (which is larger than the corresponding amplitude for the initial nucleus). Furthermore, the overlap factor $(x\bar{x} - y\bar{y})$ is always *greater* than unity, due to the indefiniteness of the RPA metric requiring $x^2 - y^2 = 1 = \bar{x}^2 - \bar{y}^2$.

We have mentioned above that the Hamiltonian (7.3) is not realistic because it lacks the spin-orbit splitting present in nuclei. Nevertheless, we believe that the present model is illuminating for two reasons. First, it demonstrates unequivocally the existence of a zero in $M_{\rm GT}$. Such a zero must occur even in realistic situations, for any addition to the Hamiltonian (7.3) can only change details of the dependence of $M_{\rm GT}$ on $g_{\rm pp}$ and not the overall qualitative behavior. Second, the model shows the QRPA to be a useful approximation scheme for the study of double-beta decay. In particular, we observe that the QRPA predicts the exact location of the zero in $M_{\rm GT}$ for the model. Figure 3 also provides some justification for our procedure of averaging the matrix element over the initial and final nucleus solutions.

To conclude this section, we mention that the vanishing of $M_{\rm GT}$ for $g_{\rm pp} = g_{\rm pair}$ is the consequence of a dynamical SU(4) symmetry. Using group-theoretical

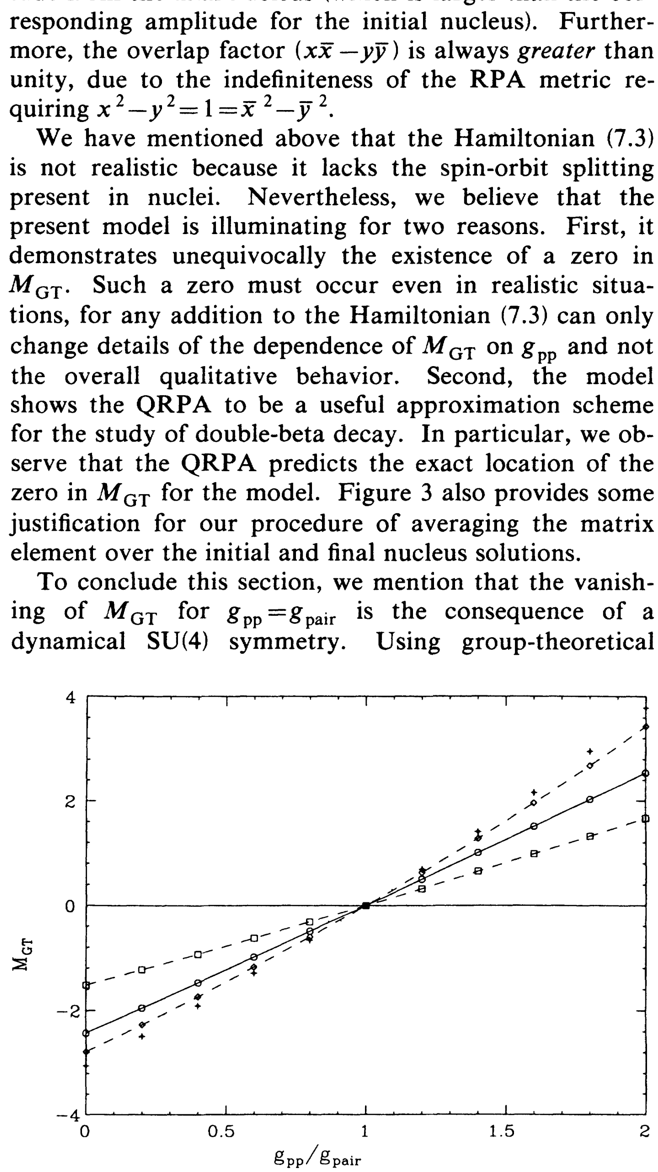

FIG. 3. Dependence of $M_{\rm GT}$ (in arbitrary units) on $g_{\rm pp}/g_{\rm pair}$ for the exactly solvable model with the parameters given in the text. The solid line connecting circles is the exact solution, the dashed line connecting squares (diamonds) is the QRPA result for the initial (final) nucleus, and the dotted line connecting crosses is the CFT result.

methods, we can derive closed algebraic expressions for energies and wave functions in this limit. These do not contribute to our understanding of double-beta decay, and therefore we do not expand on them here.

VIII. 0ν MODE OF DOUBLE-BETA DECAY

2ν double-beta decay is often regarded as a proving ground for any scheme of calculation that hopes to establish a reliable connection between 0ν decay rates and the Majorana mass of the electron neutrino. Having developed and tested our methods in the context of 2ν decay, we turn now to the evaluation of the matrix elements $M_{\mathrm{GT}}^{0\nu}$ and $M_F^{0\nu}$ defined in (2.6a) and (2.6b). Once these are known, the half-lives in terms of the neutrino mass are determined from (2.5). [The half-lives for majoron emission can be calculated from (2.7).]

The QRPA expression for the 0ν matrix elements (2.6) is given in Eqs. (3.10) and (3.12). It involves a sum over quantum numbers J and π and, consequently, the QRPA equations must be solved separately for each spin and parity of the intermediate odd-odd nucleus. Therefore, unlike the case of the 2ν decay, we need to know the value of α_0' as well as of α_1', α_0, and α_1. When evaluating the half-lives, we again use $g_A=1.0$ for consistency. Also, as before, we evaluate expression (3.10) for the initial and final nuclei separately, and average the results. The CFT procedure[15] mentioned above can be generalized to the 0ν case. However, the results of Sec. VII suggest that our simpler averaging procedure is more accurate, and we therefore use it throughout.

We begin the discussion by presenting the dependence of the matrix elements governing 0ν decay on the coupling constants of the residual interaction. (Remember that for the particle-particle force the two parameters are connected because α_0' is fixed at $0.6\alpha_1'$.) The relevant quantity here is $M^{0\nu}\equiv M_{\mathrm{GT}}^{0\nu}-M_F^{0\nu}$ [see (2.5) and remember that, in our case, $g_V/g_A=1.0$], resulting from summation over all combinations $p,n;p',n'$ as well as over the states in the intermediate nucleus labeled by J^π and m. As the neutrino potential $H(\bar{E},r)$ (2.3) is only weakly singular, behaving like $1/r$ for small r, a common approximation used in evaluating expressions (2.6) is to ignore the short-range correlations, i.e., to assume $\rho=1$ in (2.8). Table II contains matrix elements calculated under this assumption, where we have taken $\bar{E}$ in (2.3) to be the energy of the Gamow-Teller giant resonance. The entries show that the ground-state correlations associated with the particle-hole part of the interaction reduce $M^{0\nu}$ by approximately a factor of 1.5 to 2. When the particle-particle interaction is included, the matrix element is reduced by an additional factor of 2 to 3.

In Table III we present the same quantities calculated with the short-range correlations included. The cutoff at short internucleon spacings reduces the matrix elements still further. The reduction is particularly dramatic when the particle-particle interaction is turned on. As with the 2ν mode, the matrix element passes through zero for some value of the interaction constant, leading to a vanishing 0ν double-beta decay rate at that point. While in 2ν decay the matrix elements crossed zero before entering our window, here they cross only after passing through the window. The matrix elements calculated without the short-range correlation also pass through zero eventually, but in that case the zero is even further after our window.

To gain insight into the behavior of the 0ν matrix elements we show in Figs. 4 and 5 the contributions of individual multipoles to the Gamow-Teller and Fermi matrix elements in ^{76}Ge (a typical case). For independent quasiparticles, or when only the particle-hole interaction is included, the different multipolarities contribute essentially coherently to both matrix elements; the dominant contribution to $M_{\mathrm{GT}}^{0\nu}$ comes from $J^\pi=1^+$ and to $M_F^{0\nu}$ from 0^+. The overall ratio $M_{\mathrm{GT}}^{0\nu}/M_F^{0\nu}$ is roughly -3, in accordance with the expectation that the two nucleons in (3.12) are dominantly in the spin-singlet state. When the particle-particle interaction is included, the contribution of the 1^+ multipole passes through zero, as does its analog in 2ν decay. Therefore, $M_{\mathrm{GT}}^{0\nu}$ is reduced faster than $M_F^{0\nu}$, and changes sign for a certain value of the particle-particle coupling constant, so that the ratio of the two matrix elements is no longer close to -3. The short-range correlations affect the different multipolarities differently. The higher multipolarities are affected more and speed up the variation of $M_{\mathrm{GT}}^{0\nu}$ with the strength of the particle-particle coupling constant.

Tables I and IV show that the matrix elements of the 2ν and 0ν double-beta decays are not proportional in general. This does not mean, however, that knowledge of the 2ν decay rate, and thus of the corresponding nuclear matrix element $M_{\mathrm{GT}}^{2\nu}$, is not useful. On the con-

TABLE II. The quantity $M_{\mathrm{GT}}^{0\nu}-M_F^{0\nu}$ for different values of α_1' (in units of MeV fm^3), neglecting the effects of short-range nucleon-nucleon correlations. The column labeled "pairing" again gives the independent quasiparticle result. The other particle-particle parameter α_0' is fixed at $0.6\alpha_1'$, as discussed in the text.

	Pairing	0	−390	−432
^{76}Ge	−14.4	−9.5	−4.7	−3.7
^{82}Se	−10.7	−7.2	−3.6	−2.7
^{96}Zr	−14.1	−8.6	−2.9	
^{100}Mo	−21.5	−13.2	−5.5	
^{128}Te	−21.7	−12.9	−6.2	−5.2
^{130}Te	−19.3	−11.4	−5.5	−4.7
^{136}Xe	−8.9	−5.2	−2.5	−2.1

TABLE III. The quantity $M_{\mathrm{GT}}^{0\nu}-M_F^{0\nu}$ for different values of α_1' (in units of MeV fm^3), including the effects of short-range correlations.

	Pairing	0	−390	−432
^{76}Ge	−11.7	−6.9	−2.0	−0.9
^{82}Se	−8.7	−5.0	−1.5	−0.7
^{96}Zr	−11.2	−6.5	−1.2	
^{100}Mo	−17.2	−10.0	−2.8	
^{128}Te	−17.5	−9.9	−3.8	−2.8
^{130}Te	−15.6	−8.7	−3.4	−2.5
^{136}Xe	−7.2	−4.0	−1.5	−1.1

TABLE IV. The 0ν phase space function $G^{0\nu}(E_{max},Z)$, Eq. (2.5), and 0ν double-beta decay half-lives (in yr) for different values of α_1' (in units of MeV fm^3). The dimension of $G^{0\nu}$ is yr^{-1} if $\langle m_\nu \rangle$ is given in units of eV.

	$G^{0\nu}$	Pairing	0	-390	-432
^{76}Ge	9.68×10^{-27}	7.6×10^{23}	2.2×10^{24}	2.7×10^{25}	1.4×10^{26}
^{82}Se	4.29×10^{-26}	3.1×10^{23}	9.4×10^{23}	1.1×10^{25}	4.5×10^{25}
^{96}Zr	8.89×10^{-26}	9.0×10^{22}	2.7×10^{23}	7.8×10^{24}	
^{100}Mo	6.94×10^{-26}	4.9×10^{22}	1.5×10^{23}	1.9×10^{24}	
^{128}Te	2.77×10^{-27}	1.2×10^{24}	3.7×10^{24}	2.5×10^{25}	4.5×10^{25}
^{130}Te	6.79×10^{-26}	6.1×10^{22}	2.0×10^{23}	1.3×10^{24}	2.3×10^{24}
^{136}Xe	7.18×10^{-26}	2.7×10^{23}	8.8×10^{23}	6.3×10^{24}	1.1×10^{25}

trary, our results indicate that the suppression of the matrix element $M_{GT}^{0\nu}$ is caused to a large extent by the cancellation of the contribution from the multipolarity 1^+ by that from all others, and the size of the 1^+ term is indeed almost proportional to the value of the matrix element $M_{GT}^{2\nu}$.

It is of interest to compare our results to those of other authors. The matrix elements of Grotz and Klapdor[8] are calculated via a QRPA method that resembles our own with several important differences: particle-particle interactions and short-range correlations are not included while quadrupole correlations are. Table XI of Ref. 8 presents results that, with only pairing (no residual interaction), are similar to, though slightly larger than, our own. The table also indicates that a reduction in the matrix elements is obtained when the particle-hole interaction is switched on, although it is smaller than in our case. Of course, our matrix elements are still further reduced by the particle-particle interaction, an effect that is missing altogether from the treatment of Ref. 8.

The calculations of Haxton and Stephenson, presented in Table XII of Ref. 2, yield matrix elements that generally lie between the values we obtain with $\alpha_1'=0$ and $\alpha_1'=-390$ MeV fm^3, the lower limit of our window. Direct comparison of the matrix elements is difficult because a significantly smaller shell-model space was used in Ref. 2 than here. Reducing our space to that of Ref. 2 in ^{76}Ge decreases our matrix elements by a factor of 2 or so, and prevents them from passing through zero as the particle-particle interaction strength is increased. While a more thorough study would require the renormalization of our interaction in the smaller space, it appears that the levels omitted from the calculations in Ref. 2 contribute to the decay rate in a nontrivial way.

After this paper was submitted for publication, we became aware of unpublished work by Tomoda and Faessler.[27] In the same way that Ref. 15 extended the techniques of I by using a G-matrix-based calculation for 2ν decay, Tomoda and Faessler, in Ref. 27, apply the techniques presented here to the 0ν mode. It is apparent from Fig. 1 of Ref. 27 that $M_{GT}^{0\nu}$ in ^{76}Ge is substantially reduced by the particle-particle interaction; a simple extrapolation suggests that it passes through zero at a value of g_{pp} around 1.15, just beyond the value of 1.0

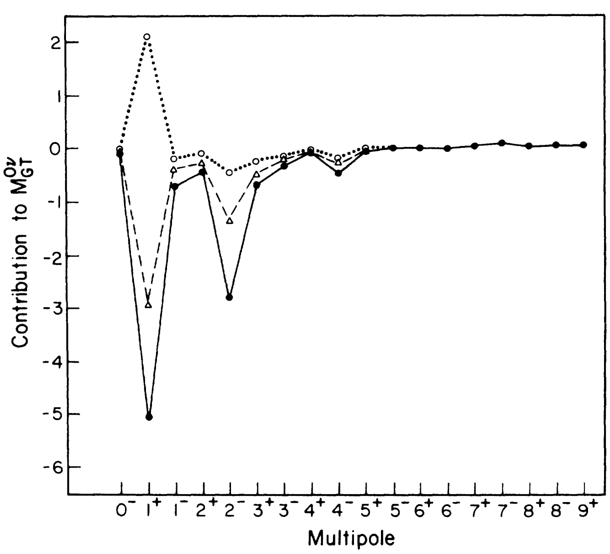

FIG. 4. Contributions of individual multipoles to $M_{GT}^{0\nu}$ for the nucleus ^{76}Ge. The solid circles are the independent quasiparticle result, the triangles result from including the residual particle-hole interaction, but not the particle-particle interaction, and the open circles result from the additional inclusion of the particle-particle force with $\alpha_1'=-432$ MeV fm^3, the upper limit of our window. The parameter α_0' is fixed at $0.6\alpha_1'$. Note the change in sign of the contribution from the multipole 1^+. The multipole 0^+ does not contribute to $M_{GT}^{0\nu}$.

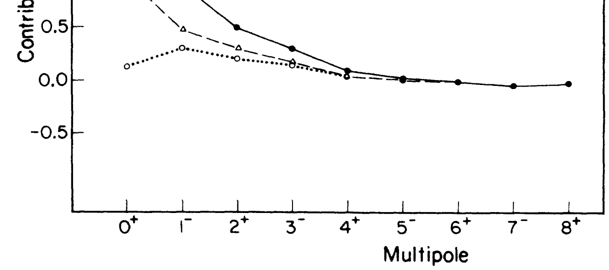

FIG. 5. Same as Fig. 4, but for the Fermi matrix element $M_F^{0\nu}$. Only multipoles with natural parity contribute to $M_F^{0\nu}$.

chosen by the authors. In fact, the main source of difference in the limits on $\langle m_\nu \rangle$ quoted in Ref. 27 and here is the different ways in which the strength of the particle-particle interaction was determined. If we use here the condition $M_{\rm GT}^{2\nu}=0$, as in Ref. 27 (as well as $g_A=1.25$ and $T_{1/2}>4.7\times10^{23}$ yr), we obtain an upper limit on $\langle m_\nu \rangle$ of about 3 eV, a factor only 1.5 times larger than the limit quoted in Ref. 27.

Finally, the work of Tomoda *et al.*[4] indicates that reductions caused by short-range correlations are approximately 20% of the "uncorrelated" matrix element. Here we find a larger effect. When the QRPA is included, the reductions due to short-range effects are quite dramatic, as shown by Table III.

Our results are summarized in Table IV, which presents predicted half-lives for several double-beta decay candidates under the assumption that the electron neutrino has a Majorana mass of 1 eV. (The half-life scales as the inverse square of $\langle m_\nu \rangle$.) (The entries in Tables II–IV in the preliminary preprint version of this paper were affected by a numerical error and therefore differ from the correct values here.)

It is obvious that near the point where the matrix element passes through zero other nuclear structure phenomena, not included here, will have a disproportionally large effect on the resulting decay rate. For example, deviation of the nuclear axial-vector coupling constant g_A from our nominal value of unity or inclusion of the ground-state correlations associated with the low-lying vibrational 2^+ or 3^- states should be included in the calculation. To illustrate the degree of suppression of the nuclear matrix elements calculated here, we evaluate the upper limit on the neutrino mass derived from the experimental lower limit for the 0ν half-life in ^{76}Ge. From the value $T_{1/2}^{0\nu}>4.1\times10^{23}$ yr of Caldwell *et al.*[24] we find a lower bound on $\langle m_\nu \rangle$ of 2.2 eV when the nuclear matrix elements of Ref. 2 are used. This limit becomes worse, 5.8 eV, if the "scaling" procedure[1] is used, i.e., if the matrix elements are scaled in such a way that the experimental 2ν half-life in ^{82}Se is obtained. (We noted above that this scaling is of dubious value according to our results.) On the other hand, using the matrix elements calculated here we obtain an upper limit of 2.3 eV if we use the particle-hole interaction only, 8 eV if we use $M^{0\nu}=-2.0$ calculated with $\alpha_1'=-390$ MeV fm^3 (the lower edge of our window), and 10 eV if we use $M^{0\nu}=-1.6$ obtained for the optimal $\alpha_1'=-405$ MeV fm^3.

The results in Table IV seem to suggest that the heavier double-beta decay candidates, for example ^{136}Xe or ^{130}Te, are from the point of view of nuclear structure somewhat better candidates for 0ν double-beta decay. The geochemical data[21] on double beta decay in Te isotopes determine total lifetimes and thus upper limits for the 0ν decay rates. From the limits in ^{130}Te we find $\langle m_\nu \rangle <26$ eV, and from those on ^{128}Te $\langle m_\nu \rangle <2.2$ eV for α_1' in the middle of our window. These limits are very close to those obtained by Tomoda and Faessler[27] for the same decays, provided one takes into account that we use $g_A=1.0$ while Ref. 27 uses $g_A=1.25$. Experimental determination of suppressed quantities, e.g. 2ν decay rates and β^+ strengths, would help greatly to reduce the uncertainties in the calculations presented in this section.

IX. CONCLUSIONS

The main result of this paper is that a consistent treatment of particle-particle interactions alongside particle-hole interactions in the framework of the quasiparticle random phase approximation yields rates for both 2ν and 0ν double-beta decay that are highly suppressed. The persistence of this phenomenon in our solvable model, and the accuracy of the QRPA in reproducing it, give us some confidence that the suppression obtained in real nuclei is not an artifact of the approximations inherent in our treatment. In the case of 2ν decay, this suppression is observed experimentally in several nuclei, and our calculations are in reasonably close agreement with the data. When extended to 0ν decay, our methods result also in substantial suppression, the implication of which is that limits on the neutrino mass deduced from limits on the half-lives obtained in experimental searches for 0ν double-beta decay are less stringent than commonly thought. We also find that short-range correlations due to hard-core repulsion have a larger effect on calculated half-lifes than has been obtained in other work. We plan to study the effect of the short-range correlations in greater detail elsewhere.[28]

In spite of our conclusion that the nuclear matrix elements for double-beta decay are suppressed and therefore difficult to calculate with precision, the great importance of the experimental search for 0ν decay remains undiminished. Experimental detection of 0ν decay would show the electron neutrino to be a massive Majorana particle, a result that would have far-reaching consequences, independent of the nuclear structure effects we have considered.

Continued investigation within the framework outlined here would be useful. The effects of quadrupole collective motion, for example, ought to be studied carefully and incorporated into our approach in a consistent way. Perhaps most important is an explanation of the discrepancy between the suppressed, parameter-sensitive results obtained here and those of calculations performed in different approaches (e.g., those of Refs. 2 and 4) that do not contain parameters directly analogous to our particle-particle interaction strength α_1'. Work in this direction is currently in progress.

ACKNOWLEDGMENTS

We would like to acknowledge useful discussions with G. E. Brown, S. E. Koonin, and J. P. Vary. One of the authors (P.V.) would like to thank Prof. H. C. Pauli and Prof. H. A. Weidenmüller for the warm hospitality extended to him at the Max-Planck-Institut für Kernphysik in Heidelberg, where parts of this manuscript were written. This work was supported by the U.S. Department of Energy under Contract No. DE-AS03-81ER40002 and by the National Science Foundation under Grants PHY86-04197 and PHY85-05682.

APPENDIX

We give expressions for the two-body matrix elements of a δ-function interaction, $V^{J;S,T}$, with total angular momentum J, spin S, and isospin T. The particle-particle matrix elements are

$$V^{J;0,1}_{pn,p'n'}=\frac{\hat{j}}{4\pi}R_{pn,p'n'}(-)^{j_n+j_{n'}+l_p+l_{p'}-1}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\tfrac{1}{2}[1+(-)^{l_p+l_n-J}] , \tag{A1}$$

$$V^{J;1,0}_{pn,p'n'}=\frac{\hat{j}}{4\pi}R_{pn,p'n'}\left[\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}+(-)^{j_n+j_{n'}+l_p+l_{p'}-1}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\right.$$
$$\left.\times\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\tfrac{1}{2}[1-(-)^{l_p+l_n-J}]\right] . \tag{A2}$$

The corresponding particle-hole matrix elements are obtained by angular momentum recoupling,

$$\tilde{V}^{J;S,T}_{pn,p'n'}\equiv-(-)^{j_p+j_n+j_{p'}+j_{n'}}\sum_{J'}(2J'+1)W(j_pj_nj_{n'}j_{p'};JJ')V^{J';S,T}_{p'n,pn'} , \tag{A3}$$

which yields

$$\tilde{V}^{J;0,1}_{pn,p'n'}=-\frac{\hat{j}}{4\pi}R_{pn,p'n'}\left[\tfrac{1}{2}(-)^{l_p+l_{p'}}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}\right.$$
$$\left.-(-)^{j_n+j_{n'}-1}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\tfrac{1}{2}(-)^{l_p+l_n-J}\right] , \tag{A4}$$

$$\tilde{V}^{J;1,0}_{pn,p'n'}=-\frac{\hat{j}}{4\pi}R_{pn,p'n'}\left[\tfrac{1}{2}(-)^{l_p+l_{p'}}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{bmatrix}\right.$$
$$\left.+(-)^{j_n+j_{n'}-1}\begin{bmatrix} j_p & j_n & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\begin{bmatrix} j_{p'} & j_{n'} & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}[1+\tfrac{1}{2}(-)^{l_p+l_n-J}]\right] . \tag{A5}$$

$R_{pn,p'n'}$ is the integral over radial wave functions,

$$R_{pn,p'n'}=\int_0^\infty r^2dr\,\psi_p(r)\psi_n(r)\psi_{p'}(r)\psi_{n'}(r) , \tag{A6}$$

and

$$\hat{j}=[(2j_p+1)(2j_n+1)(2j_{p'}+1)(2j_{n'}+1)]^{1/2} . \tag{A7}$$

The two-body matrix elements V^J and $\tilde{V}^J$ appearing in Sec. III are isospin-uncoupled matrix elements given by

$$V^J_{pn,p'n'}=\tfrac{1}{2}(\alpha_0'V^{J;0,1}_{pn,p'n'}+\alpha_1'V^{J;1,0}_{pn,p'n'}) , \tag{A8a}$$

$$\tilde{V}^J_{pn,p'n'}=\tfrac{1}{2}(\alpha_0\tilde{V}^{J;0,1}_{pn,p'n'}+\alpha_1\tilde{V}^{J;1,0}_{pn,p'n'}) , \tag{A8b}$$

where we have used the fact that the Clebsch-Gordan coefficient $\langle\frac{1}{2}+\frac{1}{2},\frac{1}{2}-\frac{1}{2}\,|\,T0\rangle^2=\frac{1}{2}$ for $T=0,1$. The matrix element (A8b) is identical to that of Speth *et al.*[25] provided we make the identifications $\alpha_0+\alpha_1=-4C_0g_0^{np}$ and $3\alpha_1-\alpha_0=-4C_0f_0^{np}$ and take into account that the neutron and proton quantum numbers are arranged in reverse order in this reference.

[1]M. Doi, T. Kotani and E. Takasugi, Prog. Theor. Phys. Suppl. **83**, 1 (1985).

[2]W. C. Haxton and G. J. Stephenson Jr., Prog. Part. Nucl. Phys. **12**, 409 (1984).

[3]B. Kayser, in *Nuclear Beta Decays and Neutrino,* edited by T. Kotani, H. Ejiri, and E. Takasugi (World-Scientific, Singapore, 1986), p. 473.

[4]T. Tomoda, A. Faessler, K. W. Schmid, and F. Grümmer, Nucl. Phys. **A452**, 591 (1986).

[5]J. D. Vergados, Phys. Rev. C **24**, 640 (1981).

[6]P. Vogel and M. R. Zirnbauer, Phys. Rev. Lett. **57**, 3148 (1986).
[7]G. A. Miller and J. E. Spencer, Ann. Phys. **100**, 562 (1976).
[8]H. V. Klapdor and K. Grotz, Phys. Lett. **142B**, 323 (1984); K. Grotz and H. V. Klapdor, Nucl. Phys. **A460**, 395 (1986).
[9]C. Gaarde, Nucl. Phys. **A396**, 127 (1983).
[10]R. Madey *et al.*, in *Weak and Electromagnetic Interactions in Nuclei*, edited by H. V. Klapdor (Springer-Verlag, Heidelberg, 1987), p. 280.
[11]P. Vogel and P. Fisher, Phys. Rev. C **32**, 1362 (1985).
[12]J. A. Halbleib and R. A. Sorensen, Nucl. Phys. **A98**, 542 (1967).
[13]D. Cha, Phys. Rev. C **27**, 2269 (1983); Ph.D. thesis, Michigan State University, 1982 (unpublished).
[14]A. de Shalit and I. Talmi, *Nuclear Shell Theory* (Academic, New York, 1963).
[15]O. Civitarese, A. Faessler, and T. Tomoda, Phys. Lett. **196**, 11 (1987).
[16]G. Bertsch, *The Practitioner's Shell Model* (American Elsevier, New York, 1972).
[17]F. D. Becchetti, Jr. and G. W. Greenless, Phys. Rev. **182**, 1190 (1969).
[18]J. P. Vary, private communication.
[19]P. Kleinheinz *et al.*, Phys. Rev. Lett. **55**, 2664 (1985); W. Kurcewicz *et al.*, Z. Phys. A **308**, 21 (1982); E. Nolte *et al.*, *ibid.* **306**, 233 (1983); **309**, 33 (1982).
[20]I. S. Towner, Nucl. Phys. **A444**, 402 (1985).
[21]T. Kirsten *et al.*, in *Nuclear Beta Decays and Neutrino*, edited by T. Kotani, H. Ejiri, and E. Takasugi (World-Scientific, Singapore, 1986), p. 81.
[22]B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. London **84**, 673 (1964); S. C. Pang, Nucl. Phys. **A128**, 497 (1969).
[23]E. Wigner, Phys. Rev. **51**, 106 (1937).
[24]D. O. Caldwell *et al.*, Phys. Rev. D **33**, 2737 (1986).
[25]J. Speth, E. Werner, and W. Wild, Phys. Rep. **33**, 127 (1977).
[26]H. Liu and L. Zamick, Nucl. Phys. **A467**, 29 (1987).
[27]T. Tomoda and A. Faessler (unpublished).
[28]J. Engel, P. Vogel, O. Civitarese, and M. Zirnbauer (submitted to Phys. Lett.)