Ambiguity-free polarization measurements from mixture initial preparations

G. Ramachandran
Department of Physics, University of Mysore, Mysore 570 006, India

V. Ravishankar*
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012, India
(Received 24 June 1987)

Starting from beam and target spin systems which are polarized in the usual way by applying external magnetic fields, measurements of appropriate final state tensor parameters, viz., $\{t^k_{0,1}, k=1,\ldots,2j\}$ of particle d with spin j in a reaction $a+b\to d+c_1+c_2+\ldots$, are suggested to determine the reaction amplitudes in spin space free from any associated discrete ambiguity.

Considering the general reaction

$$a+b\to d+c_1+c_2+\cdots , \qquad (1)$$

where the particles have respective spins $j_a, j_b, j, j_1, j_2, \ldots$, it has been shown recently[1] that the reaction amplitudes $A^j_m(\alpha,i)$ defined through

$$\langle jm;\alpha \mid T \mid i\rangle = (-1)^m A^j_{-m}(\alpha,i) \qquad (2)$$

may be determined, except for an overall phase and certain discrete ambiguities, by measuring the tensor parameters t^k_q with $k=2j$ alone and the differential cross section. Here, $|i\rangle$ denotes the initial spin state and α denotes collectively the spin state of the coproduced companions c_i of particle d. This result provides an extension, to arbitrary spin j, of a theorem obtained for spin-1 particles by Goldstein and Moravcsik[2] using their optimal formalism. A basic difficulty that is met in using these theorems in a practical situation is that the initial system is required to be prepared in a pure state $|i\rangle$. This requirement appears to be rather too stringent to be capable of experimental realization in the near future. Even in reactions initiated by neutrinos (which are intrinsically in a pure spin state to the extent that they are massless) or muons (which are produced in a pure spin state in π decay but which nevertheless suffer depolarization before they get absorbed by a nucleus), it is, however, almost impossible at present to prepare particle or nuclear targets which are populated entirely in the same magnetic substate.

The purpose of this paper is thus twofold: (1) We suggest a possible method by which the information that is obtained by using hypothetical pure initial spin states could still be deduced by employing mixture state preparations such as are usually realized by subjecting the system to external magnetic fields. (2) Since the results of Refs. 1 and 2 leave a large number (see below) of discrete ambiguities unresolved, we examine alternative sets of polarization observables as candidates for complete sets[3] and demonstrate the existence of several such, each of which is equally capable of determining the amplitudes A^j_m without any ambiguity. In particular, the observables $\{\mathrm{Tr}\rho, t^k_{0,1}, k=1,\ldots,2j\}$ are shown to be complete.

If ρ^I denotes the $n_I\times n_I$ density matrix, where $n_I=(2j_a+1)(2j_b+1)$, specifying the spin state of the initial system, it is clear that, ρ^I being Hermitian, it is canonically characterized by its eigenvalues p_i and the corresponding eigenstates $|i\rangle, i=1,\ldots,n_I$. In practice, the spin states of a and b are uncorrelated; ρ^I is a direct product of density matrices ρ^a and ρ^b characterizing the beam and the target. If, in particular, they are oriented systems (i.e., uniaxial[4] or cylindrically symmetric), $p_i=p_{m_a}p_{m_b}$ and $|\psi_i\rangle=|j_am_a\rangle|j_bm_b\rangle$ with respect to their respective axes of orientation. Thus, the measured t^k_q of the particle d in the final state are given by generalizing Eq. (30) of Ref. 1 as

$$\mathrm{Tr}\rho t^k_q=(-1)^{j+k}[j]\sum_{i=1}^{n_I} p_i[A^j(\alpha,i)\otimes A^{\dagger j}(\alpha,i)]^k_q \qquad (3)$$

where ρ is the density matrix of the spin j particle. If we repeat the experiment with different initial preparations characterized by the statistical weights p^r_i, with the index r taking values $r=1,\ldots,n_I$, it is clear that

$$(-1)^{j+k}[j][A^j(\alpha,i)\otimes A^{\dagger j}(\alpha,i)]^k_q = \sum_{r=1}^{n_I}(P^{-1})_{ir}\mathrm{Tr}\rho(r)t^k_q(r) , \qquad (4)$$

provided, the matrix P with elements $(P)_{ri}=p^{(r)}_i$ is invertible. In particular, when $P=P^a\times P^b$ where, $(P^a)_{rm_a}=p^{(r)}_{m_a}$, $(P^b)_{r,m_b}=p^{(r)}_{m_b}$, the existence of P^{-1} is assured if $\det(P^a)\neq 0$, $\det(P^b)\neq 0$. If the target and beam are polarized by applying external magnetic fields H_a and H_b,

$$(P^a)_{rm_a}=\exp(m_ag_aH^{(r)}_a/kT_a)/Z^{(r)}_a , \qquad (5)$$

where $Z_a^{(r)}$ is the associated partition function, and likewise for P^b. Keeping the direction as well as the temperature fixed, the conditions on the invertibility of P^a may be translated to those on the allowed values of $H_a^{(r)}$. To that end, we observe that $D=\det(P^a)=0$, if and only if, the determinant D' of the associated equivalent matrix $(P'^a)_{rm_a}=(x_r)^{j_a+m_a}$ vanishes, where $x_r=\exp(g_a H_a^{(r)}/kT_a)$. But then, D' is the well known Vandermonde determinant and has the factorization $D'=\prod_{i>j}(x_i-x_j)$. Thereby, we conclude that any $(2j_a+1)$ unequal fields, which may be chosen at will, may be used to prepare the spin states of the particle a. The same argument holds for the values of H_b, *ad verbum*. It may be noted that the choice $H_a=H_b=0$, corresponding to unpolarized state for both a and b, is an allowed one. Thus, experimentally one needs to only use $2j_a(2j_b)$ nonvanishing magnetic fields for $a\,(b)$.

Having accomplished the first objective of relaxing the purity conditions on the initial state, we now turn our attention to identification of spin observables which determine the reaction amplitudes without any ambiguity. The following preliminary remarks are in order here. No doubt, the discrete ambiguities which survive in the earlier analyses[1,2] may be eliminated by relatively crude experiments such as the sign of t_0^1 for spin-1 particles, whence the importance of the tensor parameters t_q^{2j}. However, in general, the nature and number of these additional experiments is unspecified. Moreover, as it follows from Eq. (47a) of Ref. 1, the number of discrete ambiguities has an inflationary character with increasing j values and is given by $N(2^{2j-1}-1)/(2j+1)$, where N is the number of complex amplitudes that describe the process (1). Note that this number is zero only in one case, viz., $j=\frac{1}{2}$. So far, the resolution of these discrete ambiguities has remained a vexing problem in spite of several attempts.[5-6] To our knowledge, the problem of identifying complete sets seems to have been tackled satisfactorily only in the case of N-N scattering, first by Schumacher and Bethe[7] and more recently, by France and Winternitz.[8] In the course of their study, the latter have aptly observed that even in this simple case where only five complex amplitudes have to be determined, one could injudiciously choose as many as 27 observables which would still not completely determine the reaction amplitudes. This feature, in juxtaposition with the report[9] that there are ongoing programs in at least three laboratories, LAMPF, KEK, and SACLAY, to measure as large as 30 spin observables to study pd scattering, shows one cannot overemphasize the importance of obtaining optimal choices of observables which are truly complete—given the great effort (and cost) that goes into planning each experiment.

The approach that we take here is based on exceedingly simple but equally general considerations. Our analysis adheres to the choice of pure initial states, purely for simplicity's sake (as we have already seen that mixture states may be used with equal utility). The results we present are completely general, since we assume no symmetry such as parity or property such as unitarity. A given kinematical configuration is understood.

Consider, then, the process (1). The particle d may be isolated from its companions c_p quite arbitrarily. The reaction is, in general, characterized by $N=nn_an_b\prod_p n_p$ (complex) amplitudes. We have set $n_p=(2j_p+1)$. Starting from the expression

$$\rho^f=T\rho^iT^\dagger\ , \tag{6}$$

the observables in ρ^f are to be chosen so as to determine T completely. Since ρ^I is (assumed) prepared in a pure state $|i\rangle$, we obtain

$$\rho(\alpha,i)=\sum_{f,f'}A_f(\alpha,i)A_{f'}^*(\alpha,i)\,|f\rangle\langle f'| \tag{7}$$

relative to some basis $|f\rangle$ for particle d. Note that if $\{|f\rangle\}=\{|jm\rangle\}$, $A_f=(-1)^mA^j_{-m}$, and that (7) implies that d is produced in a pure state $\sum_f A_f(\alpha,i)\,|f\rangle$. We examine how $A_f(\alpha,i)$ (which are at once the probability amplitudes for the states $|f\rangle$ of particle d as well as the transition amplitudes for the process) may be determined from (7). Indeed, measurements of diagonal elements of ρ fix the values of $|A_f(\alpha,i)|$. To determine the $2j$ relative phases, one may choose any row or column of ρ. Since a relative phase by itself is not an observable, but some (trigonometric) function of it is, it is obvious that $2\times2j$ measurements have to be performed. Thus, for a given (α,i), the number of measurements is $(3n-2)$ so that as all (α,i) are covered the number of observables totals to $N(3n-2)/n$. Remembering that one has to determine, in addition, $N/n-1$ relative phases of the amplitudes with different (α,i) (a discussion of which we do not enter here in view of the detailed attention it has received in Ref. 1), it follows that the total number of observables that we have obtained for completely determining the amplitudes is given by[10]

$$N_c=3N-\frac{2N}{n}+2\left[\frac{N}{n}-1\right]=3N-2\ . \tag{8}$$

Thus the number of redundant measurements, with this optimal choice of observables, is given by

$$N'=N^2-(3N-2)=(N-1)(N-2)\ . \tag{9}$$

The significance of (9) for $N=1$ and 2 is quite clear.

To concentrate on the tensor parameters as a choice for the observables, let us express ρ in a $|jm\rangle$ basis. We advocate a sequential determination of the phase of $A^j_m(\alpha,i)$ relative to $A^j_{m-1}(\alpha,i)$. This is easily achieved by determining the $2j$ entities $\rho_{m,m+1}$, $m=-j,\ldots,j-1$. To wit, the measurement of $\{\rho_{mm},\rho_{m,m+1}\}$ is equivalent to measuring $\{\mathrm{Tr}\rho, t_{0,1}^{k=1,\ldots,2j}\}$; the latter set consequently turns out to be capable of determining the reaction amplitudes without any ambiguity. Thus, the observables $t_{q>1}^{k>1}$ are truly redundant, and may play a role only in minimizing uncertainties and errors associated with the measurements. Only the overall phase (which is not in any way observable) remains undetermined.

One of us (V.R.) recalls with pleasure the hospitality extended to him by Professor Y. R. Waghmare at the Indian Institute of Technology (Kanpur) and Professor N. Mukunda at the Centre for Theoretical Studies, Indian Institute of Science (Bangalore) during his visits. He is thankful to them. A major part of the work was done when he was holding a fellowship of the Council of Scientific and Industrial Research, which is duly acknowledged.

---

*Present address: Theoretical Physics Group, Tata Institute of Fundamental Research, Bombay 400 005, India.

[1]V. Ravishankar and G. Ramachandran, Mod. Phys. Lett. A **1**, 333 (1986); Phys. Rev. C **35**, 62 (1987), to which all equations in the text refer.

[2]Gary R. Goldstein and Michael J. Moravcsik, Phys. Rev. Lett. **53**, 1885 (1984); for references on the optimal formalism, see *Nuclear Instruments and Methods in Physics Research* (North-Holland, Amsterdam, 1984), Vol. 227, pp. 108-114.

[3]In the optimal formalism (Ref. 2) a set of observables is said to be complete if only discrete ambiguities survive (in the determination of reaction amplitudes) and fully complete if no ambiguity survives. Our complete sets are fully complete in this sense.

[4]G. Ramachandran and V. Ravishankar, J. Phys. G **12**, L143 (1986).

[5]Michael J. Moravcsik, Phys. Rev. D **29**, 2625 (1984).

[6]K. Nam *et al.*, Phys. Rev. Lett. **52**, 2305 (1984).

[7]C. R. Schumacher and H. A. Bethe, Phys. Rev. **121**, 1534 (1961).

[8]P. La France and P. Winternitz, Phys. Rev. D **27**, 112 (1983).

[9]Michael J. Moravcsik *et al.*, Ann. Phys. (N.Y.) **171**, 205 (1986).

[10]Not any set of $3N-2$ observables is complete, however.