^{3}H and ^{3}He solutions for momentum-dependent potentials

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The configuration-space Faddeev equations for the trinucleon ground states are solved for the Paris, Nijmegen, and (coordinate-space version) Bonn potentials, both with and without a Coulomb interaction. These momentum-dependent forces are structurally identical, but have different functional forms parametrizing their radial dependence. Our triton binding energies agree with some previous results for the Paris potential and with all previous Bonn results, and our new Nijmegen solutions exhibit binding commensurate with all other "realistic" potentials except the Bonn. Other model properties are also calculated, including such observables as the rms radii, asymptotic normalization constants, and charge densities.

The first triton bound-state solutions[1] of the Faddeev equations[2] for local potentials were obtained for model central potentials which contained no tensor force component. These were soon superseded by solutions for "realistic" two-body potentials with strong tensor forces,[3] and were followed by the calculations of Kim and coworkers.[4] Approximately three years ago the first complete solutions with errors less than 20 keV ($\frac{1}{4}$%) in the binding energy were obtained for such forces.[5,6] These calculations required the use of 34 channels (all nucleon-nucleon partial waves with $j \leq 4$). Subsequently, complete solutions were also obtained for diverse combinations of two-body and three-body force models.[6–9] This underscores the recent attention that the nuclear physics community has focused on understanding the few-body bound-state problem in quantitative terms. For example, the first complete solution for the alpha particle[10] was recently generated, made possible by the resolution[11] of the "ghost state" problem in the Green's function Monte Carlo (GFMC) method.[12]

In addition to investigating the effects of three-nucleon forces,[13] the effect of the charge dependence of the nucleon-nucleon force on the triton[14] has been recently examined. If one ignores the tiny isoquartet state induced by any isospin dependence (known to be an excellent approximation) the effect of such a dependence is given[14] by the "$\frac{2}{3}$-$\frac{1}{3}$" rule. The effective force in $T=1$ channels is $\frac{2}{3}V_{\rm NN}+\frac{1}{3}V_{\rm np}$, where NN is nn (pp) for ^{3}H (^{3}He). (The prescription used in early separable potential three-body calculations is similar to this.[15]) Forces fit to the (stronger) np data alone are too strong [i.e., the potential generated in the ($T=1$) channel is too attractive]; conversely, forces fit to the pp data alone are too weak. Each $\frac{1}{3}$ in the relationship above accounts for roughly 100 keV in the triton binding energy.[14,16] Thus, using an np ($T=1$) force could lead to roughly 200 keV too much binding compared to using the appropriate combination of ($T=1$) forces, while using a pp ($T=1$) force would lead to roughly 100 keV too little binding. This prediction has been confirmed by Ref. 17.

Most of the potential models which have been used to generate solutions for the triton can be summarized as having binding energies of $E_B=7.5\pm0.2$ MeV. The lowest binding corresponds to the Reid soft-core model,[18] where $E_B=7.36$ MeV, while the highest binding in this group corresponds to the Argonne V_{14} model,[19] where $E_B=7.68$ MeV. The former is a pp ($T=1$) potential, while the latter is an np ($T=1$) potential; thus, when allowance is made for the $\frac{2}{3}$-$\frac{1}{3}$ rule discussed above, they predict very similar triton binding energies. Folklore for years has attributed special properties to nonlocal potentials, which decouple the D-state probability in the deuteron from D-state observables such as the quadrupole moment and the asymptotic D/S ratio.[16] It has long been known that special relativity produces essential momentum dependence in the potential, and to order $(v/c)^2$ there must be $\mathbf{p}^2$ terms ($\mathbf{p}$ is the nucleon-nucleon relative momentum) and other momentum-dependent terms.[20]

Contemporary potentials of this type include the Paris,[21] Nijmegen,[22] and (coordinate-space version) Bonn[23] potentials. All of these models contain a single momentum-dependent component of the form $\{\mathbf{p}^2,\phi(r)\}/M$, in addition to more conventional terms, where M is the nucleon mass and ϕ is a spin- and isospin-dependent (central) radial function. In the case of the latter two potentials such terms arise from relativity, while in the case of the Paris potential the origin is more obscure. We will subsequently see that the force component of this form is much stronger for the Paris than for the other two potentials.

The value of the triton binding energy for the Paris potential is somewhat controversial (see Table I). A hybrid momentum-space, configuration-space calculation[24,6] produced a binding energy of 7.64 MeV for 34 channels, 7.56 MeV for 18 channels ($j \leq 2$), and 7.48 MeV for 5 channels (positive parity $j \leq 1$); the momentum space calculation of Hajduk and Sauer[25] produced $E_B=7.38$ MeV for 18 channels, while the 5-channel approximation yield-

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TABLE I. Summary of triton binding energy results (E_F in MeV) for the Paris potential model for n_c Faddeev channels.

Ref.	n_c	E_F
Sendai[a]	34	7.64
	18	7.56
	5	7.48
Hannover[b]	18	7.38
	5	7.30
Osaka[c]	5	7.31
Bochum[d]	18	7.33
Grenoble[e]	5	6.83
Graz[f]	5	7.05
Los Alamos-Iowa	34	7.47
	18	7.39
	5	7.31

[a]References 6 and 24.
[b]Reference 25.
[c]Reference 26.
[d]Reference 27.
[e]Reference 28.
[f]Reference 29; PEST1-6 with tensor.

ed 7.30 MeV. Our own results for this potential are 7.47 MeV for 34 channels, 7.39 MeV for 18 channels, and 7.31 MeV for 5 channels, confirming those of Ref. 25. Our results also agree well with Refs. 26 and 27, but less well with Refs. 28 and 29. [We note that this potential contains a pp ($T=1$) force.]

More recently, Sasakawa[30] found a much enhanced triton binding energy for the Bonn potential. This potential in its detailed publication[23] comes in three different versions: (1) the full potential (which is energy dependent), (2) a momentum-space, energy-independent potential, and (3) a coordinate space (CS), energy-independent potential. The latter two forms are approximate parametrizations of the first. Sasakawa's complete (34-channel) triton result was $E_B=8.33$ MeV, for an unpublished version of the CS potential. Subsequent calculations using the momentum-space version of the potential have agreed quite closely with that large value of the binding.[31,17] We find 8.32 MeV for the Sasakawa version and 8.29 MeV for the published version of the Bonn CS force. [This potential has an (np) $T=1$ force component.]

A mechanism for achieving a large triton binding energy is quite simple and has been well known for at least two decades: a reduction in the strength of the tensor force. Models with realistic low-energy S-wave scattering parameters and no tensor force produce an overbound triton.[1,4] The parametrized Bonn potentials have been criticized recently[32,33] since they do not reproduce the proper energy dependence of the $^3S_1$-$^3D_1$ mixing parameter ϵ_1, and this could account for the large triton binding energy of these models.

In addition to the Paris and Bonn (CS) potentials, the Nijmegen potential model[22] is momentum dependent and has heretofore not been used to generate a solution for the triton. We find that $E_B=7.63$ MeV for this model with a pp ($T=1$) force. [We use twice the np-reduced mass for the nucleon mass in various pieces of the force.] This result agrees with the lower range of triton binding energies.

The Paris potential proved to be (by far) the most difficult potential model that we have investigated. It has an almost pathological behavior at small nucleon-nucleon separations. The $\mathbf{p}^2$-term in the central force is exceptionally strong and accounts for about half of the potential energy. This strong momentum dependence means that the wave function and its first two derivatives must be modeled very accurately. Consequently, the mesh in configuration space must be finely spaced for small values of x (pair separation) even for large values of y (spectator coordinate). Our final results were checked by using the wave functions from the Faddeev solution in the Raleigh-Ritz variational procedure.[34] The other two models were much less sensitive. The fraction of the total potential energy generated by the $\mathbf{p}^2$ term was 49% for the Paris potential, 18% for the Bonn potential, and 11% for the Nijmegen potential.

By working in configuration space, we can easily include a Coulomb force[35] to model ^{3}He as well as ^{3}H. In addition, we have calculated various model properties and physical observables. Some observables are indicated in Table II, together with the Faddeev eigenvalues (E_F) and the variational upper bound for the channel Hamiltonian ($\langle H \rangle$):[36] the point-nucleon rms charge radii for ^{3}He and ^{3}H, the point-nucleon (E_C^0) and dipole-modified (E_C^D) Coulomb energies, the S-wave asymptotic normalization constant (C_S), and asymptotic D/S ratio (η). The D-state percentage for each model is also given. The subscript "C" on the number of channels indicates that a (point-nucleon) Coulomb interaction has been added to the strong interaction. In addition, a prime for the Bonn potential indicates the model uses Sasakawa's unpublished version of the Bonn potential. The relatively large (10 keV) difference between $\langle H \rangle$ and E_F for the Paris case is a reflection of our difficulty in obtaining an accurate solution for this model.

These results are much more meaningful when placed within the context of previous model calculations. The calculated observables for a specific triton model depend on the binding energy for that model. This has been amply demonstrated by the "scaling" analysis of the momentum-independent potential model calculations performed by the Los Alamos-Iowa group.[35,37] By plotting (and fitting) the calculated values of observables versus the model triton (or ^{3}He, in the Coulomb case) binding energies, one can predict values of these observables at the physical trinucleon binding energies.

The first quantity we consider is the asymptotic S-wave normalization constant C_S, which agrees well with the fits for all the models.[38] The asymptotic D/S ratio for the Nijmegen and Bonn models, contrariwise, are somewhat lower than the best fit to all models; the fit agrees rather well for the Paris model. In addition, the rms charge radii of ^{3}H and ^{3}He agree well with the fits (both with and without a Coulomb interaction), and the fits pass through the experimental data.[35]

TABLE II. Model results for ^{3}H and ^{3}He. The number of Faddeev channels is n_c, E_F and $\langle H \rangle$ are the Faddeev eigenvalues and variational upper bound in MeV, $\langle r^2 \rangle^{1/2}$ is the point-nucleon rms charge radius in fm, E_C^0 and E_C^D are the point-nucleon and dipole-modified Coulomb energies of ^{3}He calculated in perturbation theory, P_D is the D-state percentage, C_S is the S-wave asymptotic normalization constant, and η is the ratio of D-wave to S-wave asymptotic constants.

Model	n_c	E_F	$\langle H \rangle$	$\langle r^2 \rangle^{1/2}_{^3\mathrm{H}}$	$\langle r^2 \rangle^{1/2}_{^3\mathrm{He}}$	E_C^0	E_C^D	P_D	C_S	η
Paris	5	7.308	7.302	1.68	1.87	648	617	8.43	1.798	0.0381
	9	7.416	7.408	1.67	1.86	651	620	8.46	1.809	0.0387
	18	7.388	7.378	1.68	1.86	651	621	8.41	1.797	0.0382
	34	7.467	7.457	1.67	1.85	654	623	8.46	1.802	0.0389
	34_C	6.817	6.809	1.70	1.89	643	614	8.43	1.812	0.0367
Nijm	5	7.489	7.485	1.67	1.85	657	624	7.87	1.810	0.0384
	9	7.597	7.594	1.67	1.83	660	628	7.91	1.821	0.0390
	18	7.541	7.537	1.68	1.84	659	627	7.85	1.809	0.0384
	34	7.625	7.620	1.67	1.83	662	629	7.89	1.814	0.0391
	34_C	6.967	6.964	1.69	1.87	651	620	7.87	1.824	0.0369
Bonn	5	8.228	8.226	1.59	1.74	696	657	7.02	1.828	0.0422
	9	8.317	8.317	1.59	1.74	699	659	7.05	1.839	0.0427
	18	8.236	8.234	1.60	1.74	698	658	7.01	1.823	0.0424
	34	8.288	8.286	1.60	1.74	699	660	7.03	1.825	0.0429
	34′	8.319	8.317	1.59	1.74	697	659	6.80	1.829	0.0422
	34_C	7.592	7.592	1.62	1.77	689	651	7.02	1.836	0.0407

Next we consider the Coulomb energy. It has long been known that this energy can be predicted accurately[37] (at the 1% level) by the hyperspherical formula.[39,40] This approximate relationship between the trinucleon charge densities (or form factors) and the Coulomb energy allows one to use the experimental values of the form factors to predict the Coulomb energy. This procedure yields a value of approximately 640 keV. By calculating both E_C^D and its hyperspherical approximation for the three momentum-dependent potentials, we find the same (1%) discrepancy as was found for potential models with no $\mathbf{p}^2$ dependence. The first order (in perturbation theory) Coulomb energies for the Paris and Nijmegen models agree well with all other models, while the Bonn result is approximately 10–12 keV higher than the best fit. The second-order Coulomb energy is also higher for the last model. This immediately suggests that the ^{3}He charge density of the Bonn model (and consequently the charge form factor) is different from the others.

Another quantity we calculate is the D-state percentage, which is not an observable. The ratios of the triton D-state percentage to that of the deuteron are 1.47 for the Paris model and 1.46 for the Nijmegen and Bonn models, which agree well with previous results.[37] Counting $T=0$ pairs leads to the naive estimate of 1.5. (The ratio for the D-state percentage reported in Ref. 17 for the momentum-space Bonn model is only 1.40, which might be related to the differences between our asymptotic normalization results.)

The charge densities for the Reid soft-core model and the three momentum-dependent models are shown in Fig. 1, together with the quasiexperimental data (i.e., the Fourier transform of the point Coulomb form factor). The "hole" in the latter generates a lower (hyperspherical) Coulomb energy than any of the potential models extrapolated to the physical triton binding energy.[35,37] The obviously larger overall size of the Bonn result inside 1 fm is due entirely to the larger binding energy and the correspondingly compressed size. The negative slope inside 1 fm is greater than that of the other models, in disagreement with the data. In momentum space this more negative slope results in a smaller secondary maximum in the form factor; the experimental data require just the opposite. One finds that the diffraction minimum of the point-nucleon ^{3}He form factor occurs at $q^2=(14.3,14.4,16.5)$ fm^{-2} for the (Paris, Nijmegen, Bonn) potentials with no Coulomb interaction, while the

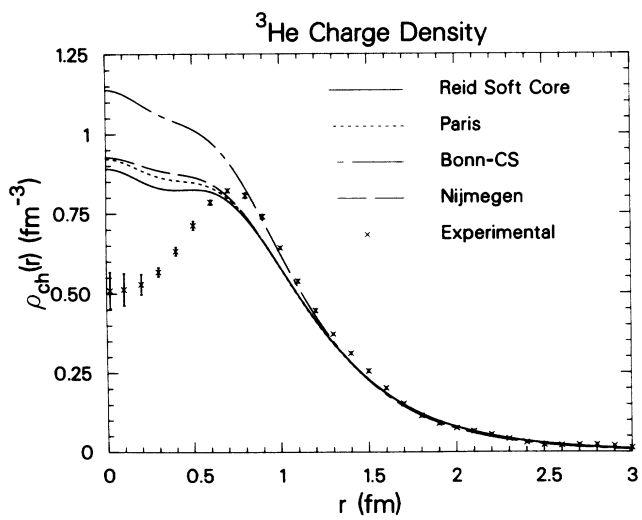

FIG. 1. ^{3}He point-nucleon charge densities for the RSC, Paris, Nijmegen, and Bonn potential models, together with the quasiexperimental data. The calculations were performed without a p-p Coulomb interaction, which changes $\rho(r)$ insignificantly.

corresponding values of the secondary maximum are $(-6.5,-6.1,-5.3)\times10^{-3}$ at $q^2=(21.5,21.5,24.4)$ fm^{-2}. Including the Coulomb interaction lowers the q^2 value where the minimum occurs by roughly 0.1 fm^{-2}, and the absolute value of the form factor value at the secondary maximum by 0.2×10^{-3}. The smaller (negative) values of the Bonn form factor in the region of the secondary maximum account for that model's larger Coulomb energy; the Bonn form factor does not agree as well with experiment as the others. (Although the presence of meson exchange currents in uncertain amounts[20,41] can alter this argument somewhat, they also will contribute to the Coulomb energy.[42]) The charge densities are little affected by the Coulomb interaction. The shapes remain the same, while $\rho(0)$ is lowered by 3%. The form factors of ^{3}H have a diffraction minimum at $q^2=(15.9,15.7,17.9)$ fm^{-2} for the same models and have a secondary maximum of $(-6.7,-6.4,-5.6)\times10^{-3}$.

In summary, we have solved the configuration-space Faddeev equations for the eigenvalues and wave functions of ^{3}H and ^{3}He corresponding to the Paris, Nijmegen, and Bonn potential models. Using these wave functions we have calculated the rms charge radii, the asymptotic normalization constants, the Coulomb energies, and the charge densities. Our triton results agree quite well with those of the Sendai group,[30] although our Paris model binding energy is about 200 keV smaller. Adding the proton-proton Coulomb interaction to the Hamiltonian to calculate ^{3}He yields results that are in line with expectations based upon our earlier work. Most of the calculated ^{3}H and ^{3}He observables agree well with our previous fits to calculations for potential models having no $\mathbf{p}^2$ term, although the Bonn results for η and E_C are slightly anomalous.

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