Extended thermal random phase approximation equation for nuclear collective excitation at finite temperature

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To take systematic account of thermal correlation effect on the mean field, the thermal random phase approximation formalism is extended based on the thermofield dynamics scheme. The random phase approximation phonon operator is expressed in terms of the thermofield dynamics quasiparticle operators with a tilde as well as those without a tilde, and the increase of the dimensionality of variational parameter manifold enables us to derive the extended thermal random phase approximation equation from deeper minimum of grand potential. It is shown that the extended thermal random phase approximation equation is justified also from viewpoint of time-dependent formalism. The method of constrained variation applied to the first extended thermal random phase approximation can be straightforwardly generalized to the second, or higher extended thermal random phase approximation. Extended expressions are given also for the stability condition of thermal Hartree-Fock-Bogoliubov approximation and the energy-weighted sum rule.

I. INTRODUCTION

Accumulated experimental data of giant resonances (GR),[1–3] as universal nuclear phenomena studied for a long time, have offered excellent tests of the microscopic theory describing nuclear excitation mechanism.[4,5] It has turned out that various types of GR and their damping mechanism are essentially well understood in terms of particle-hole excitations from nuclear ground states, which can be, in most cases, successfully described within the framework of random phase approximation (RPA) at zero temperature.[4–8] Such studies are, however, restricted to collective excitations of "cold" nuclei carrying only small angular momentum and little thermal energy.

Recent experiments have been successful to separate γ rays associated with decays of the giant dipole resonances (GDR)[9–21] and the giant quadrupole resonances (GQR)[19,20] build on excited states of high energy and high spin from the continuum spectra of high-energy γ rays following heavy-ion fusion reactions for various combinations of target and projectile nuclei at various bombarding energies. The experiments have found several interesting features of excited nuclear system such as downward shift of centroid energy and broadening of the GDR spectrum with increasing temperature T (or excitation energy)[10] and angular momentum I,[12,14] splitting of resonance peak which gives information on deformation,[9,10,17,19,20] their mass number dependence,[10,17,19,20] and neutron emission rate.[22] Those shed light on the nuclear structure in the new domain far above the yrast line, and provide further tests of our microscopic theory.

Experimental evidences have suggested that a nuclear system is thermally equilibrated rapidly after formation of a compound nucleus from deep-inelastic nuclear collision.[9,10,20,21] Because of a large nuclear state density, individual highly-excited states are equally populated in such processes, and usually average properties of the system are measured. Therefore, statistical extension of the RPA, which will be called the thermal RPA (TRPA)[23–26] also in the present paper, must be appropriate to be applied to collective states such as GR at finite temperature. In fact, a number of calculations have been performed essentially based on the TRPA,[25–32] and their results gave partial explanations for the above characteristics of the GDR. Since the monopole-pairing interaction and higher multipole-pairing interactions[33] are expected to play significant roles in structure change, or phase transition under the influence of both temperature and rotation, the quasiparticle TRPA must be formulated on top of the cranked thermal Hartree-Fock-Bogoliubov (THFB) solution.[34–38] This formulation provides a method not only comprehensive, but also practicable.

The present paper attempts to establish an unambiguous relation between assumed microscopic interaction Hamiltonian and observed properties of GR. For this purpose the TRPA will be extended by means of the thermofield dynamics (TFD) formalism.[39,40] It will be shown that the extended TRPA (ETRPA) equation is capable of taking into account possible correlations caused by thermal effects in a much more systematic way. This possibility is due to the dual extension of single-particle operator space by the inclusion of tilde operators, with which an ensemble average of any operator can be written in the form of vacuum expectation value. Accordingly, the dimensionality of parameter manifold introduced through the definition of the ETRPA phonon operator becomes much larger than the TRPA, and such extended parameter space is subject to the variation in deriving the ETRPA equation. The ETRPA is a natural generalization of the RPA in a sense that any quantity given in the ETRPA goes to the one in the RPA at zero-temperature limit. Variational derivation applied to the TRPA equation[23,24] will be extensively adopted in the present paper. The variational method has advantages that (i) a set of the ETRPA equations is derived without assuming *a priori* a set of separate relations as in the equation-of-motion method,[7] and also (ii) the higher ETRPA equation can be derived on the same footing as for the simple ETRPA equation. The latter provides a formal justification of the higher RPA formalism.

After brief summary of the derivation of the TRPA equation from constrained variation in Sec. II, the method will be applied to the TFD to obtain the ETRPA in Sec. III. The ETRPA will be compared with the TRPA in Sec. IV. The second ETRPA equation will be derived, and an argument about possible modification of constraints will be given in Sec. V. Relation of the ETRPA to time-dependent formalism will be investigated, and the formal energy-weighted sum rule will be derived in Sec. VI. Implications of our results will be discussed in Sec. VII.

II. QUASIPARTICLE TRPA BUILT ON THE THFB SOLUTION

A. The THFB equation

To facilitate extension of the TRPA formalism in the following sections, the TRPA constructed in the quasiparticle picture of the THFB solution[34,35] is summarized in the present section. Our Hamiltonian contains a general pairing interaction, which, together with Coriolis force in the rotating nucleus, controls alignment of spins and nuclear shapes along and off the yrast line and is responsible for phase transitions in off-yrast region as predicted by the cranked THFB calculations.[34–38] Therefore, it is interesting to know if the pairing interaction plays an important role in determining properties of hot and rotating nuclear medium which propagates collective vibrations. Given a generic model Hamiltonian H with two-body interactions written in terms of single-nucleon operators c_k and $c_k^\dagger$ for kth single-particle level in the shell-model space, then the cranked Hamiltonian is given by

$$H' \equiv H - \omega_{\rm rot}\hat{J}_X - \lambda_p \hat{Z} - \lambda_n \hat{N} \tag{2.1a}$$

$$= U_0 + \sum_{\mu\nu} (H_{11})_{\mu\nu}\alpha_\mu^\dagger \alpha_\nu + \tfrac{1}{2}\sum_{\mu\nu} [(H_{20})_{\mu\nu}\alpha_\mu^\dagger\alpha_\nu^\dagger + \text{H.c.}] + \sum_{\mu\nu\rho\sigma}(H_{22})_{\mu\nu\rho\sigma}\alpha_\mu^\dagger\alpha_\nu^\dagger\alpha_\sigma\alpha_\rho + \sum_{\mu\nu\rho\sigma}[(H_{31})_{\mu\nu\rho\sigma}\alpha_\mu^\dagger\alpha_\nu^\dagger\alpha_\rho^\dagger\alpha_\sigma + \text{H.c.}] + \sum_{\mu\nu\rho\sigma}[(H_{40})_{\mu\nu\rho\sigma}\alpha_\mu^\dagger\alpha_\nu^\dagger\alpha_\rho^\dagger\alpha_\sigma^\dagger + \text{H.c.}] , \tag{2.1b}$$

where $\hat{J}_X$ is the X component of angular momentum operator and $\hat{Z}$ ($\hat{N}$) the proton (neutron) number operator. The Lagrange multipliers $\omega_{\rm rot}$, λ_p, and λ_n are to be determined from the requirements for ensemble averages,

$$\langle \hat{J}_X \rangle = I, \quad \langle \hat{Z} \rangle = Z - Z_0, \quad \langle \hat{N} \rangle = N - N_0 , \tag{2.2}$$

where Z_0 (N_0) is the number of protons (neutrons) in the core, which is irrelevant to calculation. The quasiparticle operators α_μ and $\alpha_\mu^\dagger$ appearing in (2.1b) are related to single-particle operators through the Bogoliubov transformation

$$c_k = \sum_\mu (A_{k\mu}\alpha_\mu + B_{k\mu}^*\alpha_\mu^\dagger) , \quad c_k^\dagger = \sum_\mu (A_{k\mu}^*\alpha_\mu^\dagger + B_{k\mu}\alpha_\mu) , \tag{2.3}$$

with coefficients satisfying the unitarity relations

$$(AA^\dagger + B^*B^{\rm tr})_{kl} = \delta_{kl} , \quad (AB^\dagger + B^*A^{\rm tr})_{kl} = 0 , \tag{2.4a}$$

$$(A^\dagger A + B^\dagger B)_{\mu\nu} = \delta_{\mu\nu} , \quad (A^\dagger B + B^{\rm tr}A)_{\mu\nu} = 0 . \tag{2.4b}$$

We anticipate that the main part of H' will be diagonalized in the quasiparticle picture provided by the self-consistent solution to the THFB equation, i.e.,

$$H^{\rm eff} = U_0^{\rm eff} + \sum_\mu E_\mu \alpha_\mu^\dagger \alpha_\mu . \tag{2.5}$$

Introducing the approximate trial density matrix

$$W \equiv \frac{\exp(-\beta H^{\rm eff})}{\text{Tr}[\exp(-\beta H^{\rm eff})]} \tag{2.6}$$

with $\beta = 1/(kT)$ and the Boltzmann constant k, we have the approximate entropy

$$S' \equiv -k\,\text{Tr}(W\ln W) = -k\sum_\mu [f_\mu \ln f_\mu + (1-f_\mu)\ln(1-f_\mu)] \tag{2.7}$$

and the ensemble average for any operator O

$$\langle O \rangle \equiv \text{Tr}(WO) . \tag{2.8}$$

Application of the Bloch-de Dominicis theorem (or generalized Wick theorem) enables us to write down any ensemble average in terms of the products of quasiparticle distribution function f_μ, i.e.,

$$\langle \alpha_\mu^\dagger \alpha_\nu \rangle = f_\mu \delta_{\mu\nu}, \quad f_\mu \equiv \frac{1}{\exp(\beta E_\mu)+1} , \tag{2.9a}$$

$$\langle \alpha_\mu^\dagger \alpha_\nu^\dagger \rangle = \langle \alpha_\nu \alpha_\mu \rangle = 0 . \tag{2.9b}$$

Thus, the approximate grand potential (or thermodynamical potential) for our problem is given by

$$F \equiv \langle H' \rangle - S'T = \langle H \rangle - \omega_{\rm rot}\langle \hat{J} \rangle - \lambda_p \langle \hat{Z} \rangle - \lambda_n \langle \hat{N} \rangle - S'T . \tag{2.10}$$

On minimizing F under the constraints in (2.2) and (2.4a), we regard $A_{k\mu}$, $B_{k\mu}$, $A^*_{k\mu}$, $B^*_{k\mu}$, and E_μ as independent variational parameters to obtain

$$(H_{11}^{\rm eff})\equiv(H_{11})_{\mu\nu}+4\sum_\rho (H_{22})_{\mu\rho\nu\rho}=E_\mu\delta_{\mu\nu}\ , \qquad (2.11a)$$

$$(H_{20}^{\rm eff})_{\mu\nu}\equiv(H_{20})_{\mu\nu}+6\sum_\rho (H_{31})_{\mu\nu\rho\rho}=0\ , \qquad (2.11b)$$

$$U_0^{\rm eff}\equiv U_0-2\sum_{\mu\nu}(H_{22})_{\mu\nu\mu\nu}f_\mu f_\nu\ . \qquad (2.11c)$$

Note that (2.11a) and (2.11b) together with the orthonormality relations in (2.4b) yield a set of the THFB equation in the standard form[35] similar to the zero-temperature case (i.e., the HFB equation).

The THFB equation is to be solved under the constraints (2.2) and orthonormalities (2.4a). However, the orthogonality relations in (2.4b) are automatically satisfied between different solutions to the THFB equation. A complete set is composed of the eigenvectors $(A_{k\mu},B_{k\mu})$ belonging to eigenvalue E_μ together with $(B^*_{k\mu},A^*_{k\mu})$ to negative eigenvalue $-E_\mu$.

B. The stability condition

In the second step, the residual interaction ΔH defined by

$$H'\equiv H^{\rm eff}+\Delta H \qquad (2.12)$$

is taken into account in terms of the TRPA. In the variational derivation of the TRPA equation,[23] the grand potential is shifted by the small unitary transformation applied to the density matrix,

$$W\to\widetilde{W}=\exp(iR)W\exp(-iR),\quad R=Q^\dagger+Q\ . \qquad (2.13)$$

Then, the corresponding shift of the grand potential is given by

$$\Delta F={\rm Tr}(\widetilde{W}H')+kT\,{\rm Tr}(\widetilde{W}\ln\widetilde{W})-F \qquad (2.14a)$$

$$\simeq i\langle[H',R]\rangle+\tfrac{1}{2}\langle[R,[H',R]]\rangle\ , \qquad (2.14b)$$

where we have neglected the third and higher orders of R in the series expansion.

When the most general bilinear forms of $\alpha^\dagger_\mu$ and α_μ is assumed for $Q^\dagger$,

$$Q^\dagger=\sum_{\mu<\nu}(X_{\mu\nu}\alpha^\dagger_\mu\alpha^\dagger_\nu-Y_{\mu\nu}\alpha_\nu\alpha_\mu)+\sum_{\mu\nu}Z_{\mu\nu}\alpha^\dagger_\mu\alpha_\nu\ , \qquad (2.15)$$

the first term in (2.14b) vanishes due to the relations (2.11a) and (2.11b) because the following relations hold:

$$\langle[H',\alpha_\nu\alpha_\mu]\rangle=-(1-f_\mu-f_\mu)\times\left[(H_{20})_{\mu\nu}+6\sum_\rho(H_{31})_{\mu\nu\rho\rho}f_\rho\right]=0\ , \qquad (2.16a)$$

$$\langle[H',\alpha^\dagger_\mu\alpha_\nu]\rangle=(f_\nu-f_\mu)\times\left[(H_{11})_{\nu\mu}+4\sum_\rho(H_{22})_{\nu\rho\mu\rho}f_\rho\right]=(f_\nu-f_\mu)E_\nu\delta_{\mu\nu}=0\ . \qquad (2.16b)$$

In the higher RPA we generalize (2.15) to include higher orders of $\alpha^\dagger_\mu$ and α_μ, and the first term in (2.14b) does not vanish in general, i.e.,

$$\langle[H',R]\rangle=\langle[H',Q^\dagger]\rangle-\langle[H',Q^\dagger]\rangle^*\neq0\ . \qquad (2.17)$$

It vanishes only when $\langle[H',Q^\dagger]\rangle$ is real.

To investigate stability of our grand potential ΔF against the transformation (2.13), we multiply $Q^\dagger$ by an arbitrary small complex number $z=x+iy$ (x,y are real) in the definition of R, i.e.,

$$R=zQ^\dagger+z^*Q\ . \qquad (2.18)$$

Then, ΔF is rewritten as

$$\Delta F=\tfrac{1}{2}(Ax^2+2Bxy+Cy^2) \qquad (2.19)$$

with

$$A\equiv\left.\frac{\partial^2\Delta F}{\partial x^2}\right|_{x=y=0}=\langle[Q^\dagger+Q,[H',Q^\dagger+Q]]\rangle\ , \qquad (2.20a)$$

$$B\equiv\left.\frac{\partial^2\Delta F}{\partial x\partial y}\right|_{x=y=0}=\langle[Q,[H',Q]]-[Q^\dagger,[H',Q^\dagger]]\rangle\ , \qquad (2.20b)$$

$$C\equiv\left.\frac{\partial^2\Delta F}{\partial y^2}\right|_{x=y=0}=\langle[i(Q^\dagger-Q),[H',i(Q^\dagger-Q)]]\rangle\ . \qquad (2.20c)$$

Thus, when A and C are positive, ΔF has a minimum at $x=y=0$ provided that

$$B^2-AC=4|\langle[Q^\dagger,[H',Q^\dagger]]\rangle|^2-\langle[Q^\dagger,[H',Q]]+[Q,[H',Q^\dagger]]\rangle|^2<0\ . \qquad (2.21)$$

This inequality holds sufficiently if

$$\langle[Q^\dagger,[H',Q^\dagger]]\rangle=\langle[Q,[H',Q]]\rangle=0\ . \qquad (2.22)$$

Therefore, relations in (2.22) may be regarded as the constraints for the stability of TRPA oscillation about the THFB solution, and the TRPA solution violating such conditions becomes unstable. When the condition (2.22) holds, positive-definiteness of A and C can be expressed as

$$\langle[Q,[H',Q^\dagger]]+[Q^\dagger,[H',Q]]\rangle>0\ . \qquad (2.23)$$

C. The TRPA equation

The normalization condition imposed on the TRPA eigenvector

$$\mathbf{X}_{\mu\nu}\equiv\begin{bmatrix}X\\Y\\Z\end{bmatrix}_{\mu\nu} \qquad (2.24)$$

is given by

$$\langle [Q,Q^\dagger]\rangle = \mathbf{X}^\dagger \mathbf{M}\mathbf{X} = 1 \tag{2.25}$$

with the TRPA metric matrix $\mathbf{M}$ taking diagonal form

$$\mathbf{M}_{\mu\nu,\mu\nu} = \begin{bmatrix} 1-f_\mu-f_\nu & 0 & 0 \\ 0 & -(1-f_\mu-f_\nu) & 0 \\ 0 & 0 & f_\nu-f_\mu \end{bmatrix} . \tag{2.26}$$

Multiplying Lagrange multipliers into quantities in (2.22) and (2.25) and subtracting those from ΔF, we perform the constrained variation

$$\delta\left[\Delta F - \hbar\omega\langle [Q,Q^\dagger]\rangle - \frac{a}{2}\langle [Q,[H',Q]]\rangle - \frac{b}{2}\langle [Q^\dagger,[H',Q^\dagger]]\rangle\right] = 0 \tag{2.27}$$

to obtain the TRPA equation

$$\boldsymbol{\Omega}\mathbf{M}\mathbf{X}^{(n)} = \hbar\omega_n \mathbf{X}^{(n)} , \tag{2.28a}$$

$$\boldsymbol{\Omega}\mathbf{M}\mathbf{X}^{(-n)} = -\hbar\omega_n \mathbf{X}^{(-n)} \quad (\hbar\omega_{-n} = -\hbar\omega_n < 0) . \tag{2.28b}$$

At the same time we can show that (2.22) is satisfied provided that

$$a = b = 1 . \tag{2.29}$$

In (2.28), $\mathbf{X}^{(n)}$ and $\mathbf{X}^{(-n)}$ are, respectively, the nth eigensolution for positive energy $\hbar\omega_n$ and the one for negative energy $-\hbar\omega_n$, and those are, respectively, given by

$$\mathbf{X}^{(n)}_{\mu\nu} = \begin{bmatrix} X^{(n)}_{\mu\nu} \\ Y^{(n)}_{\mu\nu} \\ Z^{(n)}_{\mu\nu} \end{bmatrix} , \quad \mathbf{X}^{(-n)}_{\mu\nu} = \begin{bmatrix} -Y^{(n)*}_{\mu\nu} \\ -X^{(n)*}_{\mu\nu} \\ Z^{(n)*}_{\nu\mu} \end{bmatrix} . \tag{2.30}$$

For later convenience we cite here the matrix elements of the stability matrix $(\mathbf{M}\boldsymbol{\Omega}\mathbf{M})$ rather than the TRPA matrix $\boldsymbol{\Omega}$ $(=\boldsymbol{\Omega}^\dagger)$ appearing in (2.28):

$$(\mathbf{M}\boldsymbol{\Omega}\mathbf{M})_{\mu\nu,\rho\sigma} = (\mathbf{M}\boldsymbol{\Omega}\mathbf{M})^*_{\rho\sigma,\mu\nu} = \begin{bmatrix} A_{\mu\nu,\rho\sigma} & B_{\mu\nu,\rho\sigma} & C_{\mu\nu,\rho\sigma} \\ B^*_{\mu\nu,\rho\sigma} & A^*_{\mu\nu,\rho\sigma} & C^*_{\nu\mu,\sigma\rho} \\ C^*_{\rho\sigma,\mu\nu} & C_{\sigma\rho,\nu\mu} & D_{\mu\nu,\rho\sigma} \end{bmatrix} \tag{2.31}$$

with

$$A_{\mu\nu,\rho\sigma} = (E_\mu+E_\nu)(1-f_\mu-f_\nu)(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}) + 4(1-f_\mu-f_\nu)(H_{22})_{\mu\nu\rho\sigma}(1-f_\rho-f_\sigma) , \tag{2.32a}$$

$$B_{\mu\nu,\rho\sigma} = 24(1-f_\mu-f_\nu)(H_{40})_{\mu\nu\rho\sigma}(1-f_\rho-f_\sigma) , \tag{2.32b}$$

$$C_{\mu\nu,\rho\sigma} = 6(1-f_\mu-f_\nu)(H_{31})_{\mu\nu\sigma\rho}(f_\sigma-f_\rho) , \tag{2.32c}$$

$$D_{\mu\nu,\rho\sigma} = (E_\mu-E_\nu)(f_\nu-f_\mu)\delta_{\mu\rho}\delta_{\nu\sigma} + 4(f_\nu-f_\mu)(H_{22})_{\mu\sigma\nu\rho}(f_\sigma-f_\rho) . \tag{2.32d}$$

The orthonormality relation

$$\langle [Q_m,Q_n^\dagger]\rangle = \mathbf{X}^{(m)\dagger}\mathbf{M}\mathbf{X}^{(n)} = \delta_{mn} \tag{2.33}$$

is proved directly from (2.28)

Due to the stability condition (2.22) and the normalization condition (2.34), the excitation energy of the system for the nth positive energy eigensolution is given by the simple expression,

$$\Delta F = \tfrac{1}{2}\langle [Q_n^\dagger+Q_n,[H',Q_n^\dagger+Q_n]]\rangle = \hbar\omega_n . \tag{2.34}$$

The completeness condition

$$\sum_{n>0} [(\mathbf{X}^{(n)}\mathbf{X}^{(n)\dagger} - \mathbf{X}^{(-n)}\mathbf{X}^{(-n)\dagger})\mathbf{M}]_{\mu\nu,\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} \tag{2.35}$$

guarantees that the whole operator space composed of bilinear forms in quasiparticle operators can be spanned by the TRPA operators $[Q_n^\dagger, Q_n(=Q_{-n}^\dagger)]$ which correspond to the eigensolutions of the TRPA equation. Therefore, any one-body operator F can be expanded in terms of $Q_n^\dagger$ and Q_n, i.e.,

$$F = \sum_{n>0} (\langle [Q_n,F]\rangle Q_n + \langle [F,Q_n^\dagger]\rangle Q_n) , \tag{2.36}$$

so that we obtain the thermal energy-weighted sum rule in the form

$$\tfrac{1}{2}\langle [F,[H',F]]\rangle = \sum_{n>0} \hbar\omega_n \,|\langle [Q_n^\dagger,F]\rangle|^2 . \tag{2.37}$$

Every formula in the above continuously approaches the ordinary RPA formulas in the zero-temperature limit (i.e., $f_\mu \to 0$).

III. DERIVATION OF THE EXTENDED TRPA EQUATION IN THE TFD FORMALISM

A. The TFD formalism

From the viewpoint of the RPA at finite temperature the thermofield dynamics (TFD) formalism proposed by Takahashi and Umezawa[39,40] has several attractive features. In the TFD, (i) a state vector called the temperature-dependent "vacuum" is introduced to describe an unperturbed thermal equilibrium "state," and (ii) with this vacuum an ensemble average of any operator is expressed in the form of vacuum expectation value. (iii) Since such a scheme is attained by introducing single-particle operators with a tilde in addition to those without a tilde, this enlarged space provides us with the new possibility of supplying more variational parameters than in the variational derivation of the TRPA equation in the previous section. Furthermore, (iv) since the formalism has nothing to do with trace calculation and is rather parallel with the field theory without temperature, there is in principle no difficulty in constructing time-dependent formalism in the framework of TFD.

Given a set of fermion quasiparticle operators $\alpha_\mu^\dagger$ and α_μ, which will be identified with the quasiparticles in the THFB solution throughout the present paper and quasivacuum $|0\rangle$, i.e.,

$$\{\alpha_\mu,\alpha_\nu^\dagger\}=\delta_{\mu\nu},\quad \{\alpha_\mu,\alpha_\nu\}=\{\alpha_\mu^\dagger,\alpha_\nu^\dagger\}=0\ , \tag{3.1}$$

$$\alpha_\mu\,|\,0\rangle=0\ . \tag{3.2}$$

In the TFD scheme, the operator space is extended by including another set of fermion operators $\tilde{\alpha}_\mu^\dagger,\tilde{\alpha}_\mu$ and corresponding vacuum $|\,\tilde{0}\rangle$ characterized by

$$\{\tilde{\alpha}_\mu,\tilde{\alpha}_\nu^\dagger\}=\delta_{\mu\nu},\quad \{\tilde{\alpha}_\mu,\tilde{\alpha}_\nu\}=\{\tilde{\alpha}_\mu^\dagger,\tilde{\alpha}_\nu^\dagger\}=0\ , \tag{3.3}$$

$$\{\tilde{\alpha}_\mu,\alpha_\nu^\dagger\}=\{\alpha_\mu,\tilde{\alpha}_\nu^\dagger\}=\{\tilde{\alpha}_\mu,\alpha_\nu\}=\{\alpha_\mu^\dagger,\tilde{\alpha}_\nu^\dagger\}=0\ , \tag{3.4}$$

$$\tilde{\alpha}_\mu\,|\,\tilde{0}\rangle=0\ . \tag{3.5}$$

Imaging a direct product of two equivalent operator spaces, we extend meaning of notations without changing any appearance of formulas from (3.1) to (3.4) simply by the following replacements:

$$\alpha\otimes 1\to\alpha,\quad \alpha^\dagger\otimes 1\to\alpha^\dagger\ , \tag{3.6}$$

$$1\otimes\tilde{\alpha}\to\tilde{\alpha},\quad 1\otimes\tilde{\alpha}^\dagger\to\tilde{\alpha}^\dagger\ , \tag{3.7}$$

$$|\,0\rangle\otimes|\,\tilde{0}\rangle\to|\,0\rangle\ . \tag{3.8}$$

Then, we disregard (3.5) and take

$$\tilde{\alpha}_\mu\,|\,0\rangle=0 \tag{3.9}$$

in addition to (3.2)

It is easily shown that the statistical ensemble average of an operator 0, which is written only with $\alpha_\mu^\dagger$ and α_μ, can be expressed as the vacuum expectation value,

$$\langle 0\rangle=\langle 0(\beta)\,|\,0\otimes 1\,|\,0(\beta)\rangle\ , \tag{3.10}$$

if the temperature-dependent vacuum $|\,0(\beta)\rangle$ is defined by

$$|\,0(\beta)\rangle=\prod_\mu\frac{\left[\sum_{n=0,1}\exp(-\beta nE_\mu/2)(\alpha_\mu^\dagger\tilde{\alpha}_\mu^\dagger)^n\right]|\,0\rangle}{[1+\exp(\beta E_\mu)]^{1/2}}$$

$$\equiv\exp(-G)\,|\,0\rangle\ . \tag{3.11}$$

In the above and hereafter we denote

$$G\equiv i\sum_\mu\theta_\mu(\alpha_\mu^\dagger\tilde{\alpha}_\mu^\dagger-\tilde{\alpha}_\mu\alpha_\mu)\ , \tag{3.12}$$

$$\sin\theta_\mu=f_\mu^{1/2}\equiv g_\mu,\quad \cos\theta_\mu=\bar{f}_\mu^{1/2}\equiv\bar{g}_\mu\ , \tag{3.13}$$

with

$$\bar{f}_\mu\equiv 1-f_\mu\ . \tag{3.14}$$

By the unitary transformation defined by (3.11), new quasiparticle operators $\beta_\mu^\dagger(\tilde{\beta}_\mu^\dagger)$ and $\beta_\mu(\tilde{\beta}_\mu)$ are related to $\alpha_\mu^\dagger(\tilde{\alpha}_\mu^\dagger)$ and $\alpha_\mu(\tilde{\alpha}_\mu)$:

$$\beta_\mu\equiv\exp(-iG)\alpha_\mu\exp(iG)$$

$$=\bar{g}_\mu\alpha_\mu-g_\mu\tilde{\alpha}_\mu^\dagger\ , \tag{3.15a}$$

$$\tilde{\beta}_\mu\equiv\exp(-iG)\tilde{\alpha}_\mu\exp(iG)$$

$$=\bar{g}_\mu\tilde{\alpha}_\mu+g_\mu\alpha_\mu^\dagger\ , \tag{3.15b}$$

or inversely

$$\alpha_\mu=\bar{g}_\mu\beta_\mu+g_\mu\tilde{\beta}_\mu^\dagger\ , \tag{3.16a}$$

$$\tilde{\alpha}_\mu=\bar{g}_\mu\tilde{\beta}_\mu-g_\mu\beta_\mu^\dagger\ . \tag{3.16b}$$

Hence, we have the same commutation relations also for $\beta_\mu^\dagger(\tilde{\beta}_\mu^\dagger)$ and $\beta_\mu(\tilde{\beta}_\mu)$ as for $\alpha_\mu^\dagger(\tilde{\alpha}_\mu^\dagger)$ and $\alpha_\mu(\tilde{\alpha}_\mu)$, and

$$\beta_\mu\,|\,0(\beta)\rangle=\tilde{\beta}_\mu\,|\,0(\beta)\rangle=0\ . \tag{3.17}$$

Notice that it follows from (3.15) and (3.16) that

$$\tilde{\tilde{\alpha}}_\mu=-\alpha_\mu,\quad \tilde{\tilde{\beta}}_\mu=-\beta_\mu\ . \tag{3.18}$$

B. The ETRPA equation

For later convenience we employ the following notations for quasiparticle operators:

$$\alpha_{1\mu}\equiv\alpha_\mu,\quad \alpha_{2\mu}\equiv\tilde{\alpha}_\mu\ , \tag{3.19a}$$

$$\beta_{1\mu}\equiv\beta_\mu,\quad \beta_{2\mu}\equiv\tilde{\beta}_\mu\ . \tag{3.19b}$$

Then, applying transformations in (3.16) to the Hamiltonian (2.1b), we get

$$H'=h_0+\sum_{i,j=1}^{2}\sum_{\mu\nu}\{(h_{11})_{i\mu,j\nu}\beta_{i\mu}^\dagger\beta_{j\nu}+\tfrac{1}{2}[(h_{20})_{i\mu j\nu}\beta_{i\mu}^\dagger\beta_{j\nu}^\dagger+\text{H.c.}]\}$$

$$+\sum_{ijkl=1}^{2}\sum_{\mu\nu\rho\sigma}\{(h_{22})_{i\mu j\nu,k\rho l\sigma}\beta_{i\mu}^\dagger\beta_{j\nu}^\dagger\beta_{l\sigma}\beta_{k\rho}+[(h_{31})_{i\mu j\nu k\rho,l\sigma}\beta_{i\mu}^\dagger\beta_{j\nu}^\dagger\beta_{k\rho}^\dagger\beta_{l\sigma}+\text{H.c.}]$$

$$+[(h_{40})_{i\mu j\nu k\rho l\sigma}\beta_{i\mu}^\dagger\beta_{j\nu}^\dagger\beta_{k\rho}^\dagger\beta_{l\sigma}^\dagger+\text{H.c.}]\}\ , \tag{3.20}$$

where

$$h_0\equiv U_0^{\text{eff}}+\sum_\mu E_\mu f_\mu \tag{3.21a}$$

$$(h_{11})_{i\mu,j\nu}\equiv E_\mu(\bar{f}_\mu\delta_{i1}\delta_{j2}-f_\mu\delta_{i2}\delta_{j1})\delta_{\mu\nu}\ , \tag{3.21b}$$

$$\tfrac{1}{2}(h_{20})_{i\mu j\nu}\equiv E_\mu\bar{g}_\mu g_\mu\delta_{i1}\delta_{j2}\delta_{\mu\nu}\ , \tag{3.21c}$$

$$
\begin{aligned}
(h_{22})_{i\mu j\nu,k\rho l\sigma} \equiv\; & (H_{22})_{\mu\nu\rho\sigma}(\bar{g}_\mu \bar{g}_\nu g_\rho \bar{g}_\sigma \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + g_\mu g_\nu g_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2}) + 4(H_{22})_{\mu\sigma\rho\nu}\bar{g}_\mu g_\nu \bar{g}_\rho g_\sigma \delta_{i1}\delta_{j2}\delta_{k1}\delta_{l2} \\
& + 3[(H_{31})_{\mu\nu\sigma\rho}\bar{g}_\mu \bar{g}_\mu \bar{g}_\rho g_\sigma \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l2} - (H_{31})_{\rho\sigma\mu\nu}\bar{g}_\mu g_\nu g_\rho g_\sigma \delta_{i1}\delta_{j2}\delta_{k2}\delta_{l2} \\
& \quad - (H_{31})^*_{\mu\nu\rho\sigma} g_\mu g_\nu \bar{g}_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l2} + (H_{31})^*_{\rho\sigma\nu\mu}\bar{g}_\mu g_\nu \bar{g}_\rho \bar{g}_\sigma \delta_{i1}\delta_{j2}\delta_{k1}\delta_{l1}] \\
& \quad - 6[(H_{40})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu g_\rho g_\sigma \delta_{i1}\delta_{j1}\delta_{k2}\delta_{l2} + (H_{40})^*_{\mu\nu\rho\sigma} g_\mu g_\nu \bar{g}_\rho \bar{g}_\sigma \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l1}] \;, \qquad (3.21\text{d})
\end{aligned}
$$

$$
\begin{aligned}
(h_{31})_{i\mu j\nu k\rho,l\sigma} \equiv\; & 2[(H_{22})_{\mu\rho\nu\sigma}\bar{g}_\mu g_\nu \bar{g}_\rho \bar{g}_\delta \delta_{i1}\delta_{j2}\delta_{k1}\delta_{l1} - (H_{22})^*_{\mu\nu\rho\sigma} g_\mu g_\nu \bar{g}_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l2}] \\
& + (H_{31})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu \delta_{i1}\delta_{j1}(\bar{g}_\rho \bar{g}_\sigma \delta_{k1}\delta_{l1} + 3 g_\rho g_\sigma \delta_{k2}\delta_{l2}) + (H_{31})^*_{\mu\nu\rho\sigma} g_\mu g_\nu \delta_{i2}\delta_{j2}(g_\rho g_\sigma \delta_{k2}\delta_{l2} + 3\bar{g}_\rho \bar{g}_\sigma \delta_{k1}\delta_{l1}) \\
& + 4[(H_{40})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu \bar{g}_\rho \bar{g}_\sigma \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + (H_{40})^*_{\mu\nu\rho\sigma} g_\mu g_\nu g_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2}] \;, \qquad (3.21\text{e})
\end{aligned}
$$

$$
\begin{aligned}
(h_{40})_{i\mu j\nu,k\rho l\sigma} \equiv\; & -(H_{22})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu g_\rho g_\sigma \delta_{i1}\delta_{j1}\delta_{k2}\delta_{l2} + (H_{31})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu \bar{g}_\rho g_\sigma \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + (H_{31})^*_{\mu\nu\rho\sigma} g_\mu g_\nu \bar{g}_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k1}\delta_{l2} \\
& + (H_{40})_{\mu\nu\rho\sigma}\bar{g}_\mu \bar{g}_\nu \bar{g}_\rho \bar{g}_\sigma \delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + (H_{40})^*_{\mu\nu\rho\sigma} g_\mu g_\nu g_\rho g_\sigma \delta_{i2}\delta_{j2}\delta_{k2}\delta_{l2} \;. \qquad (3.21\text{f})
\end{aligned}
$$

To construct RPA phonon states excited on the temperature-dependent vacuum state $|0(\beta)\rangle$, we apply small unitary transformation to the vacuum, i.e.,

$$|0(\beta)\rangle \rightarrow |\tilde{0}(\beta)\rangle = \exp(iR)|0(\beta)\rangle, \quad R \equiv Q^\dagger + Q \;, \qquad (3.22)$$

and analogously to (2.14b) we have a corresponding shift of grand potential,

$$\Delta F \cong i\langle 0(\beta)|[H',R]|0(\beta)\rangle + \tfrac{1}{2}\langle 0(\beta)|[R,[H',R]]|0(\beta)\rangle \;. \qquad (3.23)$$

For the first RPA, the most general bilinear form composed of $\beta^\dagger_{i\mu}$ and $\beta_{i\mu}$ can be assumed for $Q^\dagger$, but the terms like $\beta^\dagger_{i\mu}\beta_{j\nu}$ do not contribute to ΔF. Therefore, we put

$$
\begin{aligned}
Q^\dagger &= \sum_{\mu\nu} \boldsymbol{\beta}^{\text{tr}}_\mu \mathbf{R}_{\mu\nu}(\beta)\boldsymbol{\beta}_\nu \\
&= \sum_{\mu<\nu} (X^{(1)<}_{\mu\nu}\beta^\dagger_{1\mu}\beta^\dagger_{1\nu} + X^{(3)<}_{\mu\nu}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} - Y^{(1)<}_{\mu\nu}\beta_{1\nu}\beta_{1\mu} - Y^{(3)<}_{\mu\nu}\beta_{2\nu}\beta_{2\mu}) + \sum_{\mu\nu}(X^{(2)}_{\mu\nu}\beta^\dagger_{1\mu}\beta^\dagger_{2\nu} + Y^{(2)}_{\mu\nu}\beta_{2\nu}\beta_{1\mu}) \;, \qquad (3.24)
\end{aligned}
$$

where we have denoted

$$
\boldsymbol{\beta}_\mu \equiv \begin{bmatrix} \beta_{1\mu} \\ \beta^\dagger_{1\mu} \\ \beta_{2\mu} \\ \beta_{2\mu} \end{bmatrix} ,
\qquad
\mathbf{R}_{\mu\nu}(\beta) \equiv \begin{bmatrix} X^{(1)}_{\mu\nu} & X^{(2)}_{\mu\nu} & & \\ -X^{(2)}_{\nu\mu} & X^{(3)}_{\mu\nu} & & 0 \\ & & Y^{(1)}_{\mu\nu} & -Y^{(2)}_{\nu\mu} \\ & 0 & Y^{(2)}_{\mu\nu} & Y^{(3)}_{\mu\nu} \end{bmatrix} , \qquad (3.25)
$$

$$
X^{(1)<}_{\mu\nu} \equiv \begin{cases} X^{(1)}_{\mu\nu} = -X^{(1)}_{\nu\mu}, & \text{for } \mu<\nu \\ 0, & \text{otherwise} \;, \end{cases} \qquad (3.26)
$$

and similar definitions also for $X^{(3)<}_{\mu\nu}$, $Y^{(1)<}_{\mu\nu}$, and $Y^{(3)<}_{\mu\nu}$. It is easiiy shown that the first term in (3.23) vanishes also here. The operator R can be rewritten as a linear combination of bilinear normal products of $\alpha^\dagger_{i\mu}$ and $\alpha_{i\mu}$ $(i=1,2)$, while H' is expressed in terms of $\alpha^\dagger_{1\mu}$ and $\alpha_{1\mu}$ only. Therefore, the terms with $\alpha^\dagger_{2\mu}(=\tilde{\alpha}_\mu)$ and/or $\alpha_{2\mu}(=\tilde{\alpha}_\mu)$ annihilate all the bra or ket vectors appearing in trace, and the other terms without these tilde fields also vanish due to relations in (2.6). Thus, only the second term remains in (3.23). The normalization condition is given by

$$\langle 0(\beta)|[Q,Q^\dagger]|0(\beta)\rangle = (\mathbf{X}^\dagger\mathbf{Y}^\dagger)\mathbf{M}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} = 1 \;, \qquad (3.27)$$

where the vector $(\mathbf{X},\mathbf{Y})$ and the metric matrix $\mathbf{M}$ taking diagonal form are, respectively, given by

$$
\mathbf{X}_{\mu\nu} \equiv \begin{bmatrix} X^{(1)<} \\ X^{(2)} \\ X^{(3)<} \end{bmatrix}_{\mu\nu} , \quad
\mathbf{Y}_{\mu\nu} \equiv \begin{bmatrix} Y^{(1)<}_{\mu\nu} \\ -Y^{(2)}_{\nu\mu} \\ Y^{(3)<}_{\mu\nu} \end{bmatrix} , \qquad (3.28)
$$

and

$$
\mathbf{M}_{\mu\nu,\rho\sigma} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}_{\mu\nu,\rho\sigma} \qquad (3.29)
$$

with the formal 3×3 matrices of unity $\mathbf{1}$ and zero $\mathbf{0}$.

The extended TRPA (ETRPA) equation is derived from the variational principle

$$
\delta\langle 0(\beta)| \left[\tfrac{1}{2}[R,[H',R]] - \hbar\omega[Q,Q^\dagger] - \frac{a}{2}[Q,[H',Q]] - \frac{a}{2}[Q^\dagger,[H'Q^\dagger]] \right] |0(\beta)\rangle = 0 \;, \qquad (3.30)
$$

which is analogous to (2.27), but is considered in the parameter manifold of much larger dimensionality $(\mathbf{X}_{\mu\nu},\mathbf{Y}_{\mu\nu},\mathbf{X}^*_{\mu\nu},\mathbf{Y}^*_{\mu\nu})$. Constraints are required for stability

$$\langle 0(\beta) | [Q^\dagger,[H',Q^\dagger]] | 0(\beta)\rangle = \langle 0(\beta) | [Q,[H',Q]] | 0(\beta)\rangle = 0 \quad (3.31)$$

as well as the normalization condition (3.27). Hence,

$$(1-a)\mathbf{\Omega M X}^{(-)} = -(\mathbf{\Omega M} - \hbar\omega)\mathbf{X}^{(+)} , \quad (3.32a)$$

$$(1-b)\mathbf{\Omega M X}^{(+)} = -(\mathbf{\Omega M} + \hbar\omega)\mathbf{X}^{(-)} , \quad (3.32b)$$

where

$$\mathbf{X}^{(+)}_{\mu\nu} \equiv \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}_{\mu\nu}, \quad \mathbf{X}^{(-)}_{\mu\nu} \equiv \begin{bmatrix} -\mathbf{Y}^* \\ -\mathbf{X}^* \end{bmatrix}_{\mu\nu} , \quad (3.33)$$

and

$$\Omega_{i\mu j\nu,k\rho l\sigma} \equiv \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{A}^* \end{bmatrix}_{i\mu j\nu,k\rho l\sigma} \quad (3.34)$$

with

$$\mathcal{A}_{1\mu 1\nu,1\rho 1\sigma} = (E_\mu \bar{f}_\mu + E_\nu \bar{f}_\nu)(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) + 4\bar{g}_\mu \bar{g}_\nu (H_{22})_{\mu\nu\rho\sigma} \bar{g}_\rho \bar{g}_\sigma , \quad (3.35a)$$

$$\mathcal{A}_{1\mu 1\nu,1\rho 2\sigma} = 6\bar{g}_\mu \bar{g}_\nu (H_{31})_{\mu\nu\sigma\rho} \bar{g}_\rho g_\sigma , \quad (3.35b)$$

$$\mathcal{A}_{1\mu 1\nu,2\rho 2\sigma} = -24\bar{g}_\mu \bar{g}_\nu (H_{40})_{\mu\nu\rho\sigma} g_\rho g_\sigma , \quad (3.35c)$$

$$\mathcal{A}_{1\mu 2\nu,1\rho 1\sigma} = 6\bar{g}_\mu g_\nu (H_{31})^*_{\rho\sigma\nu\mu} \bar{g}_\rho \bar{g}_\sigma , \quad (3.35d)$$

$$\mathcal{A}_{1\mu 2\nu,1\rho 2\sigma} = (E_\mu \bar{f}_\mu - E_\nu f_\nu)\delta_{\mu\rho}\delta_{\nu\sigma} + 4\bar{g}_\mu g_\nu (H_{22})_{\mu\sigma\nu\rho} \bar{g}_\rho g_\sigma , \quad (3.35e)$$

$$\mathcal{A}_{1\mu 2\nu,2\rho 2\sigma} = -6\bar{g}_\mu g_\nu (H_{31})_{\rho\sigma\mu\nu} g_\rho g_\sigma , \quad (3.35f)$$

$$\mathcal{A}_{2\mu 2\nu,1\rho 1\sigma} = -24 g_\mu g_\nu (H_{40})^*_{\mu\nu\rho\sigma} \bar{g}_\rho \bar{g}_\sigma , \quad (3.35g)$$

$$\mathcal{A}_{2\mu 2\nu,1\rho 2\sigma} = -6 g_\mu g_\nu (H_{31})^*_{\mu\nu\rho\sigma} \bar{g}_\rho g_\sigma , \quad (3.35h)$$

$$\mathcal{A}_{2\mu 2\nu,2\rho 2\sigma} = -(E_\mu f_\mu + E_\nu f_\nu)(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) + 4 g_\mu g_\nu (H_{22})^*_{\mu\nu\rho\sigma} g_\rho g_\sigma , \quad (3.35i)$$

and

$$\mathcal{B}_{1\mu 1\nu,1\rho 1\sigma} = -24\bar{g}_\mu \bar{g}_\nu (H_{40})_{\mu\nu\rho\sigma} \bar{g}_\rho \bar{g}_\sigma , \quad (3.36a)$$

$$\mathcal{B}_{1\mu 1\nu,1\rho 2\sigma} = -6\bar{g}_\mu \bar{g}_\nu (H_{31})_{\mu\nu\rho\sigma} \bar{g}_\rho g_\sigma , \quad (3.36b)$$

$$\mathcal{B}_{1\mu 1\nu,2\rho 1\sigma} = 4\bar{g}_\mu \bar{g}_\nu (H_{22})_{\mu\nu\rho\sigma} g_\rho g_\sigma , \quad (3.36c)$$

$$\mathcal{B}_{1\mu 2\nu,1\rho 1\sigma} = -6\bar{g}_\mu g_\nu (H_{31})_{\rho\sigma\mu\nu} \bar{g}_\rho \bar{g}_\sigma , \quad (3.36d)$$

$$\mathcal{B}_{1\mu 2\nu,1\rho 2\sigma} = -4\bar{g}_\mu g_\nu (H_{22})_{\mu\rho\nu\sigma} \bar{g}_\rho g_\sigma , \quad (3.36e)$$

$$\mathcal{B}_{1\mu 2\nu,2\rho 2\sigma} = 6\bar{g}_\mu g_\nu (H_{31})^*_{\rho\sigma\nu\mu} g_\rho g_\sigma , \quad (3.36f)$$

$$\mathcal{B}_{2\mu 2\nu,1\rho 1\sigma} = 4 g_\mu g_\nu (H_{22})_{\rho\sigma\mu\nu} \bar{g}_\rho \bar{g}_\sigma , \quad (3.36g)$$

$$\mathcal{B}_{2\mu 2\nu,1\rho 2\sigma} = 6 g_\mu g_\nu (H_{31})^*_{\mu\nu\sigma\rho} \bar{g}_\rho g_\sigma , \quad (3.36h)$$

$$\mathcal{B}_{2\mu 2\nu,2\rho 2\sigma} = -24 g_\mu g_\nu (H_{40})^*_{\mu\nu\rho\sigma} g_\rho g_\sigma . \quad (3.36i)$$

Thus, the TRPA matrix $\mathbf{\Omega}$ defined by (3.34) is a formal 6×6 matrix in which $\mathcal{A}$ and $\mathcal{B}$ are 3×3 matrices like

$$\mathcal{A}_{i\mu j\nu,k\rho l\sigma} = \begin{bmatrix} \mathcal{A}_{1\mu 1\nu,1\rho 1\sigma} & \mathcal{A}_{1\mu 1\nu,1\rho 2\sigma} & \mathcal{A}_{1\mu 1\nu,2\rho 2\sigma} \\ \mathcal{A}_{1\mu 2\nu,1\rho 1\sigma} & \mathcal{A}_{1\mu 2\nu,1\rho 2\sigma} & \mathcal{A}_{1\mu 2\nu,2\rho 2\sigma} \\ \mathcal{A}_{2\mu 2\nu,1\rho 1\sigma} & \mathcal{A}_{2\mu 2\nu,1\rho 2\sigma} & \mathcal{A}_{2\mu 2\nu,2\rho 2\sigma} \end{bmatrix} . \quad (3.37)$$

In the zero-temperature limit ($\beta \to \infty$)

$$\bar{f}_\mu, \bar{g}_\mu \to 1 \text{ and } f_\mu, g_\mu \to 0 , \quad (3.38)$$

so that all the ETRPA matrix elements vanish except for

$$\mathcal{A}_{1\mu 1\nu,1\rho 1\sigma} \to (E_\mu + E_\nu)(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) + 4(H_{22})_{\mu\nu\rho\sigma} , \quad (3.39a)$$

$$\mathcal{B}_{1\mu 1\nu,1\rho 1\sigma} \to -24(H_{40})_{\mu\nu\rho\sigma} , \quad (3.39b)$$

which exactly coincide with ordinary RPA matrix elements.

In virtue of the relations resulting from (3.31),

$$\mathbf{X}^{(+)\dagger}\mathbf{M}\mathbf{X}^{(-)} = \mathbf{X}^{(-)\dagger}\mathbf{M}\mathbf{X}^{(+)} = 0 , \quad (3.40)$$

Eqs. (3.32a) and (3.32b) become

$$(1-a)\mathbf{X}^{(-)\dagger}\mathbf{M\Omega M X}^{(-)} = -\mathbf{X}^{(-)\dagger}\mathbf{M\Omega M X}^{(+)} , \quad (3.41a)$$

$$(1-b)\mathbf{X}^{(+)\dagger}\mathbf{M\Omega M X}^{(+)} = -X^{(+)\dagger}\mathbf{M\Omega M X}^{(-)} , \quad (3.41b)$$

or equivalently

$$(1-a)\langle 0(\beta) | [Q^\dagger,[H',Q]] | 0(\beta)\rangle = -\langle 0(\beta) | [Q^\dagger,[H',Q^\dagger]] | 0(\beta)\rangle , \quad (3.42a)$$

$$(1-b)\langle 0(\beta) | [Q,[H',Q^\dagger]] | 0(\beta)\rangle = -\langle 0(\beta) | [Q,[H',Q]] | 0(\beta)\rangle . \quad (3.42b)$$

The r.h.s. of (3.42a) and (3.42b) vanish on account of the requirements in (3.31), but $\langle 0(\beta) | [Q,[H',Q^\dagger]] | 0(\beta)\rangle$ is finite in general, so that we have

$$a = b = 1 . \quad (3.43)$$

Consequently, Eqs. (3.32a) and (3.32b) yield a set of the ETRPA equation

$$\mathbf{\Omega M X}^{(n)} = \hbar\omega_n \mathbf{X}^{(n)} , \quad (3.44a)$$

$$\mathbf{\Omega M X}^{(-n)} = \hbar\omega_{-n} \mathbf{X}^{(-n)} \quad (\hbar\omega_{-n} = -\hbar\omega_n < 0) , \quad (3.44b)$$

where the nth positive energy eigensolution $\mathbf{X}^{(n)}$ and the nth negative energy eigensolution $\mathbf{X}^{(-n)}$ take forms corresponding to (3.33),

$$\mathbf{X}^{(n)}_{\mu\nu} = \begin{bmatrix} \mathbf{X}(n) \\ \mathbf{Y}(n) \end{bmatrix}_{\mu\nu}, \quad \mathbf{Y}^{(n)}_{\mu\nu} = \begin{bmatrix} -\mathbf{Y}^*(n) \\ -\mathbf{X}^*(n) \end{bmatrix}_{\mu\nu} . \quad (3.45)$$

Solution of the ETRPA equation determines

$$Q_n^\dagger = \sum_{\mu<\nu} [X^{(1)<}_{\mu\nu}(n)\beta^\dagger_{1\mu}\beta^\dagger_{1\nu} + X^{(3)<}_{\mu\nu}(n)\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} - Y^{(1)<}_{\mu\nu}(n)\beta_{1\nu}\beta_{2\mu} - Y^{(3)<}_{\mu\nu}(n)\beta_{2\nu}\beta_{2\mu}] + \sum_{\mu\nu}[X^{(2)}_{\mu\nu}(n)\beta^\dagger_{1\mu}\beta^\dagger_{2\nu} + Y^{(2)}_{\mu\nu}(n)\beta_{2\mu}\beta_{1\nu}] , \quad (3.46)$$

from which a relation

$$Q_n = Q^\dagger_{-n} \tag{3.47}$$

is proved for any n either positive or negative.

The orthonormality relation

$$\langle 0(\beta) | [Q_m, Q_n^\dagger] | 0(\beta) \rangle = \mathbf{X}^{(m)\dagger}\mathbf{M}\mathbf{X}^{(n)} = \frac{\omega_m}{|\omega_m|}\delta_{mn} \tag{3.48}$$

follows from (3.27) and (3.47). This leads to

$$\langle 0(\beta) | [Q_m, [H', Q_n^\dagger]] | 0(\beta) \rangle = \mathbf{X}^{(m)\dagger} M \boldsymbol{\Omega} \mathbf{M}\mathbf{X}^{(n)} = \hbar |\omega_m| \delta_{mn} , \tag{3.49}$$

which includes also the required stability condition (3.31) because of (3.47), i.e.,

$$\langle 0(\beta) | [Q_m^\dagger, [H', Q_n^\dagger]] | 0(\beta) \rangle = \langle 0(\beta) | [Q_{-m}, [H', Q_n^\dagger]] | 0(\beta) \rangle = \hbar |\omega_{-m}| \delta_{-m,n} . \tag{3.50}$$

The completeness condition of the set $[\mathbf{X}^{(n)}, \mathbf{X}^{(-n)} (n > 0)]$ is expressed as

$$\left[\sum_{n>0} (\mathbf{X}^{(n)}\mathbf{X}^{(n)\dagger} - \mathbf{X}^{(-n)}\mathbf{X}^{(-n)\dagger})\mathbf{M} \right]_{\mu\nu,\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} . \tag{3.51}$$

IV. COMPARISON OF THE ETRPA WITH THE TRPA

In this section the ETRPA discussed in Sec. III will be compared with the TRPA result in Sec. II. For this purpose we transform the ETRPA equation (3.44) given in the β-quasiparticle picture into the one in the α-quasiparticle picture. Insertion of expressions (3.15a) and (3.15b) into (3.24) gives

$$\begin{aligned} Q^\dagger &= \tfrac{1}{2} \sum_{\mu\nu} \boldsymbol{\alpha}_\mu^{\mathrm{tr}} \mathbf{R}_{\mu\nu}(\alpha) \boldsymbol{\alpha}_\nu \\ &= \sum_{\mu<\nu} (x_{\mu\nu}^{(1)<} \alpha_{1\mu}^\dagger \alpha_{1\nu}^\dagger + x_{\mu\nu}^{(3)<} \alpha_{2\mu}^\dagger \alpha_{2\nu}^\dagger - y_{\mu\nu}^{(1)<} \alpha_{1\nu}\alpha_{1\mu} - y_{\mu\nu}^{(3)<} \alpha_{2\nu}\alpha_{2\mu}) \\ &\quad + \sum_{\mu\nu} (x_{\mu\nu}^{(2)} \alpha_{1\mu}^\dagger \alpha_{2\nu}^\dagger + y_{\mu\nu}^{(2)} \alpha_{2\nu}\alpha_{1\mu} + z_{\mu\nu}^{(1)} \alpha_{1\mu}^\dagger \alpha_{1\nu} + z_{\mu\nu}^{(2)} \alpha_{1\mu}^\dagger \alpha_{2\nu} - z_{\mu\nu}^{(3)} \alpha_{2\mu}^\dagger \alpha_{1\nu} + z_{\mu\nu}^{(4)} \alpha_{2\mu}^\dagger \alpha_{2\nu}) , \end{aligned} \tag{4.1}$$

where we have denoted

$$\boldsymbol{\alpha}_\mu \equiv \begin{bmatrix} \alpha_{1\mu}^\dagger \\ \alpha_{2\mu}^\dagger \\ \alpha_{1\mu} \\ \alpha_{2\mu} \end{bmatrix}, \quad \mathbf{R}_{\mu\nu}(\alpha) \equiv \begin{bmatrix} x_{\mu\nu}^{(1)} & x_{\mu\nu}^{(2)} & z_{\mu\nu}^{(1)} & z_{\mu\nu}^{(2)} \\ -x_{\nu\mu}^{(2)} & x_{\mu\nu}^{(3)} & z_{\mu\nu}^{(4)} & -z_{\nu\mu}^{(3)} \\ -z_{\nu\mu}^{(1)} & -z_{\nu\mu}^{(4)} & y_{\mu\nu}^{(1)} & -y_{\nu\mu}^{(2)} \\ -z_{\nu\mu}^{(2)} & z_{\mu\nu}^{(3)} & y_{\mu\nu}^{(2)} & y_{\mu\nu}^{(3)} \end{bmatrix}, \tag{4.2}$$

$$x_{\mu\nu}^{(1)<} \equiv \begin{cases} x_{\mu\nu}^{(1)} = -x_{\nu\mu}^{(1)}, & \text{for } \mu<\nu \\ 0, & \text{otherwise ,} \end{cases} \tag{4.3}$$

and similar definitions also for $x_{\mu\nu}^{(3)<}$, $y_{\mu\nu}^{(1)<}$, and $y_{\mu\nu}^{(3)<}$. Two pictures are related through the transformation

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y}' \end{bmatrix}_{\mu\nu} = \mathbf{U}(\mu\nu) \begin{bmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{bmatrix}_{\mu\nu} , \tag{4.4}$$

where

$$\mathbf{x}_{\mu\nu} \equiv \begin{bmatrix} x^{(1)<} \\ y^{(1)<} \\ z^{(1)} \end{bmatrix}_{\mu\nu}, \quad \tilde{\mathbf{x}}_{\mu\nu} \equiv \begin{bmatrix} x^{(3)<} \\ y^{(3)<} \\ z^{(3)} \end{bmatrix}_{\mu\nu}, \tag{4.5}$$

and

$$\mathbf{Y}'_{\mu\nu} \equiv \begin{bmatrix} Y^{(1)<} \\ Y^{(2)} \\ Y^{(3)<} \end{bmatrix}_{\mu\nu}, \tag{4.6}$$

which differs from $\mathbf{Y}_{\mu\nu}$ in (3.28); and

$$\mathbf{U}(\mu\nu) = \mathbf{V}(\mu\nu) \mathcal{M}^{-1}(\mu\nu) \tag{4.7}$$

with

$$\mathbf{V}(\mu\nu)\equiv\begin{bmatrix} \bar{g}_\mu\bar{g}_\nu & 0 & 0 & 0 & -g_\mu g_\nu & 0 \\ 0 & 0 & \bar{g}_\mu g_\nu & 0 & 0 & g_\mu\bar{g}_\nu \\ 0 & -g_\mu g_\nu & 0 & \bar{g}_\mu\bar{g}_\nu & 0 & 0 \\ 0 & \bar{g}_\mu\bar{g}_\nu & 0 & -g_\mu g_\nu & 0 & 0 \\ 0 & 0 & -g_\mu\bar{g}_\nu & 0 & 0 & -\bar{g}_\mu g_\nu \\ -g_\mu g_\nu & 0 & 0 & 0 & \bar{g}_\mu\bar{g}_\nu & 0 \end{bmatrix}, \tag{4.8}$$

$$\mathcal{M}(\mu\nu)\equiv\begin{bmatrix} 1-f_\mu-f_\nu & & & & & \\ & -(1-f_\mu-f_\nu) & & & 0 & \\ & & f_\nu-f_\mu & & & \\ & & & 1-f_\mu-f_\nu & & \\ & 0 & & & -(1-f_\mu-f_\nu) & \\ & & & & & -(f_\nu-f_\mu) \end{bmatrix}. \tag{4.9}$$

The other four coefficients appearing in (4.1) and (4.2), $x^{(2)}_{\mu\nu}$, $y^{(2)}_{\mu\nu}$, $z^{(2)}_{\mu\nu}$, and $z^{(4)}_{\mu\nu}$, are not linearly independent and can be expressed in terms of six quantities in (4.5), $x^{(1)<}_{\mu\nu}$, $y^{(1)<}_{\mu\nu}$, $z^{(1)}_{\mu\nu}$, $x^{(3)<}_{\mu\nu}$, $y^{(3)<}_{\mu\nu}$, and $z^{(3)}_{\mu\nu}$, i.e.,

$$x^{(2)}_{\mu\nu}=\frac{1}{f_\nu-f_\mu}(\bar{g}_\nu g_\nu z^{(1)}_{\mu\nu}+\bar{g}_\mu g_\mu z^{(3)}_{\mu\nu})\ , \tag{4.10a}$$

$$y^{(2)}_{\mu\nu}=\frac{-1}{f_\nu-f_\mu}(\bar{g}_\mu g_\mu z^{(1)}_{\mu\nu}+\bar{g}_\nu g_\nu z^{(3)}_{\mu\nu})\ , \tag{4.10b}$$

$$z^{(2)}_{\mu\nu}=\frac{-1}{1-f_\mu-f_\nu}[\bar{g}_\nu g_\nu(x^{(1)<}_{\mu\nu}-x^{(1)<}_{\nu\mu})-\bar{g}_\mu g_\mu(y^{(3)<}_{\mu\nu}-y^{(3)<}_{\nu\mu})]\ , \tag{4.10c}$$

$$z^{(4)}_{\mu\nu}=\frac{-1}{1-f_\mu-f_\nu}[\bar{g}_\mu g_\mu(y^{(1)<}_{\mu\nu}-y^{(1)<}_{\nu\mu})-\bar{g}_\nu g_\nu(z^{(3)<}_{\mu\nu}-z^{(3)<}_{\nu\mu})]\ . \tag{4.10d}$$

To rewrite the ETRPA equation (3.44) in the form comparable with the TRPA equation (2.28), we define a new vector

$$\begin{bmatrix}\mathbf{u}\\ \tilde{\mathbf{u}}\end{bmatrix}_{\mu\nu}\equiv\begin{bmatrix}u^{(1)<}\\ v^{(1)<}\\ w^{(1)}\\ u^{(3)<}\\ v^{(3)<}\\ w^{(3)}\end{bmatrix}_{\mu\nu}=\mathcal{M}(\mu\nu)\begin{bmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{bmatrix}_{\mu\nu} \tag{4.11}$$

whose normalization condition is derived from (3.27) by applying (4.4), (4.7), and (4.11) one after another, i.e.,

$$\begin{aligned} 1=(\mathbf{X}^\dagger\mathbf{Y}^\dagger)\mathbf{M}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}&=(\mathbf{X}^\dagger\mathbf{Y}'^\dagger)\mathbf{M}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}'\end{bmatrix}\\ &=(\mathbf{x}^\dagger\tilde{\mathbf{x}}^\dagger)\mathbf{U}^\dagger\mathbf{M}\mathbf{U}\begin{bmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{bmatrix}\\ &=(\mathbf{u}^\dagger\tilde{\mathbf{u}}^\dagger)\mathcal{M}\begin{bmatrix}\mathbf{u}\\ \tilde{\mathbf{u}}\end{bmatrix}\\ &=\sum_{\mu<\nu}(1-f_\mu-f_\nu)(\,|u^{(1)<}_{\mu\nu}|^2+|u^{(3)<}_{\mu\nu}|^2-|v^{(1)<}_{\mu\nu}|^2-|v^{(3)<}_{\mu\nu}|^2)\\ &\quad+\sum_{\mu\nu}(f_\nu-f_\mu)(\,|w^{(1)}_{\mu\nu}|^2-|w^{(3)}_{\mu\nu}|\,)^2\ , \end{aligned} \tag{4.12}$$

where the metric matrix

$$\mathcal{M}(\mu\nu)=\mathbf{V}^\dagger(\mu\nu)\mathbf{M}\mathbf{V}(\mu\nu) \tag{4.13}$$

coincides with (4.9) and is an extension of the TRPA metric matrix given by (2.26). Applying the transformation (4.4) together with (4.11) to the ETRPA equation in (3.44), we get

$$\tilde{\mathbf{\Omega}}\mathcal{M}\begin{bmatrix}\mathbf{u}\\ \tilde{\mathbf{u}}\end{bmatrix}=\hbar\omega\begin{bmatrix}\mathbf{u}\\ \tilde{\mathbf{u}}\end{bmatrix} \tag{4.14}$$

with the transformed ETRPA matrix defined by

$$\tilde{\mathbf{\Omega}}\equiv\mathbf{V}^{-1}\mathbf{\Omega}'(\mathbf{V}^{\dagger})^{-1}\ . \tag{4.15}$$

In the above $\mathbf{\Omega}'$ is obtained from $\mathbf{\Omega}$ defined in (3.34)–(3.37) only by interchanging suffixes and supplying minus signs for some matrix elements as explicitly shown by, without touching up the other matrix elements as indicated by dots, i.e.,

$$\mathbf{\Omega}'_{i\mu j\nu,k\rho l\sigma}\equiv\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & -B_{1\mu 2\nu,1\sigma 2\rho} & \cdot \\ \cdot & \cdot & \cdot & \cdot & -B_{1\mu 2\nu,1\sigma 2\rho} & \cdot \\ \cdot & \cdot & \cdot & \cdot & -B_{1\mu 2\nu,1\sigma 2\rho} & \cdot \\ \cdot & \cdot & \cdot & \cdot & -A^*_{1\mu 2\nu,1\sigma 2\rho} & \cdot \\ -B^*_{1\nu 2\mu,1\rho 2\sigma} & -B^*_{1\nu 2\mu,1\rho 2\sigma} & -B^*_{1\nu 2\mu,1\rho 2\sigma} & -A^*_{1\nu 2\mu,1\rho 2\sigma} & A^*_{1\nu 2\mu,1\sigma 2\rho} & -A_{1\nu 2\mu,1\rho 2\sigma} \\ \cdot & \cdot & \cdot & \cdot & -A^*_{1\mu 2\nu,1\sigma 2\rho} & \cdot \end{bmatrix}. \tag{4.16}$$

Then, the transformed stability matrix is given by

$$\mathcal{M}\tilde{\mathbf{\Omega}}\mathcal{M}=\mathbf{V}^{\dagger}\mathbf{M}\mathbf{\Omega}'\mathbf{M}\mathbf{V}=\mathbf{0}^{(1)}+\mathbf{0}^{(2)}\ , \tag{4.17}$$

where

$$\mathbf{0}^{(1)}_{\mu\nu,\rho\sigma}=\begin{bmatrix} p_{\mu\nu,\rho\sigma} & 0 & 0 & 0 & t_{\rho\sigma,\mu\nu} & 0 \\ 0 & p_{\mu\nu,\rho\sigma} & 0 & t_{\rho\sigma,\mu\nu} & 0 & 0 \\ 0 & 0 & q_{\mu\nu,\rho\sigma} & 0 & 0 & u_{\rho\sigma,\mu\nu} \\ 0 & t_{\mu\nu,\rho\sigma} & 0 & r_{\mu\nu,\rho\sigma} & 0 & 0 \\ t_{\mu\nu,\rho\sigma} & 0 & 0 & 0 & r_{\mu\nu,\rho\sigma} & 0 \\ 0 & 0 & u_{\mu\nu,\rho\sigma} & 0 & 0 & s_{\mu\nu,\rho\sigma} \end{bmatrix} \tag{4.18}$$

with

$$p_{\mu\nu,\rho\sigma}\equiv[E_\mu(\bar{f}^{\,2}_\mu\bar{f}_\nu-f^2_\mu f_\nu)+E_\nu(\bar{f}_\mu\bar{f}^{\,2}_\nu-f_\mu f^2_\nu)](\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\ ,$$
$$q_{\mu\nu,\rho\sigma}\equiv[E_\mu(\bar{f}^{\,2}_\mu f_\nu-f^2_\mu\bar{f}_\nu)+E_\nu(f_\mu\bar{f}^{\,2}_\nu-\bar{f}_\mu f^2_\nu)]\delta_{\mu\rho}\delta_{\nu\sigma}\ ,$$
$$r_{\mu\nu,\rho\sigma}\equiv-[E_\mu\bar{f}_\mu f_\mu(\bar{f}_\nu-f_\nu)+E_\nu\bar{f}_\nu f_\nu(\bar{f}_\mu-f_\mu)](\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\ ,$$
$$s_{\mu\nu,\rho\sigma}\equiv[E_\mu\bar{f}_\mu f_\mu(\bar{f}_\nu-f_\nu)+E_\nu\bar{f}_\nu f_\nu(\bar{f}_\mu-f_\mu)]\delta_{\mu\rho}\delta_{\nu\sigma}\ ,$$
$$t_{\mu\nu,\rho\sigma}\equiv-\bar{g}_\mu g_\mu\bar{g}_\nu g_\nu[E_\mu(\bar{f}_\mu-f_\mu)+E_\nu(\bar{f}_\nu-f_\nu)](\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\ ,$$
$$u_{\mu\nu,\rho\sigma}\equiv\bar{g}_\mu g_\mu\bar{g}_\nu g_\nu[E_\mu(\bar{f}_\mu-f_\mu)+E_\nu(\bar{f}_\nu-f_\nu)]\delta_{\mu\rho}\delta_{\nu\sigma}\ ,$$

and

$$\mathbf{0}^{(2)}_{\mu\nu,\rho\sigma}=(\mathbf{NQN})_{\mu\nu,\rho\sigma} \tag{4.19}$$

with

$$\mathbf{N}_{\mu\nu,\rho\sigma}\equiv\begin{bmatrix} \bar{f}_\mu\bar{f}_\nu+f_\mu f_\nu & & & & & \\ & \bar{f}_\mu\bar{f}_\nu+f_\mu f_\nu & & & 0 & \\ & & \bar{f}_\mu f_\nu+f_\mu\bar{f}_\nu & & & \\ & & & \bar{g}_\mu g_\mu\bar{g}_\nu g_\nu & & \\ & 0 & & & \bar{g}_\mu g_\mu\bar{g}_\nu g_\nu & \\ & & & & & \bar{g}_\mu g_\mu\bar{g}_\nu g_\nu \end{bmatrix}, \tag{4.20}$$

$$\mathbf{Q}_{\mu\nu,\rho\sigma} \equiv \begin{bmatrix} 4(H_{22})_{\mu\nu\rho\sigma} & 24(H_{40})_{\mu\nu\rho\sigma} & 6(H_{31})_{\mu\nu\sigma\rho} & -48(H_{40})_{\mu\nu\rho\sigma} & -8(H_{22})_{\mu\nu\rho\sigma} & 12(H_{31})_{\mu\nu\sigma\rho} \\ 24(H_{40})^*_{\mu\nu\rho\sigma} & 4(H_{22})^*_{\mu\nu\rho\sigma} & 6(H_{31})^*_{\mu\nu\rho\sigma} & -8(H_{22})^*_{\mu\nu\rho\sigma} & -48(H_{40})^*_{\mu\nu\rho\sigma} & 12(H_{31})^*_{\mu\nu\rho\sigma} \\ 6(H_{31})^*_{\rho\sigma\nu\mu} & 6(H_{31})_{\rho\sigma\mu\nu} & 4(H_{22})_{\mu\sigma\nu\rho} & -12(H_{31})_{\rho\sigma\mu\nu} & -12(H_{31})^*_{\rho\sigma\nu\mu} & 8(H_{22})_{\mu\sigma\nu\rho} \\ -48(H_{40})^*_{\mu\nu\rho\sigma} & -8(H_{22})^*_{\mu\nu\rho\sigma} & -12(H_{31})^*_{\mu\nu\rho\sigma} & 16(H_{22})^*_{\mu\nu\rho\sigma} & 96(H_{40})^*_{\mu\nu\rho\sigma} & -24(H_{31})_{\mu\nu\rho\sigma} \\ -8(H_{22})_{\mu\nu\rho\sigma} & -48(H_{40})_{\mu\nu\rho\sigma} & -12(H_{31})_{\mu\nu\sigma\rho} & 96(H_{40})_{\mu\nu\rho\sigma} & 16(H_{22})_{\mu\nu\rho\sigma} & -24(H_{31})_{\mu\nu\sigma\rho} \\ 12(H_{31})^*_{\rho\sigma\nu\mu} & 12(H_{31})_{\rho\sigma\mu\nu} & 8(H_{22})_{\mu\sigma\nu\rho} & -24(H_{31})_{\rho\sigma\mu\nu} & -24(H_{31})^*_{\rho\sigma\nu\mu} & 16(H_{22})_{\mu\sigma\nu\rho} \end{bmatrix} . \tag{4.21}$$

The upper-left corner of the matrix (4.17) is more explicitly rewritten as

$$\mathcal{M}\tilde{\Omega}\mathcal{M} = \begin{bmatrix} A'_{\mu\nu,\rho\sigma} & B'_{\mu\nu,\rho\sigma} & C'_{\mu\nu,\rho\sigma} & \cdots \\ B'^{*}_{\mu\nu,\rho\sigma} & A'^{*}_{\mu\nu,\rho\sigma} & C'^{*}_{\nu\mu,\sigma\rho} & \\ C'^{*}_{\rho\sigma,\mu\nu} & C'_{\sigma\rho,\nu\mu} & D'_{\mu\nu,\rho\sigma} & \\ & \cdots & & \cdots \end{bmatrix} , \tag{4.22}$$

where

$$A'_{\mu\nu,\rho\sigma} \equiv \{E_\mu(\bar{f}^{\,2}_\mu \bar{f}_\nu - f^2_\mu f_\nu) + E_\nu(\bar{f}_\mu \bar{f}^{\,2}_\nu - f_\mu f^2_\nu)\}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) + 4(\bar{f}_\mu \bar{f}_\nu + f_\mu f_\nu)(H_{22})_{\mu\nu\rho\sigma}(\bar{f}_\rho \bar{f}_\sigma + f_\rho f_\sigma) , \tag{4.23a}$$

$$B'_{\mu\nu,\rho\sigma} \equiv 24(\bar{f}_\mu \bar{f}_\nu + f_\mu f_\nu)(H_{40})_{\mu\nu\rho\sigma}(\bar{f}_\rho \bar{f}_\sigma + f_\rho f_\sigma) , \tag{4.23b}$$

$$C'_{\mu\nu,\rho\sigma} \equiv 24(\bar{f}_\mu \bar{f}_\nu + f_\mu f_\nu)(H_{31})_{\mu\nu\sigma\rho}(\bar{f}_\rho f_\sigma + f_\rho \bar{f}_\sigma) , \tag{4.23c}$$

$$D'_{\mu\nu,\rho\sigma} \equiv [E_\mu(\bar{f}^{\,2}_\mu f_\nu - f^2_\mu \bar{f}_\nu) + E_\nu(f_\mu \bar{f}^{\,2}_\nu - \bar{f}_\mu f^2_\nu)]\delta_{\mu\rho}\delta_{\nu\sigma} + 4(\bar{f}_\mu f_\nu + f_\mu \bar{f}_\nu)(H_{22})_{\mu\sigma\nu\rho}(\bar{f}_\rho f_\sigma + f_\rho \bar{f}_\sigma) . \tag{4.23d}$$

The ETRPA equation in the alternative form (4.14) and the stability matrix (4.22) are obviously extensions of those in (2.28) and (2.31), respectively. Though the upper-left block of larger matrix (4.22) corresponds to the previous one (2.31), matrix elements in (4.23) differ from those in (2.32). In the zero-temperature limit the other matrix elements in (4.22) vanish, and both (4.22) and (2.31) coincide with each other and become an ordinary RPA stability matrix. However, the difference between those matrices is marked at high temperature. It is observed that all the matrix elements in (3.23) vanish in the high-temperature limit (i.e., $\bar{f}_\mu, f_\mu \to \frac{1}{2}$), while in (4.23) some terms arising from residual interactions gain a common factor $\frac{1}{4}$ from the factors composed of distribution functions f_μ and $\bar{f}_\mu$, and remain finite, i.e.,

$$A'_{\mu\nu,\rho\sigma} \to (H_{22})_{\mu\nu\rho\sigma} , \tag{4.24a}$$

$$B'_{\mu\nu,\rho\sigma} \to 6(H_{40})_{\mu\nu\sigma\rho} , \tag{4.24b}$$

$$C'_{\mu\nu,\rho\sigma} \to 6(H_{31})_{\mu\nu\sigma\rho} , \tag{4.24c}$$

$$D'_{\mu\nu,\rho\sigma} \to (H_{22})_{\mu\sigma\nu\rho} . \tag{4.24d}$$

It is also remarked that all the other matrix elements which are not explicitly given in (4.22) are also finite, so that the effective dimensionality of the stability matrix is not reduced in the same limit. Those imply that our ETRPA equation (3.44) will be effective in taking account of the thermal-correlation effect on the mean field especially at high temperatures.

V. THE SECOND ETRPA

Usefulness of the second RPA at zero temperature is widely known in taking account of $2p$-$2h$ excitations to describe spreading widths of GR,[41–44] for instance. Our variational method of deriving the ETRPA equation in Sec. III is in principle applicable to any form of the ETRPA operator $Q^\dagger$. Therefore, the first ETRPA can be generalized to the second (or even higher) ETRPA only if we extend definitions of the vectors $\mathbf{X}^{(\pm)}$, the matrices $\mathbf{\Omega}$ and $\mathbf{M}$ according to a given form of $Q^\dagger$. To systematize complicated features of the second ETRPA, we employ notations as self-evident as possible. Greek indices $(\kappa,\lambda,\mu,\nu,\rho,\sigma,\tau,\upsilon)$ specify quasiparticle states and latin indices $(k,l,m,n,r,s,t,u=1$ or $2)$, nontilde or tilde operators as before. Then, the second ETRPA operator $Q^\dagger$ takes the form

$$Q^\dagger = \sum_{\kappa\lambda} \left[\frac{1}{2!} X^{20}_{\kappa\lambda} \beta^\dagger_{1\kappa}\beta^\dagger_{l\lambda} + X^{11}_{\kappa\lambda}\beta^\dagger_{1\kappa}\beta^\dagger_{2\lambda} + \frac{1}{2!} X^{02}_{\kappa\lambda}\beta^\dagger_{2\kappa}\beta^\dagger_{2\lambda} - \frac{1}{2!} Y^{20}_{\kappa\lambda}\beta_{1\lambda}\beta_{1\kappa} - Y^{11}_{\kappa\lambda}\beta_{2\lambda}\beta_{1\kappa} - \frac{1}{2!} Y^{02}_{\kappa\lambda}\beta_{2\lambda}\beta_{2\kappa} \right]$$
$$+ \sum_{\kappa\lambda\mu\nu} \left[\frac{1}{4!} X^{40}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{1\mu}\beta^\dagger_{1\nu} + \frac{1}{3!} X^{31}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{1\mu}\beta^\dagger_{2\nu} + \frac{1}{2!2!} X^{22}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} \right.$$
$$+ \frac{1}{3!} X^{13}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{2\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} + \frac{1}{4!} X^{04}_{\kappa\lambda\mu\nu}\beta^\dagger_{2\kappa}\beta^\dagger_{2\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} - \frac{1}{4!} Y^{40}_{\kappa\lambda\mu\nu}\beta_{1\nu}\beta_{1\mu}\beta_{1\lambda}\beta_{1\kappa} - \frac{1}{3!} Y^{31}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{1\mu}\beta_{1\lambda}\beta_{1\kappa}$$
$$\left. - \frac{1}{2!2!} Y^{22}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{1\lambda}\beta_{1\kappa} - \frac{1}{3!} Y^{13}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{2\lambda}\beta_{1\kappa} - \frac{1}{4!} Y^{04}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{2\lambda}\beta_{2\kappa} \right] \quad (5.1a)$$

$$= \sum_{\kappa<\lambda} (X^{20<}_{\kappa\lambda}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda} + X^{02<}_{\kappa\lambda}\beta^\dagger_{2\kappa}\beta^\dagger_{2\lambda} - Y^{20<}_{\kappa\lambda}\beta_{1\lambda}\beta_{1\kappa} - Y^{02<}_{\kappa\lambda}\beta_{2\lambda}\beta_{2\kappa}) + \sum_{\kappa\lambda} (X^{11}_{\kappa\lambda}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda} - Y^{11}_{\kappa\lambda}\beta_{1\lambda}\beta_{1\kappa})$$
$$+ \sum_{\kappa<\lambda<\mu<\nu} (X^{40<}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{1\mu}\beta^\dagger_{1\nu} + X^{04<}_{\kappa\lambda\mu\nu}\beta^\dagger_{2\kappa}\beta^\dagger_{2\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} - Y^{40<}_{\kappa\lambda\mu\nu}\beta_{1\nu}\beta_{1\mu}\beta_{1\lambda}\beta_{1\kappa} - Y^{04<}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{2\lambda}\beta_{2\kappa})$$
$$+ \sum_{\kappa<\lambda<\mu} (X^{31<}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\nu}\beta^\dagger_{1\mu}\beta^\dagger_{1\lambda}\beta^\dagger_{2\kappa} + X^{13<}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\nu}\beta^\dagger_{2\mu}\beta^\dagger_{2\lambda}\beta^\dagger_{2\kappa} - Y^{31<}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{1\mu}\beta_{1\lambda}\beta_{1\kappa} - Y^{13<}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{2\lambda}\beta_{1\kappa})$$
$$+ \sum_{\kappa<\lambda,\mu<\nu} (X^{22<}_{\kappa\lambda\mu\nu}\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu} - Y^{22<}_{\kappa\lambda\mu\nu}\beta_{2\nu}\beta_{2\mu}\beta_{1\lambda}\beta_{1\kappa}) , \quad (5.1b)$$

which includes all the possible fourth-order terms in $\beta^\dagger_{k\mu}$ and $\beta_{k\mu}$ as well as the second-order terms whose coefficients are redefined as

$$X^{20}_{\kappa\lambda} \equiv \begin{cases} X^{(1)}_{\kappa\lambda} = -X^{(1)}_{\kappa\lambda}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} , \end{cases} \quad (5.2a)$$

$$X^{11}_{\kappa\lambda} \equiv X^{(2)}_{\kappa\lambda} , \quad (5.2b)$$

$$X^{02}_{\kappa\lambda} \equiv \begin{cases} X^{(3)}_{\kappa\lambda} = -X^{(3)}_{\lambda\kappa}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} , \end{cases} \quad (5.2c)$$

$$Y^{20}_{\kappa\lambda} \equiv \begin{cases} Y^{(1)}_{\kappa\lambda} = -Y^{(1)}_{\lambda\kappa}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} , \end{cases} \quad (5.2d)$$

$$Y^{11}_{\kappa\lambda} \equiv -Y^{(2)}_{\kappa\lambda} , \quad (5.2e)$$

$$Y^{02}_{\kappa\lambda} \equiv \begin{cases} Y^{(3)}_{\kappa\lambda} = -Y^{(3)}_{\lambda\kappa}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} . \end{cases} \quad (5.2f)$$

In (5.1a), all the coefficients are totally or partially antisymmetric in Greek suffixes according to partitions indicated by indices, while the coefficients introduced in (5.1b) have ordered suffixes as defined by

$$X^{20<}_{\kappa\lambda} \equiv \begin{cases} X^{20}_{\kappa\lambda} = -X^{20}_{\lambda\kappa}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} , \end{cases} \quad (5.3a)$$

$$X^{02<}_{\kappa\lambda} \equiv \begin{cases} X^{02}_{\kappa\lambda} = -X^{02}_{\lambda\kappa}, & \text{for } \kappa<\lambda , \\ 0, & \text{otherwise} , \end{cases} \quad (5.3b)$$

$$X^{40<}_{\kappa\lambda\mu\nu} \equiv \begin{cases} X^{40}_{\kappa\lambda\mu\nu}, & \text{for } \kappa<\lambda<\mu<\nu \ (X^{40}_{\kappa\lambda\mu\nu} \text{ totally antisymmetric}) , \\ 0, & \text{otherwise} , \end{cases} \quad (5.3c)$$

$$X^{31<}_{\kappa\lambda\mu\nu} \equiv \begin{cases} X^{31}_{\kappa\lambda\mu\nu}, & \text{for } \kappa<\lambda<\mu \ (X^{31}_{\kappa\lambda\mu\nu} \text{ antisymmetric in } \kappa,\lambda,\mu) , \\ 0, & \text{otherwise} , \end{cases} \quad (5.3d)$$

$$X^{22<}_{\kappa\lambda\mu\nu} \equiv \begin{cases} X^{22}_{\kappa\lambda\mu\nu} = X^{22}_{\lambda\kappa\nu\mu} = -X^{22}_{\lambda\kappa\mu\nu} = -X^{22}_{\kappa\lambda\nu\mu}, & \text{for } \kappa<\lambda \text{ and } \mu<\nu , \\ 0, & \text{otherwise} , \end{cases} \quad (5.3e)$$

etc., so that each linearly-independent term appears once in (5.1b).

Retaining only the lowest-order terms, we obtain an expression for the shift of grand potential analogous to (3.23),

$$\begin{aligned}\Delta F &= \langle 0(\beta) \mid \exp[-i(Q^\dagger+Q)]H'\exp[i(Q^\dagger+Q)] \mid 0(\beta)\rangle \\ &\cong i\langle 0(\beta) \mid [H',Q^\dagger+Q] \mid 0(\beta)\rangle + \tfrac{1}{2}\langle 0(\beta) \mid [Q^\dagger+Q,[H',Q^\dagger+Q]] \mid 0(\beta)\rangle \\ &= i(\mathbf{C}^{\rm tr}\,\mathbf{C}) \begin{bmatrix} \mathbf{X}-\mathbf{Y}^* \\ \mathbf{Y}-\mathbf{X}^* \end{bmatrix} + \tfrac{1}{2}(\mathbf{X}-\mathbf{Y}^{\rm tr}\,\mathbf{Y}-\mathbf{X}^{\rm tr})\mathbf{M}\boldsymbol{\Omega}\mathbf{M} \begin{bmatrix} \mathbf{X}-\mathbf{Y}^* \\ \mathbf{Y}-\mathbf{X}^* \end{bmatrix} ,\end{aligned} \tag{5.4}$$

where we have introduced the following notations for generalized vectors $\mathbf{X},\mathbf{Y}$, generalized matrices $\mathbf{M},\boldsymbol{\Omega}$, and a new column matrix $\mathbf{C}$:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^{(2)} \\ \mathbf{X}^{(4)} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}^{(2)} \\ \mathbf{Y}^{(4)} \end{bmatrix} , \tag{5.5}$$

with

$$\mathbf{X}^{(2)}_{\kappa\lambda} \equiv \begin{bmatrix} X^{20<} \\ X^{11} \\ X^{02<} \end{bmatrix}_{\kappa\lambda} , \quad \mathbf{Y}^{(2)}_{\kappa\lambda} \equiv \begin{bmatrix} Y^{20<} \\ Y^{11} \\ Y^{02<} \end{bmatrix}_{\kappa\lambda} , \tag{5.6a}$$

$$\mathbf{X}^{(4)}_{\kappa\lambda\mu\nu} \equiv \begin{bmatrix} X^{40<} \\ X^{31<} \\ X^{22<} \\ X^{13<} \\ X^{04<} \end{bmatrix}_{\kappa\lambda\mu\nu} , \quad \mathbf{Y}^{(4)}_{\kappa\lambda\mu\nu} \equiv \begin{bmatrix} Y^{40<} \\ Y^{31<} \\ Y^{22<} \\ Y^{13<} \\ Y^{04<} \end{bmatrix}_{\kappa\lambda\mu\nu} , \tag{5.6b}$$

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{1}^{(22)} & & 0 & \\ & \mathbf{1}^{(44)} & & \\ 0 & & -\mathbf{1}^{(22)} & \\ & & & -\mathbf{1}^{(44)} \end{bmatrix} , \tag{5.7}$$

$$\boldsymbol{\Omega} \equiv \begin{bmatrix} A^{(22)} & A^{(24)} & B^{(22)} & 0 \\ A^{(42)} & A^{(44)} & 0 & 0 \\ B^{(22)*} & 0 & A^{(22)*} & A^{(24)*} \\ 0 & 0 & A^{(42)*} & A^{(44)*} \end{bmatrix} , \tag{5.8}$$

with

$$A^{(22)}_{k\kappa l\lambda,r\rho s\sigma} \equiv \langle 0(\beta) \mid [\beta_{l\lambda}\beta_{k\kappa}\, ,[H',\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma}]] \mid 0(\beta)\rangle , \tag{5.9a}$$

$$A^{(24)}_{k\kappa l\lambda,r\rho s\sigma t\tau u v} \equiv \langle 0(\beta) \mid [\beta_{l\lambda}\beta_{k\kappa},[H',\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma},\beta^\dagger_{t\tau}\beta^\dagger_{uv}]] \mid 0(\beta)\rangle , \tag{5.9b}$$

$$A^{(42)}_{k\kappa l\lambda m\mu n\nu,r\rho s\sigma} \equiv \langle 0(\beta) \mid [\beta_{n\nu}\beta_{m\mu}\beta_{l\lambda}\beta_{k\kappa},[H',\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma}]] \mid 0(\beta)\rangle , \tag{5.9c}$$

$$A^{(44)}_{k\kappa l\lambda m\mu n\nu,r\rho s\sigma t\tau u v} \equiv \langle 0(\beta) \mid [\beta_{n\nu}\beta_{m\mu}\beta_{l\lambda}\beta_{k\kappa},[H',\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma}\beta^\dagger_{t\tau}\beta^\dagger_{uv}]] \mid 0(\beta)\rangle , \tag{5.9d}$$

$$B^{(22)}_{k\kappa l\lambda,r\rho s\sigma} \equiv \langle (\beta) \mid [\beta_{l\lambda}\beta_{k\kappa},[H',\beta_{s\sigma}\beta_{r\rho}]] \mid 0(\beta)\rangle; \tag{5.9e}$$

and

$$\mathbf{C}_{\kappa\lambda\mu\nu} = \begin{bmatrix} C^{(40)} \\ C^{(31)} \\ C^{(22)} \\ C^{(13)} \\ C^{(04)} \end{bmatrix}_{\kappa\lambda\mu\nu} \tag{5.10}$$

with

$$C^{(40)}_{\kappa\lambda\mu\nu} \equiv \langle 0(\beta) \mid [H',\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{1\mu}\beta^\dagger_{1\nu}] \mid 0(\beta)\rangle , \tag{5.11a}$$

$$C^{(31)}_{\kappa\lambda\mu\nu} \equiv \langle 0(\beta) \mid [H',\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{1\mu}\beta^\dagger_{2\nu}] \mid 0(\beta)\rangle , \tag{5.11b}$$

$$C^{(22)}_{\kappa\lambda\mu\nu} \equiv \langle 0(\beta) \mid [H',\beta^\dagger_{1\kappa}\beta^\dagger_{1\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu}] \mid 0(\beta)\rangle , \tag{5.11c}$$

$$C^{(13)}_{\kappa\lambda\mu\nu} \equiv \langle 0(\beta) \mid [H', \beta^\dagger_{1\kappa}\beta^\dagger_{2\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu}] \mid 0(\beta) \rangle , \tag{5.11d}$$

$$C^{(04)}_{\kappa\lambda\mu\nu} \equiv \langle 0(\beta) \mid [H', \beta^\dagger_{2\kappa}\beta^\dagger_{2\lambda}\beta^\dagger_{2\mu}\beta^\dagger_{2\nu}] \mid 0(\beta) \rangle . \tag{5.11e}$$

All the B elements, except for the one in (5.9e), vanish as indicated in (5.8), since H' does not include any term higher than the quadratic one in β. Formal generalizations of the normalization condition (3.27) and the stability condition (3.31) are, respectively, given by

$$\langle 0(\beta) \mid [Q, Q^\dagger] \mid 0(\beta) \rangle = (\mathbf{X}^{(2)\dagger}\mathbf{X}^{(4)\dagger}\mathbf{Y}^{(2)\dagger}\mathbf{Y}^{(4)\dagger})\mathbf{M} \begin{bmatrix} \mathbf{X}^{(2)} \\ X^{(4)} \\ \mathbf{Y}^{(2)} \\ Y^{(4)} \end{bmatrix} = 1 , \tag{5.12}$$

$$\langle 0(\beta) \mid [Q^\dagger, [H', Q^\dagger]] \mid 0(\beta) \rangle = (-\mathbf{Y}^{(2)^{\rm tr}} - \mathbf{Y}^{(4)^{\rm tr}} - \mathbf{X}^{(2)^{\rm tr}} - \mathbf{X}^{(4)^{\rm tr}})\mathbf{M}\boldsymbol{\Omega}\mathbf{M} \begin{bmatrix} \mathbf{X}^{(2)} \\ X^{(4)} \\ \mathbf{Y}^{(2)} \\ Y^{(4)} \end{bmatrix} = 0 , \tag{5.13}$$

and its complex conjugation. However, these conditions must be required for $Q^\dagger$ and Q which are so redefined as to eliminate the first term in (5.4). To cope with this linear term we replace $Q^\dagger$ by $Q^\dagger + Q_0^\dagger$, which corresponds to the shifts of $\mathbf{X}^{(4)}$ and $\mathbf{Y}^{(4)}$, i.e.,

$$\mathbf{X}^{(4)} \rightarrow \mathbf{X}^{(4)} + \mathbf{X}_0^{(4)}, \quad \mathbf{Y}^{(4)} \rightarrow \mathbf{Y}^{(4)} + \mathbf{Y}_0^{(4)} . \tag{5.14}$$

Thus, we get a newly defined grand potential without a linear term,

$$\Delta F' = \Delta F + \Delta F_0 , \tag{5.15}$$

provided that

$$\mathbf{X}_0^{(4)} - \mathbf{Y}_0^{(4)*} = i(A^{(44)})^{-1}\mathbf{C}^* , \tag{5.16}$$

where

$$\Delta F_0 \equiv -\tfrac{1}{2}[\mathbf{C}^{\rm tr}(A^{(44)})^{-1}\mathbf{C}^* + {\rm c.c.}] . \tag{5.17}$$

This expression represents an additional contribution to the ground-state correlation at finite temperature, which is accounted for in the second ETRPA.

Minimizing ΔF under the constraints in (5.12) and (5.13), we derive the second ETRPA equation

$$\boldsymbol{\Omega}\mathbf{M}\mathbf{X} = \hbar\omega\mathbf{X} , \tag{5.18}$$

where

$$\mathbf{X} \equiv \begin{bmatrix} \mathbf{X}^{(2)} \\ \mathbf{X}^{(4)} \\ \mathbf{Y}^{(2)} \\ \mathbf{Y}^{(4)} \end{bmatrix} \tag{5.19}$$

and

$$\boldsymbol{\Omega}\mathbf{M} = \begin{bmatrix} A^{(22)} & A^{(24)} & -B^{(22)} & 0 \\ A^{(42)} & A^{(44)} & 0 & 0 \\ B^{(22)} & 0 & -A^{(22)*} & A^{(24)*} \\ 0 & 0 & -A^{(42)*} & -A^{(44)*} \end{bmatrix} . \tag{5.20}$$

Since eigensolutions to (5.18) for positive eigenvalues $\hbar\omega_n$ and negative ones $\hbar\omega_{-n}(=-\hbar\omega_n<0)$ compose a complete set $[\mathbf{X}^{(n)}, \mathbf{X}^{(-n)}(n>0)]$, any operator can be expanded in terms of $Q_n^\dagger$ and Q_n within the extent of a second ETRPA. The orthonormality, completeness, and stability conditions are expressed in the generalized forms similar to (3.48), (3.50), and (3.51), respectively.

If we eliminate the second ETRPA amplitude $\mathbf{X}^{(4)}$ and $\mathbf{Y}^{(4)}$ from (5.18), we get the equation pretending the first ETRPA, but with frequency-dependent matrix elements,

$$\begin{bmatrix} A^{(22)} + A^{(24)}(\hbar\omega - A^{(44)})^{-1}A^{(42)} & -B^{(22)} \\ B^{(22)*} & -A^{(22)*} + A^{(24)}(\hbar\omega + A^{(44)*})^{-1}A^{(42)*} \end{bmatrix} \begin{bmatrix} \mathbf{X}^{(2)} \\ \mathbf{Y}^{(2)} \end{bmatrix} = \hbar\omega \begin{bmatrix} \mathbf{X}^{(2)} \\ \mathbf{Y}^{(2)} \end{bmatrix} . \tag{5.21}$$

This type of equation is useful to estimate contributions from the second ETRPA to the spreading widths of one-phonon states calculated in the first ETRPA as in the case of the RPA applied to the GDR[43] and the giant Gamow-Teller resonances.[44]

Finally we consider the excitation energy of a collective state described by the nth solution to the first ETRPA equation (3.44), or the second ETRPA equation (5.18),

$$\Delta E = \langle 0(\beta) | \exp(-iR_n)(H' + \omega_{\rm rot}\hat{J}_X + \lambda_p \hat{Z} + \lambda_n \hat{N})\exp(iR_n) | 0(\beta)\rangle - \langle 0(\beta) | H' | 0(\beta)\rangle$$
$$\cong \hbar\omega_n + i\langle 0(\beta) | [\omega_{\rm rot}\hat{J}_X + \lambda_p\hat{Z} + \lambda_n\hat{N}, R_n] | 0(\beta)\rangle + \tfrac{1}{2}\langle 0(\beta) | [R_n, [\omega_{\rm rot}\hat{J}_X + \lambda_p\hat{Z} + \lambda_n\hat{N}, R_n]] | 0(\beta)\rangle \quad (5.22)$$

with

$$R_n = Q_n^\dagger + Q_n \ . \quad (5.23)$$

Excluding the trivial case such that Q_n is characterized by negative parity, for example, the second term in (5.22) vanishes due to Hermiticity of R_n, $\hat{J}_X$, $\hat{Z}$, and $\hat{N}$, as long as coefficients X and Y are chosen to be real. An alternative way of proving this is to use the fact that any operator F can be expanded in terms of a complete set $\{Q_n^\dagger, Q_n\}$ within the ETRPA, i.e.,

$$F = \sum_{n>0} \{\langle 0(\beta) | [Q_n, F] | 0(\beta)\rangle Q_n^\dagger + \langle 0(\beta) | [F, Q_n^\dagger] | 0(\beta)\rangle Q_n\} \ . \quad (5.24)$$

Only if we apply this expansion to the Hermitian one-body operator $\omega_{\rm rot}\hat{J} + \lambda_p\hat{Z} + \lambda_n\hat{N}$, we find that two terms calculated from orthonormality condition similar to (3.48) cancel each other. Since the constraints for angular momentum and particle numbers in (2.2) must be reset as

$$\langle 0(\beta) | \hat{J}_X + \tfrac{1}{2}[R_n, [\hat{J}_X, R_n]] | 0(\beta)\rangle = I \ , \quad (5.25a)$$

$$\langle 0(\beta) | \hat{Z} + \tfrac{1}{2}[R_n, [\hat{Z}, R_n]] | 0(\beta\rangle = Z - Z_0 \ , \quad (5.25b)$$

$$\langle 0(\beta) | \hat{N} + \tfrac{1}{2}[R_n, [\hat{N}, R_n]] | 0(\beta)\rangle = N - N_0 \ , \quad (5.25c)$$

the excitation energy (5.22) becomes

$$\Delta E = \hbar\omega_n + \omega_{\rm rot}[I - \langle 0(\beta) | \hat{J}_X | 0(\beta)\rangle] + \lambda_p[Z - Z_0 - \langle 0(\beta) | \hat{Z} | 0(\beta)\}] + \lambda_n[N - N_0 - \langle 0(\beta) | \hat{N} | 0(\beta)\rangle] \ . \quad (5.26)$$

Therefore, it is expected that changes of the constraints in (5.25) may result in a certain shift of centroid energy of GR in a hot rotating nucleus, and this may give a partial explanation of the change of resonance spectrum in a hot rotating nucleus.

VI. THE ETRPA EQUATION IN THE TIME-DEPENDENT FORMALISM AND ENERGY-WEIGHTED SUM RULE

In this section it will be shown that the ETRPA equation can be justified also from a viewpoint of linear response theory, whose TFD version has a rather simplified feature. Electromagnetic decay of a GR state emitting a γ-ray is described by the interaction of electromagnetic field coupled to nuclear charge current, which is a Hermitian one-body operator. The relevant part of the interaction Hamiltonian can be expressed in a β-quasiparticle picture as

$$H^{\rm ext}(t) = i\phi(t)\left[\sum_{\mu<\nu}(p_{\mu\nu}^{20<}\beta_{1\mu}^\dagger\beta_{1\nu}^\dagger + p_{\mu\nu}^{02<}\beta_{2\mu}^\dagger\beta_{2\nu}^\dagger - p_{\mu\nu}^{20<*}\beta_{1\nu}\beta_{1\mu} - p_{\mu\nu}^{02<*}\beta_{2\nu}\beta_{2\mu}) + \sum_{\mu\nu}(p_{\mu\nu}^{11}\beta_{1\mu}^\dagger\beta_{2\nu}^\dagger - p_{\mu\nu}^{11*}\beta_{2\nu}\beta_{1\mu})\right] \ , \quad (6.1)$$

in which all the coefficients p are related to shell-model single-particle matrix elements. This interaction is in general time dependent since not only the decay of a resonance state with finite lifetime can occur after its formation, but also its existence probability is affected by other competing processes such as neutron emission.

Suppose that $H^{\rm ext}(t)$ is switched on at $t = t_2$ later than the residual interaction $\Delta H(t)$, which is responsible for collective excitation and is switched on at $t = t_1$ $(t_0 < t_1 < t_2)$ in the unperturbed Hamiltonian

$$H' = H^{\rm eff} + \Delta H(t) \ . \quad (6.2)$$

Following the argument in Ref. 24, we start from the Liouville equation for time-dependent density matrix $W(t)$,

$$i\hbar\frac{\partial W(t)}{\partial t} = [H' + H^{\rm ext}(t), W(t)] \ . \quad (6.3)$$

Then, the density matrix in the interaction picture

$$W_I(t) = \exp\left[\frac{i}{\hbar}H'(t - t_0)\right]W(t)\exp\left[-\frac{i}{\hbar}H'(t - t_0)\right] \quad (6.4)$$

obeys the equation of motion derived from (6.3), i.e.,

$$i\hbar\frac{\partial}{\partial t}W_I(t) = \left[\exp\left[\frac{i}{\hbar}H'(t - t_0)\right]H^{\rm ext}(t)\exp\left[-\frac{i}{\hbar}H'(t - t_0)\right], W_I(t)\right] \ , \quad (6.5)$$

which can be converted to the integral equation as follows:

$$W(t)=W+\frac{1}{i\hbar}\int_{t_0}^{t}dt'\exp\left[-\frac{i}{\hbar}H'(t-t_0)\right][H^{\text{ext}}(t'),W(t')]\exp\left[\frac{i}{\hbar}H'(t-t_0)\right] \tag{6.6}$$

with the boundary value at $t=t_0$ ($<t_1$)

$$\exp\left[-\frac{i}{\hbar}H'(t-t_0)\right]W_I(t_0)\exp\left[\frac{i}{\hbar}H'(t-t_0)\right]=W(t_0)=W\ , \tag{6.7}$$

where W is defined for H^{eff} as seen in (2.6).

Now, we retain only the lowest-order terms with respect to $H^{\text{ext}}(t)$, multiply both sides of (6.6) by $\beta_{n\nu}\beta_{m\mu}$, for instance, and take the trace over the α-Fock space. If we make use of the invariance under the cyclic permutation of factors in trace and definitions for time-dependent operators,

$$\beta_\mu(t)=\exp\left[\frac{i}{\hbar}H't\right]\beta_\mu\exp\left[-\frac{i}{\hbar}H't\right]\ , \tag{6.8}$$

etc., we obtain

$$\text{Tr}[W\beta_{n\nu}(t)\beta_{m\mu}(t)]-\text{Tr}(W\beta_{n\nu}\beta_{m\mu})=\frac{1}{i\hbar}\int_{-\infty}^{t}dt'\,\text{Tr}\{W[\beta_{n\nu}(t-t')\beta_{m\mu}(t-t'),H^{\text{ext}}(t')]\}\ , \tag{6.9}$$

etc. Application of the TFD formula (3.10) and insertion of the explicit form of $H^{\text{ext}}(t)$ in (6.1) reduces (6.9) to

$$\langle 0(\beta)|\begin{bmatrix}\beta_{n\nu}(t)\beta_{m\mu}(t)\\ \beta^\dagger_{m\mu}(t)\beta^\dagger_{n\nu}(t)\end{bmatrix}|0(\beta)\rangle=-i\int_{-\infty}^{t}dt'\,R^{<}_{m\mu n\nu,r\rho s\sigma}(t-t')\phi(t')\begin{bmatrix}\mathbf{p}_{r\rho s\sigma}\\ \mathbf{p}^*_{r\rho s\sigma}\end{bmatrix}\ , \tag{6.10}$$

where

$$R^{<}_{m\mu n\nu,r\rho s\sigma}(t)\equiv\frac{1}{i\hbar}\langle 0(\beta)|\begin{bmatrix}[\beta_{n\nu}(t)\beta_{m\mu}(t),\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma}] & -[\beta_{n\nu}(t)\beta_{m\mu}(t),\beta_{s\sigma}\beta_{r\rho}]\\ [\beta^\dagger_{m\mu}(t)\beta^\dagger_{n\nu}(t),\beta^\dagger_{r\rho}\beta^\dagger_{s\sigma}] & -[\beta^\dagger_{m\mu}(t)\beta^\dagger_{n\nu}(t),\beta_{s\sigma}\beta_{r\rho}]\end{bmatrix}|0(\beta)\rangle,\quad \text{for } t\ge 0\ , \tag{6.11}$$

is the response function, and

$$\mathbf{p}_{r\rho s\sigma}\equiv\begin{bmatrix}p^{20<}\\ p^{11}\\ p^{02<}\end{bmatrix}_{\rho\sigma}\ . \tag{6.12}$$

On the other hand, we assume the time-dependent form for the density matrix as

$$W(t)=\exp[iR(t)]W\exp[-iR(t)]\ , \tag{6.13}$$

with

$$R(t)\equiv Q^\dagger(t)+Q(t)\ , \tag{6.14}$$

where the coefficients X, Y, X^*, and Y^* in $Q^\dagger(t)$ and $Q(t)$ are time dependent. Retaining only the lowest-order terms with respect to $R(t)$ or $H^{\text{ext}}(t)$ in the Liouville equation (6.3), we get

$$\text{Tr}\left[W\left[F,[H',R(t)]+i\hbar\frac{d}{dt}R(t)\right]\right]=-i\,\text{Tr}\{W[F,H^{\text{ext}}(t)]\} \tag{6.15}$$

for an arbitrary operator F, which is assumed here to represent any operator within the operator sector composed of bilinear forms for the first ETRPA and up to quadratic forms for the second ETRPA in β's. In the TFD, the linear-response equation (6.15) is rewritten as

$$\langle 0(\beta)|F,[H',R(t)]]+i\hbar\left[F,\frac{d}{dt}R(t)\right]|0(\beta)\rangle=-i\langle 0(\beta)|[F,H^{\text{ext}}(t)]|0(\beta)\rangle\ . \tag{6.16}$$

If $H^{\text{ext}}(t)=0$, the set of equations is the same as the one assumed in the equation-of-motion method.[7]

For simplicity, here we restrict ourselves to the first ETRPA. If we introduce Fourier transforms of relevant time-dependent functions by

$$X^{20<}_{\mu\nu}(t)-Y^{20<*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)X^{20<}_{\mu\nu}(\omega)\ , \tag{6.17a}$$

$$X^{11}_{\mu\nu}(t)-Y^{11*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)X^{11}_{\mu\nu}(\omega)\ , \tag{6.17b}$$

$$X^{02<}_{\mu\nu}(t)-Y^{02<*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)X^{02<}_{\mu\nu}(\omega)\ , \tag{6.17c}$$

$$Y^{20<}_{\mu\nu}(t)-X^{20<*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)Y^{20<}_{\mu\nu}(\omega)\ , \tag{6.17d}$$

$$Y^{11}_{\mu\nu}(t)-X^{11*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)Y^{11}_{\mu\nu}(\omega)\ , \tag{6.17e}$$

$$Y^{02<}_{\mu\nu}(t)-X^{02<*}_{\mu\nu}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)Y^{02<}_{\mu\nu}(\omega)\ , \tag{6.17f}$$

and

$$\phi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\exp(i\omega t)\phi(\omega)\ , \tag{6.18}$$

the Fourier transform of (6.15) gives

$$(\mathbf{\Omega M}-\hbar\omega)\begin{bmatrix}\mathbf{X}(\omega)\\ \mathbf{Y}(\omega)\end{bmatrix}=-\phi(\omega)\begin{bmatrix}\mathbf{p}\\ \mathbf{p}^*\end{bmatrix}\ , \tag{6.19a}$$

$$(\mathbf{\Omega M}+\hbar\omega)\begin{bmatrix}-\mathbf{Y}^*(\omega)\\ -\mathbf{X}^*(\omega)\end{bmatrix}=-\phi(-\omega)\begin{bmatrix}\mathbf{p}\\ \mathbf{p}^*\end{bmatrix}\ , \tag{6.19b}$$

where

$$\mathbf{X}_{\mu\nu}(\omega)\equiv\begin{bmatrix}X^{20<}(\omega)\\ X^{11}(\omega)\\ X^{02<}(\omega)\end{bmatrix}_{\mu\nu}\ ,\quad \mathbf{Y}_{\mu\nu}(\omega)\equiv\begin{bmatrix}Y^{20<}(\omega)\\ Y^{11}(\omega)\\ Y^{02<}(\omega)\end{bmatrix}_{\mu\nu}\ . \tag{6.20}$$

From reality of $\phi(t)$ and (6.17) we can show

$$\mathbf{X}(\omega)=-\mathbf{Y}^*(-\omega) \tag{6.21}$$

and

$$\phi(\omega)=\phi^*(-\omega)\ , \tag{6.22}$$

so that (6.19b) is obtained from (6.19a) by the replacement $\hbar\omega\to-\hbar\omega$. Thus, we have seen that the equation given by (6.19) coincides with the ETRPA equation (3.44) when $H^{\text{ext}}(t)=0$. Formal solution of the inhomogeneous equation (6.19),

$$\begin{bmatrix}\mathbf{X}(\omega)\\ \mathbf{Y}(\omega)\end{bmatrix}=R^{<}_{m\mu n\nu,r\rho s\sigma}(\omega)\phi(\omega)\begin{bmatrix}\mathbf{p}\\ \mathbf{p}^*\end{bmatrix}\ , \tag{6.23}$$

defines the Fourier transform of the response function (6.11) as

$$R^{<}_{m\mu n\nu,r\rho s\sigma}(\omega)\equiv[(\hbar\omega-\mathbf{\Omega M})^{-1}]_{m\mu n\mu,r\rho s\sigma}\ . \tag{6.24}$$

To confirm directly that the Fourier transform of (6.11) coincides with (6.24), we extensively apply the expansionlike (5.24) to the time-dependent operator $F(t)$, i.e.,

$$F(t)=\sum_{n>0}\{\langle 0(\beta)\,|\,[Q_n,F]\,|\,0(\beta)\rangle Q_n^\dagger(t)+\langle 0(\beta)\,|\,[F,Q_n^\dagger]\,|\,0(\beta)\rangle Q_n(t)\}\ , \tag{6.25}$$

where time dependences of $Q_n^\dagger(t)$ and $Q_n(t)$ are defined by (6.17) as

$$Q_n^\dagger(t)=Q_n^\dagger\exp(i\omega_n t),\quad Q_n(t)=Q_n\exp(-i\omega_n t)\ . \tag{6.26}$$

In (6.11) we identify $F(t)$ with $\beta_{n\nu}(t)\beta_{m\mu}(t)$ or $\beta^\dagger_{m\mu}(t)\beta^\dagger_{n\nu}(t)$, and apply the identity (6.25) together with (6.26) to obtain

$$R^{<}_{m\mu n\nu,r\rho s\sigma}(t)=\frac{1}{i\hbar}\left[\sum_{n>0}[\mathbf{X}^{(n)}\mathbf{X}^{(n)\dagger}\exp(-i\omega_n t)-\mathbf{X}^{(-n)}\mathbf{X}^{(-n)\dagger}\exp(i\omega_n t)]\mathbf{M}\right]_{m\mu n\nu,r\rho s\sigma}\ ,\quad \text{for } t\geq 0\ , \tag{6.27}$$

whose Fourier transform is calculated as

$$R^{<}_{m\mu n\nu,rps\sigma}(\omega)=\int_0^{\infty}dt\,\exp(i\omega t)R^{<}_{m\mu n\nu,rps\sigma}(t)$$
$$=\frac{1}{\hbar}\left[\sum_{n>0}\left\{\frac{\mathbf{X}^{(n)}\mathbf{X}^{(n)\dagger}}{\omega-\omega_n+i\epsilon}-\frac{\mathbf{X}^{(-n)}\mathbf{X}^{(-n)\dagger}}{\omega+\omega_n+i\epsilon}\right\}\mathbf{M}\right]_{m\mu n\nu,rps\sigma}$$
$$=\left[(\hbar\omega-\mathbf{\Omega M})^{-1}\sum_{n>0}(\mathbf{X}^{(n)}X^{(n)\dagger}-\mathbf{X}^{(-n)}\mathbf{X}^{(-n)\dagger})\mathbf{M}\right]_{m\mu n\nu,rps\sigma}$$
$$=[(\hbar\omega-\mathbf{\Omega M})^{-1}]_{m\mu n\nu,rps\sigma}\ . \tag{6.28}$$

We have used the completeness condition in (3.51) to get the last expression, which is nothing but the one given by (6.24).

In order to derive the energy-weighted sum rule (EWSR) for an operator F, we first apply the expansion in (6.25) to F and make use of the relation (3.47) and the condition in the form (3.5) representing both orthonormality and stability to obtain the formula expressed in terms of the TFD vacuum expectation value,

$$\langle 0(\beta)\,|\,[F,[H',F]]\,|\,0(\beta)\rangle=\sum_{n>0}\hbar\omega_n\{\,|\,\langle 0(\beta)\,|\,[F,Q_n^{\dagger}]\,|\,0(\beta)\rangle\,|^2+|\,\langle 0(\beta)\,|\,[F^{\dagger},Q_n^{\dagger}]\,|\,0(\beta)\rangle\,|^2\}\ . \tag{6.29}$$

When F is Hermitian , we have

$$\tfrac{1}{2}\langle 0(\beta)\,|\,[F,[H',F]]\,|\,0(\beta)\rangle=\sum_{n>0}\hbar\omega_n\langle 0(\beta)\,|\,[F,Q_n^{\dagger}]\,|\,0(\beta)\rangle\,|^2\ , \tag{6.30}$$

which is the formal EWSR generalized by the ETRPA. All the above arguments in this section can be generalized to any higher ETRPA without any difficulty except for increasing complication.

VII. SUMMARY AND DISCUSSION

Importance of thermal correlation increases with increasing temperature as a natural consequence of many-particle–many-hole excitations. Such a correlation effect becoming a problem through the change of a one-body field is partially taken into account by the THFB solution, on which the TRPA is built. On the other hand, the TRPA calculations have shown that additional thermal effect on residual interactions causes some physical effects such as the shift of centroid energy and the broadening of resonance spectral width of GDR observed in hot rotating nuclei.

In order to attain a more precise relation between a given microscopic model and experimentally observable quantities, it is necessary to provide a theoretical framework having enough possibility of dealing with a thermal effect as well as a rotational effect systematically. For the purpose of applying the TFD to the RPA formalism, we have extended the TRPA. Since in the TFD, a thermal-ensemble average can be expressed as an expectation value sandwiched between the TFD vacua defined for the enlarged Fock space, not only does the relevant expression become independent of trace calculation, but also there exists a state vector which is the TFD vacuum describing the thermal equilibrium state for a given temperature. Though we have not argued about the ETRPA state vectors, it is well known that there exists the ETRPA state $|\,g(\beta)\rangle$ which is different from $|\,0(\beta)\rangle$ and annihilated by $Q_n(=Q^{\dagger}_{-n})$, i.e.,

$$Q_n\,|\,g(\beta)\rangle=0, \tag{7.1}$$

and then $Q_n^{\dagger}\,|\,g(\beta)\rangle$ describes an ETRPA single-phonon state.[45] This is because a parallel holds between the formal aspect of the ETRPA and the ordinary RPA at zero temperature. The extended single-particle space of the TFD allows for an enlargement of the manifold of variational parameter space. Making full use of such modification, we have derived the first ETRPA equation in Sec. III, and it was generalized to the second ETRPA in Sec. V.

The structure of our ETRPA matrix implies that in contrast to the TRPA, its nondiagonal elements become more important at high temperatures, so that decay strengths are distributed into many levels. As a consequence of these characteristics, more rapid increase of resonance width of GDR, for instance, will be predicted by the ETRPA. On the other hand, higher ETRPA will become important to explain the shift of centroid energy and increase of spreading width of GDR at high temperature. It is quite interesting to investigate whether modifications of constraints for angular momentum and particle numbers have any influence on resonance spectrum. The ETRPA equation will be applied to some realistic problems in future study.

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