Intrinsic operators and wave functions for the interacting boson model

F. G. Scholtz
Institute of Theoretical Nuclear Physics, University of Stellenbosch, Stellenbosch 7600, South Africa
(Received 26 May 1987)

We use the eigenstate of the shape operators in second quantized form to give a geometrical interpretation to each boson excitation in the interacting boson model. For well-deformed systems a method is presented by which the intrinsic components of any boson number preserving operator can be calculated in second quantized form and to leading order in the boson number. In this way vibrational and rotational modes are identified and the corresponding excitation energies and moments of inertia are calculated to leading order in the boson number. The intrinsic components of the $E2$ transition operator are also calculated. Using these results several $E2$ transition rates are calculated to leading order in the boson number.

I. INTRODUCTION

The interacting boson model has been subjected to many investigations, all having basically two aims in mind. The first is to relate it to the geometrical model, i.e., to determine the shapes and shape phase transitions described by the interacting boson model, and to give a geometrical interpretation to each boson excitation. The second aim is to obtain information on the excitation energies, moments of inertia and transition rates of the interacting boson model without resorting to a diagonalization of the IBM Hamiltonian. The techniques employed in these investigations include the usual Ritz variational principle,[1–6] classical approximations,[7] the generator coordinate method,[3,6,8–11] the random phase approximation,[12,13] and the self-consistent cranking model.[14–17]

In a previous paper[18] we illustrated by means of a U(3) interacting boson model how the eigenstates of the position operators in second quantized form can be used to achieve the two goals set above. Using this state it was not only possible to give a geometrical interpretation to the boson model without leaving the domain of quantum mechanics, but a procedure was also developed by which the parameters of the geometrical description for the deformed case can be calculated in a series expansion controlled by the inverse of the boson number. In particular, excitation energies and moments of inertia can be calculated to different orders in the inverse of the boson number. Another advantage of this approach was that it allowed for a proper treatment of the symmetry of the system. As we show in Secs. V and VI this property is extremely advantageous as it avoids the introduction of spurious excitations.

The present paper is organized in the following way. In Sec. II we discuss the eigenstate of the shape operators in second quantized form for the U(6) interacting boson model. Section III is devoted to the case of a deformed axially symmetric system. In Sec. IV a well deformed system is considered and some useful approximations are introduced. These results are applied to a quadrupole-quadrupole interaction in Sec. V to calculate excitation energies and moments of inertia. In Sec. VI we calculate $E2$-transition rates, and finally we present a discussion and conclusions.

II. EIGENSTATE OF THE SHAPE OPERATORS IN SECOND QUANTIZED FORM

The interacting boson model attempts to describe the low-lying collective modes of an even-even nucleus by assuming that the valence nucleons tend to form pairs coupled to angular momentum 0 and 2. It is furthermore assumed that these pairs can be treated as exact monopole and quadrupole bosons.[19] Boson creation operators $s^\dagger, d^\dagger_\mu$ $(\mu=-2,-1,\ldots,2)$ and annihilation operators s, d_μ are introduced for the six levels. As is well known, the dynamical group associated with this model is a U(6) group generated by the 36 bilinear combinations[19] (Schwinger representation) $s^\dagger d_\mu$, $d^\dagger_\mu s$, $d^\dagger_\mu d_\nu$, and $s^\dagger s$. Any one-plus two-body operator which preserves the total number of bosons, such as the interacting boson model (IBM) Hamiltonian or electromagnetic transition operators, can be written in terms of the generators of the U(6) group. It is also well known that the U(6) group contains an SO(3) subgroup generated by the Cartan-Weyl basis[19]

$$L_\pm = \pm\sqrt{10}(d^\dagger \tilde{d})^1_{\pm 1} \,, \qquad L_0=\sqrt{10}(d^\dagger \tilde{d})^1_0 \,. \tag{2.1}$$

Here the parentheses denote angular momentum coupling and $\tilde{d}_\mu$ denotes the covariant components of the annihilation operators.

The boson Fock space in which the U(6) algebra discussed above is realized, can carry only completely symmetric U(6) representations labeled by the eigenvalues of the first order Casimir operator, $\hat{N}=s^\dagger s + d^\dagger\cdot\tilde{d}$, which is just the total number of s and d bosons $N=n_s+n_d$. To introduce shape operators and their eigenstates we employ a generalized Holstein-Primakoff realization[20] for the symmetric U(6) representations instead of the Schwinger representation. Eliminating the s boson, which is a scalar under the SO(3) group, leads to the following Holstein-Primakoff realization for the symmetric U(6) representation $[N]$:

$$
\begin{aligned}
s^\dagger d_\mu &\to (N-\hat{n}_d)^{1/2} d_\mu \ ,\\
d_\mu^\dagger s &\to d_\mu^\dagger (N-\hat{n}_d)^{1/2} \ ,\\
d_\mu^\dagger d_\nu &\to d_\mu^\dagger d_\nu \ ,\\
s^\dagger s &\to N-\hat{n}_d \ ,\\
\hat{n}_d &= d^\dagger \cdot \tilde{d} = \sum_\mu d_\mu^\dagger d_\mu \ .
\end{aligned}
\tag{2.2}
$$

For this realization the carrier space is the subspace

$$\left\{ |n_2,n_1,n_0,n_{-1},n_{-2}\rangle : n_d = \sum_\mu n_\mu \le N \right\}$$

of the d boson Fock space.

Shape operators can now be introduced in the usual way:

$$\hat{\alpha}_\mu = \frac{1}{\sqrt{2}}(d_\mu^\dagger + \tilde{d}_\mu) \ . \tag{2.3}$$

Note that the shape operators form the components of a self-adjoint tensor operator, i.e., $\hat{\alpha}_\mu^\dagger = (-1)^\mu \hat{\alpha}-\mu$. In the Appendix we prove that the state

$$|\bar{\alpha}\rangle = \frac{2}{\pi^{5/4}} \exp(-\tfrac{1}{2}\alpha\cdot\alpha)\exp(-\tfrac{1}{2}d^\dagger\cdot d^\dagger + \sqrt{2}\alpha\cdot d^\dagger)|0\rangle \ , \tag{2.4}$$

where the complex numbers α_μ satisfy $\alpha_\mu^* = (-1)^\mu \alpha_{-\mu}$, is a simultaneous eigenstate of the shape operators. We use the notation $\bar{\alpha}$ to denote the set of complex variables $\{\alpha_\mu\}_{\mu=-2}^2$. The bra state, $\langle\bar{\alpha}|$, is given by

$$\langle\bar{\alpha}| = \langle 0|\exp(-\tfrac{1}{2}\tilde{d}\cdot\tilde{d} + \sqrt{2}\alpha\cdot\tilde{d})\exp(-\tfrac{1}{2}\alpha\cdot\alpha)\frac{2}{\pi^{5/4}} \tag{2.5}$$

In the Appendix we also prove the identity

$$\int_{R^5} d\bar{\alpha}\, |\bar{\alpha}\rangle\langle\bar{\alpha}| = 1 \ , \tag{2.6a}$$

where we have introduced

$$d\bar{\alpha} = \left[\prod_{\mu>0} dx_\mu dy_\mu\right] d\alpha_0 \tag{2.6b}$$

and

$$
\begin{aligned}
\alpha_\mu &= x_\mu + i y_\mu, \quad \mu>0 \ ,\\
\alpha_\mu &= (-1)^\mu (x_{-\mu} - i y_{-\mu}), \quad \mu<0 \ .
\end{aligned}
\tag{2.6c}
$$

The integration is performed over the entire five-dimensional real space R^5. In the Appendix we also show that

$$\langle\bar{\alpha}|\bar{\alpha}'\rangle = \left[\prod_{\mu>0}\delta(x_\mu - x'_\mu)\delta(y_\mu - y'_\mu)\right]\delta(\alpha_0 - \alpha'_0) \ . \tag{2.7}$$

The completeness relation (2.6) involves the identity on the d boson Fock space, and hence it involves a direct sum of all the symmetric U(6) representations $[N]$, $N=0,1,2,\ldots$. To consider a particular symmetric U(6) representation, $[N]$, we introduce the projector

$$
\begin{aligned}
P_N &= \sum_{n_\mu}{}' |n_2,n_1,n_0,n_{-1},n_{-2}\rangle\langle n_2,n_1,n_0,n_{-1},n_{-2}|\\
&= 1_N \ ,
\end{aligned}
\tag{2.8}
$$

where the primed sum denotes that the summation is carried out over all values of n_μ $(\mu=-2,-1,\ldots,2)$ subject to the restriction $n_d = \sum_\mu n_\mu \le N$. Furthermore 1_N denotes the identity on the symmetric U(6) representation $[N]$. The completeness relation (2.6) becomes

$$\int d\bar{\alpha}\, P_N|\bar{\alpha}\rangle\langle\bar{\alpha}|P_N = 1_N \ . \tag{2.9}$$

It is important to note that all the U(6) generators of Eq. (2.2) commute with P_N.

A transformation to intrinsic variables and Euler angles can be performed in the usual way[21,22]

$$\alpha_\mu = \sum_\nu D_{\mu\nu}^{2*}(\Omega) a_\nu, \quad a_2 = a_{-2}, \ a_1 = a_{-1} = 0 \ . \tag{2.10}$$

Note also the transformation properties of the d boson creation operators under rotations

$$R(\Omega) d_\mu^\dagger R^{-1}(\Omega) = \sum_\nu D_{\nu\mu}^2 d_\nu^\dagger \ . \tag{2.11}$$

Here the rotation operator, $R(\Omega)$, is given by

$$R(\Omega) = \exp(-i\theta_1 L_z)\exp(-i\theta_2 L_y)\exp(-i\theta_3 L_z) \ , \tag{2.12}$$

and the Cartesian components of the angular momentum operators are defined in the usual way [see Eq. (2.1)]:

$$
\begin{aligned}
L_x &= \frac{1}{\sqrt{2}}(L_+ + L_-) \ ,\\
L_y &= \frac{-i}{\sqrt{2}}(L_+ - L_-) \ ,\\
L_z &= L_0 \ .
\end{aligned}
\tag{2.13}
$$

Introducing the transformation (2.10) in the state $|\bar{\alpha}\rangle$ and using (2.11) gives

$$
\begin{aligned}
|\bar{\alpha}\rangle = |\Omega,a_0,a_2\rangle &= \frac{2}{\pi^{5/4}}\exp[-\tfrac{1}{2}(a_0^2 + 2a_2^2)]R(\Omega)\exp\{-\tfrac{1}{2}d^\dagger\cdot d^\dagger + \sqrt{2}[a_0 d_0^\dagger + a_2(d_2^\dagger + d_{-2}^\dagger)]\}|0\rangle\\
&\equiv R(\Omega)|a_0,a_2\rangle \ .
\end{aligned}
\tag{2.14}
$$

Performing the transformation (2.1) in the completeness relation (2.9) leads to (see for instance Ref. 21)

$$\int d\Omega \int_s d\tau(a_0,a_2) R(\Omega) P_N |a_0,a_2\rangle\langle a_0,a_2| P_N R^{-1}(\Omega) = \int d\Omega\, R(\Omega) Q_{\rm intr} R^{-1}(\Omega) = 1_N \;, \tag{2.15a}$$

with

$$Q_{\rm intr} = \int_s d\tau(a_0,a_2) P_N |a_0,a_2\rangle\langle a_0,a_2| P_N \;,$$
$$d\tau(a_0,a_2) = \sqrt{2}a_2(3a_0^2 - 2a_2^2)da_0 da_2 \;, \tag{2.15b}$$
$$d\Omega = \sin\theta_2 d\theta_1 d\theta_2 d\theta_s \;.$$

Here $d\tau$ is the usual Bohr-Mottelson volume element[21,22] and $d\Omega$ the usual volume element for the Euler angles.[21,22] The region of integration in (2.15) is $(a_0,a_2)\in\{[0,\infty)\times[0,\sqrt{3}a_0/\sqrt{2}]\}\equiv S$. The operator $Q_{\rm intr}$ can be interpreted as a projector onto an intrinsic subspace. Rotating this subspace over all possible orientations gives back the whole space.

In the usual β and γ representation with $a_0 = \beta\cos\gamma$ and $a_2 = (\beta/\sqrt{2})\sin\gamma$, Eq. (2.15) reads

$$\int d\Omega \int_0^\infty \beta^4 d\beta \int_0^{\pi/3} d\gamma \sin 3\gamma R(\Omega) P_N |\beta,\gamma\rangle\langle\beta,\gamma| P_N R^{-1}(\Omega) = \int d\Omega\, R(\Omega) Q_{\rm intr} R^{-1}(\Omega) = 1_N \;, \tag{2.16a}$$

with

$$|\beta,\gamma\rangle = \frac{2}{\pi^{5/4}}\exp(-\tfrac{1}{2}\beta^2)\exp\{-\tfrac{1}{2}d^\dagger\cdot d^\dagger + \sqrt{2}[\beta\cos\gamma d_0^\dagger + \frac{1}{\sqrt{2}}\beta\sin\gamma(d_2^\dagger + d_{-2}^\dagger)]\}\,|0\rangle \;. \tag{2.16b}$$

The region of integration here is $(\beta,\gamma)\in\{[0,\infty)\times[0,\pi/3]\}=S$. As usual the integration is restricted to the region S. The reason for this is that the coordinates α_μ are scalars with respect to rotations of the intrinsic axes [see Eq. (2.10)]. Furthermore, any point in the a_0-a_2 or β-γ plane can be reached from a point in the region S by a combination of a rotation of the intrinsic axes through π around the x axis, a rotation of $\pi/2$ around the intrinsic z axis, and a cyclic permutation of the intrinsic axes (which can also be built up out of rotations of the intrinsic axes). To ensure that the coordinates α_μ are single valued, the integration must therefore be restricted to the region S. Integrating over the whole a_0-a_2 or β-γ plane would simply give rise to a multiplicative factor of 6 in the completeness relations (2.15) and (2.16). This can also be verified directly from the properties of the states $|a_0,a_2\rangle$ and $|\beta,\gamma\rangle$ and the SO(3) group property[23]

$$\int d\Omega\, R(\Omega) R(\alpha,\beta,\gamma) = \int d\Omega\, R(\Omega) \;. \tag{2.17}$$

The state $|\bar{\alpha}\rangle$, cast into the form of Eqs. (2.14) or (2.16b), can be used to obtain a differential operator realization of any boson operator written in terms of the U(6) generators of Eq. (2.2). Our main aim, however, is not to cast the IBM into a differential form, but rather to give a geometrical interpretation to each boson excitation and to calculate excitation energies and transition rates approximately for well deformed systems. We therefore only indicate briefly how a differential operator realization can be obtained. Suppose that the operator $\hat{O}_\mu^l$ transforms like the SO(3) representation l and that it is written in terms of the U(6) generators of Eq. (2.2). Using the completeness relation (2.15), we can write

$$\hat{O}_\mu^l = \hat{O}_\mu^l 1_N = \int d\Omega\, d\tau(a_0,a_2) R(\Omega) P_N |a_0,a_2\rangle\langle a_0,a_2| \sum_K D_{\mu K}^{l*}\hat{O}_K^l R^{-1}(\Omega) P_N \;, \tag{2.18}$$

where integration over the Euler angles and the region S is understood. A few simple relations can be derived which enables one to transform the operator $\hat{O}_K^l$ into a differential form. We have

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_2 + d_{-2}) = \frac{1}{2}\left[2a_2 + \frac{\partial}{\partial a_2}\right]\langle a_0,a_2| \;,$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_2^\dagger + d_{-2}^\dagger) = \frac{1}{2}\left[2a_2 - \frac{\partial}{\partial a_2}\right]\langle a_0,a_2| \;,$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_1^\dagger + d_{-1}^\dagger) = \langle a_0,a_2| \frac{L_x}{2(a_2 + \sqrt{3}a_0/\sqrt{2})} \;,$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_1 + d_{-1}) = \langle a_0,a_2| \frac{L_x}{2(a_2 + \sqrt{3}a_0/\sqrt{2})} \;,$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_1^\dagger - d_{-1}^\dagger) = \langle a_0,a_2| \frac{-iL_y}{2(a_2 - \sqrt{3}a_0/\sqrt{2})} \;, \tag{2.19}$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_1 - d_{-1}) = \langle a_0,a_2| \frac{iL_y}{2(a_2 - \sqrt{3}a_0/\sqrt{2})} \;,$$

$$\langle a_0,a_2| \frac{1}{\sqrt{2}}(d_2^\dagger - d_{-2}^\dagger) = \langle a_0,a_2| \frac{L_z}{4a_2} \;,$$

$$\langle a_0,a_2 | \frac{1}{\sqrt{2}}(d_2-d_{-2}) = \langle a_0,a_2 | \frac{L_z}{4a_2} ,$$

$$\langle a_0,a_2 | d_0 = \frac{1}{\sqrt{2}}\left[a_0+\frac{\partial}{\partial a_0}\right]\langle a_0,a_2 | ,$$

$$\langle a_0,a_2 | d_0^\dagger = \frac{1}{\sqrt{2}}\left[a_0-\frac{\partial}{\partial a_0}\right]\langle a_0,a_2 | .$$

Here the operators L_x, L_y, and L_z are still the boson angular momentum operators of Eq. (2.13) and the relation

$$[f(d_\mu),L_i]=\sum_\nu \frac{\partial f(d_\mu)}{\partial d_\nu}[d_\nu,L_i] , \tag{2.20}$$

where f is an analytic function, was used. This relation can easily be proved if the boson commutation relations and the commutation relations of the boson operator with the angular momentum operators are used. Finally we note the well known relations[21]

$$L_x R^{-1}(\Omega)=\tilde{L}'_z R^{-1}(\Omega) ,$$
$$L_y R^{-1}(\Omega)=\tilde{L}'_y R^{-1}(\Omega) , \tag{2.21}$$
$$L_z R^{-1}(\Omega)=\tilde{L}'_z R^{-1}(\Omega) .$$

Here $\tilde{L}'_i$ denotes differential operators acting on the Euler angles and the prime denotes that it is the component with respect to the intrinsic frame.[21] With the aid of Eqs. (2.19) and (2.21) the operator $\hat{O}^l_K$ can be transformed into a differential form. It should, however, be noted that the boson operators L_i, occurring in Eq. (2.18), have to be commuted to the immediate left of the rotation operator $R^{-1}(\Omega)$ before they can be replaced with differential operators according to Eq. (2.21). This gives rise to correction terms which can become very complicated, especially if the square root factors of Eq. (2.2) enter into the boson operator. In the case of a well deformed system these corrections are, however, small and they can be ignored in the lowest order approximation.

Finally we remark on some properties of the intrinsic wave functions. Taking the matrix element of Eq. (2.18) between two U(6) basis states transforming properly under SO(3), one can define the intrinsic wave functions

$$\phi^N_{\epsilon LK}(a_0,a_2)=\left[\frac{8\pi^2}{2L+1}\right]^{1/2}\langle a_0,a_2 | [N],\epsilon,L,K\rangle . \tag{2.22}$$

Here ϵ denotes the additional quantum numbers needed to specify the U(6) states completely. The normalization factor is chosen for convenience. With this normalization of the intrinsic wave functions, the normalization factors needed for the Wigner D function occur explicitly in the expressions for the total wave functions. Note that Eq. (2.22) does not imply that K is a good quantum number, it merely provides a convenient quantum number for labeling the intrinsic wave functions. As a matter of fact, from Eqs. (2.18), (2.19), and (2.21) it is clear that the total geometrical wave function corresponding to the U(6) state $|[N],\epsilon,L,M\rangle$, which transforms properly under SO(3), is given in terms of the intrinsic wave functions of Eq. (2.22) by

$$\psi^N_{\epsilon,L,M}(\Omega,a_0,a_2)=\langle\Omega,a_0,a_2 | [N],\epsilon,L,M\rangle=\sum_K\left[\frac{2L+1}{8\pi^2}\right]^{1/2} D^{L*}_{M,K}\phi^N_{\epsilon,L,K}(a_0,a_2) . \tag{2.23}$$

Clearly K mixing occurs. However, in the case of well deformed axially symmetric systems it turns out, as is discussed in Sec. V, that the K mixing is of a higher order in N, the total number of bosons.

The intrinsic wave functions of Eq. (2.22) has all the usual features. We immediately note from (2.14) that the intrinsic wave functions vanish if K is odd. Furthermore we note that

$$\begin{aligned}\phi^N_{\epsilon L\,-K}(a_0,a_2)&=\left[\frac{8\pi^2}{2L+1}\right]^{1/2}\langle a_0,a_2 | [N],\epsilon,L,-K\rangle\\&=(-1)^L\left[\frac{8\pi^2}{2L+1}\right]^{1/2}\langle a_0,a_2 | R(0,\pi,0) | [N],\epsilon,L,K\rangle\\&=(-1)^L\phi^N_{\epsilon LK}(a_0,a_2) ,\end{aligned} \tag{2.24}$$

where Eq. (2.11) and (2.14) were used. The only nonvanishing linear independent intrinsic wave functions are therefore those for which the angular momentum projection, K, on the intrinsic axis are non-negative even integers. Note also that if $K=0$ the intrinsic wave functions with odd values of L must vanish identically. Finally we remark that since the α's are scalars with respect to rotations of the intrinsic frame, the state $|\bar{\alpha}\rangle$ is also a scalar under rotations of the intrinsic frame. Therefore the wave functions

$$\langle\bar{\alpha} | [N],\epsilon,L,M\rangle=\langle\Omega,a_0,a_2 | [N],\epsilon,L,M\rangle$$

are invariant under rotation of the intrinsic frame as they should be.

III. DEFORMED AXIALLY SYMMETRIC SYSTEMS

Since no stable triaxial deformations occur in the IBM-1 (see Refs. 1—6), we only need to consider axial symmetric systems. Suppose therefore that the potential energy has a pronounced minimum in both the β and γ directions at $\beta=\beta_0$ and $\gamma=0$ or $\gamma=\pi/3$, corresponding to prolate or oblate deformations, respectively. The lowest excitations then correspond to small oscillations around these equilibrium values.

Instead of treating oblate deformations with the conventional choice $\gamma=\pi/3$, $\beta\in[0,\infty)$, it is often more convenient to use the choice $\gamma=0$, $\beta\in(-\infty,0]$ as is described in Refs. 21 and 22. Hence, instead of using the region of integration S in Sec. II, one must use the region

$$(\beta,\gamma)\in\{(-\infty,0]\times[0,\pi/3]\}$$

or, equivalently,

$$(a_0,a_2)\in\{(-\infty,0]\times[-\sqrt{3}a_0/\sqrt{2},0]\}\ .$$

Clearly, all the results of Sec. II also hold in this case. For axially symmetric systems one can therefore take for the deformation in the γ direction $\gamma=0$ without any loss of generality, provided that one allows for negative values of β_0. For convenience we use prolate deformation, i.e., $\beta_0>0$ to derive the results of this and the next section. However, keeping in mind the remarks made above, it is easy to see that all these results also hold for oblate deformations.

For our purpose it is more convenient to work in the a_0-a_2 representation and to introduce the coordinate transformation

$$\begin{aligned} a_2&=0+\eta,\quad \eta\in[0,(\tfrac{3}{2})^{1/2}a_0]\ ,\\ a_0&=\beta_0+\xi,\quad \xi\in[-\beta_0,\infty)\ . \end{aligned} \tag{3.1}$$

Inserting (3.1) into the state $|a_0,a_2\rangle$ gives

$$\begin{aligned} |a_0,a_2\rangle&=|\xi,\eta,\beta_0\rangle\\ &=\frac{2}{\pi^{5/4}}\exp(b_1^\dagger b_{-1}^\dagger)\exp[-\tfrac{1}{2}(\xi^2+b_0^\dagger b_0^\dagger)+\sqrt{2}\xi b_0^\dagger]\\ &\quad\times\exp[-(\eta^2+b_2^\dagger b_{-2}^\dagger)+\sqrt{2}\eta(b_2^\dagger+b_{-2}^\dagger)]\,|\bar{0}\rangle, \end{aligned} \tag{3.2a}$$

with

$$\begin{aligned} b_\mu^\dagger&=d_\mu^\dagger,\quad \forall\mu\neq0\ ,\\ b_0^\dagger&=-\frac{\beta_0}{\sqrt{2}}+d_0^\dagger\ ,\\ |\bar{0}\rangle&=\exp(-\tfrac{1}{4}\beta_0^2)\exp\left[\frac{1}{\sqrt{2}}\beta_0 d_0^\dagger\right]|0\rangle\ ,\\ \langle\bar{0}|\bar{0}\rangle&=1\ ,\\ b_\mu|\bar{0}\rangle&=0,\quad\forall\mu\ . \end{aligned} \tag{3.2b}$$

Note that the new vacuum is not an SO(3) scalar and that it breaks the SO(3) symmetry. However, because of the assumption of axial symmetry, the new vacuum is still an SO(2) scalar. Furthermore, note that the b_μ bosons do not transform properly under SO(3), but that they do transform properly under SO(2). This is once again a consequence of the assumption of axial symmetry.

On inspection of (3.2a), we not that the state $|\xi,\eta,\beta_0\rangle$ factorizes into a part associated with a one dimensional oscillator and the radial part associated with a two dimensional oscillator (see the Appendix and also Ref. 18). With the transformation (3.1) the completeness relation of Eq. (2.15) becomes

$$1_N=\int d\Omega\int_S d\tau(\xi,\eta)\times R(\Omega)P_N|\xi,\eta,\beta_0\rangle\langle\xi,\eta,\beta_0|P_N R^{-1}(\Omega), \tag{3.3a}$$

with

$$\begin{aligned} d\tau(\xi,\eta)&=3\sqrt{2}\beta_0^2\eta\left[\left[1+\frac{\xi}{\beta_0}\right]^2-\frac{2\eta^2}{3\beta_0^2}\right]d\xi\,d\eta\\ &\equiv3\sqrt{2}\beta_0^2\eta f(\xi,\eta^2)d\xi\,d\eta\ . \end{aligned} \tag{3.3b}$$

Equation (2.19) yields the following relations:

$$\begin{aligned} \langle\xi,\eta,\beta_0|b_0&=\frac{1}{\sqrt{2}}\left[\xi+\frac{\partial}{\partial\xi}\right]\langle\xi,\eta,\beta_0|\ ,\\ \langle\xi,\eta,\beta_0|b_0^\dagger&=\frac{1}{\sqrt{2}}\left[\xi-\frac{\partial}{\partial\xi}\right]\langle\xi,\eta,\beta_0|\ . \end{aligned} \tag{3.4}$$

All the other relations of Eq. (2.19) hold with the replacements $a_2=\eta$ and $a_0=\beta_0+\xi$.

Because of the form $L_z/4a_2$ occurring in Eq. (2.19), the coupling between the rotations around the intrinsic z axis and the a_2 vibrations is strong in the axially symmetric case. This coupling should therefore be treated exactly. For this purpose it is convenient to introduce the rotations around the intrinsic z axis into the state $|\xi,\eta,\beta_0\rangle$ itself and to write

$$\exp(-i\phi L_z)|\xi,\eta,\beta_0\rangle\equiv\frac{2}{\sqrt{\pi}}\exp(b_1^\dagger b_{-1}^\dagger)|\xi,\eta,\phi\rangle\ , \tag{3.5}$$

with

$$L_z=\sqrt{10}(d^\dagger\tilde{d})_0^1=2b_2^\dagger b_2+b_1^\dagger b_1-b_{-1}^\dagger b_{-1}-2b_{-2}^\dagger b_{-2}\ . \tag{3.6}$$

If one uses the SO(3) group property of Eq. (2.17) in the completeness relation (3.3a), it is easy to verify that

$$\frac{2}{\pi^2}\int_0^{2\pi}d\phi\int d\Omega\int_S d\tau(\xi,\eta)R(\Omega)P_N\exp(b_1^\dagger b_{-1}^\dagger)\,|\,\xi,\eta,\phi\rangle\langle\xi,\eta,\phi\,|\,\exp(b_1 b_{-1})P_N R^{-1}(\Omega)=1_N \; . \tag{3.7}$$

Actually, all that is done in Eq. (3.7) is that states with a fixed projection of the angular momentum are projected out of the state $|\,\xi,\eta,\beta_0\rangle$. Another important property of the states $|\,\xi,\eta,\phi\rangle$ is that they form a complete set of states for the boson Fock space

$$\{(n_0!n_2!n_{-2}!)^{-1/2}(b_0^\dagger)^{n_0}(b_2^\dagger)^{n_2}(b_{-2}^\dagger)^{n_{-2}}\,|\,\bar{0}\rangle\equiv|\,n_0,n_2,n_{-2}\rangle\}\; .$$

Indeed, from the results of Ref. 18 it is easy to verify that

$$\int_{-\infty}^{\infty}d\xi\int_0^{2\pi}d\phi\int_0^{\infty}\eta\,d\eta\,|\,\xi,\eta,\phi\rangle\langle\xi,\eta,\phi\,|=\tfrac{1}{2}\sum_{n_0,n_2,n_{-2}}|\,n_0,n_2,n_{-2}\rangle\langle n_0,n_2,n_{-2}\,|\; . \tag{3.8}$$

A final property of the state $|\,\xi,\eta,\phi\rangle$ worthwhile noting is that it is the vacuum of the b_1 and b_{-1} bosons, i.e.,

$$b_1\,|\,\xi,\eta,\phi\rangle=b_{-1}\,|\,\xi,\eta,\phi\rangle=0\; . \tag{3.9}$$

In particular this implies that

$$L_z\,|\,\xi,\eta,\phi\rangle=2(b_2^\dagger b_2-b_{-2}^\dagger b_{-2})\,|\,\xi,\eta,\phi\rangle\equiv J_z\,|\,\xi,\eta,\phi\rangle \tag{3.10}$$

and

$$\langle\xi,\eta,\phi\,|\,J_zR^{-1}(\Omega)=\langle\xi,\eta,\phi\,|\,L_zR^{-1}(\Omega)=\tilde{L}_z'\langle\xi,\eta,\phi\,|\,R^{-1}(\Omega)\;, \tag{3.11}$$

where $\tilde{L}_z'$ is a differential operator acting on the Euler angles [see Eq. (2.21)].

IV. WELL-DEFORMED AXIALLY SYMMETRIC SYSTEMS

Suppose that the ground state of the system is well deformed, hence the fluctuations of ξ and η in the ground state are small compared to β_0, i.e.,

$$\frac{\langle\xi^2\rangle}{\beta_0^2}\ll 1\;,\qquad \frac{\langle\eta^2\rangle}{\beta_0^2}\ll 1\;. \tag{4.1}$$

Here $\langle\;\rangle$ denotes the expectation value in the ground state. The lowest excited states which involve a small number of b_μ bosons and low angular momenta are then also expected to satisfy the condition (4.1). From now on we only consider the subspace of states satisfying the condition (4.1). Our aim is to obtain the excitation energies and moments of inertia of these states to leading order. Furthermore we are also seeking a prescription to calculate the leading order contribution to a transition matrix element between these states.

To do this we consider the matrix element of any U(6) operator $\hat{O}_\mu^l$, transforming like the SO(3) representation l, between two U(6) states satisfying the condition (4.1). Next we insert the identity (3.7) between the bra state and the operator. Furthermore we note that, under the assumption that the system is well deformed in the sense of (4.1), the finite integration limits can be ignored and the region of integration extended to $\xi\in(-\infty,\infty)$ and $\eta\in[0,\infty)$. Finally, using Eq. (3.8) leads to

$$\langle[N],\epsilon',L',M'\,|\,\hat{O}_\mu^l\,|\,[N],\epsilon,L,M\rangle\approx\int d\Omega\langle[N],\epsilon',L',M'\,|\,R(\Omega)T^\dagger T\sum_K D_{\mu K}^{l*}\hat{O}_K^l R^{-1}(\Omega)\,|\,[N],\epsilon,L,M\rangle\; . \tag{4.2}$$

Here we have introduced the operator

$$T=\frac{2(3\sqrt{2})^{1/2}\beta_0}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{2\pi}d\phi\int_0^{\infty}\eta\,d\eta f^{1/2}(\hat{\xi},\hat{\eta}^2)\,|\,\xi,\eta,\phi\rangle\langle\xi,\eta,\phi\,|\,\exp(b_1b_{-1})\;, \tag{4.3}$$

with $T^\dagger$ the Hermitian conjugate of T. The operators $\hat{\xi}$ and $\hat{\eta}^2$ are defined by

$$\hat{\xi}=\frac{1}{\sqrt{2}}(b_0^\dagger+b_0)\;,\qquad \hat{\eta}^2=\tfrac{1}{2}(b_2^\dagger b_{-2}^\dagger+b_2b_{-2}+b_2^\dagger b_2+b_{-2}^\dagger b_{-2}+1)\;, \tag{4.4}$$

and we have noted the properties

$$\langle\xi,\eta,\phi\,|\,\hat{\xi}=\langle\xi,\eta,\phi\,|\,\xi\;,\qquad \langle\xi,\eta,\phi\,|\,\hat{\eta}^2=\langle\xi,\eta,\phi\,|\,\eta^2\;. \tag{4.5}$$

From Eq. (4.4) it is easy to verify the following commutation relation:

$$[[b_0,\hat{\xi}],\hat{\xi}]=[[b_0^\dagger,\hat{\xi}],\hat{\xi}]=0\ ,$$
$$[b_\mu,\hat{\xi}]=[b_\mu^\dagger,\hat{\xi}]=0,\quad \forall\mu\neq 0\ ,$$
$$[b_0,\hat{\eta}^2]=[b_0^\dagger,\hat{\eta}^2]=0\ , \tag{4.6}$$
$$[[b_\mu,\hat{\eta}^2],\hat{\eta}^2]=[[b_\mu^\dagger,\hat{\eta}^2],\hat{\eta}^2]=0,\quad \mu=\pm 2\ .$$

Using Eq. (4.6) one obtains the relations

$$Tb_0^\dagger=\left[b_0^\dagger+\frac{f^{-1}(\hat{\xi},\hat{\eta}^2)}{\sqrt{2}\beta_0}\left[1+\frac{\hat{\xi}}{\beta 0}\right]\right]T\ , \tag{4.7a}$$

$$Tb_0=\left[b_0-\frac{f^{-1}(\hat{\xi},\hat{\eta}^2)}{\sqrt{2}\beta_0}\left[1+\frac{\hat{\xi}}{\beta_0}\right]\right]T\ , \tag{4.7b}$$

$$Tb_\mu^\dagger=\left[b_\mu^\dagger-\frac{f^{-1}(\hat{\xi},\hat{\eta}^2)}{6\beta_0^2}(b_\mu^\dagger+b_{-\mu})\right]T,\quad \mu=\pm 2\ , \tag{4.7c}$$

$$Tb_\mu=\left[b_\mu+\frac{f^{-1}(\hat{\xi},\hat{\eta}^2)}{6\beta_0^2}(b_{-\mu}^\dagger+b_\mu)\right]T,\quad \mu=\pm 2\ , \tag{4.7d}$$

$$Tb_{\pm 1}R^{-1}=\left[\frac{\tilde{L}'_x}{2\sqrt{3}\beta_0}B_+\pm\frac{i\tilde{L}'_y}{2\sqrt{3}\beta_0}B_-\right]TR^{-1}\ , \tag{4.7e}$$

$$Tb_\pm^\dagger R^{-1}=\left[\frac{\tilde{L}'_x}{2\sqrt{3}\beta_0}B_+\mp\frac{i\tilde{L}'_y}{2\sqrt{3}\beta_0}B_-\right]TR^{-1}\ , \tag{4.7f}$$

$$TJ_zR^{-1}=J_zTR^{-1}=TL_zR^{-1}=\tilde{L}'_zTR^{-1}\ . \tag{4.7g}$$

Here we introduced the notation

$$B_\pm=\left[\left[\frac{2\hat{\eta}^2}{3\beta_0^2}\right]^{1/2}\pm\left[1+\frac{\hat{\xi}}{\beta_0}\right]\right]^{-1}\ . \tag{4.7h}$$

Note that the action of the boson operator J_z on the wave function is the same as the action of the differential operator $\tilde{L}'_z$. With the aid of Eqs. (4.7) it is possible to calculate the intrinsic components of any operator in terms of the b_0, b_2, and b_{-2} bosons as well as the angular momentum operators $\tilde{L}'_x$, $\tilde{L}'_y$, and $\tilde{L}'_z$. To do this we expand the operator $\hat{O}^l_K$ in terms of the b_μ bosons

$$\hat{O}^l_K=A(\beta_0)+\sum_\mu[A^\mu(\beta_0)b_\mu^\dagger+A_\mu(\beta_0)b_\mu]+\sum_{\mu\geq\nu}[A^{\mu\nu}(\beta_0)b_\mu^\dagger b_\nu^\dagger+A_{\mu\nu}(\beta_0)b_\mu b_\nu]+\sum_{\mu\nu}A^\mu_\nu(\beta_0)b_\mu^\dagger b_\nu+\cdots\ . \tag{4.8}$$

As we have already remarked, the assumption of axial symmetry implies that the b bosons transform properly under SO(2) [see (3.2)]. Consequently the only nonvanishing terms on the right-hand side of Eq. (4.8) are those having the same SO(2) transformation properties as the operator $\hat{O}^l_K$. In particular, this implies that for the Hamiltonian, which is an SO(2) scalar, all terms that do not commute with L_z must vanish. Among these terms are the terms linear in the b_μ ($\mu\neq 0$) bosons. Using Eqs. (4.7) and (4.8) it is possible to cast Eq. (4.2) into the form

$$\langle[N],\epsilon',L',M'\,|\,\hat{O}^l_\mu\,|\,[N],\epsilon,L,M\rangle\approx\int d\Omega\sum_{K_1}D^{L'}_{M'K_1}\left\langle[N],\epsilon',L',K_1\,\middle|\,T^\dagger\sum_{K_2}D^{l*}_{\mu K_2}(\hat{O}^l_{K_2})_{\rm intr}(b_\nu^\dagger,b_\nu,\tilde{L}'_i)\times\sum_{K_3}D^{L*}_{MK_3}T\,\middle|\,[N],\epsilon,L,K_3\right\rangle\ . \tag{4.9}$$

The intrinsic components, $(\hat{O}^l_K)_{\rm intr}$, of the operator are obtained by replacing the b_0 and $b_{\pm 2}$ bosons in the expansion (4.8) by the corresponding expressions of Eq. (4.7). For the $b_{\pm 1}$ bosons the situation is more complicated since the boson angular momentum operators have to be commuted to the immediate left of the rotation operator before they can be replaced by differential operators acting on the Euler angles. This leads to rather complicated corrections to the intrinsic components of the operator. It is, however, simple to see that these corrections can also be expressed in terms of the $b_0,b_{\pm 2}$ bosons and the angular momentum operators $\tilde{L}'_i$. In this way one obtains for the intrinsic operator a constant term, terms which depend only on the $b_0,b_{\pm 2}$ bosons (called vibrational), terms which depend only on the angular momentum operators $\hat{L}'_i$ (called rotational), and mixed terms (called vibrational-rotational). The intrinsic component thus has the form

$$(\hat{O}^l_K)_{\rm intr}=(\hat{O}^l_K)_{\rm intr,0}+(\hat{O}^l_K)_{\rm intr,vib}(b_\nu^\dagger,b_\nu)+(\hat{O}^l_K)_{\rm intr,rot}(\tilde{L}'_i)+(\hat{O}^l_K)_{\rm intr,rot,vib}(b_\nu^\dagger,b_\nu,\tilde{L}'_i)\ . \tag{4.10}$$

The operators given in Eq. (4.10) are extremely complicated and an approximation scheme is clearly called for. However, before any calculation of the intrinsic operators can be attempted, we need a prescription to calculate the expansion coefficients of Eq. (4.8) as well as β_0. To calculate the expansion coefficients of Eq. (4.8) we use the relations

$$A(\beta_0)=\langle\bar{0}\,|\,\hat{O}^l_K\,|\,\bar{0}\rangle\ ,$$
$$A^\nu(\beta_0)=\langle\bar{0}\,|\,[b_\nu,\hat{O}^l_K]\,|\,\bar{0}\rangle\ ,$$
$$A_\nu(\beta_0)=\langle\bar{0}\,|\,[\hat{O}^l_K,b_\nu^\dagger]\,|\,\bar{0}\rangle\ ,$$

$$A^{\mu}_{\nu}(\beta_0)=\langle \bar{0} \,|\, [b_\mu[\hat{O}^{\,l}_{K},b^\dagger_\nu]] \,|\, \bar{0}\rangle \,,$$
$$A^{\mu\nu}(\beta_0)=(1+\delta_{\mu\nu})^{-1}\langle \bar{0} \,|\, [b_\mu b_\nu,\hat{O}^{\,l}_{K}] \,|\, \bar{0}\rangle \,,$$
$$A_{\mu\nu}(\beta_0)=(1+\delta_{\mu\nu})^{-1}\langle \bar{0} \,|\, [\hat{O}^{\,l}_{K},b^\dagger_\mu b^\dagger_\nu] \,|\, \bar{0}\rangle \,. \quad (4.11)$$

The coefficients of higher order terms are calculated in a similar way, the only difference being that the expectation values of commutators with more than two boson operators occur. Introducing the Glauber coherent state[24]

$$|\,\bar{z}\rangle=\exp\left[-\tfrac{1}{2}\sum_{\mu=-2}^{2} z^*_\mu z_\mu+\sum_{\mu=-2}^{2} z_\mu d^\dagger_\mu\right]|\,0\rangle \quad (4.12)$$

one has

$$|\,\bar{z}=\{\beta_0/\sqrt{2}\}\,\rangle=|\,\bar{0}\rangle \,. \quad (4.13)$$

Here we have used the short hand notation $\{\beta_0/\sqrt{2}\}$ to denote the set $\{z_0=\beta_0/\sqrt{2},\ z_\mu=0;\ \forall\mu\neq 0\}$. The expansion coefficients can now be calculated in the following way:[18]

$$\begin{aligned}
A(\beta_0)&=\{\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle_{\{\beta_0/\sqrt{2}\}}\,,\\
A^{\nu}(\beta_0)&=\left\{\frac{\partial}{\partial z^*_\nu}\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}}\,,\\
A_{\nu}(\beta_0)&=\left\{\frac{\partial}{\partial z_\nu}\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}}\,,\\
A^{\mu}_{\nu}(\beta_0)&=\left\{\frac{\partial^2}{\partial z^*_\mu\partial z_\nu}\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}}\,,\\
A^{\mu\nu}(\beta_0)&=(1+\delta_{\mu\nu})^{-1}\left\{\frac{\partial^2}{\partial z^*_\mu\partial z^*_\nu}\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle\right\}_{\{\beta/\sqrt{2}\}}\,,\\
A_{\mu\nu}(\beta_0)&=(1+\delta_{\mu\nu})^{-1}\left\{\frac{\partial^2}{\partial z_\mu\partial z_\nu}\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}}\,.
\end{aligned} \quad (4.14)$$

The variables z_μ and z^*_μ are treated as independent complex variables. For higher order terms one finds similar expressions for the expansion coefficients, the only difference being that they involve higher order derivatives. Indeed, it is easy to see that the expression for a particular A involves a derivative of an order which is equal to the total number of indices on A, i.e., the total number of boson creation and annihilation operators occurring in the term under consideration.

The value of β_0 for which the potential energy is minimized, is determined by setting the first order derivatives

$$A^{\nu}(\beta_0)=\left\{\frac{\partial}{\partial z^*_\nu}\langle \bar{z}\,|\,H\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}} =A_{\nu}(\beta_0)=\left\{\frac{\partial}{\partial z_\nu}\langle \bar{z}\,|\,H\,|\,\bar{z}\rangle\right\}_{\{\beta_0/\sqrt{2}\}}$$

zero. Here we have used the fact that the Hamiltonian, H, is Hermitian to note that $A^{\nu}(\beta_0)=A_{\nu}(\beta_0)$. If the potential energy is indeed axially symmetric, all the derivatives with respect to z^*_ν $(\nu\neq 0)$ should vanish when calculated at $\gamma=0$. This should serve as a check for axial symmetry. The value of β_0 is therefore determined by setting the derivative with respect to z^*_0 zero. This is equivalent to demanding that the terms linear in the $b^\dagger_0$ and b_0 bosons vanish in the intrinsic Hamiltonian. As is well known,[1-6] and as one can easily convince oneself, $\beta_0\sim\sqrt{N}$ (N is the total number of s and d bosons) in the deformed regions.

For a one-body operator it is simple to verify from the properties of the Glauber state that to leading order in N the expectation value $\langle \bar{z}\,|\,\hat{O}^{\,l}_{K}\,|\,\bar{z}\rangle$ is a homogeneous polynomial of degree 2 in z^*_μ, z_μ, and $(N-\sum_\mu z^*_\mu z_\mu)^{1/2}$. Similarly, for a two-body operator it is a homogeneous polynomial of degree 4. From this, Eq. (4.14) and the N dependence of β_0, it easily follows that for a one body operator $A(\beta_0)\sim N$, $A^{\nu}(\beta_0)$ and $A_\nu(\beta_0)\sim\sqrt{N}$, etc. Similarly, for a two-body operator $A(\beta_0)\sim N^2$, $A^{\nu}(\beta_0)$ and $A_\nu(\beta_0)\sim N^{3/2}$, etc.

The obvious approximation for the intrinsic operators of Eq. (4.10) would be to truncate them to leading order in $1/N$. From the discussion above it is clear that the leading order contribution to the intrinsic operator is simply the constant term, if it does not vanish. However, if one truncates the intrinsic operator at the constant term, the dynamics of the system is neglected completely. For instance, transition matrix elements between orthogonal states will vanish. Furthermore, making this approximation in the Hamiltonian would only provide one with information about the leading order behavior of the ground state energy while no information about excitation energies and moments of inertia will be obtained. To include the dynamics one has to consider higher order contributions in $1/N$ to the intrinsic operator. Precisely where one truncates the operator depends on the question under consideration. For instance, if a transition matrix element vanishes for an operator calculated to a certain order in $1/N$, one would want to check the next order as well. For our present purposes we just consider the leading order contribution to the constant term, the vibrational part and the rotational part, while we neglect the rotational-vibrational part. For the Hamiltonian this approximation implies that we consider the leading order behavior of the ground state energy, the intrinsic excitation energies and the moments of inertia, while all interactions between rotations and vibrations are neglected. This is the information required in lowest order, namely, one wants to know what are the typical vibrational energies and what governs the rotational spectrum. Only after knowing this would one start to refine the picture by including the next order including the coupling.

Using the N dependence of the expansion coefficients one can show that it is consistent in the above approximation to use the approximations

$$Tb^\dagger_\mu\approx b^\dagger_\mu T,\quad \forall\mu=0,\pm 2\,,$$
$$Tb_\mu\approx b_\mu T,\quad \forall\mu=0,\pm 2\,,$$

$$Tb_{\pm 1}R^{-1} \approx \frac{\tilde{L}'_x \mp i\tilde{L}'_y}{2\sqrt{3}\beta_0} TR^{-1} ,$$

$$Tb^\dagger_{\pm 1}R^{-1} \approx \frac{\tilde{L}'_x \pm i\tilde{L}'_y}{2\sqrt{3}\beta_0} TR^{-1} , \qquad (4.15)$$

instead of the expressions (4.7), and to neglect corrections arising from commuting the boson angular momentum operators to the left of the rotation operator before replacing them by differential operators. In this approximation the calculation of the intrinsic components of an operator therefore simplifies considerably. In the expansion (4.8) one keeps only the constant term and the lowest order terms in the b_μ bosons. To obtain the intrinsic component one then simply replaces the $b_{\pm 1}$ bosons by the corresponding expressions in Eq. (4.15), while keeping the b_0 and $b_{\pm 2}$ bosons. This gives the leading order contribution to the vibrational and rotational parts, while all rotation-vibration interactions are clearly neglected.

V. EXCITATION ENERGIES AND MOMENTS OF INERTIA FOR A QUADRUPOLE-QUADRUPOLE INTERACTION

To illustrate the use of the approach outlined in Secs. I–IV, we consider a specific example here. We take for the Hamiltonian

$$H = \kappa_1 Q\cdot Q + \kappa_2 L\cdot L , \qquad (5.1a)$$

with

$$Q^2_\mu = (s^\dagger \tilde{d})^2_\mu + (d^\dagger s)^2_\mu + \chi(d^\dagger\tilde{d})^2_\mu . \qquad (5.1b)$$

In the Holstein-Primakoff realization of Eq. (2.2) the quadrupole operator takes the form

$$Q^2_\mu = (N-\hat{n}_d)^{1/2}\tilde{d}^{\,2}_\mu + d^{2\dagger}_\mu (N-\hat{n}_d)^{1/2} + \chi(d^\dagger\tilde{d})^2_\mu . \qquad (5.2)$$

Note from Eq. (2.21) that the replacement

$$L\cdot L \to \tilde{L}'\cdot\tilde{L}' \qquad (5.3)$$

can immediately be made. We therefore only have to consider the quadrupole-quadrupole part.

To calculate the excitation energies and moments of inertia in the approximation of Sec. IV, we have to employ the expansion (4.8) to second order in b_μ and $b^\dagger_\mu$. Under the assumption that the system has axial symmetry the first order terms in b_μ and $b^\dagger_\mu$ ($\mu\neq 0$) must vanish. Indeed, all terms that do not commute with L_z must vanish. This should serve as a check that the system is indeed axially symmetric. Furthermore, the first order terms in b_0 and $b^\dagger_0$ are set zero, thereby determining the value of β_0. The first nonvanishing terms in the expansion (4.8) are therefore quadratic in the boson operators. To calculate the expansion coefficients we use Eq. (4.14). To leading order N the expectation value $\langle \bar{z} \mid Q\cdot Q \mid \bar{z}\rangle$ is given by

$$\langle \bar{z} \mid Q\cdot Q \mid \bar{z}\rangle = \sum_\mu (-1)^\mu Q_\mu(\bar{z}) Q_{-\mu}(\bar{z}) , \qquad (5.4)$$

where $Q_\mu(\bar{z})$ is the function obtained by making the replacements $b^\dagger_\mu \to z^*_\mu$ and $b_\mu \to z_\mu$ in the quadrupole operator. For the constant term one then has

$$A(\chi,\beta_0) = 2\beta_0^2\left[\Delta - \frac{\bar{\chi}\beta_0}{2}\right]^2 , \qquad (5.5a)$$

with

$$\Delta = \left[N - \frac{\beta_0^2}{2}\right]^{1/2} ,$$

$$\bar{\chi} = \frac{\chi}{\sqrt{7}} . \qquad (5.5b)$$

For the expansion coefficients of the linear terms one finds

$$A^\mu = A_\mu = 0, \quad \forall \mu\neq 0 . \qquad (5.6)$$

Setting the first order derivatives $A^0(\chi,\beta_0) = A_0(\chi,\beta_0)$ zero yields

$$\beta_0 = \frac{\chi}{|\chi|}\left[\frac{28N}{\chi^2+14}\right]^{1/2} , \qquad (5.7a)$$

or

$$\beta_0 = -\epsilon\sqrt{N}\left[1+\epsilon\frac{\chi}{(\chi^2+14)^{1/2}}\right]^{1/2} , \quad \epsilon=\pm 1 . \qquad (5.7b)$$

Which of these solutions gives a stable minimum is determined by the second order derivatives. Note that $\beta_0 \sim \sqrt{N}$ and hence if we consider a system with a large number of particles and restrict ourselves to the low-lying states where ξ and η scale like unity, we have $\langle \xi^2\rangle/\beta_0^2 \sim 1/N$ and $\langle \eta^2\rangle/\beta_0^2 \sim 1/N$. In this case conditions (4.1) are thus satisfied. From Eq. (4.14) one also finds

$$A^\mu_\nu = 0 \ \text{ if } \mu\neq\nu ,$$

$$A^{\mu\nu} = A_{\mu\nu} = 0 \ \text{ if } \mu \neq -\nu . \qquad (5.8)$$

Equations (5.6) and (5.8) confirm that the system is indeed axially symmetric. Furthermore one has

$$A^2_2 = A^{-2}_{-2} = 2(x_7^2 + x_2 x_8) ,$$

$$A^1_1 = A^{-1}_{-1} = 2(x_1^2 - x_2 x_3) ,$$

$$A^0_0 = 2x_2 x_4 ,$$

$$A^{2-2} = A_{2-2} = 2x_7^2 , \qquad (5.9a)$$

$$A^{1-1} = A_{1-1} = -2x_1^2 ,$$

$$A^{00} = A_{00} = x_2 x_6 + x_5^2 ,$$

with

$$x_1 = \Delta - \bar{\chi}\beta_0 ,$$

$$x_2 = \sqrt{2}\beta_0\left[\Delta - \frac{\bar{\chi}\beta_0}{2}\right] ,$$

$$x_3=\frac{1}{\sqrt{2}}(\Delta^{-1}\beta_0+\bar{\chi})\ ,$$
$$x_4=-\sqrt{2}\left[\Delta^{-1}\beta_0+\frac{\Delta^{-3}\beta_0^3}{8}+\bar{\chi}\right]\ ,$$
$$x_5=\left[\Delta-\frac{\Delta^{-1}\beta_0^2}{2}-\bar{\chi}\beta_0\right]\ , \qquad (5.9b)$$
$$x_6=-\frac{\Delta^{-1}\beta_0}{\sqrt{2}}\left[1+\frac{\Delta^{-2}\beta_0^2}{4}\right]\ ,$$
$$x_7=\Delta+\bar{\chi}\beta_0\ ,$$
$$x_s=-\frac{1}{\sqrt{2}}(\Delta^{-1}\beta_0-2\bar{\chi})\ .$$

Using Eqs. (4.15), (5.6), (5.8), and (5.9), one has for the intrinsic Hamiltonian

$$H_{\text{intr}}=\kappa_1\Big[A(\chi,\beta_0)+A_0^0 b_0^\dagger b_0+A^{00}(b_0^\dagger b_0^\dagger+b_0 b_0)+A_2^2(b_2^\dagger b_2+b_{-2}^\dagger b_{-2})+A^{2-2}(b_2^\dagger b_{-2}^\dagger+b_2 b_{-2})-\frac{x_2x_3}{3\beta_0^2}[(\tilde{L}'_x)^2+(\tilde{L}'_y)^2]\Big]+\kappa_2\tilde{L}'\cdot\tilde{L}'\ . \qquad (5.10)$$

From Eq. (5.10) we can identify the moment of inertia about the x and y axes. Note that the reciprocal moment of inertia about the z axis, resulting from the quadrupole-quadrupole part, vanishes. This is because of the axial symmetry of the system. The vibrational part can be diagonalized by introducing the following Bogoliubov transformations:

$$b_\mu^\dagger=a_\mu^\dagger\cosh\phi_{|\mu|}+a_{-\mu}\sinh\phi_{|\mu|},\quad \mu=0,\pm 2\ ,$$
$$b_\mu=a_{-\mu}^\dagger\sinh\phi_{|\mu|}+a_\mu\cosh\phi_{|\mu|},\quad \mu=0,\pm 2\ . \qquad (5.11)$$

Note that the SO(2) transformation properties are preserved by (5.11). Indeed, it can easily be verified that in terms of the a_μ bosons the angular momentum operator J_z simply reads $J_z=2(a_2^\dagger a_2-a_{-2}^\dagger a_{-2})$. Using (5.11) one finds for the diagonalized intrinsic Hamiltonian

$$H_{\text{intr}}=E_0(\chi,\beta_0)+\kappa_1(A_0^0\cosh 2\phi_0+2A^{00}\sinh\phi_0)a_0^\dagger a_0+\kappa_1(A_2^2\cosh 2\phi_2+A^{2-2}\sinh\phi_2)(a_2^\dagger a_2+a_{-2}^\dagger a_{-2})-\frac{x_2x_3\kappa_1}{3\beta_0^2}[(\tilde{L}'_x)^2+(\tilde{L}'_y)^2]+\kappa_2\tilde{L}'\cdot\tilde{L}'\ , \qquad (5.12)$$

where all constant terms have been absorbed in $E_0(\chi,\beta_0)$ and ϕ_0,ϕ_2 satisfy

$$\tanh\phi_0=-\frac{2A^{00}}{A_0^0}\ ,$$
$$\tanh\phi_2=-\frac{A^{2-2}}{A_2^2}\ . \qquad (5.13)$$

As a special case we consider the well known SU(3) limit of the IBM.[25] If one chooses $\bar{\chi}=-\frac{1}{2}$ the quadrupole operator of Eq. (5.1b) and the angular momentum operators of Eq. (2.1) span a SU(3) subalgebra of the U(6) algebra. In this case the Hamiltonian of Eq. (5.1a) can be written in terms of the second order SU(3) and SO(3) Casimir operators. One has[25]

$$H_{\text{SU(3)}}=\tfrac{1}{2}\kappa_1 C_{2\text{SU(3)}}+(\kappa_2-\tfrac{3}{8}\kappa_1)L\cdot L\ . \qquad (5.14)$$

The eigenstates can consequently be labeled by $|[N],(\lambda,\mu),\kappa,L,M\rangle$, where (λ,μ) labels the SU(3) representations contained within the symmetric U(6) representation $[N]$. The quantum number κ labels repeated SO(3) representations contained within a particular SU(3) representation. We remark that we always use the orthogonal Vergados basis.[26] The corresponding eigenvalues are given by[25]

$$E(\lambda,\mu,L)=\tfrac{1}{2}\kappa_1[\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)]+(\kappa_2-\tfrac{3}{8}\kappa_1)L(L+1)\ . \qquad (5.15)$$

The allowed values of the quantum numbers λ, μ, κ, and L for a particular N can be found in Ref. 25. The only states of importance to us are the ground band states, $|[N],(2N,0),\kappa=0,L,M\rangle$ $(L=0,2,\ldots,2N)$, the β-band states, $|[N],(2N-4,2),\kappa=0,L,M\rangle$ $(L=0,2,\ldots,2N-4)$, and the γ-band states, $|[N],(2N-4,2),\kappa=2,L,M\rangle$ $(L=2,3,\ldots,2N-4)$.

For this choice of $\bar{\chi}$ one finds that (5.7b) with $\epsilon=-1$ gives a stable minimum if $\kappa_1<0$. The value of β_0 is then given by

$$\beta_0=\frac{2\sqrt{N}}{\sqrt{3}}\ . \qquad (5.16)$$

The Bogoliubov transformation which diagonalizes H_{intr} is given by

$$\cosh\phi_0=\frac{2}{\sqrt{3}},\quad \sinh\phi_0=-\frac{1}{\sqrt{3}}\ ,$$
$$\cosh\phi_2=1,\quad \sinh\phi_2=0\ . \qquad (5.17)$$

The β and γ excitation energies follow from Eqs. (5.9) and (5.12),

$$E_\beta=E_\gamma=-6N\kappa_1\ . \qquad (5.18)$$

With $\kappa_2=0$ we obtain from Eqs. (5.9) and (5.12) for the moments of inertia

$$\frac{1}{2J_x}=\frac{1}{2J_y}=-\frac{3\kappa_1}{8}\ . \qquad (5.19)$$

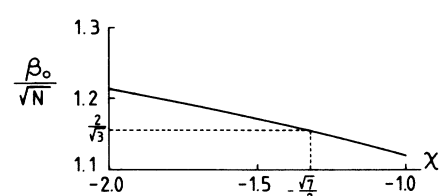

FIG. 1. The scaled equilibrium deformation $\beta_0/\sqrt{N}$ as a function of χ.

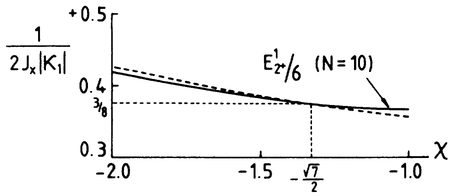

FIG. 2. The scaled reciprocal moments of inertia $1/2J_x\,|\kappa_1| = 1/2J_y\,|\kappa_1|$ as a function of χ. In these calculations $\kappa_2 = 0$ was used. The dashed line shows the approximate calculation. The solid line shows the exact result as obtained from dividing the excitation energy of the first excited 2^+ state, $E^1_{2^+}$, by $L(L+1)=6$ in a calculation with $N=10$.

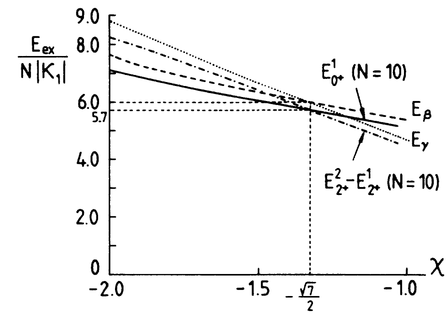

FIG. 3. The scaled β- and γ-excitation energies $E_\beta/N\,|\kappa_1|$ and $E_\gamma/N\,|\kappa_1|$ as a function of χ. The dashed and the dotted lines show the approximate β- and γ-excitation energies, respectively. The exact β-excitation energy as obtained from the excitation energy of the first excited 0^+ state, $E^1_{0^+}$, in a calculation with $N=10$, is shown by the solid line. The dashed-dotted line shows the exact γ-excitation energy as obtained from the difference between the excitation energies of the second and first excited 2^+ states, $E^2_{2^+} - E^1_{2^+}$, in a calculation with $N=10$.

Comparing these results with the exact result of Eq. (5.15) with $\kappa_2=0$, one finds agreement to leading order in N. The same results have also been obtained in several other analyses.[7,12–15]

Using Eqs. (5.7), (5.9), (5.12), and (5.13), it is easy to calculate the β- and γ-excitation energies and moments of inertia in general. This calculation is shown in Figs. 1–3 for $\chi\in[-2,-1]$ and $\kappa_2=0$. To obtain a stable minimum one must demand $\kappa_1<0$ throughout. In Fig. 1 we show the equilibrium value $\beta_0/\sqrt{N}$ as a function of χ. In Fig. 2 we show the value of $1/2J_x\,|\kappa_1| = 1/2J_y\,|\kappa_1|$ as a function of χ. We also show the exact value as obtained from dividing the excitation energy of the first excited 2^+ state, $E^1_{2^+}$, by $L(L+1)=6$ in a calculation with $N=10$. In Fig. 3 we show $E_\gamma/N\,|\kappa_1|$ and $E_\beta/N\,|\kappa_1|$ as a function of χ. The exact values as obtained from the excitation energy of the first excited 0^+ state, $E^1_{0^+}$, and the difference between the excitation energies of the second and first excited 2^+ states, $E^2_{2^+}-E^1_{2^+}$, in a calculation with $N=10$, are also shown. Good agreement between the exact and approximate results is found. However, for large values of χ and for values close to zero, the agreement starts to break down which can be understood as follows. If one increases χ, the dominating term in the quadrupole-quadrupole part is the term $(d^\dagger\tilde{d})^2\cdot(d^\dagger\tilde{d})^2$, which correspond to a spherically symmetric rather than an axially deformed system. Hence, as χ is increased a transformation from an axially deformed potential to a spherically symmetric potential occurs. In the spherically symmetric case the conditions (4.1) are not satisfied and one can therefore expect that the agreement would start to break down. On the other hand, as χ approaches zero one tends towards an SO(6) symmetry. In this limit the potential energy becomes flat in the γ direction and consequently the conditions (4.1) are once again violated.

VI. INTRINSIC COMPONENTS OF THE $E2$-TRANSITION OPERATOR

We take for the $E2$-transition operator in the IBM the quadrupole operator of Eq. (5.1b). In the approach of Sec. IV the leading order behavior of the $E2$-transition matrix elements are calculated from Eq. (4.9) using the quadrupole operator

$$\bar{Q}^2_\mu = \sum_K D^{2*}_{\mu K}(Q^2_K)_{\text{intr}}(b^\dagger_\mu, b_\mu)\ , \tag{6.1}$$

where the intrinsic components of the operators are calculated in the present approximation as described in Sec. IV, using Eq. (4.15). The necessary expansion coefficients of Eq. (4.8) can easily be calculated with the aid of Eq. (4.14). Using Eq. (4.15) we obtain for the intrinsic components, up to second order in the boson operators b_μ, the following expressions:

$$\begin{aligned}
(Q^2_2)_{\text{intr}} &= (Q^2_{-2})^\dagger_{\text{intr}} = (\Delta+\bar{\chi}\beta_0)(b^\dagger_2+b_{-2}) - \frac{1}{2\sqrt{2}}(\Delta^{-1}\beta_0-4\bar{\chi})(b^\dagger_2 b_0 + b^\dagger_0 b_{-2}) - \frac{\Delta^{-1}\beta_0}{2\sqrt{2}}(b^\dagger_2 b^\dagger_0 + b_0 b_{-2})\ ,\\
(Q^2_1)_{\text{intr}} &= (Q^2_{-1})^\dagger_{\text{intr}} = 0\ ,\\
(Q^2_0)_{\text{intr}} &= \sqrt{2}\beta_0\left[\Delta - \frac{\bar{\chi}\beta_0}{2}\right] - \sqrt{2}\left[\Delta^{-1}\beta_0 + \frac{\Delta^{-3}\beta_0^3}{8} + \bar{\chi}\right] b^\dagger_0 b_0 - \frac{\Delta^{-1}\beta_0}{2\sqrt{2}}\left[1+\frac{\Delta^{-2}\beta_0^2}{4}\right](b^\dagger_0 b^\dagger_0 + b_0 b_0)\\
&\quad - \frac{1}{\sqrt{2}}(\Delta^{-1}\beta_0 - 2\bar{\chi})(b^\dagger_2 b_2 + b^\dagger_{-2} b_{-2}) + \left[\Delta - \frac{\Delta^{-1}\beta_0^2}{2} - \bar{\chi}\beta_0\right](b^\dagger_0 + b_0)\ .
\end{aligned} \tag{6.2}$$

Introducing the SU(3) values of $\bar{\chi}$ and β_0 as well as the Bogoliubov transformation which diagonalizes $H_{\rm intr}$ in the SU(3) case [see Eqs. (5.11) and (5.17)], one finds

$$(Q_2^2)_{\rm intr}=(Q_{-2}^2)_{\rm intr}^\dagger=-(\tfrac{3}{2})^{1/2}(a_2^\dagger a_0+a_0^\dagger a_{-2})\ ,$$
$$(Q_1^2)_{\rm intr}=(Q_{-1}^2)_{\rm intr}^\dagger=0\ , \quad (6.3)$$
$$(Q_0^2)_{\rm intr}=\sqrt{2}N-\frac{3}{\sqrt{2}}(a_0^\dagger a_0+a_2^\dagger a_2+a_{-2}^\dagger a_{-2})\ .$$

The most important aspect that should be noted from Eq. (6.2) is that the $K=\pm 1$ intrinsic components of the quadrupole operator vanish. In a random-phase approximation (RPA) approach these operators coincide with the excitation operators of the spurious zero-frequency modes.[12,13] Another noteworthy point is that the leading order behavior of the intraband $E2$-transition rates are determined by the constant term in $(Q_0^2)_{\rm intr}$.

The interesting feature of the SU(3) limit [Eq. (6.3)] is that the terms which change the total number of γ and β excitations, i.e., $\bar{N}=n_0+n_2+n_{-2}$ (here n_μ denotes the number of a_μ bosons and we use $\bar{N}$ to distinguish it from N, the total number of s and d bosons) vanish up to the order of unity in the $E2$-transition operator. Consequently $E2$ transitions between states with different values of $\bar{N}$ are prohibited. In particular, this implies that $E2$ transitions between the ground band and β or γ bands are prohibited. However, transitions between the β and γ bands are allowed. From Eq. (6.3) it is clear that the interband transitions are suppressed by a factor $1/N$ with respect to the intraband transitions.

This situation is completely analogous to the original SU(3) limit of the IBM. With the choice $\bar{\chi}=-\frac{1}{2}$ the quadrupole operator is a generator of the SU(3) group, and hence it cannot couple different SU(3) representations.[25] Consequently transitions between the ground band belonging to the $(2N,0)$ representation and the β or γ bands belonging to the $(2N-4,2)$ representation are prohibited. On the other hand, transitions between the β and γ bands are allowed since they belong to the same SU(3) representation. However, these interband transitions are also strongly suppressed in comparison with the intraband transitions.[25]

This analogy can be made even more explicit if one notes that in the SU(3) limit the intrinsic Hamiltonian of Eq. (5.12) can be written in the form

$$H_{\rm intr}=E_0-6N\kappa_1 C_1({\rm U}(3))+H_{\rm rot}\ . \quad (6.4)$$

Here $C_1({\rm U}(3))$ denotes the first order Casimir operator of the U(3) algebra spanned by $\{a_\mu^\dagger a_\nu;\ \mu,\nu=0,\pm 2\}$. The operator $C_1({\rm U}(3))$ is therefore simply the boson number operator $\hat{N}=\hat{n}_0+\hat{n}_2+\hat{n}_{-2}$. Note also that the Hilbert space on which this algebra is realized can carry only completely symmetric representations which are labeled by the eigenvalues of $C_1({\rm U}(3))$, i.e., the total number of bosons $\bar{N}$. We also remark that this U(3) algebra should not be confused with the original U(3) algebra of the IBM. From Eq. (6.3) we note that the $E2$-transition operator is also written in terms of the generators of this U(3) algebra. Therefore, $E2$ transitions are only allowed between states belonging to the same symmetric U(3) representation $[\bar{N}]$. This situation is completely analogous to the original SU(3) limit of the IBM. In practice one usually considers a quadrupole operator which breaks the SU(3) symmetry slightly. This introduces terms linear in the boson operators which allow transitions between states with different numbers of β and γ excitations. The discussion of the SU(3) limit illustrates that all the qualitative features of the $E2$-transition rates are preserved if the transition operator (6.3) is used in the prescription of Eq. (4.9).

Before one can calculate $E2$-transition rates quantitatively from Eq. (4.9) and the $E2$-transition operator of Eq. (6.2), one needs the states $T\,|\,[N],\alpha,L,K\,\rangle$ occurring in Eq. (4.9). To obtain these states explicitly is clearly an extremely difficult task. Instead, we use the following approach. Suppose we wish to calculate the leading order behavior of a $E2$-transition rate between two eigenstates of the IBM Hamiltonian. Instead of using the states $T\,|\,[N],\alpha,L,K\,\rangle$ in (4.9), we use eigenstates of the intrinsic Hamiltonian (5.12) having excitation energies which agree to leading order in N with the excitation energies of the IBM eigenstates under consideration. Note, however, that the states $T\,|\,[N],\alpha,L,K\,\rangle$ already have the required invariance properties under the basic rotations of the intrinsic axes.[21,22] On the other hand, solving for the eigenstates of (5.12) one still has to impose the required invariance properties. Therefore, one should use the correctly symmetrized states in Eq. (4.9).

The excitation rates calculated in this way agree to leading order in N with the exact transition rates. We illustrate this by means of some examples. For this purpose it is again convenient to consider the SU(3) limit since the exact transition rates can be calculated analytically there. As an example we consider the transition rates $0_g^+\to 2_g^+$, $2_g^+\to 2_g^+$, and $0_\beta^+\to 2_\gamma^+$. Using the appropriate ${\rm U}(6)\supset{\rm SU}(3)$ and ${\rm SU}(3)\supset{\rm SO}(3)$ reduced Wigner coefficients,[26,27] one can calculate the leading order behavior of the exact transition rates. For the calculational procedure we refer to Ref. 27 and only list the results here:

$$|\,\langle [N],(2N,0),0,2\|Q^2\|[N],(2N,0),0,0\rangle^2\approx 2N^2\ ,$$
$$|\,\langle [N],(2N,0),0,2\|Q^2\|[N],(2N,0),0,2\rangle\,|^2\approx\frac{20N^2}{7}\ ,$$
$$|\,\langle [N],(2N-4,2),2,2\|Q^2\|[N],(2N-4,2),0,0\rangle\,|^2\approx 3\ . \quad (6.5)$$

Now we recalculate these transition rates with the aid of Eq. (4.9), using the $E2$-transition operator of Eq. (6.3), and the correctly symmetrized eigenstates of the intrinsic Hamiltonian (6.4). The eigenstates of the Hamiltonian (6.4) can easily be written down; they are

$$|\,\bar{N},n,K,L,M\,\rangle=D_{MK}^{L*}\,|\,\bar{N}=n_0+n_2+n_{-2},n=n_2+n_{-2},K=2(n_2-n_{-2})\rangle\ . \quad (6.6)$$

Here (3.11) and (4.7g) were used to note that $K=2(n_2-n_{-2})$. Imposing the required invariance properties under rotations of the intrinsic axes, one finds for the normalized wave functions[21,22]

$$|\bar{N},n,K,L,M)=\left[\frac{2L+1}{16\pi^2(\delta_{K0}+1)}\right]^{1/2}(D^{L*}_{MK}\,|\bar{N},n,K\rangle+(-1)^L D^{L*}_{M-K}\,|\bar{N},n,-K\rangle)\,. \tag{6.7}$$

First we calculate the $E2$-transition rates within the ground band. The wave functions of the ground band are

$$|\bar{N}=0,\,n=0,\,K=0,\,L,\,M)=\left[\frac{2L+1}{8\pi^2}\right]^{1/2}D^{L*}_{M0}\,|\bar{N}=0,\,n=0,\,K=0\rangle\,. \tag{6.8}$$

Using Eqs. (6.1), (6.3), (6.8), and integrating over the Euler angles, one finds for the $E2$-transition rates

$$|(\bar{N}=0,\,n=0,\,K=0,\,L'\|\bar{Q}^2\|\bar{N}=0,\,n=0,\,K=0,\,L)|^2=2N^2(2L+1)(2L'+1)\begin{pmatrix}L' & 2 & L\\ 0 & 0 & 0\end{pmatrix}^2\,. \tag{6.9}$$

In particular we have

$$\begin{aligned}&|(\bar{N}=0,n=0,K=0,2\|\bar{Q}^2\|\bar{N}=0,n=0,K=0,0)|^2=2N^2\,,\\&|(\bar{N}=0,n=0,K=0,2\|\bar{Q}^2\|\bar{N}=0,n=0,K=0,2)|^2=\frac{20N^2}{7}\,,\end{aligned} \tag{6.10}$$

which is in agreement with (6.5). Similarly one finds for the transition rates $L'^{+}_{\beta}\rightarrow L^{+}_{\gamma}$

$$\begin{aligned}&|(\bar{N}=1,\,n=1,K=2,\,L\|\bar{Q}^2\|\bar{N}=1,\,n=0,\,K=0,L')|^2\\&\qquad=\frac{3(2L+1)(2L'+1)}{4}\left[\begin{pmatrix}L & 2 & L'\\ -2 & 2 & 0\end{pmatrix}+(-1)^L\begin{pmatrix}L & 2 & L'\\ 2 & -2 & 0\end{pmatrix}\right]^2\,.\end{aligned} \tag{6.11}$$

For the transition $0^+_{\beta}\rightarrow 2^+_{\gamma}$ one has

$$|(\bar{N}=1,\,n=1,\,K=2,\,2\|\bar{Q}^2\|\bar{N}=1,\,n=0,\,K=0,0)|^2=3 \tag{6.12}$$

which again agrees with (6.5).

VII. DISCUSSION AND CONCLUSION

The eigenstate of the shape operators has enabled us to achieve four objectives. Firstly, we were able to give a geometrical interpretation to each boson excitation in the IBM. Secondly, contact has been made with the usual rotation-vibration model[21] without resorting to classical or semiclassical approximations. Thirdly, a well defined prescription has been given to calculate intrinsic excitation energies, moments of inertia and transition rates to leading order in the number of bosons in the case of a well-deformed system. Finally, the SO(3) symmetry of the system has been treated exactly even in the well deformed case. In particular this has manifest itself in the fact that the $K=\pm 1$ intrinsic components of the quadrupole operator vanished in this approach. In a normal RPA approach these operators would correspond to excitation operators of spurious modes.

The method presented here is currently being used to analyze the IBM-2 having both proton and neutron bosons. As a next step one would like to generalize the procedure described here by avoiding the explicit use of the eigenstate of the shape operators. Different ways in which this might possibly be done are currently investigated.

I gratefully acknowledge useful discussions with Fritz Hahne and Hendrik Geyer.

APPENDIX

To prove Eqs. (2.6) and (2.7) we need some properties of the eigenstates of the position and momentum operators for a one-dimensional oscillator. The position operator for a one-dimensional oscillator is given by

$$\hat{x}=\frac{1}{\sqrt{2}}(b^{\dagger}+b)\,, \tag{A1}$$

As was shown in Ref. 18, the eigenstate of this operator is given by

$$\begin{aligned}|x\rangle&=\frac{1}{\pi^{1/4}}\exp[-\tfrac{1}{2}(x^2+b^{\dagger}b^{\dagger})+\sqrt{2}xb^{\dagger}]\,|0\rangle\\&\equiv\hat{X}(b^{\dagger},x)\,|0\rangle\,,\end{aligned} \tag{A2}$$

and it has the following properties:

$$|x\rangle = \sum_{n=0}^{\infty} F_n(x)|n\rangle ,$$
$$\langle x|x'\rangle = \delta(x-x') , \quad \text{(A3)}$$
$$\int_{-\infty}^{\infty} |x\rangle\langle x|dx = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 .$$

Here $F_n(x)$ is a harmonic oscillator wave function.

Similarly one can construct an eigenstate of the momentum operator

$$\hat{p} = \frac{i}{\sqrt{2}}(b^{\dagger} - b) . \quad \text{(A4)}$$

The result is

$$|p\rangle = \frac{1}{\pi^{1/4}} \exp[-\tfrac{1}{2}(p^2 - b^{\dagger}b^{\dagger}) + \sqrt{2}ipb^{\dagger}]|0\rangle$$
$$\equiv \hat{P}(b^{\dagger},p)|0\rangle . \quad \text{(A5)}$$

This state satisfies properties similar to (A3) which can easily be proved

$$|p\rangle = \sum_{n=0}^{\infty} (i)^n F_n(p)|n\rangle ,$$
$$\langle p|p'\rangle = \delta(p-p') , \quad \text{(A6)}$$
$$\int_{-\infty}^{\infty} |p\rangle\langle p|dp = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 .$$

For an n-dimensional oscillator the eigenstates of the n position and n momentum operators are given by

$$|x_1,x_2,\ldots,x_n\rangle = \prod_{i=1}^{n} \hat{X}(b_i^{\dagger},x_i)|0\rangle ,$$
$$|p_1,p_2,\ldots,p_n\rangle = \sum_{i=1}^{n} \hat{P}(b_i^{\dagger},p_i)|0\rangle . \quad \text{(A7)}$$

The generalization of the properties (A3) and (A6) are straightforward and we do not list it here.[18]

To construct the eigenstate of the shape operators of Eq. (2.3) we introduce the following set of commuting Hermitian operators:

$$\hat{x}_\mu = \tfrac{1}{2}[\hat{\alpha}_\mu + (-1)^\mu \hat{\alpha}_{-\mu}], \quad \mu>0 ,$$
$$\hat{y}_\mu = \frac{-i}{2}[\hat{\alpha}_\mu - (-1)^\mu \hat{\alpha}_{-\mu}], \quad \mu>0 , \quad \text{(A8)}$$
$$\hat{x}_0 = \hat{\alpha}_0 .$$

Next we introduce the following set of bosons:

$$b_+^{\dagger}(\mu) = \frac{1}{\sqrt{2}}[d_\mu^{\dagger} + (-1)^\mu d_{-\mu}^{\dagger}], \quad \mu>0 ,$$
$$b_-^{\dagger}(\mu) = \frac{1}{\sqrt{2}}[d_\mu^{\dagger} - (-1)^\mu d_{-\mu}^{\dagger}], \quad \mu>0 , \quad \text{(A9)}$$
$$b_0^{\dagger} = d_0^{\dagger} .$$

In terms of these bosons the operators of Eq. (A8) can be expressed as

$$\hat{x}'_\mu \equiv \sqrt{2}\hat{x}_\mu = \frac{1}{\sqrt{2}}[b_+^{\dagger}(\mu) + b_+(\mu)] ,$$
$$\hat{y}'_\mu \equiv -\sqrt{2}\hat{y}_\mu = \frac{i}{\sqrt{2}}[b_-^{\dagger}(\mu) - b_-(\mu)] , \quad \text{(A10)}$$
$$\hat{x}_0 \equiv \frac{1}{\sqrt{2}}(b_0^{\dagger} + b_0) .$$

On comparison with Eqs. (A1) and (A4), we can immediately write down the simultaneous eigenstate of these five commuting Hermitian operators (see also Ref. 18).

$$|x'_\mu,y'_\mu,x_0\rangle = \left[\prod_{\mu>0} \hat{X}(b_+^{\dagger}(\mu),x'_\mu)\right] \times \left[\prod_{\mu>0} \hat{P}(b_-^{\dagger}(\mu),y'_\mu)\right] \hat{X}(b_0^{\dagger},x_0)|0\rangle . \quad \text{(A11)}$$

From Eqs. (A3) and (A6) it follows that this state satisfies

$$\int_{R^5} dx'_1 dx'_2 dy'_1 dy'_2 dx_0 |x'_\mu,y'_\mu,x_0\rangle\langle x'_\mu,y'_\mu,x_0| = 1 \quad \text{(A12)}$$

and

$$\langle x'_\mu,y'_\mu,x_0|x''_\mu,y''_\mu,x'_0\rangle = \left[\prod_{\mu>0} \delta(x'_\mu - x''_\mu)\delta(y'_\mu - y''_\mu)\right] \times \delta(x_0 - x'_0) . \quad \text{(A13)}$$

From Eq. (A8) we note that the state (A11) is an eigenstate of the shape operators of Eq. (2.3) with eigenvalues

$$\alpha_\mu = \frac{1}{\sqrt{2}}(x'_\mu - iy'_\mu), \quad \mu>0 ,$$
$$\alpha_{-\mu} = \frac{1}{\sqrt{2}}(-1)^\mu(x'_\mu + iy'_\mu), \quad \mu>0 , \quad \text{(A14)}$$
$$\alpha_0 = x_0 .$$

Note that the eigenvalues satisfy $\alpha_\mu^* = (-1)^\mu \alpha_{-\mu}$.

Transforming to the variables α_μ and the d bosons [see Eq. (A9)] in the state (A11) leads to the eigenstate of the shape operators

$$|\bar{\alpha}\rangle = \frac{1}{\pi^{5/4}} \exp(-\tfrac{1}{2}\alpha\cdot\alpha) \times \exp(-\tfrac{1}{2}d^{\dagger}\cdot d^{\dagger} + \sqrt{2}\alpha\cdot d^{\dagger})|0\rangle , \quad \text{(A15)}$$

with $\alpha_\mu^* = (-1)^\mu \alpha_{-\mu}$.

Note that this state differs from the state of Eq. (2.4) by a normalization factor 2. This factor is accounted for if we transform to the variables $x_\mu = x'_\mu/\sqrt{2}$ and $y_\mu = -y'_\mu/\sqrt{2}$ in the completeness relation (A12). This transformation gives rise to a multiplicative factor 4 on the left of the completeness relation. Incorporating this factor as a normalization factor of the state $|\bar{\alpha}\rangle$ leads to the state of Eq. (2.4). The identities (2.6) and (2.7) follow directly from Eqs. (A12) and (A13).

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