Delta-hole model in the local-density approximation

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The local-density approximation to the delta-hole model is investigated. Features learned from work on the momentum-space approach to the pion-nucleus optical potential are incorporated into the model. In the limit of no delta-nucleus interaction, we are able to examine the reliability of the approximations of the model by comparing it to a model-exact momentum-space calculation. The model provides a numerically efficient approach to studying delta dynamics in the nucleus.

I. INTRODUCTION

Classical nuclear physics assumes that nuclei and their interaction with an external probe can be described by neutrons and protons which interact via a potential in a Schrödinger equation. This approach assumes that the degrees of freedom associated with mesons, antinucleons, and excited states of the nucleon can be subsumed into a nucleon-nucleon potential. The interaction of a pion with a nucleus necessarily goes beyond this view of classical nuclear physics. By injecting a pion into the nucleus at energies which are comparable to or greater than the pion mass, the relativistic, field-theoretic nature of the pion must be included in the theory. In addition, near the resonance where the pi-nucleon interaction is dominated by the formation of the Δ_{33}, a unique opportunity presents itself—the opportunity to study the propagation and interaction of an excited state of the nucleon, the Δ_{33}, in the nuclear medium.

The approaches which theorists have taken toward this problem fall into three general categories. The first[1] assumes that the pion-nucleon interaction may be well approximated by a zero-range interaction. This allows one to work in coordinate space and has served as the basis for a majority of the phenomenological work in the field. The second[2,3] approach is generally termed the "momentum-space" approach as it requires the construction of the pion-nucleus optical potential, or transition amplitude, in momentum space and the subsequent solution of the Klein-Gordon or (relativistic-Schrödinger) equation in momentum space. The third approach[4] is the delta-hole approach. Each of these three general approaches has its individual strengths and weaknesses, and thus the pursuit of all of the approaches will enhance our ability to better unravel the several physical phenomena which appear to be present in pion-nucleus reactions.

A circumstance which restricts progress on this problem is that the quantitative relation between these various approaches has not yet been firmly established. This inhibits information which is learned while utilizing one approach from being effectively implemented in another and inhibits the use of one approach as a cross-check of an alternative approach. In this work we will examine in detail the relation of the "momentum-space" approaches to the delta-hole approach. We utilize a local-density approximation[5] to the delta-hole model; this approximation has been derived in Ref. 5 and found to reproduce well the results of the full delta-hole model. It is also numerically much simpler, and thus much more convenient, than the full model. Moreover, as the local-density approximation generates an optical potential in momentum space, the comparison between this version of the delta-hole model and the "momentum-space" approaches is more direct. In order to compare results, we find that certain extensions and modifications of the approach of Ref. 5 are necessary. Some of these changes are required in order to include in the local-density approximation to the delta-hole model features which have been incorporated into the momentum-space approaches over the years.

Moreover, in the absence of a delta-nucleus potential and neglecting Pauli blocking in the pion-nucleon t matrix, we are able to define a model problem which can be solved exactly utilizing the computer code[3] ROMPIN. The model problem includes a fully relativistic treatment of the kinematics and an exact treatment of the nonlocalities in the problem, including the recoil of the delta. The full delta-hole model utilizes approximate kinematics, treats approximately the recoil of the interacting pion-nucleon pair in other than the P_{33} channel, and linearizes the propagator in the P_{33} channel. In addition, the local-density delta-hole model uses a local-density approximation to treat the recoil of the delta. The model-exact calculation can thus be used to calibrate the local-density delta-hole model.

In Sec. II we derive and review the local-density delta-hole model. We do this in a form where the comparison to momentum-space optical-model potentials can be made most easily. The following section, Sec. III, presents the modifications and extensions to the local-density approximation to the delta-hole model that we have made. Section IV compares the lowest order results of the delta-hole approach with the most contemporary "momentum-space" approach[3] and outlines in detail the differences between these two approaches.

II. DERIVATION OF THE MODEL

The local-density approximation to the delta-hole model was derived in Ref. 5; we review the derivation here in order to define our notation and to present the necessary background for Sec. III. The pion-nucleon optical potential in the impulse approximation is given by

$$\langle \mathbf{k}'_\pi | \Sigma(\omega_{\rm lab}) | \mathbf{k}_\pi \rangle = \sum_B \int d^3k'_{\rm N} d^3k_{\rm N} \Psi_B^*(\mathbf{k}'_{\rm N}) \langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t(\omega_{\rm lab}+E_B, \mathbf{k}_\pi+\mathbf{k}_{\rm N}) | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle \Psi_B(\mathbf{k}_{\rm N}) \;, \tag{1}$$

where $\mathbf{k}_\pi$ ($\mathbf{k}'_\pi$) are the initial (final) pion momenta, $\mathbf{k}_{\rm N}$ ($\mathbf{k}'_{\rm N}$) are the initial (final) nucleon momenta, $\Psi_B(\mathbf{k}_{\rm N})$ is the wave function of a bound nucleon, and $t(\omega_{\rm lab})$ is the pion-nucleon scattering amplitude. When written in this form, the t matrix $t(\omega_{\rm lab})$ contains a total momentum conserving delta function. In the delta-hole model, the pion-nucleon scattering amplitude in the dominant $P_{3,3}$ channel is taken to be separable,

$$\langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t(\omega_{\rm lab}+E_B, P_{\rm op}) | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle = \lambda^2 \mathbf{k}'\cdot\mathbf{k}\, v^\dagger(\mathbf{k}') \left\langle \mathbf{P}' \left| G \left[\omega_{\rm lab}+E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} \right] \right| \mathbf{P} \right\rangle v(\mathbf{k}) \;, \tag{2}$$

where $\mathbf{k}$ is a relative pion-nucleon momentum, $\mathbf{P}$ is the total pion-nucleon momentum defined by $\mathbf{P}=\mathbf{k}_\pi+\mathbf{k}_{\rm N}$, $P_{\rm op}$ is the total momentum operator, $v(\mathbf{k})$ is the pion-nucleon-delta form factor including its spin and isospin dependence, $G(\omega_{\rm lab})$ is the delta Green's function, λ^2 is a coupling constant, $M_{\rm N}$ is the nucleon mass, and E_B is the binding energy of the state Ψ_B. The energy $\omega_{\rm lab}$ is the pion energy in the pion-*nucleus* laboratory reference frame. The mass utilized to calculate the energy associated with the pion-nucleon recoil (or delta propagation) is taken to be $\omega_{\rm lab}+M_{\rm N}$ rather than the resonance energy; this is because we wish to utilize the model at energies below and above the resonance energy, and the expression in Eq. (2) has been found to be an excellent approximation[6] to the fully relativistic expression. At this point the model is generic, i.e., all models can be viewed as variations on these general equations. The only assumption having been made is that the scattering amplitude is of a separable form. Differences between the theoretical models arise from the choice of the off-shell behavior of the t matrix, i.e., the form factor $v(\mathbf{k})$; the treatment of the integration over the momenta of the nucleons [Fermi averaging in Eq. (1)]; the definition of the relative momentum $\mathbf{k}$ and $\mathbf{k}'$ in Eq. (2); the treatment of Pauli corrections and other medium modifications to $G(\omega_{\rm lab})$; or the particular form chosen for additional phenomenological corrections.

To proceed in the delta-hole approach, a complex and energy-dependent mass for the resonance is defined for the free t matrix by

$$\left\langle \mathbf{P}' \left| G_0 \left[\omega_{\rm lab} - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} \right] \right| \mathbf{P} \right\rangle \equiv \left[\omega_{\rm lab} - \frac{\mathbf{P}^2}{2(\omega_{\rm lab}+M_{\rm N})} - M_\Delta \left[\omega_{\rm lab}+E_B - \frac{\mathbf{P}^2}{2(\omega_{\rm lab}+M_{\rm N})} \right] \right]^{-1} \delta(\mathbf{P}'-\mathbf{P}) \;. \tag{3}$$

The on-shell data together with Eqs. (2) and (3) then serve to define $M_\Delta(\omega_{\rm lab})$. At this point the delta-hole model differs from other models by focusing explicitly on the degree of freedom embodied in the delta resonance. The argument of the free propagator of Eq. (3) is modified—in addition to including the binding energy as indicated in Eq. (1)—by introducing a potential between the Δ_{33} and the nucleus according to

$$\omega_{\rm lab}+E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} \Longrightarrow \omega_{\rm lab}+E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} - U(\mathbf{R}) \;, \tag{4}$$

where $\mathbf{R}$ is the coordinate that is conjugate to $\mathbf{P}_{\rm op}$. The potential $U(\mathbf{R})$ is both subtracted from the energy on the right-hand side of Eq. (3) and also occurs in the mass term, because in a simple Lee model[7] one is adding a potential to both the bare delta and to the intermediate nucleons, as is depicted in Fig. 1. The result of Eq. (4) will then follow if one replaces the individual bare-delta nucleus and nucleon-nucleus interaction with an effective delta-nucleus interaction. To render the numerics feasible, the propagator in Eq. (4) is expanded, assuming the delta-nucleus Hamiltonian is small,

$$M_\Delta \left[\omega_{\rm lab}+E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} - U(\mathbf{R}) \right] \cong M_\Delta(\omega_{\rm lab}) + \left[E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} - U(\mathbf{R}) \right] \frac{dM_\Delta}{d\omega_{\rm lab}} \;. \tag{5}$$

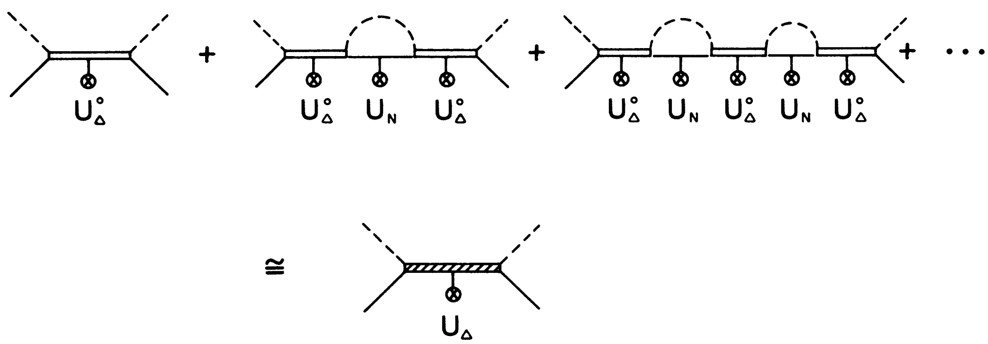

FIG. 1. The interaction of a bare delta with the nucleus, $U_\Delta^0(R_\Delta)$, and the interaction of the nucleon with the nucleus, $U_{\rm N}(R_{\rm N})$, are subsumed into an interaction which acts on the physical resonance $U(R)$.

One chooses the pion laboratory energy $\omega_{\rm lab}$ as the expansion point because the nucleon binding energy E_B will roughly cancel against the energy of the delta and thus yield a more convergent expansion. In the full delta-hole model, this Green's function is then constructed numerically by solving for the eigenstates of the delta Hamiltonian which appears in Eq. (5). One should notice that what has been conventionally termed the lowest-order approximation in the delta-hole approach is given by Eq. (5). In the other approaches one would use the term "lowest order" to imply the use of a propagator with $U(\mathbf{R})$ set to zero. We will use this latter definition.

In order to make a local-density approximation to the model, an approximation to the delta Green's function must be made. The free Green's function, Eq. (3), contains a delta function that conserves the total momentum; this delta function is absent in the delta Green's function, Eq. (4). In infinite nuclear matter the translational invariance of the problem would also produce the momentum conserving delta function. Thus, in the local-density approximation one approximates the delta potential by a spatial constant, $\bar{U}$, to recover the nuclear matter results, but at a later stage allows $\bar{U}$ to depend parametrically on $\mathbf{R}$,

$$\left\langle \mathbf{P}' \left| G_0 \left[\omega_{\rm lab} + E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} - U(\mathbf{R}) \right] \right| \mathbf{P} \right\rangle \cong \delta(\mathbf{P}'-\mathbf{P}) \left[\omega_{\rm lab} + E_B - M_\Delta(\omega_{\rm lab}) - \frac{\mathbf{P}^2}{2(\omega_{\rm lab}+M_{\rm N})} - \bar{U} - \left[E_B - \frac{\mathbf{P}^2}{2(\omega_{\rm lab}+M_{\rm N})} - \bar{U} \right] \frac{dM_\Delta}{d\omega_{\rm lab}} \right]^{-1} . \quad (6)$$

In coordinate space, the Green's function of Eq. (6) is given by

$$\left\langle \mathbf{R}' \left| G_0 \left[\omega_{\rm lab} + E_B - \frac{P_{\rm op}^2}{2(\omega_{\rm lab}+M_{\rm N})} - \bar{U} \right] \right| \mathbf{R} \right\rangle = -2M_{\rm eff} \frac{e^{iK_\Delta |\mathbf{R}'-\mathbf{R}|}}{4\pi |\mathbf{R}'-\mathbf{R}|} , \quad (7)$$

where $M_{\rm eff}$ is given by

$$M_{\rm eff} = \frac{\omega_{\rm lab}+M_{\rm N}}{1 - \dfrac{dM_\Delta}{d\omega_{\rm lab}}} \quad (8)$$

and K_Δ is given by

$$K_\Delta^2 = 2M_{\rm eff}[\omega_{\rm lab} - M_\Delta(\omega_{\rm lab})] + (\omega_{\rm lab}+M_{\rm N})(E_B - \bar{U}) . \quad (9)$$

Furthermore, the vertex functions $v(\mathbf{k})$ and $v^\dagger(\mathbf{k}')$ are factored out of the integration in Eq. (1) by choosing an approximate relative momentum $\mathbf{k}_f$ to replace $\mathbf{k}$. In order for $\mathbf{k}_f$ to occur outside of the integration it must depend only on the external pion momenta $\mathbf{k}_\pi$ and $\mathbf{k}'_\pi$ and not depend on the nucleon momenta, $\mathbf{k}_{\rm N}$ or $\mathbf{k}'_{\rm N}$. We will discuss the choice of $\mathbf{k}$ and its factored approximation $\mathbf{k}_f$ in Sec. III.

The local-density approximation obtains its name by approximating the sum of the product of target wave functions in Eq. (1) by

$$\sum_B \Psi_B^*(\mathbf{r}'_{\rm N})\Psi_B(\mathbf{r}_{\rm N}) \cong \sum_\alpha \rho_\alpha(R_{\rm av})\hat{j}_1(k_f(R_{\rm av})s) , \quad (10)$$

where

$$\mathbf{R}_{\rm av} = \frac{\mathbf{r}_{\rm N}+\mathbf{r}'_{\rm N}}{2} ,$$

$$\hat{j}_1(x) = \frac{3}{x} j_1(x) ,$$

$$k_f^3(R_{\rm av}) = 3\pi^2 \rho_\alpha(R_{\rm av}) ,$$

and where $\rho_\alpha(R_{\rm av})$ is the neutron or proton density separately.

In order to use Eq. (10) one must also approximate the binding energy E_B which occurs in the Green's function. The simplest approximation would be to replace E_B by an average binding energy $\langle E_B \rangle$. In Ref. 5 a local-density approximation was used to generate the next order correction to this approximation. This can be done by expanding Eq. (9) about an average binding energy

$$K_\Delta(E_B) \cong [K_\Delta^2(\langle E_B \rangle) + 2(\omega_{lab}+M_N)]^{1/2}$$
$$\cong K_\Delta(\langle E_B \rangle) + \frac{\omega_{lab}+M_N}{2K_\Delta(\langle E_B \rangle)}(\langle E_B \rangle - E_B) . \quad (11)$$

There is a difficulty with this expansion if we try to use the results for low-energy pion-nucleus scattering. The local delta momentum $K_\Delta(\langle E_B \rangle)$ occurs in the denominator. For some value of R_{av} the value of $K_\Delta(\langle E_B \rangle)$ is quite small, and thus the expansion can lead to large corrections. One can avoid this difficulty by neglecting the correction term in Eq. (11) but carefully choosing the value of $\langle E_B \rangle$. We do this by utilizing the momentum-space code ROMPIN (Ref. 3) to calculate results with the individual binding energies. We then recalculate utilizing an average binding energy $\langle E_B \rangle$ and adjust the average binding energy so as to produce the same total cross sections as the exact calculation. We find $|\langle E_B \rangle| = 20.5$, 20.0, and 16.0 MeV for ^{12}C, ^{16}O, and ^{40}Ca, respectively. These numbers work well at all energies. The difference that results from utilizing the average binding energy $\langle E_B \rangle$ rather than the individual energies is depicted in Fig. 2 for π^+ elastic scattering from ^{16}O at 162 MeV.

If we now utilize the approximation for the density, Eq. (10), and the approximation for the Green's function, Eq. (7), in the original expression for the impulse approximation to the optical potential, Eq. (1), we arrive at the final result

$$\langle \mathbf{k}'_\pi | \Sigma(\omega_{lab}) | \mathbf{k}_\pi \rangle = -2M_{eff} \sum_{i=p,n} \lambda_i^2 \mathbf{k}'_f\cdot\mathbf{k}_f v(\mathbf{k}'_f) v(\mathbf{k}_f) \int d^3R_{av}\, d^3s\, \rho_i(R_{av}) \hat{j}_1(k_f(R_{av})s) e^{i(\mathbf{k}'_\pi-\mathbf{k}_\pi)\cdot\mathbf{R}_{av}} e^{i[\frac{1}{2}(\mathbf{k}'_\pi+\mathbf{k}_\pi)\cdot\mathbf{s}]} \frac{e^{iK_\Delta s}}{4\pi s} , \quad (12)$$

where λ_i^2 is the appropriately spin- and isospin-averaged pion-nucleon coupling constant. Finally, the delta momentum K_Δ is allowed to depend on the variable R_{av} by replacing $\bar{U}$ in Eq. (9) by $U(R_{av})$. The final result for the $\Delta_{3,3}$ contribution to the optical potential is given by Eq. (12) with K_Δ defined by Eq. (9). This is then combined with a conventional momentum-space optical potential generated from the other pion-nucleon partial waves to form the complete potential.

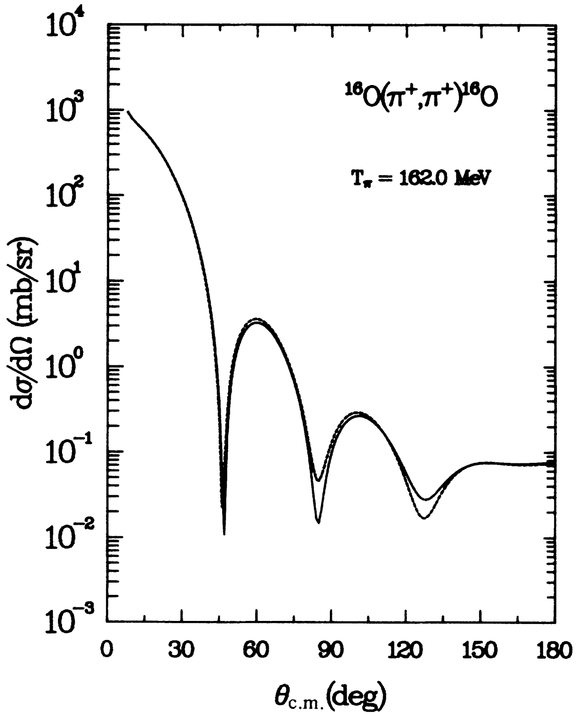

FIG. 2. Angular distribution for elastic scattering of π^+ from ^{16}O at 162 MeV. The solid curve includes the individual binding energies as in Eq. (1); the dashed curve replaces the individual binding energies by an average $\langle E_B \rangle = 20.0$ MeV.

III. MODIFICATIONS

The potential given by Eq. (12) should be quite similar to the momentum-space optical potential when the delta-nucleus potential $U(R_{av})$ is set to zero. With no delta potential the computer code ROMPIN (Ref. 3) calculates the impulse approximation to the optical potential, Eq. (1), exactly. In Sec. IV we outline the approximations which are made to this model-exact problem in the delta-hole approach and compare the results of the local-density delta-hole approach to the model-exact results. In this section we present modifications we have made to the execution of Eq. (12) which were necessary in order to make this comparison.

The first point to notice is that the energy ω_{lab} which occurs in the definition of K_Δ, Eq. (9), is the pion energy in the pion-nucleus laboratory reference frame. The transformation to the two-body, pion-nucleon reference frame is accomplished by the incorporation, into the propagator, of the recoil energy $P_{op}^2/2(\omega_{lab}+M_N)$ as in Eq. (2). If one does not treat the Fermi averaging exactly in the nonresonant partial waves, then the term $P_{op}^2/2(\omega_{lab}+M_N)$ must be treated approximately for these partial waves. This approximate treatment should not be utilized in the $P_{3,3}$ channel.

We have not yet specified the definition of the relative momenta $\mathbf{k}'$ and $\mathbf{k}$. The original discussion of how to define these momenta was carried out[8] under the appellation of "angle transformation." An internally consistent definition[6,9] is to define $\mathbf{k}$ as the momentum of the pion in the frame where the total pion-nucleon momentum is zero. This gives

$$\mathbf{k}_{\parallel}=\gamma(\beta)[\mathbf{k}_\pi\cdot\hat{\boldsymbol{\beta}}-\beta\omega_\pi(k_\pi)]\hat{\boldsymbol{\beta}}\ ,$$
$$\mathbf{k}_{\perp}=\mathbf{k}_\pi-(\mathbf{k}_\pi\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}\ , \tag{13}$$

with

$$\boldsymbol{\beta}=(\mathbf{k}_\pi+\mathbf{k}_{\rm N})/[E_{\rm N}(k_{\rm N})+\omega_\pi(k_\pi)]\ .$$

It was shown in Ref. 6 that a reasonably quantitative approximation to these expressions is

$$\mathbf{k}\cong\frac{M_{\rm N}\mathbf{k}_\pi-\omega_\pi(k_\pi)\mathbf{k}_{\rm N}}{\omega_\pi(k_\pi)+M_{\rm N}}\ . \tag{14}$$

This is the approximation to a relativistic definition of a relative momentum which is used in the full delta-hole model. However, in Ref. 5 a further approximation was used,

$$\mathbf{k}\simeq\frac{M_{\rm N}\mathbf{k}_\pi}{\omega_\pi(k_\pi)+M_{\rm N}}\ . \tag{15}$$

The physical consequence of making this further approximation is that an interaction which occurs in an orbital angular momentum state l in the pion-nucleon center-of-mass frame will occur in only the orbital angular momentum state l in the pion-nucleus reference frame. It has long been known[10] that it is important to include the mixing of the dominant p-wave interaction in the two-body center-of-mass frame into other partial waves in the transformed pion-nucleus frame. This can be accomplished without any additional numerical difficulty by utilizing the approximation of Eq. (14).

In the derivation of the local-density approximation, the form factors (and $\mathbf{k}'\cdot\mathbf{k}$ term) had to be factored out of the integral over the nucleon momenta. This was necessary so that the integral over nucleon momenta would yield the closed form expression for the Green's function given by Eq. (7). A best value for approximating this type of integration has been derived in Ref. 6. There it was argued that the range in coordinate space of the nonlocalities in the two-body t matrix was smaller in extent than the size of the nucleus. This means that in momentum space the nonlocal behavior of the t matrix is sufficiently smooth that it can be removed from the Fermi integration at the point where the density peaks, subject to the constraint that the factorization be done in a sufficiently symmetric way so as to preserve time-reversal invariance. The results of these arguments yield (neglecting the recoil of the target)

$$\mathbf{k}_{\rm N}\cong(\mathbf{k}'_\pi-\mathbf{k}_\pi)/2\ ,$$
$$\mathbf{k}'_{\rm N}\cong(\mathbf{k}_\pi-\mathbf{k}'_\pi)/2\ . \tag{16}$$

We utilize this "optimal factorization" approximation together with the approximate kinematics of Eq. (13) for the evaluation of the momenta at which the form factors are evaluated,

$$\mathbf{k}_f=\frac{M_{\rm N}+\omega_\pi(k_\pi)/2}{M_{\rm N}+\omega_\pi(k_\pi)}\mathbf{k}_\pi-\frac{\omega_\pi(k_\pi)/2}{M_{\rm N}+\omega_\pi(k_\pi)}\mathbf{k}'_\pi\ ,$$
$$\mathbf{k}'_f=\frac{M_{\rm N}+\omega_\pi(k'_\pi)/2}{M_{\rm N}+\omega_\pi(k'_\pi)}\mathbf{k}'_\pi-\frac{\omega_\pi(k'_\pi)/2}{M_{\rm N}+\omega_\pi(k'_\pi)}\mathbf{k}_\pi\ . \tag{17}$$

Nuclear physicists are not generally accustomed to utilizing invariantly normalized wave functions and phase spaces. Even at very low energies, when a pion hits a nucleon in a nucleus the intermediate states for the scattered pion are quite relativistic. Thus at all energies it is important to utilize invariant normalizations. We will norm our wave functions according to

$$\langle\mathbf{k}'_\pi\,|\,\mathbf{k}_\pi\rangle=2\omega_\pi(k_\pi)\delta(\mathbf{k}'_\pi-\mathbf{k}_\pi)\ ,$$
$$\langle\mathbf{k}'_{\rm N}\,|\,\mathbf{k}_{\rm N}\rangle=2E_{\rm N}(k_{\rm N})\delta(\mathbf{k}'_{\rm N}-\mathbf{k}_{\rm N})\ . \tag{18}$$

With this normalization the Lippmann-Schwinger equation reads, for the invariant t matrix,

$$\langle\mathbf{k}'_\pi\,|\,T\,|\,\mathbf{k}_\pi\rangle=\langle\mathbf{k}'_\pi\,|\,\Sigma\,|\,\mathbf{k}_\pi\rangle+\int\frac{d^3k''_\pi}{2\omega_\pi(k''_\pi)2E_A(k''_\pi)}\times\langle\mathbf{k}'_\pi\,|\,\Sigma\,|\,\mathbf{k}''_\pi\rangle G(W_0)\times\langle\mathbf{k}''_\pi\,|\,T\,|\,\mathbf{k}_\pi\rangle\ , \tag{19}$$

where W_0 is the total energy in the pion-nucleus center-of-mass frame. For pedagogical purposes, we here temporarily keep the finite mass of the target nucleus, while throughout the rest of this work these small corrections have been neglected. It is straightforward[3] to utilize invariant norms and Eq. (19). It is common, however, to use noninvariant normalizations and insert a phase-space factor to correct for the lack of manifest invariance. We demonstrate below that these two approaches are equivalent, as such a demonstration does not seem to exist in the literature and also because it will help elucidate the origin of the phase-space factor that occurs when using a nonrelativistic normalization.

If we were to change to a t matrix and optical potential defined without invariant normalizations (we use here the norm of Ref. 11),

$$\langle\mathbf{k}'_\pi\,|\,T^{\rm NR}\,|\,\mathbf{k}_\pi\rangle\equiv[2\omega_\pi(k'_\pi)2E_A(k'_\pi)]^{-1/2}\langle\mathbf{k}'_\pi\,|\,T\,|\,\mathbf{k}_\pi\rangle\times[2\omega_\pi(k_\pi)2E_A(k_\pi)]^{-1/2}\ ,$$
$$\langle\mathbf{k}'_\pi\,|\,\bar{\Sigma}^{\rm NR}\,|\,\mathbf{k}_\pi\rangle\equiv[2\omega_\pi(k'_\pi)2E_A(k'_\pi)]^{-1/2}\langle\mathbf{k}'_\pi\,|\,\Sigma\,|\,\mathbf{k}_\pi\rangle\times[2\omega_\pi(k_\pi)2E_A(k_\pi)]^{-1/2}\ , \tag{20}$$

then the Lippmann-Schwinger equation becomes

$$\langle\mathbf{k}'_\pi\,|\,T^{\rm NR}\,|\,\mathbf{k}_\pi\rangle=\langle\mathbf{k}'_\pi\,|\,\bar{\Sigma}^{\rm NR}\,|\,\mathbf{k}_\pi\rangle+\int d^3k''_\pi\langle\mathbf{k}'_\pi\,|\,\bar{\Sigma}^{\rm NR}\,|\,\mathbf{k}''_\pi\rangle G(W_0)\times\langle\mathbf{k}''_\pi\,|\,T^{\rm NR}\,|\,\mathbf{k}_\pi\rangle\ . \tag{21}$$

The target is clearly moving nonrelativistically so it is convenient to remove its energy factors by introducing

$$\langle\mathbf{k}'_\pi\,|\,\Sigma^{\rm NR}\,|\,\mathbf{k}_\pi\rangle\equiv2M_A\langle\mathbf{k}'_\pi\,|\,\bar{\Sigma}^{\rm NR}\,|\,\mathbf{k}_\pi\rangle\ , \tag{22}$$

and then cancelling the mass M_A against the target energies. This then yields for the optical potential

$$\langle \mathbf{k}'_\pi | \Sigma^{\rm NR} | \mathbf{k}_\pi \rangle = [2\omega_\pi(k'_\pi) 2\omega_\pi(k_\pi)]^{-1/2} \times \int \frac{d^3k_{\rm N}}{2E_{\rm N}(k_{\rm N})} \frac{d^3k'_{\rm N}}{2E_{\rm N}(k'_{\rm N})} \times \langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle \langle \mathbf{k}'_{\rm N} | \rho | \mathbf{k}_{\rm N} \rangle \ . \tag{23}$$

To reach the conventional results, we rewrite the density in terms of nonrelativistically (NR) normalized wave functions

$$\langle \mathbf{k}'_{\rm N} | \rho^{\rm NR} | \mathbf{k}_{\rm N} \rangle \equiv [2E_{\rm N}(k'_{\rm N})]^{-1/2} \langle \mathbf{k}'_{\rm N} | \rho | \mathbf{k}_{\rm N} \rangle \times [2E_{\rm N}(k_{\rm N})]^{-1/2} \ , \tag{24}$$

and must also rewrite the two-body t matrix in terms of its noninvariant counterpart as in Eq. (20),

$$\begin{aligned} \langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle &= [2\omega_\pi(k') 2E_{\rm N}(k') 2\omega_\pi(k) 2E_{\rm N}(k)]^{1/2} \delta(\mathbf{k}'_\pi + \mathbf{k}'_{\rm N} - \mathbf{k}_\pi - \mathbf{k}_{\rm N}) \langle \mathbf{k}' \| t^{\rm NR} \| \mathbf{k} \rangle \\ &= [2\omega_\pi(k') 2E_{\rm N}(k') 2\omega_\pi(k) 2E_{\rm N}(k)]^{1/2} \langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t^{\rm NR} | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle \ . \end{aligned} \tag{25}$$

Rewriting Eq. (23) in terms of the nonrelativistic quantities gives

$$\langle \mathbf{k}'_\pi | \Sigma | \mathbf{k}_\pi \rangle = \int d^3k'_{\rm N} d^3k_{\rm N} \Gamma(k',k;k'_\pi,k_\pi,k'_{\rm N},k_{\rm N}) \langle \mathbf{k}'_\pi \mathbf{k}'_{\rm N} | t^{\rm NR} | \mathbf{k}_\pi \mathbf{k}_{\rm N} \rangle \langle \mathbf{k}'_{\rm N} | \rho^{\rm NR} | \mathbf{k}_{\rm N} \rangle \ , \tag{26}$$

where

$$\Gamma(k',k;k'_\pi,k_\pi,k'_{\rm N},k_{\rm N}) = \left[\frac{\omega_\pi(k')\omega_\pi(k)E_{\rm N}(k')E_{\rm N}(k)}{\omega_\pi(k'_\pi)\omega_\pi(k_\pi)E_{\rm N}(k'_{\rm N})E_{\rm N}(k_{\rm N})} \right]^{1/2} . \tag{27}$$

Thus utilizing noninvariant normalizations [i.e., omitting the energy factors in Eq. (18)] and incorporating the extra phase-space factor Γ will produce the correct invariant result.

The original momentum-space code PIPIT (Ref. 2) incorporated this factor (following the derivation given in Ref. 11) and the full delta-hole approach[4] did also, generally by splitting Γ into two parts and incorporating the parts into the vertex operator. However, this factor was not included in the work of Ref. 5.

In constructing the optical potential in Eq. (1), a model for the off-shell behavior of the pion-nucleon t matrix is required. Following Ref. 5, we would like to use the model of Ref. 12. This presents a problem, however. How an amplitude is to be taken off-shell is a question that can only be addressed in the context of a many-body problem. There, if the two-body model derives from a Hamiltonian, many-body theory, such as the theory presented in Ref. 13, provides an unambiguous prescription for the off-shell behavior. The model of Ref. 12 does not derive from an underlying Hamiltonian, leaving the off-shell prescription ambiguous. We take the following approach to this situation.

The model of Ref. 12 is a Lee model with one change; the linear or relativistic Schrödinger propagator which occurs in the Lee model has been replaced by a quadratic or Klein-Gordon (KG) propagator. We assume that this replacement has been done for numerical convenience because the quadratic propagator allows the necessary integrations to be performed analytically. If this point of view is adopted, then the off-shell behavior that should be used is that of the original Lee model. From Ref. 14 we find

$$\langle \mathbf{k}' | t(\omega) | \mathbf{k} \rangle = \frac{1}{(2\omega_{k'})^{1/2}} \langle \mathbf{k}' | t^{\rm KG}(\omega) | \mathbf{k} \rangle \frac{1}{(2\omega_k)^{1/2}} \ , \tag{28}$$

where $t^{\rm KG}$ is the separable interaction utilizing a quadratic propagator. The square roots of the off-shell energies in Eq. (28) just cancel against the identical factors which occur in Γ, Eq. (27). The net effect of our incorporation of the relativistic phase-space factor Γ and this modification of the off-shell behavior is to replace in the optical potential a factor $(2\omega_0)^{-1}$, with ω_0 the on-shell pion energy, by a factor $(2\omega_{k'})^{1/2}(2\omega_k)^{1/2}$, with $\omega_{k'}$ and ω_k the off-shell energies.

Although we will here follow the approach of those who have used the delta-hole model and utilize a Lee model for the pion-nucleus interaction, we remind the reader that this model, like the separable-potential model, does not include the pion-nucleon pole. The neglect of this singularity in the two-body amplitude produces[15] additional cutoffs in the pion-nucleon form factors in momentum space. This results in an artificially large range in coordinate space for the pion-nucleon interaction which, in turn, produces an optical potential with a slightly increased[14] effective radius.

We also note that in PIPIT the optical potential is scaled by $(A-1)/A$ and then after the Lippmann-Schwinger equation has been solved for the T matrix, the results are scaled by $A/(A-1)$ as was derived originally by Kerman, McManus, and Thaler.[16] This scaling does not occur in the formal development of the delta-hole model, nor does it occur in the formal multiple scattering theory of Watson[17] or in the formalism of Ref. 13. There is no consistent way the Δ_{33} contribution to the optical potential can be combined with the contribution of the other partial waves if the contribution from the nonresonant partial waves is scaled by $(A-1)/A$. The simple resolution to this problem is simply to drop the scaling of the nonresonant partial wave contribution. The first order term without the scaling and without Pauli blocking on the two-body t matrix still forms the leading order term to a well-defined[13] multiple scattering theory.

Also in PIPIT the approximate relative momenta $\mathbf{k}'_f$ and $\mathbf{k}_f$ were defined by taking $\mathbf{k}_N \simeq -\mathbf{k}_\pi/A$ and $\mathbf{k}'_N \simeq -\mathbf{k}_\pi/A - \mathbf{k}'_\pi + \mathbf{k}_\pi$. It was noted in Ref. 6 that this asymmetric factorization violates time-reversal invariance. We therefore follow Ref. 6 and use the approximate relative momenta $\mathbf{k}'_f$ and $\mathbf{k}_f$ defined in Eq. (17) in the nonresonant partial waves. For these partial waves we also use Eq. (17) to approximate the recoil term in the Green's function,

$$\frac{(\mathbf{k}_\pi+\mathbf{k}_N)^2}{2(\omega_{\rm lab}+M_N)} = \frac{(\mathbf{k}'_\pi+\mathbf{k}_N)^2}{2(\omega_{\rm lab}+M_N)} \cong \frac{(\mathbf{k}'_\pi+\mathbf{k}_\pi)^2}{8(\omega_{\rm lab}+M_N)} . \quad (29)$$

In summary, we have used the "optimal factorization" approximation to the relativistic momenta as defined in Eq. (13) to remove the form factors and the $\mathbf{k}'\cdot\mathbf{k}$ terms from the Fermi integral. This maintains the mixing of the $P_{3,3}$ partial wave into other partial waves under the transformation to the pion-nucleus reference frame. We incorporate the phase-space factor Γ into the part of the optical potential generated by the $P_{3,3}$ channel. The $(A-1)/A$ scaling is removed from the part of the optical potential generated from the nonresonant partial waves. The treatment of the kinematics for the nonresonant partial waves was also modified to make use of the results of Ref. 6.

In Figs. 3 and 4 we present the results for π^+ elastic scattering from ^{12}C and ^{40}Ca at 162 MeV. The data are from Refs. 18–20. As the calculations do not include any second-order effects, agreement of any curve with the data should not be taken too seriously; we include the data only to demonstrate the qualitative success of the theory. The dashed curves represent the results of utilizing the approach of Ref. 5, while the solid curves are the results of the modified local-density approximation to the delta-hole model as derived here. We note that the modifications made here make a significant change in the predictions.

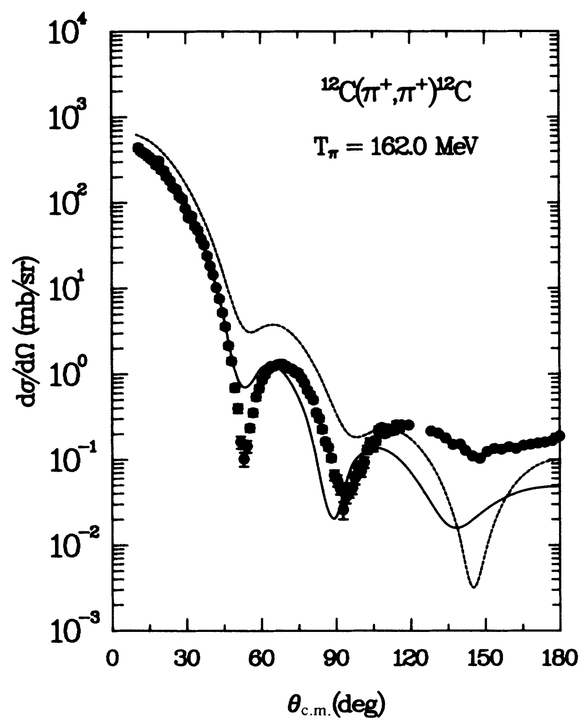

FIG. 3. Angular distribution for elastic scattering of π^+ from ^{12}C at 162 MeV. The solid curve is the result of this work, the dashed curve the result of the work of Ref. 5. In both cases, no Pauli blocking and no delta-nucleus interaction is included. The data for angles less than 120 deg are from Ref. 18; the back angle data are from Ref. 19.

IV. COMPARISON WITH THE MOMENTUM-SPACE APPROACH

If no approximations were made, then the momentum-space approach and the delta-hole approach [with $U_\Delta(R_{\rm av})=0$] would be equivalent ways of calculating Eq. (1). However, at this level the delta-hole approach contains approximations which are not present in the momentum-space approach. The full delta-hole approach contains two approximations. The first is the use of the definition of the relative momentum from Eq. (14) rather than the fully relativistic definition given in Eq. (13). The validity of this approximation has been examined in Ref. 6. The second approximation is the linearization of the propagator in Eq. (5). The local-density approximation to the delta-hole model then makes additional approximations. These are the factorization of the pion-nucleon form factors out of the Fermi integral [using Eq. (15) for the nucleon momenta], the local-density approximation to the sum over the target wave functions as given in Eq. (10), and the use of the "optimal factoriza-

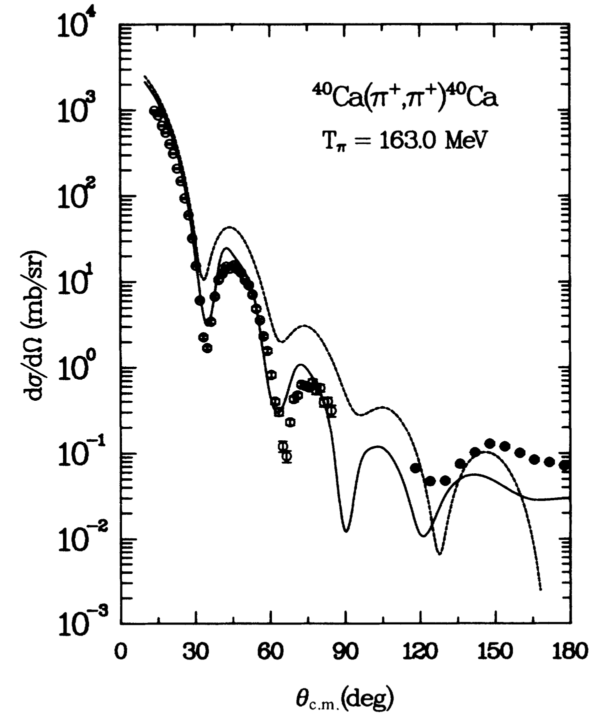

FIG. 4. The same as Fig. 3 except the target is ^{40}Ca. The data for angles less than 120 deg are from Ref. 20. The large angle data are from Ref. 19.

tion" approximation throughout the part of the optical potential coming from the nonresonant partial waves.

In Figs. 5 and 6 we present the differential cross section for elastic scattering of π^+ from ^{12}C and ^{40}Ca at 162 MeV. The solid curves represent the results from the computer code ROMPIN (Ref. 3) which serves as the model-exact solution. The dashed curves represent the results of the local-density approximation to the delta-hole model. We utilize for the purpose of comparison a separable-potential model for the pion-nucleon t matrix in the nonresonant partial waves. The optical potential is used in the relativistic Schrödinger equation to generate the pion-nucleus scattering amplitude. Neither a Pauli-blocking effect nor a delta-nucleus interaction is included.

For the case of ^{40}Ca, Fig. 6, we see that the model-exact calculation and the results from the local-density delta-hole approach are in reasonable agreement. The results for ^{12}C are surprisingly different. We can trace this difference if we also note in Fig. 5 the dash-dot curve which corresponds to our model problem but with the energy of the two-body t matrix shifted by the mean-spectral energy of Ref. 21. We see that the local-density delta-hole model lies nearly midway between the other two curves. From this we can conclude that there is an effective downward shift in the energy of the two-body t matrix when using the local-density delta-hole approach.

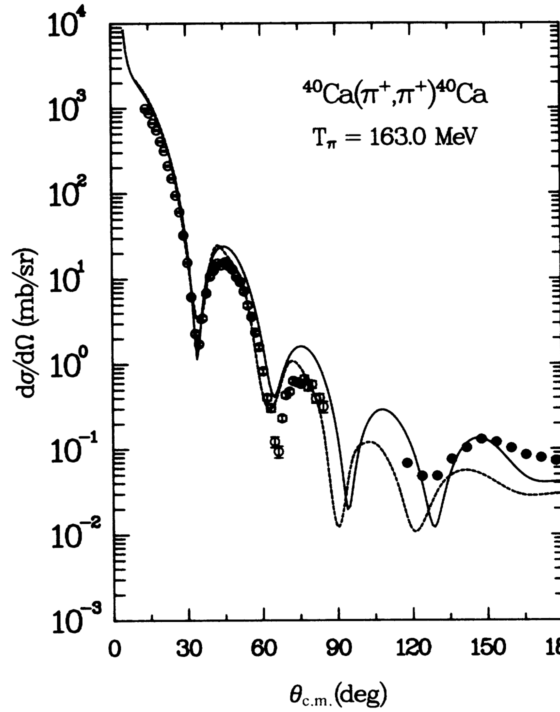

FIG. 6. The solid and dashed curve are the same as Fig. 5 except the target is ^{40}Ca.

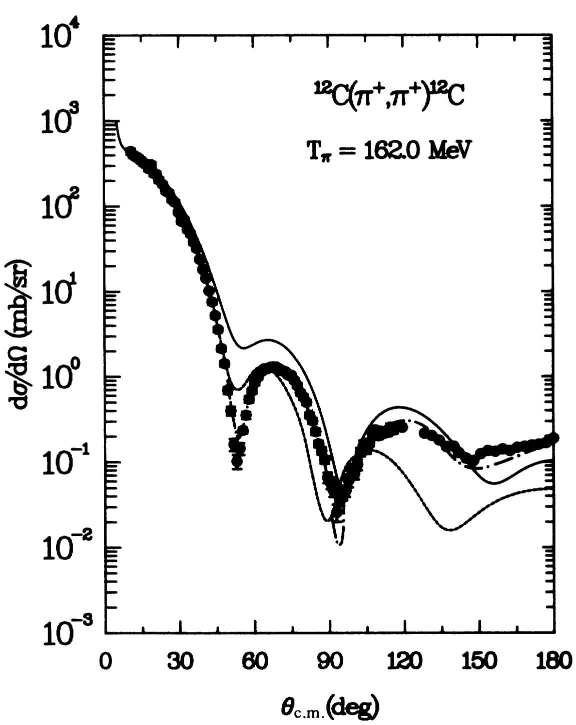

FIG. 5. Angular distribution for elastic scattering of π^+ from ^{12}C at 162 MeV. The solid curve is the model-exact result from the computer code ROMPIN; the dashed curve is the result of the local-density delta-hole model approximation; the dash-dot curve is the result of the model-exact calculation but with the energy at which the two-body t matrix is evaluated shifted by the mean spectral energy. In all cases, Pauli blocking and the delta-nucleus interaction are not included.

The shift for ^{12}C is approximately 15 MeV while it is much smaller for ^{40}Ca. The origin of this shift is under investigation. The two most likely causes would either be the linearization of the delta denominator or else the local-density approximation to the delta propagation. In either case, one must keep in mind that in using the local-density delta-hole model to do phenomenology the extracted real part of the delta-nucleus interaction will be shifted in light nuclei from its actual value.

V. CONCLUSIONS

The local-density delta-hole model provides an attractive approach to studying the physics of a pion interacting with a nucleus. It shares with the delta-hole model the desirable feature that it treats the phenomenological aspects of the problem via a delta-nucleus potential. This holds not only an aesthetic appeal but also provides the phenomenology with a connection to the true pion absorption channel. Introducing the local-density approximation makes the model computationally much simpler. This will allow the model to be applied to a large number of nuclei, including heavy nuclei. The nature of strong interaction physics is that consistency with a large body of data is necessary to fully understand the underlying dynamics.

We have incorporated features which have been learned from the momentum-space approach to modify and extend the work of Ref. 5. These include the use of the optimal factorization approximation to approximate some of the nonlocalities in the optical potential, invariant normalization of wave functions, a more consistent treatment of the off-shell behavior of the two-body t matrix, and a removal of the $(A-1)/A$ scaling factors from the nonresonant piece of the optical potential. We have also used the results from the computer code ROMPIN (Ref. 3) to calibrate the approximations used in the local-density delta-hole approach. This provides a bench mark for the accuracy of the theory and provides an estimate of the reliability of phenomenological parameters determined by the model. In particular, we have found that the local-density delta-hole model contains an effective shift in the energy at which the two-body t matrix is evaluated. This shift is largest in light nuclei.

At this point we are in a position to begin an examination of the phenomenological results of the model—i.e., what does the model imply about the behavior of the delta-nucleus interaction? This will be examined in a future work.

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