Skeleton boson realizations of collective subalgebras

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For an even number of nucleons, we consider collective motion based on any dynamical symmetry subalgebra $\hat{A}$ of the valence-shell bifermion algebra; the generators of $\hat{A}$ are coherent linear combinations of bifermion operators of the types $a^\dagger a^\dagger$, $a^\dagger a$, and aa. A boson algebra is obtained by Dyson mapping the whole valence-shell bifermion algebra, and then "skeletonizing" the resulting expressions by deleting all noncollective boson operators from them. This skeleton boson algebra is shown to have the same commutation relations as the collective bifermion algebra, provided that the bifermion algebra is self-conjugate. The resulting boson Hamiltonian is not Hermitian, but does resemble the Hamiltonian of the interacting boson model in containing only one- and two-body terms. Moreover, the skeleton boson algebra (when restricted to a collective boson subspace that is free of spurious states) is shown to be physically equivalent to the collective bifermion algebra. It is shown that by working solely with collective bosons one can identify spurious boson states unambiguously, although the procedure lacks convenience. The similarity transformation that connects conjugate versions of the skeleton boson mapping is discussed, and a general condition for hermiticity of the boson image of a Hermitian operator is given.

I. INTRODUCTION

In view of the great phenomenological success of the *s-d* interacting boson model (IBM),[1] it is of interest to inquire how well more general types of collective motion can be described by boson models. The present work is a step in this direction. We first try to understand what kinds of collective motion are "boson mappable" in the sense that they can be *exactly* described by IBM-like models, allowing as "IBM-like" all models in which the boson Hamiltonian conserves the total number of bosons and contains nothing more complicated than two-body interaction terms. Here we do not exclude models that use non-Hermitian Hamiltonians; we have previously discussed the possibility of hermitizing[2] the boson Hamiltonian.

Any shell-model problem for an even number of fermions can be formulated in terms of the Lie algebra consisting of all operators bilinear in the valence-shell creation and annihilation operators.[3] In principle, the problem can be solved by working entirely within that algebra. It can be shown that the degrees of freedom of any exactly separable collective motion (however complicated) must be describable by a "collective" *sub*algebra of the full bifermion algebra. The problem, then, is how to simulate the properties of this collective subalgebra by means of an equivalent set of operators that act on a boson space. For this purpose generalized Dyson boson mapping[4] is well suited, because it automatically gives a boson Hamiltonian of one- plus two-body form, though at the price of nonhermiticity. However, the complicated "Pauli-correction" terms of the Dyson boson images always introduce noncollective ideal-boson creation and annihilation operators. These noncollective bosons spoil the simplicity of the resulting boson representation.

For two cases [the Ginocchio SO(8) algebra and the Sp(4) algebra] Geyer and Hahne[5] have shown that if the Pauli correction terms of the boson images are mutilated[6] by omitting the noncollective bosons, *the commutation relations of the algebra are not affected,* so that the mutilated algebra is *isomorphic* to the original algebra. To such realizations purely in terms of collective bosons we give the name "skeleton realizations"; Geyer and Hahne call them "fully collective realizations." Our work is partly a systematization and extension of theirs, but arrives at several new results. First, we generalize their results to any self-conjugate collective subalgebra. Second, we insist on the need for *equivalence* of the boson realization to its fermion algebra prototype; this is a requirement that goes far beyond mere isomorphism. Mathematically, equivalence means essentially that the fermion and boson realizations can be connected by a similarity transformation; this is just what is needed to ensure equivalence of all *physical* consequences. Fortunately we are able to demonstrate this property for skeleton boson realizations that are restricted to appropriate "physical" boson subspaces. Finally we address the problem of "spurious states," i.e., boson states that have no fermion counterparts. We show that, in principle, spurious states can be identified by operations intrinsic to the collective boson subspace, without any reference to the noncollective boson operators.

The plan of the paper is as follows. Section II fixes some notation and reviews generalized Dyson boson mapping, to a great extent following the approach of Janssen *et al.*[4] Section III shows that a boson realization of the full fermion algebra trivially gives a Dyson realization of any collective subalgebra.

In Sec. IV three simple lemmas are stated, their proofs being relegated to the Appendix. We hope that reference to these lemmas will help the reader follow the rather intricate reasoning of Sec. V, in which we show that the skeletonized Dyson boson mapping of a collective algebra is still both isomorphic and equivalent to it. As a further aid, Fig. 1 gives a diagrammatic outline of the reasoning of Sec. V.

There are two versions of each skeleton boson realization, which we call "right-" and "left-Dyson" mappings. In Sec. VI we discuss the use of the left-Dyson skeleton realization of the collective subalgebra, with reference to the simple-correspondence method of Talmi and Ginocchio[7] and the spurious state problem. In Sec. VII we discuss the similarity transformation that connects the left- and right-Dyson boson realizations of the collective algebra. Section VIII summarizes the conclusions.

II. REVIEW OF DYSON BOSON MAPPING

Let $a_i^\dagger$ and a_i denote fermion creation and annihilation operators for orthonormal single-particle states numbered $i=1,\ldots,q=2\Omega$. They obey the anticommutation relations

$$\begin{aligned}&\{a_i,a_j^\dagger\}=\delta_{ij}\ ,\\&\{a_i,a_j\}=\{a_i^\dagger,a_j^\dagger\}=0\ ,\end{aligned}\tag{2.1}$$

and satisfy

$$a_i\,|\,0\rangle=0\ ,\tag{2.2}$$

where $|\,0\rangle$ is the fermion vacuum. The traceless bifermion operators

$$\begin{aligned}&\hat{A}_{ij}^\dagger=a_j^\dagger a_i^\dagger\ ,\\&\hat{A}_{ij}=a_j a_i\ ,\\&\hat{\rho}_{ij}=a_i^\dagger a_j+a_j^\dagger a_i-\delta_{ij}\ ,\end{aligned}\tag{2.3}$$

close under commutation, and give a realization of the Lie algebra[3] SO(2q). Moreover, in its action on the even-fermion-number part of the Fock space, this realization also constitutes a particular irreducible linear representation of SO$(2q)$, characterized by the vacuum relations

$$\hat{A}_{ij}\,|\,0\rangle=0,\hat{\rho}_{ij}\,|\,0\rangle=-\delta_{ij}\ .\tag{2.4}$$

Here $|\,0\rangle$ plays the role of the state of highest weight in the irreducible representation of SO$(2q)$.

The central idea of Dyson boson mapping is to simulate all physical consequences of Eqs. (2.3) and (2.4) by means of combinations of boson annihilation operators b_{ij} and their Hermitian conjugates $b_{ij}^\dagger$. These satisfy commutation relations (CR's)

$$\begin{aligned}&[b_{ij},b_{kl}^\dagger]=\delta_{ik}\delta_{jl}\\&[b_{ij},b_{kl}]=[b_{ij}^\dagger,b_{kl}^\dagger]=0\quad(i>j,k>l)\ ,\end{aligned}\tag{2.5}$$

as well as the vacuum relations

$$b_{ij}\,|\,0)=0\tag{2.6}$$

and are antisymmetric in their indices:

$$b_{kl}^\dagger=-b_{lk}^\dagger\ .\tag{2.7}$$

We follow Janssen *et al.*[4] in defining for the simulation "Pauli-corrected" boson operators (always denoted by capitals)

$$B_{ij}^\dagger=b_{ij}^\dagger-\sum_{kl}b_{ik}^\dagger b_{jl}^\dagger b_{kl}\tag{2.8}$$

with Hermitian conjugates B_{ij}, and boson transition operators

$$\rho_{ij}=-\sum_k b_{ik}^\dagger b_{kj}-\sum_k b_{jk}^\dagger b_{ki}-\delta_{ij}\ .\tag{2.9}$$

The set of operators $\{B_{ij}^\dagger,\rho_{ij},b_{ij}\}$ has the same CR's as the set

$$\{\hat{A}_{ij}^\dagger,\hat{\rho}_{ij},\hat{A}_{ij}\}\ .$$

Since both sets are linearly independent, if follows that the algebras

$$F^R=\text{span}(B_{ij}^\dagger,\rho_{ij},b_{ij})\tag{2.10}$$

and

$$\hat{F}=\text{span}(\hat{A}_{ij}{}^\dagger,\hat{\rho}_{ij},\hat{A}_{ij})\tag{2.11}$$

are *isomorphic,* denoted by

$$\hat{F}\approx F^R\tag{2.12}$$

under the unprojected Dyson mapping

$$\begin{aligned}&\hat{A}_{ij}^\dagger\to B_{ij}^\dagger\ ,\\&\hat{\rho}_{ij}\to\rho_{ij}\ ,\\&\hat{A}_{ij}\to b_{ij}\ ,\end{aligned}\tag{2.13}$$

which is linear. Thus both $\hat{F}$ and F^R can be regarded as realizations of the abstract algebra SO$(2q)$.

The isomorphism (2.12) is not yet enough to ensure that the boson description is physically equivalent to the fermion description. [As an analogy, the matrices of D^1 and D^2 for SU(2) both have the same CR's and both give faithful representations, but these representations are not equivalent.] One has to show that the mapping (2.13) is induced by a linear mapping of the fermion space

$$L_{\mathbf{F}}\equiv\text{gen}(\hat{A}^\dagger)\tag{2.14}$$

into the boson space

$$L_{\mathbf{B}}\equiv\text{gen}(\mathbf{b}^\dagger)\ .\tag{2.15}$$

We use a notation in which, for example, gen$(\mathbf{b}^\dagger)$ is the space spanned by all possible products of $b^\dagger$ operators acting on $|0)$. Here, and wherever no confusion can result, we use a boldface symbol without indices to denote the set of all generators of a given type. Since $L_{\mathbf{B}}$ is larger than $L_{\mathbf{F}}$, we actually map $L_{\mathbf{F}}$ onto a subspace, the *physical boson space*

$$L_{\mathbf{PB}}\equiv\text{gen}(\mathbf{B}^\dagger)\ .\tag{2.16}$$

This space is invariant under the algebra F^R:

$$F^R L_{\mathbf{PB}}=L_{\mathbf{PB}}\ .\tag{2.17}$$

Usui[8] has shown the existence of a linear operator U that maps L_F onto L_{PB}, with an "inverse" satisfying $U\tilde{U}=1$. These operators linearly transform $\hat{F}$ into F^R, as follows:

$$U\hat{A}_{ij}^{\dagger}\tilde{U}=B_{ij}^{\dagger}P_{PB} ,\quad U\hat{\rho}_{ij}\tilde{U}=\rho_{ij}P_{PB} ,\quad U\hat{A}_{ij}\tilde{U}=b_{ij}P_{PB} , \tag{2.18}$$

This shows[4] that not only is F^R isomorphic to $\hat{F}$, but F^R (on L_{PB}) is actually *equivalent* to $\hat{F}$ (on L_F):

$$F^R \text{ (on } L_{PB}) \equiv \hat{F} \text{ (on } L_F) . \tag{2.19}$$

This ensures that F^R will lead to the same physical results as $\hat{F}$.

In Sec. III we will extend this result to a class of *subalgebras* of $\hat{F}$ realized by Dyson mapping in terms of collective bosons. Our method generalizes the Janssen-Usui method without having to assume its chief result, Eq. (2.18). Therefore this and other major results can be precipitated from our more general result by choosing the particular case where the subalgebra is the full algebra $\hat{F}$.

The existence of an alternative mapping, here called left-Dyson mapping, has been noted by Geyer and Hahne[5] and earlier by Garbaczewski,[9] who showed that the following linear mapping of $\hat{F}$ also realizes the CR's of the algebra SO(2q):

$$\hat{A}_{ij}^{\dagger}\to b_{ij}^{\dagger},\quad \hat{\rho}_{ij}\to\rho_{ij},\quad \hat{A}_{ij}\to B_{ij} . \tag{2.20}$$

This gives an isomorphism

$$F^L\approx\hat{F} , \tag{2.21}$$

where

$$F^L=\text{span}(\mathbf{b}^{\dagger},\boldsymbol{\rho},\mathbf{B}) . \tag{2.22}$$

L_{PB} is not invariant under F^L; indeed, because $b_{ij}^{\dagger}\in F^L$ L_B is the smallest invariant subspace of F^L that contains the vacuum state. As Geyer and Hahne point out, this tends to diminish the convenience of F^L compared with F^R. Nevertheless, F^L plays an essential role in our derivations, and can be useful also in practical applications, as described in Sec. VI.

III. DYSON MAPPING OF COLLECTIVE SUBALGEBRAS

Consider a set of collective operators, i.e., linear combinations of $\hat{A}_{ij}^{\dagger}$, $\hat{\rho}_{ij}$, and $\hat{A}_{ij}$. Suppose one wants to construct a dynamical model in terms of such collective operators, and then isomorphically map the collective fermion set onto an equivalent set of operators constructed from boson operators. The collective fermion set is evidently a subset of the algebra $\hat{F}$. Since $\hat{F}$ is an algebra, the collective set is part of an algebra—either of $\hat{F}$ or of some proper subalgebra of $\hat{F}$. Let $\hat{A}$ be the smallest (proper or improper) subalgebra of $\hat{F}$ that contains the collective set of operators. Since the boson mapping preserves all CR's of $\hat{F}$, the same boson mapping of $\hat{A}$ will automatically preserve the CR's of the collective operators. Therefore we can without loss of generality suppose that the set of operators to be mapped is a subalgebra of $\hat{F}$. Accordingly, we call $\hat{A}$ the *collective subalgebra*. The necessity of treating an entire subalgebra is not entirely welcome, because it severely restricts the types of exactly decoupled collective motion that can occur for a given set of shells. For example, it does not permit collective rotation of particles in a single large-j shell to be described by an SU(3) subalgebra, though (as Halse[10] has recently emphasized) rotation is clearly exhibited by exact solutions of reasonable j^n models.

We assume that the generators of

$$\hat{A}=\text{span}(\hat{\mathbf{c}}^{\dagger},\hat{\boldsymbol{\rho}},\hat{\mathbf{c}}) \tag{3.1}$$

can be written in the forms

$$\hat{c}_{\alpha}^{\dagger}=\sum_{ij}\chi_{\alpha}^{ij}\hat{A}_{ij}^{\dagger} ,\quad \hat{c}_{\alpha}=\sum_{ij}\chi_{\alpha}^{ij*}\hat{A}_{ij} ,\quad \hat{\rho}_{\alpha}=\sum_{ij}\zeta_{\alpha}^{ij}\hat{\rho}_{ij} . \tag{3.2}$$

The collective transition operators ρ can be chosen as Hermitian linearly independent combinations of the commutators $[\hat{c}_{\alpha}^{\dagger},\hat{c}_{\alpha'}]$. Clearly $\hat{A}$ is self-conjugate in the sense that

$$\hat{g}\in\hat{A}\Longrightarrow\hat{g}^{\dagger}\in\hat{A} . \tag{3.3}$$

This self-conjugacy is essential to much of the development of this paper. It holds in many cases of interest, notably for the Ginocchio Sp(6) and SO(8) models, but it does not hold, e.g., for Lorazo's[11] definition of a quasispin SU(2) algebra with nonuniform pair destruction operators which are not the Hermitian conjugates of the pair creation operators.

The generators (3.2) have the CR's of some abstract Lie algebra,

$$[\hat{g}_i,\hat{g}_j]=\sum_k f_{ij}^k\hat{g}_k , \tag{3.4}$$

with structure constants f_{ij}^k. Here $\hat{g}_i$, $\hat{g}_j$, and $\hat{g}_k$ are members of the set $\{\hat{\mathbf{c}}^{\dagger},\hat{\boldsymbol{\rho}},\hat{\mathbf{c}}\}$. A complete set of coherent pair creation operators is obtained by supplementing the $\hat{c}_{\alpha}$ by *noncollective* (though coherent) pair-creation operators

$$\hat{n}_{\alpha}^{\dagger}\equiv\sum_{ij}\xi_{\alpha}^{ij}\hat{A}_{ij}^{\dagger} . \tag{3.5}$$

The corresponding annihilation operators are the Hermitian conjugates of these.

We introduce notations for the images of the collective bifermion operators under the linear isomorphism (2.13):

$$\hat{c}_{\alpha}^{\dagger}\to C_{\alpha}^{\dagger}\equiv\sum_{ij}\chi_{\alpha}^{ij}B_{ij}^{\dagger} ,\quad \hat{c}_{\alpha}\to c_{\alpha}\equiv\sum_{ij}\chi_{\alpha}^{ij*}b_{ij} ,\quad \rho_{\alpha}\to\rho_{\alpha}\equiv\sum_{ij}\zeta_{\alpha}^{ij}\rho ij . \tag{3.6}$$

The right-Dyson images of the noncollective pair creation and annihilation operators are defined by

$$\hat{n}^\dagger_\beta \to \hat{N}^\dagger_\beta \equiv \sum_{ij} \xi^{ij}_\beta B^\dagger_{ij} ,$$
$$\hat{n}_\beta \to n_\beta \equiv \sum_{ij} \chi^{ij*}_\beta b_{ij} . \tag{3.7}$$

The operators C_α, $c^\dagger_\alpha$, N_β, and n_β are Hermitian conjugates of those defined in Eqs (3.5) and (3.6). If we choose the coefficients χ and ξ so that the partitioned matrix $(\chi \vdots \xi)$ is unitary, the standard boson CR's hold for the collective and noncollective operators:

$$[c_{\alpha'}, c^\dagger_\alpha] = \delta_{\alpha'\alpha} ,$$
$$[n_{\beta'}, n^\dagger_\beta] = \delta_{\beta'\beta} , \tag{3.8}$$
$$[c_{\alpha'}, c^\dagger_\beta] = \delta_{\alpha'\beta} .$$

Thus the operators in Eq. (3.8) correspond to independent ideal bosons.

From the isomorphism of $\hat{F}$ and F^R [Eqs. (2.13) and (2.14)] we see that

$$\hat{A} \approx A^R , \tag{3.9}$$

where A^R is the boson algebra defined by

$$A^R \equiv \mathrm{span}(\mathbf{C}^\dagger, \boldsymbol{\rho}, \mathbf{c}) . \tag{3.10}$$

We call A^R the *unprojected right-Dyson mapping* of $\hat{A}$. The realization A^R (on L_B) can be regarded as a representation of $\hat{A}$, but is not expected to be equivalent to $\hat{A}$ (on L_F), just as F^R (on L_B) is not equivalent to $\hat{F}$ (on L_F).

If left-Dyson mapping (2.22) is specialized to the collective subalgebra $\hat{A}$, a new boson realization immediately results:

$$A^L \approx \hat{A} \tag{3.11}$$

with

$$A^L \equiv \mathrm{span}(\mathbf{c}^\dagger, \boldsymbol{\rho}, \mathbf{C}) . \tag{3.12}$$

Of some importance is the "conjugacy" relationship of A^R and A^L, namely

$$g^R \in A^R \Longrightarrow g^{R\dagger} \in A^L . \tag{3.13}$$

This is obvious for $C^\dagger_\alpha$ and c_α. For ρ_β, we need only the fact that

$$\rho^\dagger_\beta = \rho_\beta , \tag{3.14}$$

which follows from the required hermiticity of $\hat{\rho}_\beta$, together with the easily-deduced property

$$\rho^\dagger_{ij} = \rho_{ij} . \tag{3.15}$$

The conjugacy relationship does not depend on the coefficients χ and ξ being real.

The existence of the boson realizations A^R and A^L is an essentially trivial consequence of the existence of the corresponding mappings F^R and F^L of the full bifermion algebra $\hat{F}$. In Sec. IV we go beyond these results, to consider what we call "skeleton" boson realizations, which are realizations in which noncollective boson operators do not appear.

The presence of noncollective boson operators in the realizations so far considered can be seen as follows. One can express each operator $b^\dagger$ (or each operator b) in terms of the complete sets of coherent creation (or annihilation) operators $\{\mathbf{c}^\dagger, \mathbf{n}^\dagger\}$ (or $\{\mathbf{c}, \mathbf{n}\}$). When the results are inserted in (2.8) and (2.9) for use in (3.6), one finds schematically:

$$C^\dagger = c^\dagger - \sum (c^\dagger + n^\dagger)(c^\dagger + n^\dagger)(c + n) \tag{3.16}$$

and

$$\rho = \sum (c^\dagger + n^\dagger)(c + n) + \text{const.} \tag{3.17}$$

It seems by no means obvious that the noncollective operators n and $n^\dagger$ can be omitted without destroying the isomorphism of A^R and A^L. When Geyer and Hahne[5] conjectured this result for the Dyson mapping of the SO(8) bifermion algebra, they were accordingly careful to check the CR's of the skeleton boson realization. However, in Sec. V we show, among other things, that this result is in fact quite general.

IV. PROPERTIES OF CTD STRUCTURES

For convenience, we abstract some essential features shared by the realizations of the algebras to be discussed, and give the name "CTD structure" to any object that posseses these features. Thus, a CTD structure is a Lie algebra that has generators of three types, C ("creation"), T ("transition"), and D ("destruction"), such that we have the following.

(1) The commutator of generators of types C and D is of type T.

(2) The commutator of generators of types T and X is of type X, where X stands for C, D, or T.

(3) The generators are linear operators on a linear vector space that contains a simultaneous eigenvector of all D and T generators, with all D generators having zero eigenvalue. (This state is called the vacuum state, and denoted by $|0)$ or $|0\rangle$, as appropriate.)

We often need to consider two CTD structures, say

$$A' \equiv \mathrm{span}(\mathbf{g}'_C, \mathbf{g}'_T, \mathbf{g}'_D) , \tag{4.1}$$

$$A'' \equiv \mathrm{span}(\mathbf{g}''_C, \mathbf{g}''_T, \mathbf{g}''_D) , \tag{4.2}$$

that are *isomorphic,* i.e., have the same CR's. Here the subscripts C, T, and D indicate the types of the generators. Consider any matrix element involving generators of A'. It can always be written as a vacuum expectation of a function of the generators of A', in which each term contains exactly as many C factors as D factors; for example,

$$M' = (0' \,|\, f(\mathbf{g}'_C, \mathbf{g}'_T, \mathbf{g}'_D) \,|\, 0') . \tag{4.3}$$

By the *analogous* matrix element for A'', we mean

$$M'' = (0'' \,|\, f(\mathbf{g}''_C, \mathbf{g}''_T, \mathbf{g}''_D) \,|\, 0'') \tag{4.4}$$

for the same function f. We now have the following result.

Lemma M ("matrix element"). If corresponding type-T generators of two isomorphic CTD structures have equal vacuum eigenvalues, i.e.,

$$g'_{T\alpha}\,|\,0'\rangle = k'_\alpha\,|\,0'\rangle,\quad g''_{T\alpha}\,|\,0''\rangle = k''_\alpha\,|\,0''\rangle\ , \tag{4.5}$$

with $k'_\alpha = k''_\alpha$, than all analogous matrix elements are equal.

The following is an immediate corollary.

Lemma Z ("zero"). Under the conditions of Lemma *M*, if ϕ is any function of the generators of type *C*, then

$$|\,\psi'\rangle \equiv \phi(\mathbf{g}'_C)\,|\,0'\rangle = 0 \Longrightarrow P''\phi(\mathbf{g}''_C)\,|\,0''\rangle = 0\ , \tag{4.6}$$

where P'' is the orthogonal projector on the space

$$L'' \equiv \text{gen}(\mathbf{g}_D''^{\dagger})\ . \tag{4.7}$$

Finally we state a result that is often useful for deducing the invariance of certain spaces under the elements of CTD structures.

Lemma I ("invariance"). Let A' be any CTD structure, and define the space

$$L' \equiv \text{gen}(\mathbf{g}'_C) \tag{4.8}$$

generated by the *C*-type generators of A'. Then L' is invariant under A', that is

$$A'L' = L'\ . \tag{4.9}$$

The three lemmas are proved in the Appendix.

V. SKELETON BOSON REALIZATIONS OF COLLECTIVE SUBALGEBRAS

Since we need to define a variety of different spaces, we extend the system of notation so that L_x (x = F, B, PB, CF, CB, PCB, MB) is a linear vector space, P_x is the orthogonal projection operator on L_x, and d_x is the dimension of L_x. Figure 1 provides a guide to the logical flow of this section.

Our aim is to find realizations in which only collective boson operators enter. Accordingly, we first define the *collective boson space* as

$$L_{\text{CB}} \equiv \text{gen}(\mathbf{c}^\dagger)\ . \tag{5.1}$$

Lemma *I* immediately gives

$$A^L L_{\text{CB}} = L_{\text{CB}}\ . \tag{5.2}$$

Clearly, no smaller vacuum-containing subspace of L_B is invariant under A^L. Equation (5.2) can be expressed by means of projection operators, in at least two ways,

$$g^L P_{\text{CB}} = P_{\text{CB}} g^L P_{\text{CB}} \quad (g^L \in A^L) \tag{5.3}$$

and

$$P_{\text{CB}} g^R = P_{\text{CB}} g^R P_{\text{CB}} \quad (g^R \in A^R)\ , \tag{5.4}$$

through the conjugacy relation (3.13).

Equation (5.3) can be shown to imply that $\hat{A}$ can be represented by a skeleton version of A^L, free of noncollective operators, as follows. Define skeleton generators

$$\dot{g}^L = P_{\text{CB}} g^L P_{\text{CB}} \quad (g^L \in A^L) \tag{5.5}$$

spanning an algebra

$$\dot{A}^L \equiv \text{span}(\dot{\mathbf{g}}^L)\ . \tag{5.6}$$

After writing down the CR's of A^L from those of the isomorphic algebra $\hat{A}$, Eq. (3.4), we pre- and post-multiply by P_{CB}, and use Eq. (5.3) to get the CR's of $\dot{A}^L$:

$$\dot{A}^L \approx A^L\ . \tag{5.7}$$

In deducing isomorphism rather than homomorphism, we assume that the set $\{\dot{g}^L\}$ is linearly independent. It is clear that

$$\dot{A}^L L_{\text{CB}} = L_{\text{CB}}\ , \tag{5.8}$$

and that no smaller vacuum-containing subspace of L_{CB} is invariant under $\dot{A}^L$.

Now, to make a skeleton realization out of A^R, we begin by noting that the *physical collective boson space*, defined by

$$L_{\text{PCB}} \equiv \text{gen}(\mathbf{C}^\dagger)\ , \tag{5.9}$$

is invariant under A^R,

$$A^R L_{\text{PCB}} = L_{\text{PCB}}\ , \tag{5.10}$$

by lemma *I*. Typically, neither L_{CB} nor L_{PCB} is a subspace of the other, because $d_{\text{CB}} > d_{\text{PCB}}$ while noncollective bosons are present in L_{PCB} but not in L_{CB}. Again L_{PCB} is the smallest subspace of L_B that is invariant under A^R and contains the vacuum.

We now define skeleton generators

$$\dot{g}^R \equiv P_{\text{CB}} g^R P_{\text{CB}} \quad (g^R \in A^R) \tag{5.11}$$

spanning an algebra

$$\dot{A}^R \equiv \text{span}(\dot{g}^R)\ . \tag{5.12}$$

This algebra is isomorphic to $\hat{A}$, as can be seen by writing down the CR's for A^R from the CR's (3.4) of the isomorphic algebra $\hat{A}$, pre- and post-multiplying by P_{CB}, and using conjugacy as expressed by Eq. (5.4). We define the *model boson space,* which is generated by *skeleton* collective boson creation operators,

$$L_{\text{MB}} \equiv \text{gen}(\dot{\mathbf{C}}^\dagger)\ , \tag{5.13}$$

and which, by lemma *I*, is invariant under $\dot{A}^R$:

$$\dot{A}^R L_{\text{MB}} = L_{\text{MB}}\ . \tag{5.14}$$

Thus $\dot{A}^R$ is a linear representation of $\dot{A}$ on the space L_{MB}.

As we noted in Sec. II, $\hat{F}$ (on L_F) and F^R (on L_{PB}) are not merely isomorphic, but *equivalent* in the sense that an invertible linear mapping of L_F onto L_{PB} connects them. The same mapping automatically connects the collective subalgebras $\hat{A}$ (on L_{CF}) and A^R (on L_{PCB}) and establishes their equivalence:

$$\hat{A}\ (\text{on } L_{\text{CF}}) \equiv A^R\ (\text{on } L_{\text{PCB}})\ . \tag{5.15}$$

Our goal now is to extend this equivalence to the skeleton representation $\dot{A}^R$ (on L_{MB}).

The representation spaces have the same dimension,

$$d_{\text{MB}} = d_{\text{CF}}\ . \tag{5.16}$$

To see this, suppose that some linear combination

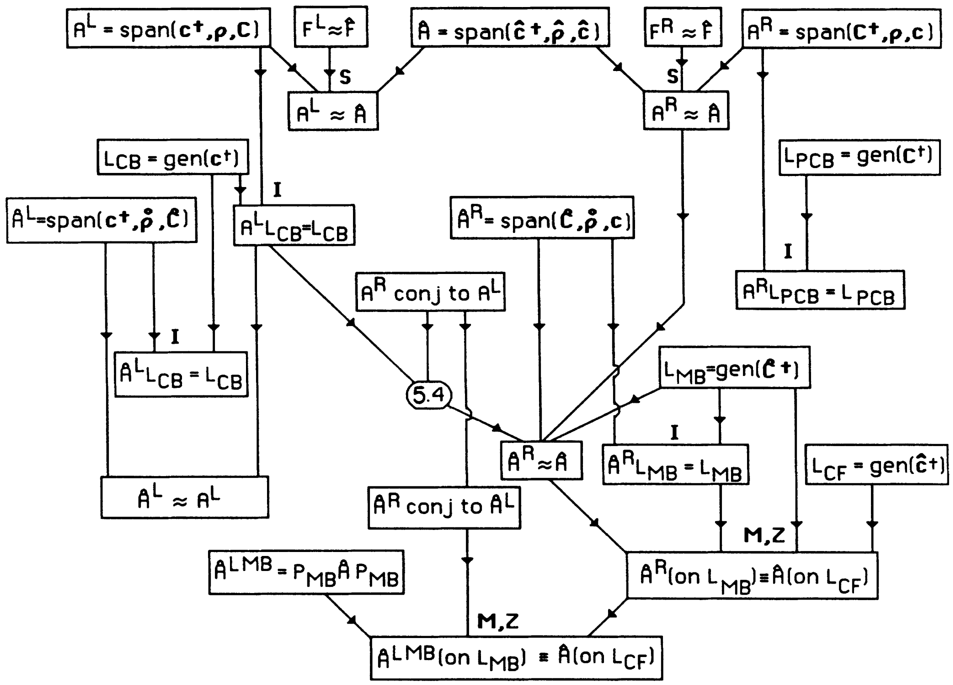

FIG. 1. Logical flow of Sec. V. M, Z, and I indicate the use of the corresponding lemmas to obtain the result in the block below. S indicates specialization of a representation to a subalgebra. (5.4) refers to Eq. (5.4). Isomorphism of algebras is denoted by $\approx$; equivalence by $\equiv$. Skeleton algebras are denoted, e.g. by $\mathring{A}^R$, rather than the $\dot{A}^R$ used in the text.

$\psi(\hat{c}^\dagger)\,|\,0\rangle$ of vectors in $L_{\rm CF}$ vanishes. Then, with $A'=\hat{A}$ and $A''=\dot{A}^R$ so that

$$L''=\text{gen}(\hat{c}^\dagger)=L_{\rm CB}\supseteq L_{\rm MB}$$

in lemma Z, it follows that $\psi(\dot{\mathbf{C}}^\dagger)\,|\,0)=0$. Consequently, $d_{\rm MB}\le d_{\rm CF}$. The reverse inequality follows by similar reasoning with $\hat{A}$ and $\dot{A}^R$ interchanged, so establishing Eq. (5.16).

Now let

$$\{\,|\,\phi_m\rangle\equiv\phi_m(\hat{c}^\dagger)\,|\,0\rangle,\quad m=1,\ldots,d_{\rm CF}\}\ , \tag{5.17}$$

be an orthonormal basis for $L_{\rm CF}$. The corresponding set of vectors in $L_{\rm MB}$ is

$$\{\,|\,\phi_m)\equiv\phi_m(\dot{\mathbf{C}}^\dagger)\,|\,0),\quad m=1,\ldots,d_{\rm CF}\}\ . \tag{5.18}$$

This set is linearly independent (and hence a basis for $L_{\rm MB}$) because if some linear combination of Eq. (5.18) vanishes, by lemma Z the same linear combination of Eq. (5.17) also vanishes, contradicting the assumed orthonormality of Eq. (5.17). Note that

$$\{(\bar{\phi}_m\,|\equiv(0\,|\,\phi_m(\mathbf{c}),\quad m=1,\ldots,d_{\rm CF}\} \tag{5.19}$$

is a basis dual to Eq. (5.18), so that the biorthonormality condition

$$(\bar{\phi}_m\,|\,\phi_{m'})=\delta_{mm'} \tag{5.20}$$

is satisfied. This result follows by applying lemma M to the orthonormality of the kets (5.17) with $A'=\hat{A}$ and $A''=\dot{A}^R$.

Transformation operators are now defined by

$$T\,|\,\phi_m\rangle=|\,\phi_m)\quad(m=1,\ldots,d_{\rm CF}) \tag{5.21}$$

$$\langle\phi_m\,|\,\tilde{T}=(\bar{\phi}_m\,|\quad(m=1,\ldots,d_{\rm CF})\ . \tag{5.22}$$

By taking matrix elements it follows that

$$\tilde{T}T=P_{\rm CF}\ . \tag{5.23}$$

By solving Eqs. (5.21) and (5.22) through the use of (5.23), we get

$$|\,\phi_m\rangle=\tilde{T}\,|\,\phi_m)\quad(m=1,\ldots,d_{\rm CF}) \tag{5.24}$$

$$\langle\phi_m\,|=(\bar{\phi}_m\,|\,T\quad(m=1,\ldots,d_{\rm CF}) \tag{5.25}$$

which then lead to

$$T\tilde{T}=P_{\rm MB}\ . \tag{5.26}$$

It is quite appropriate to consider $\tilde{T}$ as an inverse of T, even though each operator maps one space onto another space, rather than onto itself.

Now consider how generators behave under the transformation implied by (5.21) and (5.22). Let $\hat{g}$ be any generator of $\hat{A}$, and let $\dot{g}^R$ be the corresponding generator of $\dot{A}^R$ under the isomorphism. We calculate a matrix element between vectors of biorthonormal bases for $L_{\rm MB}$:

$$(\bar{\phi}_m\,|\,T\hat{g}\tilde{T}\,|\,\phi_{m'})=\langle\phi_m\,|\,\hat{g}\,|\,\phi_{m'}\rangle=(\bar{\phi}_m\,|\,\dot{g}^R\,|\,\phi_{m'})\ . \tag{5.27}$$

The first equality follows from (5.24) and (5.25), while the second equality results from lemma M by taking $A'=\hat{A}$ and $A''=\dot{A}^R$. Since both bases are complete,

$$T\hat{g}\tilde{T}=\dot{g}^{R}P_{\mathrm{MB}} \,. \tag{5.28}$$

The projection operator is needed because we make a convention that $\tilde{T}$ gives zero when it acts on the orthogonal complement of L_{MB} in L_{CB}. Equation (5.28) shows that

$$\hat{A}\ (\text{on } L_{\mathrm{CF}})\equiv \dot{A}^{R}\ (\text{on } L_{\mathrm{MB}}) \,, \tag{5.29}$$

so these representations lead to physically equivalent results. By (5.21) and (5.22) the transformation preserves products and linear combinations, so one can transform any function of generators by taking the same function of the transformed generators. It is interesting to note that the proof of Eq. (5.29) has not used Eq. (2.19). Therefore by taking $\hat{A}=\hat{F}$ in Eq. (5.29), and observing that $\dot{F}^{R}$ is the same as F^{R}, one obtains a proof of Eq. (2.19) that is independent of the Janssen *et al.* derivation[4] which used the Usui operator.

There is another approach to proving the equivalence (5.29). A theorem given by Racah states[12] that if two irreducible representations of a semisimple Lie algebra have the same highest weight, they are equivalent. It is clear that $\hat{A}$ (on L_{CF}) and $\dot{A}^{R}$ (on L_{MB}) have the same highest weight, because this is determined by the vacuum eigenvalues of the operators $\hat{\rho}_{\beta}$ and $\dot{\rho}_{\beta}$. The irreducibility of either representation (say $\hat{A}$) is easy to prove if $\hat{A}$ contains a Hermitian generator $\mathcal{N}$ whose eigenvectors (corresponding to eigenvalues N) are of the form $f_N(\hat{c}^{\dagger})\,|\,0\rangle$, where $f_N(\hat{c}^{\dagger})$ is a homogeneous polynomial of degree N in the collective creation operators. (The total fermion number operator is a natural candidate for $\mathcal{N}$, but if it is not available in $\hat{A}$ there may still be other possibilities.) If $\hat{A}$ contains $\mathcal{N}$, one can connect any two vectors in L_{CF}, say $\psi_1(\hat{c}^{\dagger})\,|\,0\rangle$ and $\psi_2(\hat{c}^{\dagger})\,|\,0\rangle$, by applying generators of $\hat{A}$ in the following way.

(1) Construct a set of polynomials in $\mathcal{N}$, say $P_N(\mathcal{N})$, each of which is an orthogonal projection operator on the subspace of vectors of homogeneous degree N in $\hat{c}^{\dagger}$.

(2) Calculate $P_N(\mathcal{N})\psi_1(\hat{c}^{\dagger})\,|\,0\rangle$ for a value of N chosen so that the result does not vanish.

(3) Apply $\psi_1(\hat{c}^{\dagger})^{\dagger}=\psi_1(\hat{c})$ to this result, to get a nonzero multiple of $|\,0\rangle$.

(4) Apply $\psi_2(\hat{c}^{\dagger})$ to obtain a nonzero multiple of $\psi_2(\hat{c}^{\dagger})\,|\,0\rangle$.

This approach also applies if $\hat{A}$ is a direct sum of semisimple algebras, each of which includes an $\mathcal{N}$ generator. The question of the irreducibility of $\hat{A}$ (on L_{CF}) is also of interest because it influences the extent to which the collective dynamics is determined by the Clebsch-Gordan coefficients of $\hat{A}$ via the Wigner-Eckart theorem.

From the transformation (5.28) a new skeleton representation, related to $\dot{A}^{L}$ rather than $\dot{A}^{R}$, can be constructed. The Hermitian conjugate of (5.28) applied to the basic generators gives

$$\begin{aligned}\tilde{T}^{\dagger}\hat{c}_{\alpha}^{\dagger}T^{\dagger}&=P_{\mathrm{MB}}\dot{c}_{\alpha}^{\dagger}\,,\\ \tilde{T}^{\dagger}\hat{c}_{\alpha}T^{\dagger}&=P_{\mathrm{MB}}\dot{C}_{\alpha}\,,\\ \tilde{T}^{\dagger}\hat{\rho}_{\beta}T^{\dagger}&=P_{\mathrm{MB}}\dot{\rho}_{\beta}\,.\end{aligned} \tag{5.30}$$

Thus the conjugacy relation between A^{R} and A^{L}, Eq. (3.13), enables us to write

$$\tilde{T}^{\dagger}\hat{g}T^{\dagger}=P_{\mathrm{MB}}\dot{g}^{L}\ \ (=P_{\mathrm{MB}}\dot{g}^{L}P_{\mathrm{MB}}) \,, \tag{5.31}$$

where $\dot{g}^{L}$ is the element of $\dot{A}^{L}$ that corresponds to $\hat{g}$ in the isomorphism of $\dot{A}^{L}$ and $\hat{A}$. It follows that the restriction of $\dot{A}^{L}$ to L_{MB},

$$\dot{A}^{\mathrm{LMB}}\equiv \mathrm{span}(P_{\mathrm{MB}}\mathbf{c}^{\dagger}P_{\mathrm{MB}},P_{\mathrm{MB}}\dot{\rho}P_{\mathrm{MB}},P_{\mathrm{MB}}\dot{\mathbf{C}}P_{\mathrm{MB}}) \,, \tag{5.32}$$

is a pseudoskeleton representation of $\hat{A}$, and is actually equivalent to $\hat{A}$:

$$\dot{A}^{\mathrm{LMB}}\ (\text{on } L_{\mathrm{MB}})\equiv \hat{A}\ (\text{on } L_{\mathrm{CF}}) \,. \tag{5.33}$$

$\dot{A}^{\mathrm{LMB}}$ is a representation in spite of the fact that L_{MB} is *not* invariant under $\dot{A}^{L}$. The representation $\dot{A}^{L}$ carried by L_{CB} does not decompose into $\dot{A}^{\mathrm{LMB}}$ and a representation on the orthogonal complement of L_{MB} in L_{CB}. Only because $\dot{A}^{L}$ is not self-conjugate can this situation arise. It would be possible in principle to make $\dot{A}^{L}$ self-conjugate by a similarity transformation, but doing so would violate the spirit of closed-form Dyson mappings and embark on the Holstein-Primakoff approach, with its operator square roots.

VI. SIMPLE CORRESPONDENCE

The isomorphism between $\hat{A}$ and $\dot{A}^{L}$ maps each fermion pair-creation operator $\hat{c}_{\alpha}^{\dagger}$ onto an ideal boson operator $c_{\alpha}^{\dagger}$. This resembles the "simple correspondence" approach of Ginocchio and Talmi.[7]

By generalizing to subalgebras some more ideas that Janssen *et al.*[4] developed for mappings of the full algebra $\hat{F}$, we will show that $\dot{A}^{L}$ (on L_{CB}) can be useful for physical calculations, even though it is not equivalent to $\hat{A}$. Suppose the fermion Hamiltonian can be written as

$$\hat{H}=h(\hat{c}^{\dagger},\hat{\rho},\hat{c}) \,, \tag{6.1}$$

where h is a function with the operators written in standard CTD order. Then we can construct an analogous boson operator

$$\dot{H}^{L}\equiv h(\mathbf{c}^{\dagger},\dot{\rho},\dot{C}) \,. \tag{6.2}$$

This is *suggested* by the isomorphism of $\dot{A}$ and $\dot{A}^{L}$, but it is not in any sense implied by it. In fact, because $\dot{A}^{L}$ is not equivalent to $\hat{A}$ there are no general rules for transforming products of the generators that appear in (6.1). Suppose we solve the eigenvalue problem for $\dot{H}^{L}$, i.e.,

$$h(\mathbf{c}^{\dagger},\dot{\rho},\dot{\mathbf{C}})\psi(\mathbf{c}^{\dagger})\,|\,0)=E\psi(\mathbf{c}^{\dagger})\,|\,0) \tag{6.3}$$

in L_{CB}. Then, by lemma Z with $A'=\dot{A}^{L}$ and $A''=\dot{A}^{R}$, Eq. (6.3) implies

$$h(\dot{C}^{\dagger},\dot{\rho},\mathbf{c})\psi(\dot{\mathbf{C}}^{\dagger})\,|\,0)=E\psi(\dot{\mathbf{C}}^{\dagger})\,|\,0) \,. \tag{6.4}$$

Since $\dot{A}^{R}$ (on L_{MB}) is equivalent to $\hat{A}$ (on L_{CF}), Eq. (5.28) applies, and so

$$h(\dot{\mathbf{C}}^{\dagger},\dot{\rho},\mathbf{c})P_{\mathrm{MB}}=Th(\hat{c}^{\dagger},\hat{\rho},\hat{c})\tilde{T}=T\hat{H}\tilde{T} \tag{6.5}$$

and

$$\psi(\dot{\mathbf{C}}^{\dagger})\,|\,0)=T\psi(\hat{c}^{\dagger})\,|\,0\rangle \,. \tag{6.6}$$

By Eqs. (6.5) and (6.6), Eq. (6.4) becomes

$$T\hat{H}\tilde{T}T\psi(\hat{c}^\dagger)\,|\,0\rangle = ET\psi(\hat{c}^\dagger)\,|\,0\rangle ,$$

i.e., by Eq. (5.23),

$$H\psi(\hat{c}^\dagger)\,|\,0\rangle = E\psi(\hat{c}^\dagger)\,|\,0\rangle . \tag{6.7}$$

This result can also be obtained directly from Eq. (6.4) by use of lemma Z, which assures us that the state in Eq. (6.7) does not vanish, provided that

$$\psi(\dot{\mathbf{C}}^\dagger)\,|\,0) \neq 0 . \tag{6.8}$$

Thus each solution of Eq. (6.3) that satisfies Eq. (6.8) belongs to a physically meaningful energy. For any two states satisfying Eq. (6.8) for energies E_s and E_t and for any operator Ω,

$$\begin{aligned}\langle 0\,|\,\psi_s(\hat{c})\Omega(\hat{c}^\dagger,\hat{\rho},\hat{c})\psi_t(\hat{c}^\dagger)\,|\,0\rangle &= (0\,|\,\psi_s(\mathbf{c})\Omega(\dot{\mathbf{C}}^\dagger,\dot{\rho},\mathbf{c})\psi_t(\dot{\mathbf{C}}^\dagger)\,|\,0) \\ &= (0\,|\,\psi_s(\dot{\mathbf{C}})\Omega(\mathbf{c}^\dagger,\dot{\rho},\dot{\mathbf{C}})\psi_t(\mathbf{c}^\dagger)\,|\,0) ,\end{aligned} \tag{6.9}$$

by lemma M applied to the mapping from A^R to A^L. Thus, whether one uses A^R or A^L, calculation of a transition matrix element seems to require construction of at least one of the states from the complicated "Pauli-corrected" operators $\dot{\mathbf{C}}^\dagger$ or $\dot{\mathbf{C}}$. However, one can avoid such explicit construction by a generalization of the method of left and right eigenvectors that was suggested by Janssen *et al.*[4] First one solves (6.3) for the eigenvalue E_s, so that $\psi_s(\dot{\mathbf{C}}^\dagger)\,|\,0)$ satisfies (6.4). Next one solves

$$(h(\dot{\mathbf{C}}^\dagger,\dot{\rho},\mathbf{c})-E_s)f_s(\mathbf{c}^\dagger)\,|\,0)=0 \tag{6.10}$$

in $L_{\rm CB}$. If E_s is nondegenerate, it follows [since Eqs. (6.10) and (6.4) are eigenvalue problems for the same operator] that

$$\psi_s(\dot{\mathbf{C}}^\dagger)\,|\,0) \propto f_s(\mathbf{c}^\dagger)\,|\,0), \quad (0\,|\,\psi_s(\dot{\mathbf{C}}) \propto (0\,|\,f_s(\mathbf{c}) . \tag{6.11}$$

If one chooses the phase and normalization of the solution of (6.10) so that

$$(0\,|\,f_s(\mathbf{c})\psi_s(\mathbf{c}^\dagger)\,|\,0)=1 , \tag{6.12}$$

one can be sure that

$$(0\,|\,f_s(\mathbf{c}) = (0\,|\,\psi_s(\dot{\mathbf{C}}) , \tag{6.13}$$

so that evaluating the right-hand side of (6.9) no longer presents any special difficulty. Note that Eq. (8) remains a necessary test for spurious states. Questions of the existence of solutions of (6.3) and the possibility of degeneracy remain to be studied.

The calculational scheme outlined here makes use only of *collective* bosons, and is in principle capable [via Eq. (6.8)] of detecting spurious states without going outside the algebra intrinsic to the collective bosons. This offers a possibility of constructing boson models with internal control of spuriosity. For this purpose, a more convenient test for spurious states is highly desirable. It seems possible that the "Majorana operator" method recently described by Park[13] for F^R could be generalized to subalgebras. If so, one could add a one- plus two-body term to the collective boson Hamiltonian to shift the spurious states to high energies so that they cannot contaminate the physical results. Perhaps one could use the Casimir operators of the group generated by all the $\dot{\rho}_\beta$ operators; but one would first have to know what eigenvalues of these Casimir operators correspond to physical states. An alternative method has been described by Geyer *et al.*[14] More work on spurious states is clearly needed.

VII. SIMILARITY OF $A^{\rm LMB}$ and A^R

Both $\dot{A}^R$ and $\dot{A}^{\rm LMB}$ are equivalent to $\dot{A}$, as we see from Eqs. (5.29) and (5.33). We show that the similarity transformation that connects them directly is

$$K \equiv \tilde{T}^\dagger\tilde{T} . \tag{7.1}$$

We solve (5.28) for $\hat{g}$ by using (5.23), and then insert the result in (5.31) to find

$$\dot{g}^L = \tilde{T}^\dagger\tilde{T}\dot{g}^R TT^\dagger . \tag{7.2}$$

The notation K^{-1} for $TT^\dagger$ is justified by the calculation [using Eqs. (5.23) and (5.26)]

$$\tilde{T}^\dagger\tilde{T}TT^\dagger = \tilde{T}^\dagger P_{\rm CF}T^\dagger = (T\tilde{T})^\dagger = P_{\rm MB} ,$$

together with the fact that K maps $L_{\rm MB}$ onto itself. Then (7.2) becomes

$$\dot{g}^L = K\dot{g}^R K^{-1} . \tag{7.3}$$

Because $\dot{A}^R$ and $\dot{A}^{\rm LMB}$ share the same transition operators $\dot{\rho}_\beta$, Eq. (7.3) immediately implies that K leaves these invariant, so that

$$[K,\dot{\rho}_\beta]=0 . \tag{7.4}$$

In the case of the Ginocchio SO(8) model, this result implies that K is invariant under SO(6), which is the algebra spanned by the generators ρ_β in the SO(8) model. We have shown elsewhere[2] how this constraint facilitates explicit construction of K. The operator K is Hermitian,

$$K = K^\dagger . \tag{7.5}$$

If $\hat{\theta}$ is any Hermitian operator constructed from the generators of $\hat{A}$, hermiticity of its simple-correspondence image

$$\theta = T\hat{\theta}\tilde{T} \tag{7.6}$$

is equivalent to

$$\theta = T\hat{\theta}\tilde{T} = \tilde{T}^\dagger\hat{\theta}T^\dagger .$$

By eliminating $\hat{\theta}$ through the action of $\tilde{T}$, T, and their Hermitian conjugates, we find

$$\theta = TT^\dagger\theta\tilde{T}^\dagger\tilde{T} = K^{-1}\theta K .$$

Since this argument is reversible,

$$\theta = \theta^\dagger \Longleftrightarrow [K,\theta]=0 . \tag{7.7}$$

This means that the hermiticity of the simple-correspondence image of a fermion Hamiltonian depends on its symmetry type, defined in a certain sense. Ginocchio and Talmi[7] have noticed this fact in some special cases. Using the simple-correspondence rule $\hat{c}^\dagger \rightarrow c^\dagger$, they ask when a fermion Hamiltonian can be replaced by a Hermitian one- plus two-body boson Hamiltonian that is equivalent in the sense of reproducing some "collective" subset of the fermion energies. They conclude that among the class of Hamiltonians considered in the SO(8) model, only those with SO(6) symmetry have Hermitian boson equivalents. We have given a more detailed treatment of the question in Ref. 2.

Within L_{MB}, every expectation value of K is positive, so K has only positive eigenvalues. The only zero eigenvalues of K correspond to states outside L_{MB}, which are annihilated by $\tilde{T}$. The eigenvectors of K have interesting transformation properties. Suppose $|k)$ is an orthonormal eigenbasis of K in L_{MB}, so that

$$K\,|k)=k\,|k)\ , \tag{7.8}$$

then

$$(k'\,|\,TT^\dagger\,|\,k)=k(k'\,|\,k)=k\delta_{k'k}\ . \tag{7.9}$$

Thus the fermion states $|k\rangle \equiv T^\dagger\,|k)$ are also orthogonal, though not normalized in general. We easily see that the orthogonal fermion set $\{\,|k\rangle\}$ is such that its boson image under T is also an orthogonal set (which happens to be normalized). Thus, although T is not unitary, there does exist a basis $\{\,|k\rangle\}$ whose orthogonality is preserved under T; we call this an *invariantly orthogonal basis* under T. The interpretation of k as the norm of $|k\rangle$ confirms that k cannot be zero or negative if $|k)$ is "physical, i.e.," lies in L_{MB}.

Although K reliably distinguishes between physical and spurious states, it is in general difficult to construct. This again highlights the need for a simple operator like the Majorana operator that Park[13] suggested for this purpose.

VIII. CONCLUSIONS AND OUTLOOK

Many types of collective motion can be kinematically described in terms of collective bifermion operators. If the collective bifermion operators close under commutation, to form a self-conjugate Lie algebra, a skeleton kinematical description—i.e., one in terms of collective bosons only—always exists. If the fermion Hamiltonian is a one- plus two-body function of the collective bifermion operators, the collective fermion dynamics can be described in terms of a one- plus two-body boson Hamiltonian, which is in general non-Hermitian. The boson dynamics is exactly equivalent to the collective fermion dynamics, provided spurious states are discarded.

Thus we see that some aspects of the IBM have a great degree of universality: *every* self-conjugate bifermion collective algebra can be exactly modeled by a system of collective bosons with a one- plus two-body Hamiltonian. However, the possibility of using a boson Hamiltonian that is also *Hermitian* is much less general, and depends on whether the fermion Hamiltonian has a certain symmetry property [see Eq. (7.7) and Ref. 2].

The derivation of these results involves restricting the Dyson mapping of the full bifermion algebra to its collective subalgebra. Subsequently the Pauli correction terms of the Dyson images must be mutilated by omitting all terms that refer to noncollective bosons [see Eqs. (3.16) and (3.17). A major surprise is that the subtle information contained in the Pauli correction terms so well survives this mutilated that it is still always possible to identify the spurious states by operations carried out entirely within the space of the collective bosons.

For any given collective algebra, there are a variety of boson realization; these are divided into two main classes, left and right, according as the burden of the Pauli corrections is carried by the boson images of the pair annihilation operators or of the pair creation operators. Left and right realizations are useful in complementary ways. We emphasize that the representation-theoretic *equivalence* of boson and fermion realizations of the collective algebra is necessary for their physical equivalence; identical commutation relations are not enough. This means that the Marumori approach, which sets up a one-to-one correspondence between fermion and boson states, plays an essential role and should not be neglected. Our insistence on equivalence of realizations makes it possible to maintain consistently the crucial distinction between physical and spurious states in the collective boson space. It might be helpful if the group-theoretical analysis of the physical and spurious boson spaces that was initiated by Park[13] could be extended to physically interesting collective subalgebras [e.g., SO(8) and Sp(4)] of the full bifermion algebra. Alternatively, the method of Geyer *et al.*[14] seems to be viable for such cases.

Heuristically, the success of any interacting boson model suggests the existence of a self-conjugate bifermion algebra whose generators suffice to express the Hamiltonian, at least approximately. The self-conjugacy requirement for boson mappability may seem innocuous, since it must presumably already be satisfied if the Hermitian fermion Hamiltonian can be expressed in terms of the generators of the algebra. However, in fact, self-conjugacy is severely restrictive. For example, it rules out small bifermion algebras that contain coherent pair creation operators in which different shells are weighted nonuniformly. Thus with realistic pair structure, self-conjugacy forces one either to use a large collective algebra or to give up the possibility of skeleton mapping the entire algebra. In these circumstances, one may resort to more *ad hoc* methods in which the fermion commutation relations are only required to hold on a selected part of the fermion space. Ginocchio and Talmi's[7] work on boson mapping of SO(8) and the work of Dukelsky *et al.*[14] on boson mapping of deformed states provide examples of this. Reference 14 uses the interesting idea of mapping only the number-conserving part of the algebra.

Of course the use of the boson mapping does not enable any calculations that could not be done in the original fermion formulation. Indeed, as the example of the fermion dynamical symmetry model[15] shows, the fermion formulation has the advantage of being able to describe noncollective states, for example broken pairs resulting from the Coriolis antipairing effect at high spin.

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APPENDIX: PROOFS OF LEMMAS

1. Proof of lemma M

One can use the CR's of A' to arrange the factors in each term of $f(\mathbf{g}'_C,\mathbf{g}'_D)$ in the standard order $g'_C g'_T g'_D$. Because of the isomorphism, the same process applied to $f(\mathbf{g}''_C,\mathbf{g}''_T,\mathbf{g}''_D)$ gives the *same* combination of products in the standard order $g''_C g''_T g''_D$. When this standard order is used, terms that contain D-type factors do not contribute. The surviving terms must be of the form

$$\begin{aligned}(0' | g'_{T1}g'_{T2}g'_{T3}\cdots g'_{Tk} | 0') &= (0' | g'_{T1} | 0')(0' | g'_{T2} | 0')\cdots(0' | g'_{Tk} | 0')\\ &= (0'' | g''_{T1} | 0'')(0'' | g''_{T2} | 0'')\cdots(0'' | g''_{Tk} | 0'')\\ &= (0'' | g''_{T1}g''_{T2}g''_{T3}\cdots g''_{Tk} | 0'') \ . \qquad \text{(A1)}\end{aligned}$$

In the second equality we have used the equality of the corresponding vacuum eigenvalues, which can also be written as expectation values. Eq. (A1) proves the lemma.

2. Proof of lemma Z

This is an immediate corollary, and holds under the same conditions as lemma M. The proof proceeds by applying lemma M to show that

$$(0'' | g''_{D_{\alpha 1}}\cdots g_{D_{\alpha_k}}\phi(\mathbf{g}''_C) | 0'') = (0' | g'_{D_{\alpha_1}}\cdots g_{D_{\alpha_k}}\phi(\mathbf{g}'_C) | 0') = 0 \ , \qquad \text{(A2)}$$

for every selection $\alpha_1,\ldots,\alpha_k$ of the D-type generators. Equation (A2) shows that $\phi(\mathbf{g}''_C) | 0'')$ is orthogonal to all vectors of gen$(\mathbf{g}_D''^{\dagger})$; this does not always imply that $\phi(\mathbf{g}''_C)$ vanishes. So the result is correctly expressed by

$$P''\phi(\mathbf{g}''_C) | 0'') = 0 \ , \qquad \text{(A3)}$$

where P'' is the orthogonal projector on the space $L'' \equiv \text{gen}(\mathbf{g}_D''^{\dagger})$.

3. Proof of lemma I

The invariance of $L = \text{gen}(\mathbf{g}_C)$ under generators of type C is obvious. Consider the action of a type-T generator on a basis state of L, schematically

$$g_T g_{C1} g_{C2},\ldots,g_{Ck} | 0) \ .$$

We can apply the CR's of the algebra to move g_T one step to the right. The commutator $[g_T, g_{C1}]$ is a type C generator, so the resulting state is in L provided that g_T acting on a product of $k-1$ factors $g_{C2},\ldots,g_{Ck}$ gives a vector in L. Since $g_T | 0)$ is in L, induction on k shows that L is invariant under generators of type T.

Finally, consider a product of the type

$$g_D g_{C1} g_{C2},\ldots,g_{Ck} | 0) \ .$$

When g_D is moved one step to the right, the commutator introduced is of type T and makes a contribution in L, by the result of the previous paragraph. Again $g_D | 0)$ is in L, and induction on k shows that L is invariant under generators of type D.

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