Approximations for double-beta-decay formulas

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The estimation of the relativistic Coulomb wave function is a time-consuming procedure, when the experimental data on the double beta decays are analyzed or the Monte Carlo calculation is performed. In order to avoid this complication, some simple approximated formulas are derived for three modes: the double beta decay with two neutrino emission, the neutrinoless double beta decay, and the neutrinoless double beta decay with Majoron emission. The errors of these approximations are of order of 1% or less in the region where decay events are measured practically.

I. INTRODUCTION

In order to get the information on the neutrino mass and/or parameters of the right-handed weak interaction, many extensive experimental investigations have been performed to measure the neutrinoless mode of the double beta decay [the $(\beta\beta)_{0\nu}$ mode]. If it is observed, the knowledge of the nuclear matrix elements is necessary. Since the double-beta decay with two neutrino emission [the $(\beta\beta)_{2\nu}$ mode] is expected to take place within the standard model as a second-order weak process, its observation offers the information on the nuclear matrix elements unambiguously. At present, there are some discrepancies between preliminary experimental results and theoretical estimates for the $(\beta\beta)_{2\nu}$ mode; see, for example, Sec. VII of our review paper,[1] which will be referred to as I.

Another recently proposed $\beta\beta$ decay mode is the decay into two electrons and a massless neutral pseudoscalar boson, B, known as the Majoron.[2] It will be referred to as the $(\beta\beta)_{0\nu B}$ mode. If the $(\beta\beta)_{0\nu B}$ mode competes with the $(\beta\beta)_{2\nu}$ mode, the experimental analysis becomes cumbersome. The electron energy spectra and the angular correlation between two emitted electrons in both modes should be compared in detail.

In addition, as one of the direct measurements, the time projection chamber (TPC) has been used by the University of California at Irvine (UCI) group for ^{82}Se and ^{100}Mo.[3] Three other (Milano, Caltech, and Moscow) groups are using the TPC or multicell proportional chambers for ^{136}Xe.[4] In these visible detector methods, the Monte Carlo calculation is helpful. However, the exact calculation of the relativistic Coulomb wave functions for electrons is complicated and is a time consuming procedure.

Thus the simpler and reliable approximate expressions for the observable quantities are required to analyze the experimental data in some higher energy region above the background. In this paper we shall try to find such simpler approximations which reproduce the result obtained from the accurate expression.

In the experiment to observe the $(\beta\beta)_{0\nu}$ mode, two other $(\beta\beta)_{0\nu B}$ and $(\beta\beta)_{2\nu}$ modes are measured too. As it is well known, the sum energy spectrum of two emitted electrons shows different characteristic features for each mode; see, for example, Fig. 6.11 of I. To see them easily, we shall express differential transition rates in compact forms. For this purpose, we shall briefly summarize the accurate theoretical expressions of the half-lives for the $0^+ \to 0^+$ transition of each mode in Sec. II. The differential transition rates as the original formulas are given and the approximated forms of them are discussed in Sec. III. In Sec. IV summary is given, and approximations which have been used to obtain the original formulas themselves will be discussed.

In order to find some simpler expressions for the measurable quantities, it is convenient to reexpress an electron wave function as a product of several factors. One of them is the pure Coulomb correction which appears even for point charge nucleus, and others are related to the finite extension of nucleus. In practice, the uniform nuclear charge distribution with radius R is assumed, and the screening correction due to the atomic electrons is not considered. Since the radial part of the electron Coulomb wave function inside the nucleus (the inner function) is taken into account to obtain the decay rate, only the normalization $A_{\pm k}(\varepsilon)$ of the inner function appears for the energy spectrum of the original formula. In addition, when the angular correlation is considered, another quantity appears; namely, the overall phase shift $\Delta^C_{\pm k}(\varepsilon)$ which is introduced to satisfy the boundary condition of the relativistic Coulomb wave at $r = \infty$. The detailed characters of $A_{\pm k}(\varepsilon)$ and $\Delta^C_{\pm k}(\varepsilon)$ are discussed in Appendix A.

Concerning the $0^+ \to 2^+$ transition, the contributions from the $(\beta\beta)_{2\nu}$ and $(\beta\beta)_{0\nu B}$ modes are negligible. The former has been discussed in Sec. 3.2.2 of I and the latter in Ref. 5 recently. Therefore, the observation of the $0^+ \to 2^+$ transition in the $\beta\beta$ decay means to measure only the $(\beta\beta)_{0\nu}$ mode, i.e., the contribution from the right-handed weak interaction, as shown in Table 1.2 of I and discussed in Sec. 3.3.2 of I. In this transition, one of

the electrons should be in the P-wave state with the total angular momentum $j=\frac{3}{2}$, which gives $A_{\pm 2}(\varepsilon)$. Thus the Coulomb wave has the more complicated character. The original formula is reexpressed in a compact form in Appendix B, from which the approximated form can be easily derived similarly to the $(0^+\to 0^+)$ transition in Sec. III.

II. THE HALF-LIFE FORMULA FOR THE $0^+\to 0^+$ TRANSITION

The half-life formulas for the $0^+\to 0^+$ transition will be summarized for the $(\beta\beta)_{2\nu}$, $(\beta\beta)_{0\nu}$, and $(\beta\beta)_{0\nu B}$ modes in order. In this $0^+\to 0^+$ transition, the emitted electrons are in the S-wave state and the P-wave state with $j=\frac{1}{2}$, so that only the normalization $A_{\pm 1}(\varepsilon)$ of the electron inner wave function appears.

The half-life of the $0^+\to 0^+$ transition in the $(\beta\beta)_{2\nu}$ mode is expressed as

$$[T_{2\nu}(0^+\to 0^+)]^{-1}=|M_{\rm GT}^{(2\nu)}/\mu_0|^2 G_{\rm GT} , \tag{2.1}$$

with the integrated kinematical factor

$$G_{\rm GT}=\left[\frac{a_{2\nu}}{\ln 2}\right]\int d\Omega_{2\nu}a(\varepsilon_1,\varepsilon_2)[\langle\mu_a\rangle(\langle K_a\rangle+\langle L_a\rangle)/2]^2 . \tag{2.2}$$

The detailed derivation of Eq. (2.1) is given in Eq. (3.2.8) and Appendix B of I.

The first factor $|M_{\rm GT}^{(2\nu)}/\mu_0|$ in Eq. (2.1) is related to the reduced nuclear matrix elements of the double Gamow-Teller (nuclear spin flip) transitions $[M_{{\rm GT}a}^{(2\nu)}$ defined in Eq. (3.2.4b) in I] by the following definition:

$$\left[\frac{M_{\rm GT}^{(2\nu)}}{\mu_0}\right]=\sum_a\left[\frac{M_{{\rm GT}a}^{(2\nu)}}{\mu_a}\right] , \tag{2.3}$$

where $\sum_a$ means the sum over the intermediate nucleus at the energy state E_a and

$$\mu_a m_e=E_a-(M_i+M_f)/2 , \tag{2.4}$$

m_e, M_i, and M_f being masses of electron, parent and daughter nuclei, respectively. Here the reduced nuclear matrix elements due to the double Fermi (non-spin-flip) transitions are omitted, because their small contribution is known theoretically.

In the factor $G_{\rm GT}$ in Eq. (2.2), $a_{2\nu}$ is the known constant defined in Eq. (3.2.2a) of I, and the phase space factor $d\Omega_{2\nu}$ is

$$\begin{aligned} d\Omega_{2\nu}=&m_e^{-11}q_1\omega_1 q_2\omega_2 p_1\varepsilon_1 p_2\varepsilon_2 \\ &\times\delta(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2+M_f-M_i) \\ &\times d\omega_1 d\omega_2 d\varepsilon_1 d\varepsilon_2 d\cos\theta , \end{aligned} \tag{2.5}$$

where $\varepsilon_k(p_k)$ and $\omega_k(q_k)$ are the energies (momenta) of the kth electron and neutrino, respectively, and θ is the opening angle between two emitted electrons. The Coulomb correction $a(\varepsilon_1,\varepsilon_2)$ is expressed as

$$a(\varepsilon_1,\varepsilon_2)=[A_{+1}^2(\varepsilon_1)+A_{-1}^2(\varepsilon_1)][A_{+1}^2(\varepsilon_2)+A_{-1}^2(\varepsilon_2)] , \tag{2.6}$$

where $A_{\pm 1}(\varepsilon)$ is defined in Eq. (A1) of Appendix A. This $a(\varepsilon_1,\varepsilon_2)$ is obtained from the one defined in Eq. (B.3.6) of I by using Eq. (B.3.7) of I, and is the origin of the cumbersome calculation. In other words, the main purpose of this paper is to take out the essential and necessary parts from $a(\varepsilon_1,\varepsilon_2)$ for the experimental analysis. The last $\langle K_a\rangle$ and $\langle L_a\rangle$ come from the intermediate energy denominators due to the second-order perturbation, and are defined as

$$\langle K_a\rangle=\frac{2\langle\mu_a\rangle}{\langle\mu_a\rangle^2-K_D^2} , \tag{2.7a}$$

and

$$\langle L_a\rangle=\frac{2\langle\mu_a\rangle}{\langle\mu_a\rangle^2-L_D^2} , \tag{2.7b}$$

where $\langle\mu_a\rangle$ is some average of μ_a in Eq. (2.4) and

$$K_D=(\varepsilon_1+\omega_1-\varepsilon_2-\omega_2)/(2m_e) , \tag{2.8a}$$

$$L_D=(\varepsilon_1-\omega_1-\varepsilon_2+\omega_2)/(2m_e) . \tag{2.8b}$$

The replacement of μ_a by $\langle\mu_a\rangle$ means to introduce an approximation into the exact formula. The reliability of this approximation will be discussed in Sec. IV. Note that in this mode, even if a neutrino has a finite mass ($m_\nu\neq 0$), its effect is considerably small. In the following we shall assume $m_\nu=0$ in the case of the $(\beta\beta)_{2\nu}$ mode. Also, the contributions from the right-handed weak interaction are negligibly small, even if it exists. The contribution from the $P_{1/2}$-wave leptons are negligible, too.

Next, let us consider the $(\beta\beta)_{0\nu}$ mode, where the right-handed weak interaction may lead to the measurable contribution (the $V+A$ part). Since this $V+A$ part gives the complicated expressions due to the $P_{1/2}$-wave electron, the full expressions will be given in Appendix A. In the text we shall show only the part proportional to the masses of virtual neutrinos (the m_ν part).

Then the half-life formula in Eq. (3.5.10) of I is

$$[T_{0\nu}(0^+\to 0^+)]^{-1}=\left[\frac{\langle m_\nu\rangle}{m_e}\right]^2|M_{\rm GT}^{(0\nu)}(1-\chi_F)|^2 G_{01} , \tag{2.9}$$

where $M_{\rm GT}^{(0\nu)}$ and χ_F are the double Gamow-Teller and double Fermi type nuclear matrix elements defined in Eqs. (3.5.1) and (3.5.2) of I, $\langle m_\nu\rangle$ is the effective neutrino mass which is defined as the sum of the virtual neutrino masses weighted with the electron neutrino mixing matrix elements squared, see Eq. (3.5.11) of I. The integrated kinematical factor G_{01} in this case is

$$G_{01}=\frac{a_{0\nu}}{(m_e R)^2\ln 2}\int d\Omega_{0\nu}a(\varepsilon_1,\varepsilon_2) , \tag{2.10}$$

where $a_{0\nu}$ is the known constant defined in Eq. (3.5.17b) of I, R is the nuclear radius, and the phase space factor

$d\Omega_{0\nu}$ is

$$d\Omega_{0\nu}=m_e^{-5}p_1\varepsilon_1 p_2\varepsilon_2\delta(\varepsilon_1+\varepsilon_2+M_f-M_i)d\varepsilon_1 d\varepsilon_2 d\cos\theta\ . \tag{2.11}$$

The Coulomb correction $a(\varepsilon_1,\varepsilon_2)$ is the same as in Eq. (2.6), because both electrons are in the S-wave state for the m_ν part. Note that $b_{01}=\alpha_+ +\beta_+$ in Eq. (3.5.18) of I is equal to $a(\varepsilon_1,\varepsilon_2)$.

Finally, in the $(\beta\beta)_{0\nu B}$ mode, the transition amplitude consists of two parts: the main and correction terms. Recently, Doi, Kotani, and Takasugi have examined the correction term in detail.[5] In this mode it is possible to have two different kinds of virtual neutrinos, both of which interact with the Majoron at one vertex. If masses of both virtually propagating neutrinos are light, say less than m_e, it is enough to consider only the main term, and the reduced nuclear matrix elements in the $(\beta\beta)_{0\nu B}$ mode are the same as in the $(\beta\beta)_{0\nu}$ mode, see Eqs. (3.11) and (5.1) of Ref. 5. Then, the half-life formula of the $0^+\to 0^+$ transition in the $(\beta\beta)_{0\nu B}$ mode is given as

$$[T_{0\nu B}(0^+\to 0^+)]^{-1}=|\langle g_B\rangle|^2|M_{\rm GT}^{(0\nu)}(1-\chi_F)|^2 G_{B0}(1)\ , \tag{2.12}$$

for the case where light neutrinos give the dominant contribution. Here $\langle g_B\rangle$ represents the effective coupling constant between neutrinos and Majoron and is defined in Eq. (5.2.3) of I, and the integrated kinematical factor[6] is

$$G_{B0}(1)=\frac{a_{0\nu}}{4\pi^2(m_eR)^2\ln 2}\int d\Omega_{0\nu B}a(\varepsilon_1,\varepsilon_2)\ , \tag{2.13}$$

where the constant factor $a_{0\nu}$ is the same constant for the $(\beta\beta)_{0\nu}$ mode in Eq. (2.10) and the phase space factor $d\Omega_{0\nu B}$ is

$$d\Omega_{0\nu B}=m_e^{-7}kp_1\varepsilon_1 p_2\varepsilon_2\delta(k+\varepsilon_1+\varepsilon_2+M_f-M_i)\times dk\, d\varepsilon_1 d\varepsilon_2 d\cos\theta\ , \tag{2.14}$$

where k is the momentum of Majoron. The Coulomb correction factor $a(\varepsilon_1,\varepsilon_2)$ is the same as for the $(\beta\beta)_{2\nu}$ mode in Eq. (2.6), because both electrons are in the S-wave states, again.

Concerning the angular correlation, the transition formula is obtained by replacing $a(\varepsilon_1,\varepsilon_2)$ with the following combination:

$$a(\varepsilon_1,\varepsilon_2)-2b(\varepsilon_1,\varepsilon_2)\cos\theta\ , \tag{2.15}$$

where

$$2b(\varepsilon_1,\varepsilon_2)=[2A_{+1}(\varepsilon_1)A_{-1}(\varepsilon_1)\cos(\Delta^C_{+1}(\varepsilon_1)-\Delta^C_{-1}(\varepsilon_1))]\times[2A_{+1}(\varepsilon_2)A_{-1}(\varepsilon_2)\cos(\Delta^C_{+1}(\varepsilon_2)-\Delta^C_{-1}(\varepsilon_2))]. \tag{2.16}$$

This is valid for the $(\beta\beta)_{2\nu}$ and $(\beta\beta)_{0\nu B}$ modes and the m_ν part of the $(\beta\beta)_{0\nu}$ mode. For the $(\beta\beta)_{2\nu}$ mode, the derivation and the definition of $b(\varepsilon_1,\varepsilon_2)$ are given in Eqs. (B.3.1), (B.3.11), and (B.3.12) of I. Derivations for the $(\beta\beta)_{0\nu}$ and $(\beta\beta)_{0\nu B}$ modes are given in Eq. (C.3.14) of I and in Eq. (B.17) of Ref. 5, respectively.

In order to get the simpler expression for measurable quantities like the electron energy spectra and the angular correlation, the factors $a(\varepsilon_1,\varepsilon_2)$ and $b(\varepsilon_1,\varepsilon_2)$ will be reexpressed by products of the important part and the nonessential part, separately.

III. ENERGY SPECTRA AND ANGULAR CORRELATION FOR THE $0^+\to 0^+$ TRANSITION

The differential decay rate for the $(0^+\to 0^+)$ transition in the $(\beta\beta)_{2\nu}$ mode is reexpressed in a compact form to take out the essential parts from $a(\varepsilon_1,\varepsilon_2)$ for the energy spectra and $b(\varepsilon_1,\varepsilon_2)$ for the angular correlation between two emitted electrons. Both the $(\beta\beta)_{0\nu B}$ and $(\beta\beta)_{0\nu}$ modes will be discussed later.

Let us start to rewrite the decay formula in the following form:

$$\frac{d^3\Gamma_{2\nu}(0^+\to 0^+)}{dT_1 dT_2 d\cos\theta}=\left|\frac{M_{\rm GT}^{(2\nu)}}{\mu_0}\right|^2 N_{2\nu}[A_{2\nu}(0)+B_{2\nu}(0)\cos\theta]\ , \tag{3.1}$$

where we have replaced $A_{\pm 1}(\varepsilon)$ in $a(\varepsilon_1,\varepsilon_2)$ and $b(\varepsilon_1,\varepsilon_2)$ in Eqs. (2.6) and (2.16) by Eqs. (A1) and (A2) of Appendix A. In Eq. (3.1), $|M_{\rm GT}^{(2\nu)}/\mu_0|$ is defined in Eq. (2.3) and the constant term $N_{2\nu}$ is

$$N_{2\nu}=(\tfrac{4}{30})a_{2\nu}(C_{F,0})^2\ , \tag{3.2}$$

where $a_{2\nu}$ comes from Eq. (2.2) and $C_{F,0}$ is

$$C_{F,0}=2\pi\alpha Z(2\alpha Z m_e R)^{2(\gamma_1-1)}\left[\frac{2}{\Gamma(2\gamma_1+1)}\right]^2\ , \tag{3.3}$$

which is the constant part of $F_0(Z,\varepsilon)$ given in Eq. (A3). Here $\gamma_1=[1-(\alpha Z)^2]^{1/2}$, Z and α are the atomic number of the daughter nuclei and the fine structure constant, respectively, and $\Gamma(x)$ is a gamma function.

The spectrum $A_{2\nu}(0)$ and angular $B_{2\nu}(0)$ parts in Eq. (3.1) are expressed by

$$\begin{Bmatrix}A_{2\nu}(0)\\B_{2\nu}(0)\end{Bmatrix}=(1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_1}(T-T_1-T_2)^5\times h_{2\nu}d_{00}\begin{Bmatrix}A_{11}\\-B_{11}\end{Bmatrix}\ , \tag{3.4}$$

where $T_j m_e=\varepsilon_j-m_e$, the kinetic energy of the jth electron, and T stands for the maximum kinetic energy release,

$$Tm_e=M_i-M_f-2m_e\ . \tag{3.5}$$

The factor $(1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_1}$ is obtained from the phase space factor of electrons in Eq. (2.5) multiplied by $(m_e/p)(\varepsilon/m_e)^{2\gamma_1-1}$ in Eq. (A2). The next $(T-T_1-T_2)^5$ and $h_{2\nu}(T_1,T_2)$ are derived from the combination of the neutrino phase space parts in Eq. (2.5) and the energy denominators $\langle K_a\rangle$ and $\langle L_a\rangle$ in Eq. (2.7), as it will become clear in Eq. (3.12). The Coulomb factor $d_{00}(T_1,T_2)$ is defined as the product of $d_0(\varepsilon)$ in Eq. (A4)

$$d_{00}(T_1,T_2)=d_0(T_1+1)\,d_0(T_2+1)\ . \tag{3.6}$$

By substituting the remaining parts of the normalization $A_{\pm 1}(\varepsilon)$ in Eq. (A1) into Eqs. (2.6) and (2.16), we define

$$A_{11}(T_1,T_2)=R_{1,1}(\varepsilon_1)R_{1,1}(\varepsilon_2)\ , \tag{3.7}$$

$$B_{11}(T_1,T_2)=\left[\frac{p_1p_2}{\varepsilon_1\varepsilon_2}\right][D_{1,-1}(\varepsilon_1)C_{1,-1}(\varepsilon_1)] \times[D_{1,-1}(\varepsilon_2)C_{1,-1}(\varepsilon_2)]\ , \tag{3.8}$$

where

$$R_{1,1}(\varepsilon)=[(\varepsilon+m_e)D_{-1,-1}(\varepsilon)+(\varepsilon-m_e)D_{+1,+1}(\varepsilon)]/(2\varepsilon)\ , \tag{3.9}$$

$$D_{\kappa,\lambda}(\varepsilon)=D_\kappa(\varepsilon)D_\lambda(\varepsilon)\ , \tag{3.10}$$

$$C_{\kappa,\lambda}(\varepsilon)=\cos[\Delta^C_\kappa(\varepsilon)-\Delta^C_\lambda(\varepsilon)]\ , \tag{3.11}$$

$D_\kappa(\varepsilon)$ being defined in Eqs. (A10) and (A11) and $\Delta^C_\kappa(\varepsilon)$ being the overall phase shift defined in Eq. (D.30) of I.

The factor $h_{2\nu}(T_1,T_2)$ related to the energy denominator such as $\langle K_a\rangle$ and $\langle L_a\rangle$ in Eq. (2.7) is defined as

$$h_{2\nu}(T_1,T_2)=(\tfrac{30}{4})\frac{\langle\mu_a\rangle^2}{\omega^5}\int_0^\omega d\omega_1\omega_1^2(\omega-\omega_1)^2H_1\ , \tag{3.12}$$

where ω means the energy carried out by two neutrino part,

$$\omega=(T-T_1-T_2)m_e\ . \tag{3.13}$$

In this definition, the neutrino masses are neglected. The factor $(T-T_1-T_2)^5$ in Eq. (3.4) comes from the fifth power of ω_1 in Eq. (3.12). Thus, $h_{2\nu}(T_1,T_2)$ is defined to be normalized as

$$\lim_{\langle\mu_a\rangle\to\infty} h_{2\nu}(T_1,T_2)=1\ . \tag{3.14}$$

The constant factor $(\frac{4}{30})$ in Eq. (3.2) has come from this normalization. The factor H_1 in Eq. (3.12) is

$$H_1=[(\langle K_a\rangle+\langle L_a\rangle)/2]^2\ , \tag{3.15}$$

where $\langle K_a\rangle$ and $\langle L_a\rangle$ are defined in Eq. (2.7) and in the notation of this section, K_D and L_D in Eq. (2.8) are expressed as

$$K_D=[2T_1-T+2(\omega_1/m_e)]/2\ , \tag{3.16a}$$

$$L_D=-[2T_2-T+2(\omega_1/m_e)]/2\ . \tag{3.16b}$$

Strictly speaking, the original expressions for $A_{2\nu}(0)$ and $B_{2\nu}(0)$ are given by using H_1+H_2 and $H_1-(\frac{1}{3})H_2$ instead of H_1 in Eq. (3.12), respectively, as seen from Eqs. (B.3.5) and (B.3.11) of I. Here, H_2 is defined as

$$H_2=\tfrac{1}{3}[(\langle K_a\rangle-\langle L_a\rangle)/2]^2\ , \tag{3.17}$$

and is proportional to

$$(T_1-T_2)^2[T_1+T_2-T+2(\omega_1/m_e)]^2\ , \tag{3.18}$$

so that this contribution is negligibly small and omitted in Eq. (2.2).

In the notation here, $G_{\rm GT}$ of the total half-life in Eq. (2.2) is expressed as

$$G_{\rm GT}=\left[\frac{2}{\ln 2}\right]N_{2\nu}\int_0^T dT_1\int_0^{T-T_1}dT_2\,A_{2\nu}(0)\ . \tag{3.19}$$

The new values of $G_{\rm GT}$ are listed in Table I, because some constants have been changed a little according to the most recent list edited by the particle data group.[7]

In the case of the $(\beta\beta)_{0\nu B}$ mode, the differential decay rate corresponding to Eq. (2.12) is expressed as

$$\frac{d^3\Gamma_{0\nu B}}{dT_1dT_2d\cos\theta}=|\langle g_B\rangle|^2|M^{(0\nu)}_{\rm GT}(1-\chi_F)|^2 \times N_{0\nu B}[A_{0\nu B}(0)+B_{0\nu B}(0)\cos\theta]\ , \tag{3.20}$$

where $|\langle g_B\rangle M^{(0\nu)}_{\rm GT}(1-\chi_F)|$ appears in Eq. (2.12), the total normalization constant $N_{0\nu B}$ is

$$N_{0\nu B}=\frac{a_{0\nu}}{4\pi^2(m_eR)^2}(C_{F,0})^2\ , \tag{3.21}$$

and

TABLE I. The integrated kinematical factors (in units of yr^{-1}) of the $0^+\to0^+$ transition of the $(\beta\beta)_{2\nu}$ mode [$G_{\rm GT}$ in Eq. (3.19)], the $(\beta\beta)_{0\nu B}$ mode [$G_{B0}(1)$ in Eq. (2.13)] (Ref. 6), and the m_ν part of the $(\beta\beta)_{0\nu}$ mode [G_{01} in Eq. (2.10)]. The normalization constant $(C_{F,0})^2$ in Eqs. (3.3) is compared with its nonrelativistic limit $(C^{\rm NR}_{F,0})^2=(2\pi\alpha Z)^2$ in Eq. (A7). Those T and $\langle\mu_a\rangle$ are defined in Eqs. (3.5) and (2.7), respectively.

	^{48}Ca	^{76}Ge	^{82}Se	^{100}Mo	^{128}Te	^{130}Te	^{136}Xe	^{150}Nd
$T(m_e)$	8.358	3.991	5.861	5.937	1.700	4.957	4.851	6.589
$G_{\rm GT}$	4.002×10^{-17}	1.317×10^{-19}	4.393×10^{-18}	9.553×10^{-18}	8.624×10^{-22}	4.849×10^{-18}	4.870×10^{-18}	1.200×10^{-16}
$G_{B0}(1)$	3.971×10^{-15}	1.212×10^{-16}	9.975×10^{-16}	1.735×10^{-15}	9.917×10^{-18}	1.308×10^{-15}	1.356×10^{-15}	1.032×10^{-14}
G_{01}	6.427×10^{-14}	6.446×10^{-15}	2.808×10^{-14}	4.625×10^{-14}	1.844×10^{-15}	4.478×10^{-14}	4.762×10^{-14}	2.116×10^{-13}
$(C_{F,0})^2$	1.498	5.726	7.049	16.07	45.23	45.16	55.75	106.7
$(C^{\rm NR}_{F,0})^2$	1.017	2.430	2.725	4.070	6.132	6.132	6.594	8.080
$\langle\mu_a\rangle$	15.10	18.42	19.73	20.0	24.53	25.98	20.0	20.0

$$\begin{Bmatrix} A_{0\nu B}(0) \\ B_{0\nu B}(0) \end{Bmatrix} = (1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_1}(T-T_1-T_2) \times d_{00} \begin{Bmatrix} A_{11} \\ -B_{11} \end{Bmatrix}, \quad (3.22)$$

and the other notation is the same as the $(\beta\beta)_{2\nu}$ mode in Eq. (3.4).

In comparison with the $(\beta\beta)_{2\nu}$ mode, there are two different features; the first power of $(T-T_1-T_2)$ and no $h_{2\nu}(T_1,T_2)$ factor. The former is due to the scalar particle (Majoron) emission instead of two fermions (neutrinos), and comes from the Majoron momentum k in Eq. (2.14). The reason for no $h_{2\nu}$ is that the intermediate energy denominator like $\langle K_a \rangle$ in Eq. (2.7) are absorbed into the neutrino exchange potential in the definition of $M_{\rm GT}^{(0\nu)}(1-\chi_F)$ through the integration over the virtual neutrino momentum, see Eqs. (3.5.1) and (3.4.1) of I. The factor $G_{B0}(1)$ of the total half-life in Eq. (2.13) is calculated by using the similar expression to Eq. (3.19). The numerical values of $G_{B0}(1)$ are given in Table I. Note that the new $G_{B0}(1)$ is twice larger than the old G_B in Table 3.1 of I.[6]

The last is the $(\beta\beta)_{0\nu}$ mode. The differential decay rate corresponding to Eq. (2.9) is expressed as

$$\frac{d^3\Gamma_{0\nu}}{dT_1 dT_2 d\cos\theta} = \left[\frac{\langle m_\nu \rangle}{m_e}\right]^2 |M_{\rm GT}^{(0\nu)}(1-\chi_F)|^2 \times N_{0\nu}[A_{0\nu}(0)+B_{0\nu}(0)\cos\theta], \quad (3.23)$$

where $(\langle m_\nu \rangle/m_e)|M_{\rm GT}^{(0\nu)}(1-\chi_F)|$ appears in Eq. (2.9), the total normalization constant $N_{0\nu}$ is

$$N_{0\nu} = \frac{a_{0\nu}}{(m_e R)^2}(C_{F,0})^2, \quad (3.24)$$

and

$$\begin{Bmatrix} A_{0\nu}(0) \\ B_{0\nu}(0) \end{Bmatrix} = (1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_1}\delta(T-T_1-T_2) \times d_{00} \begin{Bmatrix} A_{11} \\ -B_{11} \end{Bmatrix}, \quad (3.25)$$

and the other notation is the same as the $(\beta\beta)_{0\nu B}$ mode in Eq. (3.22). In comparison with the $(\beta\beta)_{0\nu B}$ mode, the appearance of $\delta(T-T_1-T_2)$ is different. This is because the $(\beta\beta)_{0\nu}$ mode is the three-body decay and the energy of the second electron is restricted by the energy conservation. The integrated kinematical factor G_{01} in Eq. (2.10) is given in Table I.

Thus the differential decay rates for three modes have been expressed in similar compact forms. It is found that there are some common factors like $C_{F,0}$ in N_j with $j=(2\nu,0\nu,0\nu B)$, d_{00}, A_{11}, and B_{11}. The factor $h_{2\nu}$ in Eq. (3.4) is the special one for the $(\beta\beta)_{2\nu}$ mode, because it comes from the two neutrino emission. The characteristic features and energy dependences of them will be discussed.

First, let us consider the constant factor $C_{F,0}$ defined in Eq. (3.3). As long as only the normalization is concerned, the use of the nonrelativistic approximation like $\gamma_1 \to 1$ in Eq. (3.3) or the plane wave one like $\alpha Z \to 0$ must not be used, because of the large corrections due to $(2\alpha Z m_e R)^{-2(1-\gamma_1)}$ in Eq. (3.3) for the large Z. This situation is shown in the sixth and seventh rows of Table I numerically. This difference is the reason why Haxton-Stephenson-Strottman and Nishiura pointed out the necessity to use the relativistic Coulomb wave functions.[8]

Next, let us consider the characters of A_{11} and B_{11} in Eqs. (3.7) and (3.8). They include both effects due to the finite de Broglie wavelength correction at the nuclear surface $[(pR)\neq 0]$ and the finite extension of nucleus $[B_{\pm k}\neq 1, h_{f,k}\neq 0$, and $h_{g,-k}\neq 0$ in Eqs. (A10) and (A11)]. Since the estimations of A_{11} and B_{11} are the origin of the time consuming procedure, let us examine their features in detail. In the limit of the plane wave for electrons $[\alpha Z \to 0]$, we have

$$A_{11}(T_1,T_2) \xrightarrow[\alpha Z\to 0]{} 1, \quad (3.26)$$

$$B_{11}(T_1,T_2) \xrightarrow[\alpha Z\to 0]{} \left[\frac{p_1 p_2}{\varepsilon_1 \varepsilon_2}\right]. \quad (3.27)$$

Here we have used Eqs. (A14), (A15), (A19), (A26), (A27), and (A31) or we may say the use of Eqs. (B8) and (B11) of Appendix B. This limit corresponds to the one which Konopinski and Uhrenbeck used to approximate the relativistic Coulomb wave function in the single-beta decay.[9]

In the normalization $A_{\pm k}$ of Eq. (A1), $D_{-1}(\varepsilon)$ and $D_{+1}(\varepsilon)$ appear with $[(\varepsilon+m_e)/2\varepsilon]^{1/2}$ and $[(\varepsilon-m_e)/2\varepsilon]^{1/2}$, respectively. Therefore, though $D_{+1}(\varepsilon)$ is proportional to the $(1/p)$ factor as seen from Eq. (A22) and Fig. 2 of Appendix A, this $1/p$ factor is canceled by the factor $(\varepsilon-m_e)^{1/2}$. This is the reason why A_{11} takes values around unity, as shown in Table II. The deviation comes mainly from terms proportional to αZ, as you see it by comparing the ^{82}Se case with ^{150}Nd in the smaller momentum region. In the higher momentum region, this contribution becomes smaller, because of the combination $y=\alpha Z\varepsilon/p$ in Eq. (A17). This character of y appears explicitly for $D_{+1}(\varepsilon)$ in the $p\to 0$ limit; that is, the leading term of $pD_{+1}(\varepsilon)$ is proportional to αZ and takes the finite value at $p=0$, see Eq. (A22).

The angular correlation factor B_{11}/A_{11} in Eq. (3.4) which appears as the coefficient of $\cos\theta$ in Eq. (3.1) can be approximated as

$$(B_{11}/A_{11}) \sim (p_1 p_2/\varepsilon_1 \varepsilon_2), \quad (3.28)$$

which is the plane wave limit given in Eqs. (3.26) and (3.27). The error of this approximation is of order of 0.5% at the smaller p region, say $p<m_e$, and less than 0.5% for $p>m_e$. The reason why B_{11}/A_{11} becomes zero

TABLE II. The T_1 dependence of $A_{11}(T_1,T_2)$ in Eq. (3.7) for the fixed T_2.

	^{82}Se ($T=5.861$)				^{100}Mo ($T=5.937$)				^{136}Xe ($T=4.851$)				^{150}Nd ($T=6.589$)			
T_1 \ T_2	0.0	2.0	3.0	4.0	0.0	2.0	3.0	4.0	0.0	1.5	2.5	3.5	0.0	2.5	3.5	4.5
0.0	1.017	1.010	1.006	1.002	1.025	1.015	1.010	1.005	1.036	1.026	1.017	1.009	1.041	1.018	1.008	0.9976
0.5	1.015	1.008	1.005	1.001	1.023	1.014	1.008	1.003	1.033	1.023	1.014	1.006	1.037	1.014	1.004	0.9940
1.0	1.013	1.007	1.003	0.9991	1.021	1.011	1.006	1.001	1.030	1.019	1.011	1.003	1.032	1.010	0.9997	0.9897
1.5	1.012	1.005	1.001	0.9973	1.018	1.009	1.004	0.9982	1.026	1.015	1.007		1.028	1.005	0.9951	0.9851
2.0	1.010	1.003	0.9994		1.015	1.006	1.001		1.022	1.011	1.003		1.023	1.000	0.9903	0.9804
2.5	1.008	1.001	0.9975		1.013	1.004	0.9984		1.017	1.007			1.018	0.9953	0.9855	
3.0	1.006	0.9994			1.010	1.001			1.013	1.003			1.013	0.9904	0.9806	
3.5	1.004	0.9974			1.007	0.9983			1.009				1.008	0.9855		
4.0	1.002				1.005				1.005				1.003	0.9805		
4.5	1.000				1.002				1.001				0.9976			
5.0	0.9982				0.9994								0.9926			
5.5	0.9962				0.9967								0.9876			
6.0													0.9826			
6.5													0.9777			

in the $p\to 0$ limit where we have $pD_{+1}(\varepsilon)\neq 0$, comes from the cosine term in Eq. (3.8). For the cosine term in Eq. (3.11), the following approximation can be used:

$$\Delta^C_{+1}(\varepsilon)-\Delta^C_{-1}(\varepsilon)\simeq\eta_{+1}(\varepsilon)-\eta_{-1}(\varepsilon)+\pi/2\ , \qquad (3.29)$$

where $\eta_\kappa(\varepsilon)$ is defined in Eq. (A29). The error due to this approximation is less than 0.1% over the practical electron energy region. The phase difference, $\eta_{+1}-\eta_{-1}$, becomes $-\pi$ only near the $p\to 0$ limit, strictly speaking in the region $y^2(m_e/\varepsilon)\gg\gamma_1$. Because of this cosine character, $C_{1,-1}(\varepsilon)$ in Eq. (3.11) has the increasing deviation from unity in the smaller T_1 region, say $T_1<0.2$, though it is almost unity in the wide region of T_1.

In order to get the simpler expression, we assume that $A_{11}=1$ and $B_{11}=(p_1p_2/\varepsilon_1\varepsilon_2)$, as in Eqs. (3.26) and (3.27). The case where these assumptions for A_{11} and B_{11} are adopted, will be called the approximation (1) and referred to as $A_{2\nu}(1)$, $B_{2\nu}(1)$ for Eq. (3.4), $A_{0\nu B}(1)$, $B_{0\nu B}(1)$ for Eq. (3.22) and $A_{0\nu}(1)$, $B_{0\nu}(1)$ for Eq. (3.25). As an example, the case of the $(\beta\beta)_{2\nu}$ mode is shown in Table III. This approximation (1) is the same as in Eqs. (3.1.24), (3.2.2C), (3.5.20), and (3.5.26) of I.

Since we would like to avoid to estimate A_{11} directly, we shall consider various combinations of the approximated forms, which compensate the effect of A_{11}, as shown in Table III.

The T_1 dependences of $h_{2\nu}$ for fixed T_2 are listed in Table IV. It shows that the deviation of $h_{2\nu}$ from unity is less than a few percent and becomes a little larger for the higher unbalanced electron energies, because of the larger values of K_D and L_D in $\langle K_a\rangle$ and $\langle L_a\rangle$, as seen from Eq. (3.16). The $\langle\mu_a\rangle$ dependence is negligible for the larger $\langle\mu_a\rangle$ value, say $\langle\mu_a\rangle>15$, as shown in Fig. 3.2 of I. The first 1^+ excited states of intermediate nuclei related to the $\beta\beta$ decay are around $\mu_a\sim 10$. Thus we may say that we can ignore the energy dependence of $h_{2\nu}$ in the region where two electrons are observed, say $T_j>0.5$. Therefore, in order to get the simpler formula, we assume $h_{2\nu}=$constant $(=1)$ for $A_{2\nu}(2)$, $A_{2\nu}(3)$, $A_{2\nu}(4)$, and $A_{2\nu}(5)$ in Table III, though it is not complicated to include the effect of $h_{2\nu}$ numerically.

Concerning $d_{00}(T_1,T_2)$ in Eq. (3.6), it is normalized to be unity in the limit $p\to 0$, as defined in Eq. (A6). The T_1 dependence of $d_{00}(T_1,T_2)$ for fixed T_2 is shown in Table V for the cases of ^{82}Se, ^{100}Mo, ^{136}Xe, and ^{150}Nd [cf. Fig. 1(a) of Appendix A]. For nuclei with smaller Z, the effect of d_{00} may not be ignored, while for larger Z nuclei, d_{00} can be treated as unity. The latter approximation will be

TABLE III. Combinations of various approximated factors for $A_{2\nu}(j)$ and $B_{2\nu}(j)$.

Approximations	Spectrum $d_0(\varepsilon)$	$h_{2\nu}(T_1,T_2)$	$A_{11}(T_1,T_2)$	γ_1	Angular corr. $B_{11}(T_1,T_2)$
$A_{2\nu}(0)$	no app.	no app.	no app.	no app.	no app.
$A_{2\nu}(1)$	no app.	no app.	1	no app.	
$A_{2\nu}(2)$	no app.	1	1	no app.	
$A_{2\nu}(3)$	1	1	1	1	$B_{11}=\dfrac{p_1p_2}{\varepsilon_1\varepsilon_2}$
$A_{2\nu}(4)$	$(1-e^{-2\pi y})^{-1}$	1	1	1	
$A_{2\nu}(5)$	1	1	1	1	

TABLE IV. The T_1 dependence of $h_{2\nu}(T_1,T_2)$ in Eq. (3.12) for the fixed T_2 and $\langle\mu_a\rangle$.

	^{82}Se ($T=5.861$, $\langle\mu_a\rangle=19.73$)				^{100}Mo ($T=5.937$, $\langle\mu_a\rangle=20.0$)				^{136}Xe ($T=4.851$, $\langle\mu_a\rangle=20.0$)				^{150}Nd ($T=6.589$, $\langle\mu_a\rangle=20.0$)			
T_1 \ T_2	0.0	2.0	3.0	4.0	0.0	2.0	3.0	4.0	0.0	1.5	2.5	3.5	0.0	2.5	3.5	4.5
0.0	1.006	1.008	1.013	1.022	1.006	1.008	1.013	1.021	1.004	1.005	1.009	1.016	1.008	1.011	1.017	1.027
0.5	1.006	1.005	1.009	1.016	1.006	1.005	1.009	1.016	1.004	1.003	1.006	1.011	1.007	1.007	1.013	1.021
1.0	1.006	1.003	1.006	1.012	1.006	1.003	1.006	1.012	1.004	1.001	1.003	1.008	1.007	1.005	1.009	1.016
1.5	1.006	1.001	1.003	1.008	1.006	1.001	1.003	1.008	1.005	1.001	1.001		1.008	1.002	1.005	1.011
2.0	1.008	1.001	1.001		1.008	1.001	1.001		1.007	1.001	1.000		1.009	1.001	1.003	1.008
2.5	1.010	1.001	1.000		1.010	1.001	1.000		1.009	1.001			1.011	1.000	1.001	
3.0	1.013	1.001			1.013	1.001			1.012	1.003			1.014	1.001	1.000	
3.5	1.017	1.003			1.017	1.003			1.016				1.017	1.001		
4.0	1.022				1.021				1.020				1.022	1.003		
4.5	1.027				1.026				1.026				1.027			
5.0	1.033				1.032								1.033			
5.5	1.040				1.039								1.039			
6.0													1.047			
6.5													1.055			

used for $A_{2\nu}(3)$ in Table III. Though d_{00} is almost constant in the practical region, as seen from Table V and Fig. 1 of Appendix A, its absolute value cannot be ignored for the smaller Z nuclei when the half-life is calculated.

If we assume $\gamma_1=1$, i.e., the nonrelativistic limit, $d_{00}(T_1,T_2)$ in Eq. (3.6) becomes

$$d_{00}^{\mathrm{NR}}(T_1,T_2)=(1-e^{-2\pi y_1})^{-1}(1-e^{-2\pi y_2})^{-1}\ , \quad (3.30)$$

with $y_j=\alpha Z\varepsilon_j/p_j$ for $j=1$ or 2, cf. Fig. 1(a) of Appendix A. This type of approximation will be called $A_{2\nu}(4)$ in Table III. In the classic paper by Primakoff and Rosen,[10] the energy dependence of d_{00}^{NR} in Eq. (3.30) is abandoned in order to get the simpler analytical expression [the case of $A_{2\nu}(5)$ in Table III].

Now we have known the energy dependences of various factors. We shall summarize the simpler approximated expressions. Various approximations for energy spectra are named as follows: The approximations in which only $A_{11}=1$ is assumed, is referred to as $A_{2\nu}(1)$, $A_{0\nu B}(1)$, and $A_{0\nu}(1)$. When $h_{2\nu}=1$ is assumed in addition, we call $A_{2\nu}(2)$ and there is no $A_{0\nu B}(2)$ and $A_{0\nu}(2)$, of course. Concerning d_{00} and γ_1, we consider three different combinations, $A_{2\nu}(j)$, $A_{0\nu B}(j)$, and $A_{0\nu}(j)$ with $j=3$, 4, and 5. As an example, only $A_{2\nu}(j)$ are listed in Table III. Similar classifications are applied to $A_{0\nu B}(j)$, $A_{0\nu}(j)$, $B_{2\nu}(j)$, $B_{0\nu B}(j)$, and $B_{0\nu}(j)$. The $A_{2\nu}(5)$ and $B_{2\nu}(5)$ are the formulae used by Primakoff and Rosen.[10]

Let us restrict our consideration to the $(\beta\beta)_{2\nu}$ mode. The single electron kinetic energy spectrum of $A_{2\nu}(0)$ for ^{76}Ge is shown in Figs. 6.1 and 6.10 of I. The results for the sum energy spectrum of two electrons are shown in Table VI. The sum spectrum of $A_{2\nu}(0)$ for ^{76}Ge is plotted in Figs. 6.4 (the $0^+\rightarrow 0^+$ transition) and 6.11 of I.[11] The dev(j) in Tables VI and VII is defined as

$$\mathrm{dev}(j)=\{1-[A_{2\nu}(j)/A_{2\nu}(0)]\}\times 100\ . \quad (3.31)$$

Note that these numerical values for dev(j) are given in

TABLE V. The T_1 dependence of $d_{00}(T_1,T_2)$ in Eq. (3.6) for the fixed T_2.

	^{82}Se ($T=5.861$)				^{100}Mo ($T=5.937$)				^{136}Xe ($T=4.851$)				^{150}Nd ($T=6.589$)			
T_1 \ T_2	0.0	2.0	3.0	4.0	0.0	2.0	3.0	4.0	0.0	1.5	2.5	3.5	0.0	2.5	3.5	4.5
0.0	1.000	1.147	1.157	1.161	1.000	1.065	1.071	1.073	1.000	0.9957	0.9992	1.001	1.000	0.9801	0.9807	0.9811
0.5	1.078	1.236	1.247	1.252	1.024	1.090	1.097	1.099	0.9856	0.9813	0.9848	0.9863	0.9767	0.9572	0.9579	0.9582
1.0	1.119	1.283	1.294	1.299	1.048	1.115	1.121	1.124	0.9919	0.9876	0.9911	0.9926	0.9774	0.9579	0.9586	0.9589
1.5	1.137	1.304	1.315	1.320	1.059	1.127	1.133	1.136	0.9957	0.9913	0.9949		0.9786	0.9591	0.9598	0.9601
2.0	1.147	1.316	1.327		1.065	1.133	1.140		0.9979	0.9935	0.9971		0.9795	0.9600	0.9606	0.9610
2.5	1.153	1.322	1.333		1.068	1.137	1.144		0.9992	0.9949			0.9801	0.9605	0.9612	
3.0	1.157	1.327			1.071	1.140			1.000	0.9958			0.9804	0.9609	0.9615	
3.5	1.159	1.330			1.072	1.141			1.001				0.9807	0.9612		
4.0	1.161				1.073				1.001				0.9809	0.9614		
4.5	1.162				1.074				1.002				0.9811			
5.0	1.163				1.075								0.9812			
5.5	1.164				1.075								0.9813			
6.0													0.9813			
6.5													0.9814			

TABLE VI. The sum energy spectrum normalized to unity for the $(\beta\beta)_{2\nu}$ mode. The dev(j) and average deviation are defined in Eqs. (3.31) and (3.32), respectively.

		^{82}Se				^{136}Xe	
T_1+T_2	$A_{2\nu}(0)$	dev(2) (%)	dev(5) (%)	T_1+T_2	$A_{2\nu}(0)$	dev(3) (%)	dev(5) (%)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2930	0.4133E-01	0.7611	−8.357	0.2425	0.8215E-01	0.7543	15.73
0.5861	0.1031	0.6331	−4.745	0.4851	0.1783	0.1045	12.21
0.8791	0.1775	0.5068	−2.457	0.7276	0.2764	−0.3550E-01	9.351
1.172	0.2538	0.3841	−1.033	0.9702	0.3634	−0.4104E-01	6.793
1.465	0.3208	0.2659	−0.1461	1.213	0.4287	−0.1582E-01	4.414
1.758	0.3692	0.1529	0.3961	1.455	0.4652	0.8029E-02	2.159
2.051	0.3928	0.4526E-01	0.7040	1.698	0.4702	0.2064E-01	−0.5261E-03
2.344	0.3892	−0.5672E-01	0.8488	1.940	0.4453	0.1969E-01	−2.081
2.637	0.3602	−0.1530	0.8737	2.183	0.3958	0.5715E-02	−4.094
2.930	0.3111	−0.2433	0.8115	2.425	0.3296	−0.1965E-01	−6.048
3.224	0.2496	−0.3277	0.6848	2.668	0.2558	−0.5543E-01	−7.949
3.517	0.1845	−0.4061	0.5095	2.911	0.1834	−0.9988E-01	−9.802
3.810	0.1239	−0.4783	0.2980	3.153	0.1197	−0.1516	−11.61
4.103	0.7391E-01	−0.5445	0.5931E-01	3.396	0.6958E-01	−0.2101	−13.38
4.396	0.3778E-01	−0.6044	−0.1994	3.638	0.3472E-01	−0.2726	−15.11
4.689	0.1556E-01	−0.6581	−0.4726	3.881	0.1398E-01	−0.3391	−16.80
4.982	0.4591E-02	−0.7054	−0.7557	4.123	0.4036E-02	−0.4085	−18.46
5.275	0.7443E-03	−0.7461	−1.045	4.366	0.6413E-03	−0.4801	−20.09
5.568	0.2838E-04	−0.7808	−1.339	4.608	0.2399E-04	−0.5532	−21.68
5.861	0.0000	0.0000	0.0000	4.851	0.0000	0.0000	0.0000
Average deviation (%)		0.3951	0.6127	Average deviation (%)		0.1488	10.38

percentage. In order to simplify the comparison, the average deviation is calculated for the cases satisfying the following two conditions: (a) Since the spectrum is normalized to unity, the counting rate should be greater than 0.02, and (b) the kinetic energy $T_k > 0.9785$ (=0.5 MeV) for the single electron spectrum or $T_1+T_2 > 2.153$ (=1.1 MeV) for the sum spectrum. Namely, when there are $n+1$ dev(j)'s which satisfy the above requirements, say $m=p, p+1, \ldots, p+n$, the average deviation is defined as

TABLE VII. The sum energy spectrum normalized to unity for $(\beta\beta)_{0\nu B}$ mode.

		^{82}Se				^{136}Xe	
T_1+T_2	$A_{0\nu B}(0)$	dev(1) (%)	dev(5) (%)	T_1+T_2	$A_{0\nu B}(0)$	dev(3) (%)	dev(5) (%)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2930	0.1937E-02	0.1913E-02	−8.2296	0.2425	0.4784E-02	0.4736E-02	24.5048
0.5861	0.6004E-02	0.5933E-02	−4.5713	0.4851	0.1290E-01	0.1285E-01	21.3803
0.8791	0.1299E-01	0.1285E-01	−2.2484	0.7276	0.2513E-01	0.2506E-01	18.8404
1.172	0.2368E-01	0.2344E-01	−0.7981	0.9702	0.4211E-01	0.4199E-01	16.5662
1.465	0.3876E-01	0.3840E-01	0.1095	1.213	0.6432E-01	0.6411E-01	14.4471
1.758	0.5879E-01	0.5832E-01	0.6636	1.455	0.9199E-01	0.9166E-01	12.4350
2.051	0.8413E-01	0.8354E-01	0.9764	1.698	0.1251	0.1246	10.5039
2.344	0.1148	0.1141	1.1190	1.940	0.1631	0.1625	8.6385
2.637	0.1505	0.1497	1.1348	2.183	0.2053	0.2046	6.8283
2.930	0.1903	0.1895	1.0567	2.425	0.2503	0.2496	5.0659
3.224	0.2326	0.2320	0.9072	2.668	0.2961	0.2953	3.3456
3.517	0.2753	0.2749	0.7023	2.911	0.3399	0.3393	1.6629
3.810	0.3153	0.3151	0.4543	3.153	0.3785	0.3781	0.0143
4.103	0.3483	0.3484	0.1721	3.396	0.4073	0.4073	−1.6034
4.396	0.3690	0.3695	−0.1370	3.638	0.4213	0.4217	−3.1925
4.689	0.3708	0.3718	−0.4679	3.881	0.4138	0.4147	−4.7553
4.982	0.3455	0.3468	−0.8158	4.123	0.3775	0.3788	−6.2938
5.275	0.2834	0.2848	−1.1776	4.366	0.3035	0.3049	−7.8094
5.568	0.1728	0.1738	−1.5503	4.608	0.1816	0.1826	−9.3038
5.861	0.0000	0.0000	0.0000	4.851	0.0000	0.0000	0.0000
Average deviation (%)		0.378	0.908	Average deviation (%)		0.300	5.298

TABLE VIII. Average deviations of $A_{2\nu}(j)$ from $A_{2\nu}(0)$ in Eq. (3.32) for the $(\beta\beta)_{2\nu}$ mode.

	^{48}Ca	^{76}Ge	^{82}Se	^{100}Mo	^{130}Te	^{136}Xe	^{150}Nd
	Average deviation (%) from $A_{2\nu}(0)$ for the single electron spectrum						
$A_{2\nu}(1)$	0.204	0.259	0.335	0.474	0.596	0.673	0.966
$A_{2\nu}(2)$	0.681	0.133	0.217	0.322	0.515	0.535	0.754
$A_{2\nu}(3)$	3.06	3.44	2.23	1.18	0.190	0.0239	0.658
$A_{2\nu}(4)$	0.997	2.90	3.01	4.59	7.09	8.06	9.11
$A_{2\nu}(5)$	2.40	1.60	1.06	2.63	5.44	6.41	8.19
	Average deviation (%) from $A_{2\nu}(0)$ for the sum energy spectrum						
$A_{2\nu}(1)$	0.270	0.371	0.468	0.665	0.904	0.955	1.364
$A_{2\nu}(2)$	0.142	0.305	0.395	0.588	0.863	0.887	1.269
$A_{2\nu}(3)$	5.31	6.79	3.96	2.03	0.227	0.149	1.224
$A_{2\nu}(4)$	1.64	5.51	5.18	7.83	12.64	13.56	15.54
$A_{2\nu}(5)$	4.11	3.32	0.613	3.75	9.25	10.38	13.62

$$\text{average deviation } (j)=\left[\frac{1}{n+1}\sum_{m=p}^{p+n}(\text{dev}(j))^2\right]^{1/2}. \tag{3.32}$$

Average deviations for various $A_{2\nu}(j)$ with $j=1-5$ are summarized in Table VIII. It is found that the $A_{2\nu}(1)$ case where all factors except A_{11} are taken into account is not the best for all nuclei. It was very fortunate for the Irvine group[3] that they have used the Primakoff-Rosen approximation[10] [our $A_{2\nu}(5)$ case] for their analysis on ^{82}Se. The ^{82}Se case has the minimum average deviation among nuclei considered here. For example, this $A_{2\nu}(5)$ case is not good for the analysis of the ^{136}Xe data in comparison with $A_{2\nu}(3)$ or $A_{2\nu}(2)$.

The similar analyses are performed for the $(\beta\beta)_{0\nu B}$ and $(\beta\beta)_{0\nu}$ modes. The sum energy spectrum and deviations are listed in Table VII, and average deviations for energy spectra are summarized in Tables VIII, IX, and X [cf. Figs. 6.5, 6.10, and 6.11 of I].

Concerning the angular correlation, the approximation given in Eq. (3.28) is extremely good, as mentioned before. The error is less than 0.5% for the practical energy region. The integrated angular correlation coefficient

$$\alpha_{2\nu}(j)=\frac{\int_0^T dT_1\int_0^{T-T_1}dT_2 B_{2\nu}(j)}{\int_0^T dT_1\int_0^{T-T_1}dT_2 A_{2\nu}(j)}, \tag{3.33}$$

has a tiny difference for various j. Two extreme cases are listed in Table XI. In practice, the range of integration should be limited by the experimental conditions. Other integrated angular correlation coefficients $\alpha_{0\nu B}(j)$ and $\alpha_{0\nu}(j)$ defined similarly to Eq. (3.33) are listed in Table XI for comparison.

IV. DISCUSSION

Concerning the half-lives of the double-beta decays, the constant normalization factor should include the relativistic Coulomb effect. This conclusion is clearly seen by comparing $(C_{F,0})^2$ in Eq. (3.3) with its nonrelativistic value $(C_{F,0}^{\rm NR})^2$ in Eq. (A7) of Appendix A, as listed in Table I for various nuclei.

However, as for the energy spectra, very rough and simple formulas can reproduce the spectra obtained from the original ones which are $A_{2\nu}(0)$ in Eq. (3.4), $A_{0\nu B}(0)$ in Eq. (3.22), and $A_{0\nu}(0)$ in Eq. (3.25). For example, in the $(\beta\beta)_{2\nu}$ mode, the nonrelativistic approximation

TABLE IX. Average deviations of $A_{0\nu B}(j)$ from $A_{0\nu B}(0)$ for the $(\beta\beta)_{0\nu B}$ mode.

	^{48}Ca	^{76}Ge	^{82}Se	^{100}Mo	^{130}Te	^{136}Xe	^{150}Nd
	Average deviation (%) from $A_{0\nu B}(0)$ for the single electron spectrum						
$A_{0\nu B}(1)$	0.161	0.146	0.221	0.311	0.512	0.427	0.572
$A_{0\nu B}(3)$	2.608	2.271	1.640	1.044	0.610	0.554	0.682
$A_{0\nu B}(4)$	0.505	1.117	1.251	1.883	2.793	3.124	3.981
$A_{0\nu B}(5)$	2.568	1.563	1.442	1.644	2.626	2.883	3.829
	Average deviation (%) from $A_{0\nu B}(0)$ for the sum energy spectrum						
$A_{0\nu B}(1)$	0.327	0.206	0.378	0.527	0.596	0.626	1.202
$A_{0\nu B}(3)$	3.960	2.026	1.478	0.617	0.179	0.300	1.117
$A_{0\nu B}(4)$	1.803	2.193	3.080	4.536	5.938	6.355	10.13
$A_{0\nu B}(5)$	2.650	0.555	0.908	2.915	4.814	5.298	9.269

TABLE X. Average deviations of $A_{0\nu}(j)$ from $A_{0\nu}(0)$ for the $(\beta\beta)_{0\nu}$ modes.

	^{48}Ca	^{76}Ge	^{82}Se	^{100}Mo	^{130}Te	^{136}Xe	^{150}Nd
	Average deviation (%) from $A_{0\nu}(0)$ for the single electron spectrum						
$A_{0\nu}(1)$	0.012	0.028	0.036	0.049	0.062	0.067	0.077
$A_{0\nu}(3)$	3.975	4.530	3.987	1.998	0.578	0.458	0.457
$A_{0\nu}(4)$	0.703	2.200	2.739	3.727	4.621	4.957	6.476
$A_{0\nu}(5)$	3.421	3.398	2.314	0.573	2.725	3.197	5.227

$A_{2\nu}(5)$ in Table III may be used for ^{76}Ge and ^{82}Se, while another simple approximation $A_{2\nu}(3)$ is better for ^{150}Nd, ^{136}Xe, ^{130}Te, and ^{100}Mo, as seen from Tables VIII, IX, and X. A little more complicated one $A_{2\nu}(2)$ offers the better approximation for all nuclei considered here in the $(\beta\beta)_{2\nu}$ mode.[17] The quite similar tendencies can be seen for the $(\beta\beta)_{0\nu B}$ and $(\beta\beta)_{0\nu}$ modes.[17] Anyhow, the deviations from the original ones are not serious for various approximations, except for the large Z nuclei like ^{136}Xe and ^{150}Nd.

The decomposition of the relativistic Coulomb wave given in Eqs. (A1) and (A2) is useful to analyze the data on the single β decay, too.

Concerning the angular correlation, the simplest approximation $(B_{11}/A_{11})\sim(p_1p_2/\varepsilon_1\varepsilon_2)$ in Eq. (3.28) is good enough and independent of decay modes. However, Table XI shows that the integrated angular correlation coefficients in the $(\beta\beta)_{2\nu}$ and $(\beta\beta)_{0\nu B}$ modes and in the m_ν part of the $(\beta\beta)_{0\nu}$ mode are different, though they are almost independent of various approximations.

Next let us consider the approximation related to the energy denominators K_a and L_a in Eq. (2.7) for the $(\beta\beta)_{2\nu}$ mode. Our original expression in Eq. (2.1) or Eq. (3.1) has been obtained after applying the following approximation: That is, the second-order perturbation gives, as shown in Eq. (3.2.1) of I,

$$H_{2\nu}(T_1,T_2)=\left[\frac{30}{4}\right]\frac{1}{\omega^5}\int_0^\omega d\omega_1\omega_1^2(\omega-\omega_1)^2 \times\left|\sum_a M_{\mathrm{GT}a}^{(2\nu)}(K_a+L_a)/2\right|^2, \tag{4.1}$$

where K_a and L_a are obtained from Eq. (2.7) by replacing $\langle\mu_a\rangle$ with μ_a in Eq. (2.4), cf. the definition of $h_{2\nu}$ in Eq. (3.12). Our original expression $A_{2\nu}(0)$ is obtained by assuming the separation

$$H_{2\nu}(T_1,T_2)=\left|\sum_a M_{\mathrm{GT}a}^{(2\nu)}/\mu_a\right|^2 h_{2\nu}(T_1,T_2)\,. \tag{4.2}$$

This kind of approximation is allowed, if the ε_j and ω_j dependences of K_a and L_a can be ignored, i.e., $\mu_a \gg |K_D|$ and $|L_D|$ in Eq. (2.7). The kinetic energy spectra of electron or neutrino have peaks at $(\varepsilon_1-m_e)\sim(\varepsilon_2-m_e)\sim\omega_1\sim\omega_2<(Tm_e)/4$, where Tm_e is the maximum kinetic energy release defined in Eq. (3.5). While the lowest μ_a for the known nuclei is of order of 10, as shown in Fig. 3.2 of I. Therefore, we expect that Eq. (4.2) is a good approximation.

Tsuboi, Muto, and Horie have examined this kind of approximation for the ^{48}Ca case by performing all integrations over ω_1, T_1, and T_2.[12] According to their result, the approximation seems to be allowed. It is desir-

TABLE XI. Integrated angular correlation coefficient $\alpha_{2\nu}(j)$ in Eq. (3.33) and the corresponding $\alpha_{0\nu B}(j)$ and $\alpha_{0\nu}(j)$.

	^{48}Ca	^{76}Ge	^{82}Se	^{100}Mo	^{130}Te	^{136}Xe	^{150}Nd
				$(\beta\beta)_{2\nu}$ mode			
$\alpha_{2\nu}(0)$	0.7613	0.5308	0.6490	0.6464	0.5838	0.5758	0.6653
$\alpha_{2\nu}(5)$	0.7468	0.5227	0.6437	0.6477	0.5914	0.5846	0.6792
				$(\beta\beta)_{0\nu B}$ mode			
$\alpha_{0\nu B}(0)$	0.8739	0.7041	0.7989	0.7982	0.7509	0.7446	0.8136
$\alpha_{0\nu B}(5)$	0.8668	0.6982	0.7957	0.7986	0.7555	0.7500	0.8213
				$(\beta\beta)_{0\nu}$ mode			
$\alpha_{0\nu}(0)$	0.9288	0.8152	0.8821	0.8827	0.8520	0.8479	0.8953
$\alpha_{0\nu}(5)$	0.9254	0.8110	0.8796	0.8816	0.8522	0.8483	0.8966

able for other nuclei to examine the constancy of the T_k and $\langle \mu_a \rangle$ dependences of $H_{2\nu}(T_1, T_2)$ theoretically in order to confirm our original expression in Eq. (2.1) or Eq. (3.1).

Concerning the $(\beta\beta)_{0\nu}$ and $(\beta\beta)_{0\nu B}$ modes, the above electron energy dependence due to the energy denominator can be neglected, because the average energy of virtual neutrino is about $80m_e$. But there is another problem. The original formulas in the text are obtained by assuming the $2n$ mechanism where the successive β transitions of two neutrons trigger the $\beta\beta$ decay. Fazely and Liu[13] pointed out that there is an appreciable contribution to the $0^+ \to 0^+$ transition in the $(\beta\beta)_{0\nu}$ mode from the N^* mechanism where the $\beta\beta$ decay occurs through the transition of $\Delta(1232)$ which interacts with other nucleons. However, Tomoda[14] and Watanabe-Toki[15] concluded that its contribution is negligible in comparison with the $2n$ mechanism. There is no definite estimation of the $0^+ \to 2^+$ transition in the N^* mechanism. Even if the N^* mechanism is taken into account, the transition formulas need not be changed, except the nuclear matrix elements, see for example Eq. (4.5) of I for the $(0^+ \to 2^+)$ transition in the $(\beta\beta)_{0\nu}$ mode.

Note added in proof. Recently we received a paper by T. Tomoda, $0^+ \to 2^+$ *Neutrinoless* $\beta\beta$ *Decay of* ^{76}*Ge* (University of Tübingen Report) which discussed the contributions from both two nucleon and N^* mechanisms.

APPENDIX A: CHARACTERS OF THE RELATIVISTIC COULOMB WAVE FUNCTION

We shall discuss the features of the normalization $[A_{\pm k}(\varepsilon)]$ and the phase $[\Delta^C_{\pm k}(\varepsilon)]$ of the electron wave function inside the nucleus with the uniform charge distribution. Their original definitions are given in Eqs. (D.18) and (D.30) of I.

It is convenient to express the normalization $A_{\pm k}(\varepsilon)$ by a product of three factors;

$$A_{\pm k}(\varepsilon)=[(\varepsilon \mp m_e)/2\varepsilon]^{1/2}[F_{k-1}(Z,\varepsilon)]^{1/2}D_{\pm k}(\varepsilon) . \quad \text{(A1)}$$

Here A_{+k} and A_{-k} correspond to the small and large components of the radial wave function, respectively. The first factors on the right-hand side of Eq. (A1) are the normalization appearing even for the plane wave solution of the Dirac equation. The second factor $F_{k-1}(Z,\varepsilon)$ is the relativistic Coulomb (Fermi) factor introduced by Konopinski and Uhlenbeck,[9] see Eq. (3.1.25) of I, where Z is the atomic number of the daughter nucleus. This $F_{k-1}(Z,\varepsilon)$ expresses the normalization of the outer regular solution for the point charge and is evaluated at the nuclear surface. The final factor $D_{\pm k}(\varepsilon)$ is related with the continuity condition between the inner and outer solutions at the nuclear surface for the case of the extended nuclear charge.

First let us examine the characters of $F_{k-1}(Z,\varepsilon)$ by expressing it as follows:

$$F_{k-1}(Z,\varepsilon)=C_{F,k-1}d_{k-1}(\varepsilon)(m_e/p)^{2k-1}(\varepsilon/m_e)^{2\gamma_k-1} , \quad \text{(A2)}$$

where $\gamma_k=[k^2-(\alpha Z)^2]^{1/2}$. The constant factor $C_{F,k-1}$ is

$$C_{F,k-1}=\pi(2\alpha Z)^{2\gamma_k-1}(m_eR)^{2(\gamma_k-k)}\left[\frac{2k(2k-1)!!}{\Gamma(2\gamma_k+1)}\right]^2 , \quad \text{(A3)}$$

where R is the nuclear radius, $\Gamma(x)$ is a gamma function, and

$$d_{k-1}(\varepsilon)=\left[\frac{1}{2\pi}\right]y^{1-2\gamma_k}e^{\pi y}\,|\,\Gamma(\gamma_k+iy)\,|^2 , \quad \text{(A4)}$$

with

$$y=\alpha Z(\varepsilon/p) . \quad \text{(A5)}$$

This $d_{k-1}(\varepsilon)$ is normalized to be

$$\lim_{p\to 0} d_{k-1}(\varepsilon)=1 \quad \text{for} \quad Z>0 . \quad \text{(A6)}$$

Since there is the factor p^{-2k+1} in F_{k-1} of Eq. (A2), one may wonder whether the transition probability has an infinity at $p=0$. But it is not so. This is because the suffix k means the electron $(k-1)$ wave with the total angular momentum $j=k-\frac{1}{2}$ so that the electron $(k-1)$-wave offers the factor p^{k-1} which results in $p^{2(k-1)}$ for the differential probability. In addition, we have one more p from the fermion phase space. Thus totally we have a finite value at $p=0$ for the single electron spectrum. This is due to the attractive force on electrons by the nuclear charge. Note that for the positron emission case $(Z<0)$, we have $\lim_{p\to 0}d_{k-1}(\varepsilon)=0$, which means the repulsive force due to the nuclear charge.

If we assume $\gamma_k \simeq k$, i.e., $\alpha Z \Longrightarrow 0$, $d_0(\varepsilon)$ becomes the nonrelativistic Coulomb Fermi factor which appears in the solution of the Schrödinger equation, i.e.,

$$d_0^{\rm NR}(\varepsilon)=\frac{1}{1-e^{-2\pi y}} \quad \text{and} \quad C^{\rm NR}_{F,0}=2\pi\alpha Z . \quad \text{(A7)}$$

Similarly, for the $k=2$ case, we have

$$d_1^{\rm NR}(\varepsilon)=d_0^{\rm NR}(\varepsilon)(1+y^{-2}) \quad \text{and} \quad C^{\rm NR}_{F,1}=2\pi(\alpha Z)^3 . \quad \text{(A8)}$$

Note that $F_{k-1}(Z,\varepsilon)$ is normalized as

$$\lim_{\alpha Z\to 0} F_{k-1}(Z,\varepsilon)=1 . \quad \text{(A9)}$$

The momentum (p) dependences of $d_{k-1}(\varepsilon)$ are shown in Fig. 1 with $d^{\rm NR}_{k-1}(\varepsilon)$ for the cases of $Z=22$, 44, and 62, corresponding to the $\beta\beta$ decay of ^{48}Ca, ^{100}Mo, and ^{150}Nd. As you see, the approximation $d_0(\varepsilon)\approx 1$ is good for the higher Z nuclei, but $d_0(\varepsilon)\approx d_0^{\rm NR}(\varepsilon)$ is better for the lower Z. For $d_1(\varepsilon)$, we should take it into account properly, at least by the approximation $d_1(\varepsilon)\sim d_1^{\rm NR}(\varepsilon)$, when we analyze experimental data.

Next, let us discuss the feature of $D_{\pm k}(\varepsilon)$ in Eq. (A1),

$$D_{+k}(\varepsilon)=[F_k^{(o)}(R)/F_k^{(i)}(R)]B_{+k}(1+h_{f,k}) , \quad \text{(A10)}$$

$$D_{-k}(\varepsilon)=[G_{-k}^{(o)}(R)/G_{-k}^{(i)}(R)]B_{-k}(1+h_{g,-k}) . \quad \text{(A11)}$$

Here $F_k^{(o)}(G_{-k}^{(o)})$ and $F_k^{(i)}(G_{-k}^{(i)})$ are the outer and inner regular solutions, respectively, and B_κ and $(1+h_{f(g),\kappa})$ defined in Eqs. (D.24) and (D.26) of I come from the nor-

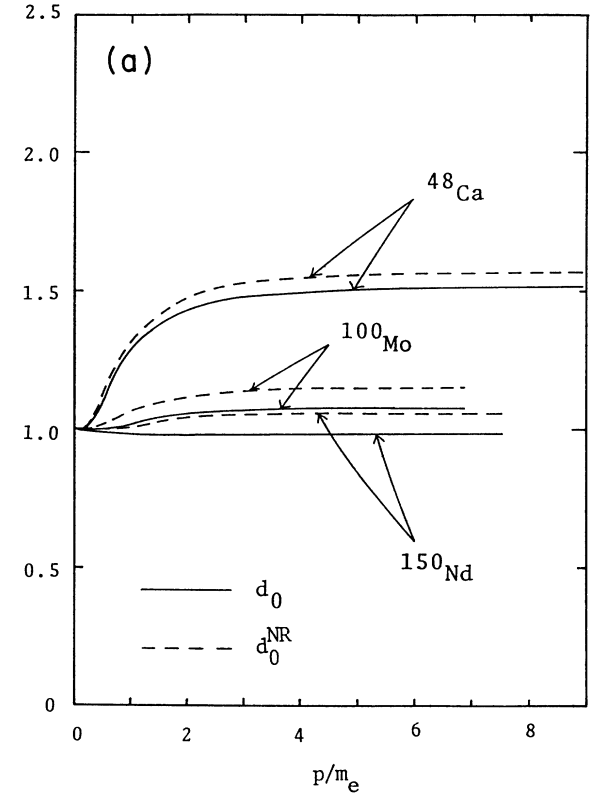

FIG. 1. The momentum dependences of $d_{k-1}(\varepsilon)$ in Eq. (A4) and its nonrelativistic approximation $d_{k-1}^{\mathrm{NR}}(\varepsilon)$ in Eqs. (A7) and (A8).

malization of the outer (regular and irregular) solutions and the continuity condition at the nuclear surface, respectively.

The outer regular solution is given as

$$F_{+k}^{(o)}(R)=[R_k(1+\zeta_k)^{1/2}+I_k(1-\zeta_k)^{1/2}]/\sqrt{2}\ , \tag{A12}$$

$$G_{-k}^{(o)}(R)=[R_k(1+\xi_k)^{1/2}+I_k(1-\xi_k)^{1/2}]/\sqrt{2}\ . \tag{A13}$$

Note that $G_{-k}^{(o)}$ is obtained from $F_{+k}^{(o)}$ by replacing m_e with $(-m_e)$, that is,

$$\begin{Bmatrix}(1+\zeta_k)/2\\(1+\xi_k)/2\end{Bmatrix}=(k+\gamma_k)(\varepsilon k\pm m_e\gamma_k)(\varepsilon\mp m_e)/(2N_p)\xrightarrow[p\to 0]{}\begin{Bmatrix}0\\1\end{Bmatrix}, \tag{A14}$$

$$\begin{Bmatrix}(1-\zeta_k)/2\\(1-\xi_k)/2\end{Bmatrix}=(k-\gamma_k)(\varepsilon k\pm m_e\gamma_k)(\varepsilon\pm m_e)/(2N_p)\xrightarrow[p\to 0]{}\begin{Bmatrix}1\\0\end{Bmatrix}, \tag{A15}$$

where

$$N_p=(\alpha Z\varepsilon)^2+\gamma_k^2p^2\ . \tag{A16}$$

R_k and I_k are defined as

$$(R_k+iI_k)=\frac{1}{k}(\gamma_k+iy)e^{ipR}F(\gamma_k-iy,2\gamma_k+1;-2ipR)\ , \tag{A17}$$

where $F(\alpha,\gamma;z)$ is a hypergeometric function. In the $p\to 0$ limit, we have

$$(R_k+iI_k)\xrightarrow[p\to 0]{}(\gamma_k/k)+i(y/k)\ , \tag{A18}$$

and $G_{-k}^{(o)}(\varepsilon)\sim[1-(\alpha Z)^2/2k^2]$ and $F_{+k}^{(o)}(\varepsilon)\to\alpha Zm_e/pk$. It should be noted that $A_{+k}(\varepsilon)$ in Eq. (A1) has a factor $p/[2\varepsilon(\varepsilon+m_e)]^{1/2}$, so that $pF_{+k}^{(o)}(\varepsilon)$ at $p=0$ is a finite value proportional to αZ.

Also it will be useful to notice that in the plane wave limit ($\alpha Z\to 0$), we have $(1+\zeta_k)/2=(1+\xi_k)/2=1$ and $(1-\zeta_k)/2=(1-\xi_k)/2=0$ in Eqs. (A14) and (A15), and

$$(R_k+iI_k)\xrightarrow[\alpha Z\to 0]{}[j_{k-1}(pR)+ij_k(pR)](2k-1)!!(pR)^{1-k}\approx 1+ipR/(2k+1)\ , \tag{A19}$$

where $j_k(x)$ is a spherical Bessel function. In other words, in this $\alpha Z\to 0$ limit, we have $F_{+k}^{(o)}\approx G_{-k}^{(o)}\approx 1$ if we ignore the finite de Broglie wavelength correction, $(pR)\ll 1$.

Next, the inner wave $F_{+k}^{(i)}(R)$ is given by

$$F^{(i)}_{+k}(R)=\sum_n a_{k,2n}$$
$$=1-\frac{(\bar{p}R)^2}{2(2k+1)}$$
$$+\frac{(\bar{p}R)^4+2\alpha Z[2(k+1)\bar{E}+m_e]R}{8(2k+1)(2k+3)}-\cdots, \tag{A20}$$

where

$$\bar{p}^{\,2}=\bar{E}^{\,2}-m_e^2 \quad \text{and} \quad \bar{E}=\varepsilon+(3\alpha Z/2R)\ . \tag{A21}$$

The other $G^{(i)}_{-k}(R)=\sum_n b_{-k,2n}$ is obtained from $F^{(i)}_{+k}$ by replacing m_e with $(-m_e)$. The exact solutions of $F^{(i)}_{+k}$ and $G^{(i)}_{-k}$ are obtained from the recurrence relations for $a_{k,2n}$ and $b_{-k,2n}$ given in Eq. (D.11) of I.

In the $p\to 0$ limit, by neglecting the m_eR term, we have

$$[F^{(o)}_{+1}(R)/F^{(i)}_{+1}(R)] \xrightarrow[p\to 0]{} (\alpha Z m_e/p)[1+\tfrac{17}{60}(\alpha Z)^2+\cdots], \tag{A22}$$

$$[F^{(o)}_{+2}(R)/F^{(i)}_{+2}(R)]\to(\alpha Z m_e/2p)[1+\tfrac{1}{6}(\alpha Z)^2+\cdots], \tag{A23}$$

$$[G^{(o)}_{-1}(R)/G^{(i)}_{-1}(R)]\to[1-\tfrac{13}{60}(\alpha Z)^2+\cdots], \tag{A24}$$

$$[G^{(o)}_{-2}(R)/G^{(i)}_{-2}(R)]\to[1+\tfrac{1}{24}(\alpha Z)^2+\cdots]. \tag{A25}$$

On the other hand, in the $\alpha Z\to 0$ limit, we have again similarly to Eq. (A19)

$$F^{(i)}_{+k} \text{ or } G^{(i)}_{-k} \xrightarrow[\alpha Z\to 0]{} j_{k-1}(pR)(2k-1)!!(pR)^{1-k}\ . \tag{A26}$$

The exact expression of $B_\kappa[1+h_{f(g),\kappa}]$ is rather complicated, as seen from Eqs. (D.24) and (D.26) of I, but its numerical value is less than unity. The deviation from unity is proportional to $(\alpha Z)^2$ like Eq. (A24), and less than 2% even for the ^{150}Nd case, i.e.,

$$B_{+k}(1+h_{f,k}) \text{ or } B_{-k}(1+h_{g,-k})\simeq 1-O((\alpha Z)^2)\ . \tag{A27}$$

In summary, $D_{+k}(\varepsilon)$ and $D_{-k}(\varepsilon)$ take the values around unity within a few percent errors, except the case of $D_{+k}(\varepsilon)$ for $p<m_e$. As an example, this situation is shown in Fig. 2 for the ^{100}Mo case. In the energy region where electrons are observed experimentally, say $p>0.5m_e$, these $D_{\pm k}(\varepsilon)$ may be treated as unity in order to avoid the complicated calculation.

Finally, let us consider the approximation for the overall phase shift $\Delta^C_\kappa(\varepsilon)$. In practice, this phase shift appears only in the interference term which gives the angular correlation between two emitted electrons, that is, a combination like the cosine (or sine) of the difference $\Delta^C_\kappa(\varepsilon)-\Delta^C_\lambda(\varepsilon)$.

In the case of the $0^+\to 0^+$ transition, only the $k=1$ wave contributes. Therefore, only one combination

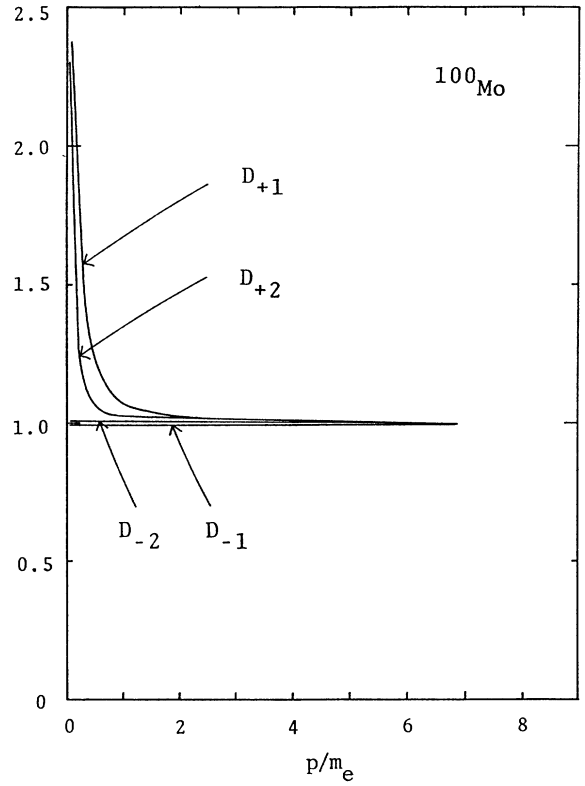

FIG. 2. The momentum dependences of $D_{\pm k}(\varepsilon)$ defined in Eqs. (A10) and (A11).

$\theta_1=\Delta^C_{+1}(\varepsilon)-\Delta^C_{-1}(\varepsilon)$ appears and can be approximated within 0.1% errors as

$$\theta_k\simeq\eta_{+k}(\varepsilon)-\eta_{-k}(\varepsilon)+\pi/2\ , \tag{A28}$$

$$\cos\eta_k=-[(1-\xi_k)/2]^{1/2},\quad \cos\eta_{-k}=[(1+\xi_k)/2]^{1/2}\ ,$$
$$\sin\eta_k=-[(1+\xi_k)/2]^{1/2},\quad \sin\eta_{-k}=-[(1-\xi_k)/2]^{1/2}\ . \tag{A29}$$

As it is easily confirmed from Eqs. (A14) and (A15), we have

$$\theta_k \xrightarrow[p\to 0]{} -\pi/2\ , \tag{A30}$$

$$\theta_k \xrightarrow[\alpha Z\to 0]{} 0\ . \tag{A31}$$

These limiting values are exact, independent of the approximation in (A28). However, as soon as p becomes nonzero, strictly speaking, in the region $y^2(m_e/\varepsilon)\ll\gamma_1$, $\cos\theta_1$ tends to become near unity rapidly. In the practical region, say $p>0.5m_e$, the deviation from $\cos\theta_1=1$ is less than 1% even for the ^{150}Nd case.

In the case of the $0^+\to 2^+$ transition in the $(\beta\beta)_{0\nu}$ mode, all six possible combinations of $(\Delta^C_\kappa-\Delta^C_\lambda)$ for λ, $\kappa=\pm 1$ and ± 2 appear. However, in the practical region

say $p > 0.5m_e$, we may assume the following limiting values:

$$\lim_{\alpha Z \to 0} \Delta_\kappa^C(\varepsilon) = 2\pi \ . \tag{A32}$$

APPENDIX B: THE $(\beta\beta)_{0\nu}$ MODE

The differential decay rates in the $(\beta\beta)_{0\nu}$ mode are given for the $0^+ \to 0^+$ and $0^+ \to 2^+$ transitions.

In the case of the $0^+ \to 0^+$ transition, by using the notation $T_2 = T - T_1$, we have the following full expression corresponding to Eq. (3.23):

$$\frac{d^2\Gamma_{0\nu}(0^+ \to 0^+)}{dT_1 d\cos\theta} = |M_{\rm GT}^{(0\nu)}|^2 N_{0\nu}[A_{0\nu}(0) + B_{0\nu}(0)\cos\theta] \ , \tag{B1}$$

where the constant factor $N_{0\nu}$ is defined in Eq. (3.24) and,

$$\begin{Bmatrix} A_{0\nu}(0) \\ B_{0\nu}(0) \end{Bmatrix} = (1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_1} d_{00} \begin{Bmatrix} A_0^{(0\nu)\prime} \\ B_0^{(0\nu)\prime} \end{Bmatrix} , \tag{B2}$$

$$A_0^{(0\nu)\prime} = C_1' \left[\frac{\langle m_\nu \rangle}{m_e}\right]^2 + C_2' \langle\lambda\rangle \frac{\langle m_\nu \rangle}{m_e}\cos\psi_1 + C_3' \langle\eta\rangle \frac{\langle m_\nu \rangle}{m_e}\cos\psi_2 + C_4' \langle\lambda\rangle^2 + C_5' \langle\eta\rangle^2 + C_6' \langle\lambda\rangle\langle\eta\rangle\cos(\psi_1 - \psi_2) \ , \tag{B3}$$

$$B_0^{(0\nu)\prime} = \left[\frac{p_1 p_2}{\varepsilon_1 \varepsilon_2}\right] \left[C_1^{(B)\prime} \left[\frac{\langle m_\nu \rangle}{m_e}\right]^2 + C_2^{(B)\prime} \langle\lambda\rangle \frac{\langle m_\nu \rangle}{m_e}\cos\psi_1 + C_3^{(B)\prime} \langle\eta\rangle \frac{\langle m_\nu \rangle}{m_e}\cos\psi_2 + C_4^{(B)\prime} \langle\lambda\rangle^2 + C_5^{(B)\prime} \langle\eta\rangle^2 + C_6^{(B)\prime} \langle\lambda\rangle\langle\eta\rangle\cos(\psi_1 - \psi_2) \right] . \tag{B4}$$

Here λ and η are the parameters due to the right-handed weak interaction defined in Eq. (3.1.3) of I and $A_{0\nu}(0)$ and $B_{0\nu}(0)$ are obtained from Eqs. (3.5.10) and (C.3.14) of I, respectively.

Six coefficients C_j' in the spectrum part $A_0^{(0\nu)\prime}$, are

$$\begin{aligned}
C_1' &= (1-\chi_F)^2 A_{11} \ , \\
C_2' &= 2(1-\chi_F)(E_{-+}P_{11}\chi_{3-} + E_{+-}Q_{11}\chi_{3+}) \ , \\
C_3' &= -2(1-\chi_F)[E_{-+}P_{11}\chi_{4-} + E_{+-}Q_{11}\chi_{4+} \\
&\qquad - E_{++}M_{11}\chi'_{PR-} + E_{--}N_{11}\chi'_{PR+}] \ , \\
C_4' &= E_{-+}P_{11}\chi_{3-}^2 + E_{+-}Q_{11}\chi_{3+}^2 \ , \\
C_5' &= E_{-+}P_{11}\chi_{4-}^2 + E_{+-}Q_{11}\chi_{4+}^2 + E_{++}M_{11}\chi'^2_{PR-} \\
&\qquad + E_{--}N_{11}\chi'^2_{PR+} \ , \\
C_6' &= -2(E_{-+}P_{11}\chi_{3-}\chi_{4-} + E_{+-}Q_{11}\chi_{3+}\chi_{4+}) \ ,
\end{aligned} \tag{B5}$$

where A_{11} is defined in Eq. (3.7),

$$\begin{aligned}
E_{++} &= (\varepsilon_1 + m_e)(\varepsilon_2 \pm m_e)/(4\varepsilon_1\varepsilon_2) \ , \\
E_{-\pm} &= (\varepsilon_1 - m_e)(\varepsilon_2 \pm m_e)/(4\varepsilon_1\varepsilon_2) \ ,
\end{aligned} \tag{B6a}$$

and, by using $D_{\kappa,\lambda}$ in Eq. (3.10), we define

$$\begin{aligned}
M_{jk}(\varepsilon_1,\varepsilon_2) &= D_{-j,-k}(\varepsilon_1) D_{-j,-k}(\varepsilon_2) \ , \\
N_{jk}(\varepsilon_1,\varepsilon_2) &= D_{+j,+k}(\varepsilon_1) D_{+j,+k}(\varepsilon_2) \ , \\
P_{jk}(\varepsilon_1,\varepsilon_2) &= D_{+j,+k}(\varepsilon_1) D_{-j,-k}(\varepsilon_2) \ , \\
Q_{jk}(\varepsilon_1,\varepsilon_2) &= D_{-j,-k}(\varepsilon_1) D_{+j,+k}(\varepsilon_2) \ .
\end{aligned} \tag{B6b}$$

The combination of nuclear parameters are defined as

$$\begin{aligned}
\chi_{3\pm} &= \tfrac{2}{9}\chi_{1+} \pm \frac{\varepsilon_1 - \varepsilon_2}{m_e}\chi_{2-} \ , \\
\chi_{4\pm} &= \tfrac{2}{9}\chi_{1-} \pm \frac{\varepsilon_1 - \varepsilon_2}{m_e}\chi_{2+} \ , \\
\chi'_{PR\pm} &= \frac{4}{m_e R}[\chi'_R - \tfrac{1}{6}(\zeta \pm 2m_e R)\chi'_P] \ , \\
\zeta &= 3\alpha Z + (T+2)m_e R \ ,
\end{aligned} \tag{B7}$$

where the nuclear parameters χ_F, $\chi_{1\pm}$, $\chi_{2\pm}$, χ'_P, and χ'_R are defined in Eqs. (3.5.2)–(3.5.9) of I and assumed to be real. The approximated form $A_0^{(0\nu)\prime}(j)$ for $j > 0$ means to assume that

$$A_{11} = M_{jk} = N_{jk} = P_{jk} = Q_{jk} = 1 \ . \tag{B8}$$

Similarly, $C_j^{(B)\prime}$ in the angular part $B_0^{(0\nu)\prime}$ are

$$
\begin{aligned}
C_1^{(B)\prime} &= -(1-\chi_F)^2 B'_{11} \ , \\
C_2^{(B)\prime} &= -(1-\chi_F)(\tfrac{2}{9}\chi_{1+}) D_{0\nu}(C_{0\nu}+S_{0\nu}) \ , \\
C_3^{(B)\prime} &= (1-\chi_F)[\tfrac{2}{9}\chi_{1-}(C_{0\nu}+S_{0\nu}) - \tfrac{4}{3}\chi'_p(C_{0\nu}-S_{0\nu})] D_{0\nu} \ , \\
C_4^{(B)\prime} &= \frac{1}{2}\left[\left[\frac{\varepsilon_1-\varepsilon_2}{m_e}\chi_{2-} \right]^2 - (\tfrac{2}{9}\chi_{1+})^2 \right] D_{0\nu}(C_{0\nu}+S_{0\nu}) \ , \\
C_5^{(B)\prime} &= \frac{1}{2}\left[\left[\frac{\varepsilon_1-\varepsilon_2}{m_e}\chi_{2+} \right]^2 - (\tfrac{2}{9}\chi_{1-})^2 \right] D_{0\nu}(C_{0\nu}+S_{0\nu}) + \left[\frac{8}{(m_e R)^2}(\tfrac{1}{6}\xi\chi'_p - \chi'_R)^2 - \tfrac{8}{9}\chi'^2_p \right] D_{0\nu}(C_{0\nu}-S_{0\nu}) \ , \\
C_6^{(B)\prime} &= -\left[\left[\frac{\varepsilon_1-\varepsilon_2}{m_e}\chi_{2-} \right] \left[\frac{\varepsilon_1-\varepsilon_2}{m_e}\chi_{2+} \right] - (\tfrac{2}{9}\chi_{1+})(\tfrac{2}{9}\chi_{1-}) \right] D_{0\nu}(C_{0\nu}+S_{0\nu}) \ ,
\end{aligned}
\tag{B9}
$$

where $B'_{11}=(\varepsilon_1\varepsilon_2/p_1p_2)B_{11}$ with B_{11} in Eq. (3.8) and, by using $D_{\kappa,\lambda}(\varepsilon)$ and $C_{\kappa,\lambda}(\varepsilon)$ in Eqs. (3.10) and (3.11), we define

$$
\begin{aligned}
D_{0\nu} &= D_{+1,-1}(\varepsilon_1) D_{+1,-1}(\varepsilon_2) \ , \\
C_{0\nu} &= C_{+1,-1}(\varepsilon_1) C_{+1,-1}(\varepsilon_2) \ , \\
S_{0\nu} &= S_{+1,-1}(\varepsilon_1) S_{+1,-1}(\varepsilon_2) \ , \\
S_{\kappa,\lambda}(\varepsilon) &= \sin[\Delta^C_\kappa(\varepsilon) - \Delta^C_\lambda(\varepsilon)] \ .
\end{aligned}
\tag{B10}
$$

The approximated form $B_{0\nu}(j)$ for $j>0$ means to assume

$$
B'_{11} = D_{0\nu} = C_{0\nu} = 1 \quad \text{and} \quad S_{0\nu} = 0 \ . \tag{B11}
$$

The spectrum and angular correlation are shown in Figs. 6.6–6.9 of I and Ref. 16.

In the $\beta\beta_{0\nu}(0^+ \to 2^+)$ mode, there are two nuclear matrix elements $|Z_{22}|$ and $|Z_{21}|$ defined in Eq. (3.5.23) of I. The main origin of the complication comes from the fact that one electron is in the S-wave state ($k=1$) and the other in the P-wave ($j=\frac{3}{2}$) state ($k=2$).

The differential decay rate in this case is

$$
\frac{d^2\Gamma_{0\nu}(0^+ \to 2^+)}{dT_1 d\cos\theta} = N_{0\nu 2}[A_{0\nu 2}(0) + B_{0\nu 2}(0)\cos\theta + C_{0\nu 2}(0)(\cos^2\theta - \tfrac{1}{3})] \ , \tag{B12}
$$

where the constant factor is

$$
N_{0\nu 2} = \frac{a_{0\nu}}{3(m_e R)^2} C_{F,0} C_{F,1} \ . \tag{B13}
$$

Here $a_{0\nu}$ is the constant factor appearing in the $\beta\beta_{0\nu}(0^+ \to 0^+)$ mode [Eq. (3.5.17) of I] and $C_{F,k}$ is defined in Eq. (A3).

The spectrum part $A_{0\nu 2}(0)$ is

$$
\begin{aligned}
A_{0\nu 2}(0) = {} & (1+T_1)^{2\gamma_2}(1+T_2)^{2\gamma_1} d_{10}[(E_{++}M_{21} + E_{--}N_{21})|Z_{21}|^2 + (E_{-+}P_{21} + E_{+-}Q_{21})|Z_{22}|^2] \\
& + (1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_2} d_{01}[(E_{++}M_{12} + E_{--}N_{12})|Z_{21}|^2 + (E_{-+}P_{12} + E_{+-}Q_{12})|Z_{22}|^2] \ ,
\end{aligned}
\tag{B14}
$$

where, by using $d_{k-1}(\varepsilon)$ defined in Eq. (A4), we define

$$
d_{jk} = d_j(\varepsilon_1) d_k(\varepsilon_2) \ , \tag{B15}
$$

and $E_{+\pm}$, M_{jk}, etc., are defined in Eq. (B6). The approximated form $A_{0\nu 2}(j)$ for $j>0$ is obtained by assuming Eq. (B8).

The first angular part $B_{0\nu 2}(0)$ is

$$
\begin{aligned}
B_{0\nu 2}(0) = {} & -\frac{1}{10}\left[\frac{p_1 p_2}{\varepsilon_1 \varepsilon_2} \right] \{ (1+T_1)^{2\gamma_2}(1+T_2)^{2\gamma_1} d_{10}[\,|Z_{21}|^2 \alpha_R(-2,2;-1,1) + |Z_{22}|^2 \alpha_R(-2,2;1,-1)] \\
& + (1+T_1)^{2\gamma_1}(1+T_2)^{2\gamma_2} d_{01}[\,|Z_{21}|^2 \alpha_R(-1,1;-2,2) + |Z_{22}|^2 \alpha_R(-1,1;2,-2)]\} \\
& - 2[(1+T_1)(1+T_2)]^{\gamma_1+\gamma_2}(d_{01}d_{10})^{1/2}\{[E_{++}\alpha_R(-2,-1;-1,-2) + E_{--}\alpha_R(2,1;1,2)]\,|Z_{21}|^2 \\
& - [E_{+-}\alpha_R(-2,-1;1,2) + E_{-+}\alpha_R(2,1;-1,-2)]\,|Z_{22}|^2\} \ ,
\end{aligned}
\tag{B16}
$$

where, by using $D_{\kappa,\lambda}$ in Eq. (3.10), we define

$$\alpha_R(\kappa,\lambda;\sigma,\eta)=D_{\kappa,\lambda}(\varepsilon_1)D_{\sigma,\eta}(\varepsilon_2)C(\kappa,\lambda;\sigma,\eta)\ ,$$
$$C(\kappa,\lambda;\sigma,\eta)=\cos[\Delta^C_\kappa(\varepsilon_1)-\Delta^C_\lambda(\varepsilon_1)+\Delta^C_\sigma(\varepsilon_2)-\Delta^C_\eta(\varepsilon_2)]\ . \tag{B17}$$

The second angular part $C_{0\nu 2}(0)$ is

$$C_{0\nu 2}(0)=\frac{3}{20}\left[\frac{p_1p_2}{\varepsilon_1\varepsilon_2}\right][(1+T_1)(1+T_2)]^{\gamma_1+\gamma_2}(d_{01}d_{10})^{1/2}$$
$$\times\{|Z_{21}|^2[\alpha_R(-2,+1;-1,+2)+\alpha_R(+2,-1;+1,-2)]$$
$$-|Z_{22}|^2[\alpha_R(-2,+1;+1,-2)+\alpha_R(+2,-1;-1,+2)]\}\ . \tag{B18}$$

The approximated forms $B_{0\nu 2}(1)$ and $C_{0\nu 2}(1)$ are obtained by assuming

$$\alpha_R(\kappa,\lambda;\sigma,\eta)=1\ . \tag{B19}$$

The approximations $A_{0\nu 2}(1)$, $B_{0\nu 2}(1)$, and $C_{0\nu 2}(1)$ in this paper are the same as in Eq. (C.4.15) of I. Note that in the $p\to 0$ limit, $A_{0\nu 2}(0)$ and $B_{0\nu 2}(0)$ are finite and $C_{0\nu 2}(0)$ is proportional to $(\alpha Z)^2$ [cf. Figs. 6.8 and 6.9 of I]. The reason for $A_{0\nu 2}(0)$ is clear from the leading terms such as $D_k(\varepsilon)\to 1$ and $pD_{-k}(\varepsilon)\to\alpha Zm_e$ in Eqs. (A22)–(A27). In the case of $B_{0\nu 2}(0)$, the first term in Eq. (B16) becomes negligible in the $p\to 0$ limit because of the small $pD_{-k}(\varepsilon)$ and Eq. (A30), but the second term is finite, because the limiting values are

$$[\Delta^C_{-2}(\varepsilon)-\Delta^C_{-1}(\varepsilon)]\to(\gamma_1-\gamma_2+\tfrac{1}{2})\pi\ ,$$
$$[\Delta^C_2(\varepsilon)-\Delta^C_1(\varepsilon)]\to(\gamma_1-\gamma_2+\tfrac{1}{2})\pi\ . \tag{B20}$$

For $C_{0\nu 2}(0)$, the momentum p is eaten by $pD_{-k}(\varepsilon)\to\alpha Zm_e$ again and the $p\to 0$ limits of the phase differences are

$$[\Delta^C_{-2}(\varepsilon)-\Delta^C_1(\varepsilon)]\to(\gamma_1-\gamma_2+1)\pi\ ,$$
$$[\Delta^C_2(\varepsilon)-\Delta^C_{-1}(\varepsilon)]\to(\gamma_1-\gamma_2)\pi\ . \tag{B21}$$

Note that in this notation, the total half-life is expressed by

$$[T_{0\nu}(0^+\to 2^+)]^{-1}$$
$$=\left[\frac{2}{\ln 2}\right]N_{0\nu 2}\int_0^T dT_1\int_0^{T-T_1}dT_2\,\delta(T-T_1-T_2)$$
$$\times A_{0\nu 2}(0)\ . \tag{B22}$$

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[11]In Ref. 1, the spectra in Figs. 6.2 and 6.4 should be exchanged; namely, Figs. 6.2 and 6.4 correspond to the $(0^+\to 2^+)$ and $(0^+\to 0^+)$ transitions, respectively. We would like to express our thanks to Dr. M. K. Moe at Irvine for pointing out this mistake.

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[17]For more detailed lists and also T_1 dependences of $A_{2\nu}(0)$ and $A_{0\nu B}(0)$ for the fixed T_2, send correspondence to T. Kotani at Osaka Gakuin University.