Time-delay analysis of the interpretation of narrow Σ-hypernuclear states as bound states in the continuum

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The interpretation of the observed narrow Σ-hypernuclear states in terms of poles of the S matrix in the second quadrant of the complex wave-number plane is considered. The value obtained for the time delay originated by the corresponding optical potential reinforces such an interpretation.

I. INTRODUCTION

The possibility of associating the observed narrow Σ-hypernuclear states with poles of the S matrix lying in the second quadrant of the complex wave-number plane was suggested a few years ago.[1] Since Imk is positive for such poles, the corresponding external wave function would be exponentially decreasing with the distance, a typical feature of bound states; as long as those poles are below the bisector of the second quadrant, the real part of the energy is positive. These characteristics of the poles led to Gal, Toker, and Alexander to call them unstable bound states (UBS's) embedded in the continuum. Second quadrant poles appear, for a complex potential, as coming from the third quadrant and crossing the real axis as the absorptive part of the potential increases.[1-3] Using a separable potential model, Johnstone and Thomas[4] have found $1p$ poles just above the negative real semiaxis for light Σ^0 hypernuclei.

The explanation of sharp Σ-hypernuclear states as UBS's has been, however, questioned in a paper by Morimatsu and Yazaki,[5] showing in the simple case of a square-well potential that positive energy UBS's do not produce observable effects neither in the scattering nor in the formation process of Σ-hypernuclear states. Moreover, from general principles of reaction theory, Feshbach[6] has recently demonstrated that UBS's could hardly be produced in the (K,π) reaction leading to hypernuclear states. Against this statement Gal[7] argues that negative-energy UBS's (poles above the bisector in the second quadrant) certainly produce an observable effect in the calculations of Morimatsu and Yazaki.

The issue seems far from being settled. Aiming to contribute to the clarification of the situation, we discuss in this paper the time delay originated by the optical potential in the cases considered in Ref. 5. Time delay is a concept introduced by Eisenbud[8] in the analysis of the scattering process undergone by a wave packet and used by Wigner[9] to interpret in terms of causality an inequality satisfied by the derivative of the phase shift as a function of the energy.

We propose in Sec. II a plausible generalization to complex potentials of the definition of time delay currently accepted for real potentials. In Sec. III we compute the time delay for the square-well potentials discussed by Morimatsu and Yazaki[5] in connection with narrow Σ-hypernuclear states. A very large positive value of the generalized time delay is found in the case of a UBS embedded in the continuum. This fact supports the interpretation of Gal and collaborators.[1] The time delay turns out to be also positive for negative-energy UBS's. Section IV is devoted to the discussion of the possibility, considered by several authors,[1,6,7] of negative-energy UBS's becoming positive-energy ones as the absorptive part of the complex potential increases. Finally, Sec. V contains some conclusions.

II. GENERALIZED TIME DELAY

For a spherically symmetric real potential, the time delay suffered by a wave of angular momentum l is given either in terms of the derivative of the phase shift δ_l with respect to the energy E,

$$\Delta t = 2\hbar d\delta_l(E)/dE \ , \tag{2.1}$$

or in terms of the S matrix,

$$\Delta t = -i\hbar S_l^* dS_l/dE \ . \tag{2.2}$$

The first definition was deduced[8-10] from a wave packet analysis. The second one was obtained by Smith,[11] from the steady-state solution of the time-independent Schrödinger equation for a single energy, and by Jauch and Marchand,[12] in a time-dependent formalism. Both time-independent and time-dependent approaches to Eq. (2.2) are equivalent, as shown by Bollé and Osborn.[13]

The familiar relation between the S matrix and the phase shift,

$$S_l = \exp\{2i\delta_l\} \ , \tag{2.3}$$

allows one to check immediately that definition (2.2) implies (2.1), and vice versa, in the case of a real potential (real δ_l). However, if one considers a complex potential, an ambiguity appears in the definition of time delay through a naive generalization of Eq. (2.1) or (2.2). As is well known, the scattering formalism can be extended to complex potentials by merely considering complex phase shifts, $\delta_l + i\eta_l$. The time delay defined in analogy with Eq. (2.1) turns out to be complex,

$$\Delta t_1 = 2\hbar d(\delta_l + i\eta_l)/dE \ , \tag{2.4}$$

but the wave packet analysis reveals that its real part could be interpreted as a time delay. On the other hand, direct generalization of Eq. (2.2) gives a time delay

$$\Delta t_2 = \exp\{-4\eta_l\} 2\hbar d(\delta_l + i\eta_l)/dE \tag{2.5}$$

different from Δt_1, as long as the S matrix is not unitary,

$$\Delta t_2 = |S_l|^2 \Delta t_1 \ . \tag{2.6}$$

In order to remove that ambiguity, let us apply to our case of the complex potential the method developed by Smith in his classical paper.[11] Let T and V denote, respectively, the kinetic and potential (energy) operators. For simplicity we assume a finite-range potential non-dependent on the energy. From the derivative of the Schrödinger equation

$$(T+V-E)\psi = 0 \tag{2.7}$$

with respect to the energy,

$$(T+V-E)\partial\psi/\partial E - \psi = 0 \ , \tag{2.8}$$

one obtains

$$\psi^* T \partial\psi/\partial E + \psi^*(V-E)\partial\psi/\partial E - \psi^*\psi = 0 \tag{2.9}$$

that, by using the complex conjugate of Eq. (2.7), can be written in the form

$$\psi^* T \partial\psi/\partial E - (\partial\psi/\partial E) T\psi^* + (V-V^*)\psi^*\partial\psi/\partial E - \psi^*\psi = 0 \ , \tag{2.10}$$

or, explicitly in coordinate representation,

$$-(\hbar^2/2\mu)\nabla[\psi^*\nabla(\partial\psi/\partial E) - (\partial\psi/\partial E)\nabla\psi^*] + (V-V^*)\psi^*\partial\psi/\partial E - \psi^*\psi = 0 \ . \tag{2.11}$$

By integrating in a sphere of radius R and using the Gauss theorem one gets

$$-(2\pi\hbar^2 R^2/\mu)[\psi^*\partial(\partial\psi/\partial E)/\partial r - (\partial\psi/\partial E)\partial\psi^*/\partial r]_{r=R} + \int_{r<R} d\mathbf{r}(V-V^*)\psi^*\partial\psi/\partial E - \int_{r<R} d\mathbf{r}\,\psi^*\psi = 0 \ . \tag{2.12}$$

Let us consider a scattering solution of the Schrödinger equation corresponding to a given angular momentum. Its radial part reads asymptotically

$$\psi \simeq \tfrac{1}{2}(\mu/\hbar k\pi)^{1/2}\frac{1}{r}[\exp(-ikr) - S\exp(ikr)] \ , \tag{2.13}$$

where the normalization constant has been fixed so as to have unit inward flux, neglecting terms $O(1/r)$, over a spherical surface of radius r. For R tending to infinity, Eq. (2.12) gives

$$-i\hbar S^* dS/dE + \lim_{R\to\infty}\left\{(\mu/\hbar k)R(1+|S|^2) - (\mu/\hbar k^2)\,\mathrm{Im}[S\exp(2ikR)] + \int_{r<R} d\mathbf{r}(V-V^*)\psi^*\partial\psi/\partial E - \int_{r<R} d\mathbf{r}\,\psi^*\psi\right\} = 0 \ . \tag{2.14}$$

By separating real and imaginary parts in the preceding equation one obtains

$$\mathrm{Re}(-i\hbar S^* dS/dE) = \lim_{R\to\infty}\left\{\int_{r<R} d\mathbf{r}\,\psi^*\psi - (\mu/\hbar k)R(1+|S|^2) + \int_{r<R} d\mathbf{r}(i\psi\partial\psi^*/\partial E - i\psi^*\partial\psi/\partial E)\,\mathrm{Im}V\right\} + (\mu/\hbar k^2)\,\mathrm{Im}[S\exp(2ikR)] \ , \tag{2.15}$$

$$\mathrm{Im}(-i\hbar S^* dS/dE) = -\lim_{R\to\infty}\left\{\frac{\partial}{\partial E}\int_{r<R} d\mathbf{r}\,\psi^*\psi\,\mathrm{Im}V\right\} . \tag{2.16}$$

The left-hand sides of Eqs. (2.15) and (2.16) are, respectively, the real and imaginary parts of the time delay obtained by direct generalization of Eq. (2.2). In what follows we shall examine the physical meaning of the right-hand sides of those equations.

In the case of a real potential the average time of residence in a region, for a particle described by a steady-state wave function, is the integrated density divided by the total flux in (or out). The time delay originated in that region is then defined as the difference of times of residence with and without interaction.[11] Equation (2.15) reads in that case

$$-i\hbar S^* dS/dE = \lim_{R\to\infty}\left\{\int_{r<R} d\mathbf{r}\,\psi^*\psi - (\mu/\hbar k)2R + (\mu/\hbar k^2)\,\mathrm{Im}[S\exp(2ikR)]\right\} . \tag{2.17}$$

The first and second terms on the right-hand side can be immediately recognized as the above-mentioned times of residence in the sphere of radius R. Their difference, for R larger than the range of the interaction, does not depend on R and turns out to be finite and equal to the time delay. The last term oscillates as R tends to infinity. It can be eliminated[11] by taking the average value of the integral I in the sphere,

$$\langle I \rangle = \lim_{R \to \infty} \frac{1}{R} \int_R^{2R} dR' I(R') \ . \tag{2.18}$$

For a complex potential the interpretation is not so evident. First, the in and out fluxes are not equal, due to absorption in the interaction region. Second, the time delay cannot be defined merely as the increase (with respect to the no-interaction case) of integrated density, since such increase is the resultant effect of two different causes: the delay originated by the real part and the absorption due to the imaginary part of the potential. The first term on the right-hand side of Eq. (2.15) obviously represents the integrated density, in a sphere of radius R, of the particle suffering interaction. The second one corresponds to such integrated density for a free particle with in (and out) flux equal to the mean of the in and out fluxes of the interacting particle. (Alternatively, it could be interpreted as the integrated density for a free particle that flows in and out like the particle described asymptotically by ψ and is partially absorbed at the origin.) The time delay is not the difference of those two terms since the first one includes the contribution to the time of residence coming from absorbed particles. But this contribution is compensated by the third term, that, written in the form

$$-\left[-\int_{r<R} d\mathbf{r}\, \psi^* \psi 2\, \mathrm{Im} V \partial(\arg\psi)/\partial E\right] , \tag{2.19}$$

can be interpreted as an average time of residence $\hbar\partial(\arg\psi)/\partial E$ weighted by the local probability of absorption $-(1/\hbar)\psi^*\psi 2\,\mathrm{Im}V$. The fourth term disappears by averaging as indicated in Eq. (2.18). Therefore, it seems reasonable to define

$$\Delta t = \mathrm{Re}(-i\hbar S^* dS/dE) \tag{2.20}$$

as the time delay for the complex potential.

The interpretation of the right-hand side of Eq. (2.16) offers no difficulty. From the continuity equation, integrated in the sphere of radius R, one immediately sees that

$$-(1/\hbar)\int_{r<R} d\mathbf{r}\, \psi^* \psi 2\, \mathrm{Im} V \tag{2.21}$$

is the number of particles absorbed inside the sphere. Therefore, $\mathrm{Im}(-iS^* dS/dE)$ corresponds to the derivative of the absorption coefficient with respect to the energy, as it should be.

Our definition of time delay, Eq. (2.20), can be immediately extended to negative energies. In the case of real potential such an extension does not provide much information, since Δt is zero for all negative energies with the exception of bound and virtual-state values. For these values S is not an analytical function of the complex variable E and, consequently, its derivative does not exist. It is interesting, however, to notice the difference between the bound-state and the virtual-state cases. The value of $\arg S$ changes suddenly from 0 to π at the bound states and from π to 0 at the virtual states as the energy increases through real values. The extended time delay, written in the form

$$\Delta t = \hbar\, |S|^2 d(\arg S)/dE \ ,$$

is seen to become $+\infty$ for bound states and $-\infty$ for virtual states. In the case of a complex potential there are no poles of the S matrix on the negative energy semiaxis, but in its neighborhood (unstable bound states). The time delay is therefore well defined for negative energies and must present maxima at the (real) energies associated to UBS's.

III. THREE EXAMPLES

In their analysis of the viability of the interpretation of narrow Σ-hypernuclear states as second quadrant poles (in the wave-number complex plane) of the S matrix, Morimatsu and Yazaki[5] have considered three different square-well potentials,

$$U(r) = U\theta(R-r) \ . \tag{3.1}$$

The range R and the (complex) intensity

$$U = V + iW \tag{3.2}$$

are given, in dimensionless units, in Table I (corresponding to Table 1 of Ref. 5) together with the position of one of the poles of the S matrix for each potential.

The square-well model is analytically solvable. The scattering function for angular momentum l reads

$$S_l = -\frac{\alpha h_l^{2'}(\alpha) j_l(\beta) - \beta h_l^2(\alpha) j_l'(\beta)}{\alpha h_l^{1'}(\alpha) j_l(\beta) - \beta h_l^1(\alpha) j_l'(\beta)} \ . \tag{3.3}$$

Here j_l', T, h_l^1 and h_l^2 are the spherical Bessel functions of the first and third kind;[14] the primes denote derivatives with respect to their arguments. We have represented by α and β the dimensionless wave numbers

$$\alpha = kR, \quad \beta = (\alpha^2 - 2\mu U R^2/\hbar^2)^{1/2} \ , \tag{3.4}$$

respectively, outside and inside the potential. The time delay for the l wave can be obtained immediately from Eq. (2.20) written as

$$\Delta t = \mathrm{Re}[-i\mu R^2 |S_l|^2 (dS_l/d\alpha)/\hbar\alpha S_l] \ . \tag{3.5}$$

In Figs. 1–3 the time delay is shown, as a function of the dimensionless wave number α, for the three typical cases discussed in Ref. 5. For convenience in drawing we have in fact represented $\Delta t/|S|^2$. Case a presents a quasibound state associated to a pole in the second quadrant of the k plane, above the bisector. The time delay, in Fig. 1, clearly shows a maximum in coincidence with the peak in the strength function reported by Morimatsu and Yazaki. Case b shows a conventional resonance in a real potential. The time delay, in Fig. 2, reproduces the resonant structure. Case c corresponds to a pole in the second quadrant, below the bisector. This is the kind of poles that Gal and collaborators[1] proposed to be associat-

TABLE I. Intensities of the square-well potentials and positions of the poles considered in the text.

Case	$V2\mu R^2/\hbar^2$	$W2\mu R^2/\hbar^2$	RekR	ImkR
a	−12.98	−1.00	−0.251	1.341
b	−7.99	0	0.795	−0.225
c	−4.49	−4.49	−2.200	0.017

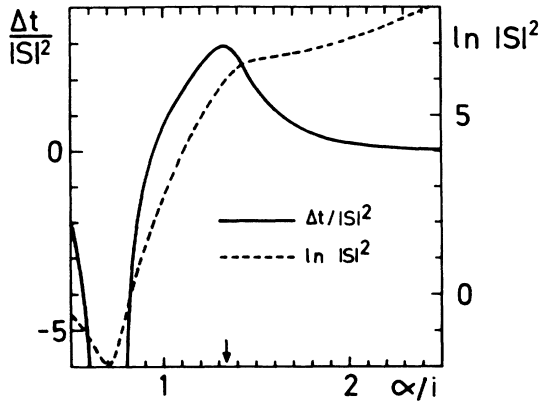

FIG. 1. Time delay at negative energies in the neighborhood of a decaying bound state. The small arrow indicates the imaginary part of the pole (case *a*) reported by Morimatsu and Yazaki (Ref. 5). The unit on the left vertical axis is $\mu R^2/\hbar$.

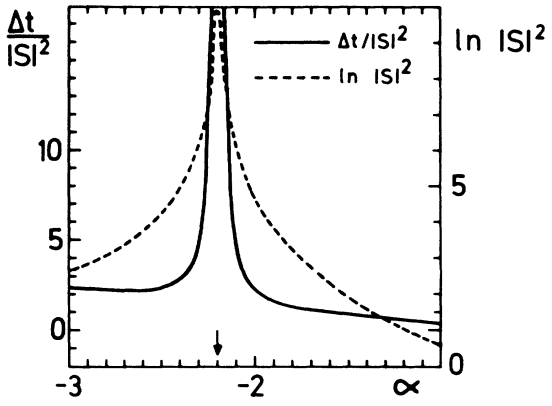

FIG. 3. Time delay at positive energies in the neighborhood of an UBS embedded in the continuum. The real part of the pole (case *c*) found in Ref. 5 is indicated by the arrow.

ed to Σ-hypernuclear states. The time delay, represented in Fig. 3, shows a very narrow structure (as the pole lies near the real axis) reaching values of $\Delta t/|S|^2$ so high as 130 in the units shown. Such a time delay suggests a very long lived state in accordance with the interpretation given in Ref. 1.

As discussed by Gal and collaborators,[1] no peak, neither in the elastic nor in the reaction cross sections, is to be expected at energies in correspondence with case *c* poles, i.e., poles of the S matrix at points k_c such that $\mathrm{Re}(k_c)<0$, $\mathrm{Im}(k_c)>0$, $|\mathrm{Im}(k_c)| \ll |\mathrm{Re}(k_c)|$. The absence of structure in the cross section, in spite of the fact that the phase shift varies rapidly with the energy and passes the value $\pi/2$, can be understood if one considers that the wave number, and consequently also the S matrix, is a double-valued function of the energy, the two values of k being opposite and those of S reciprocal to each other, according to the well-known property

$$S(k)S(-k)=1 \ . \tag{3.6}$$

In the calculation of the cross section, the value of S to be taken is that corresponding to outgoing waves, i.e., to positive wave number, $k \simeq \mathrm{Re}(-k_c)$. Since the S matrix presents a zero at $-k_c$, the value of $|S(k)|$ is very small and the only effect that could be expected is a broad bump in the elastic cross section for the corresponding partial wave. This is shown in Fig. 4. When all angular momentum waves are summed up, the resulting cross sections decrease monotonously as the energy increases and no effect is observed.

The situation should be very different if UBS's embedded in the continuum would correspond to poles in the first quadrant of the wave number plane. This happens, for instance, in the case of the potential being the complex conjugate of that in case *c*. According to the property[2]

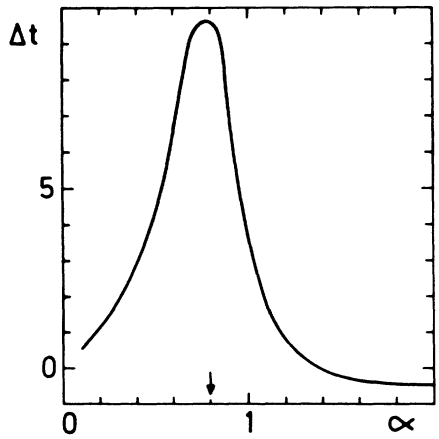

FIG. 2. Time delay at positive energies near a resonance. The arrow points at the real part of the pole (case *b*) reported in Ref. 5.

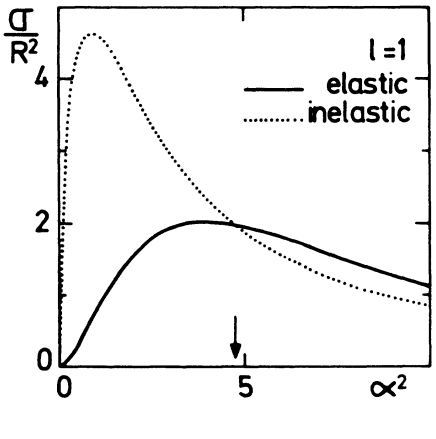

FIG. 4. *P*-wave elastic and inelastic cross sections for the potential considered in case *c* of Ref. 5. The arrow indicates the value of the energy of the UBS embedded in the continuum.

$$S_l^*(-k^*;U^*)=S_l(k;U)\ , \tag{3.7}$$

there is a pole at $-k_c^*$ and $|S|$ presents a high peak at the corresponding physical energy. This produces a narrow structure in the elastic cross section. [Notice that in the case we are now discussing, namely $\mathrm{Im}(U)>0$, the potential is *productive,* not *absorptive.* Conventional expressions could give negative values for the inelastic and total cross sections. The modifications to be introduced, in order to get the correct expressions for these quantities, are straightforward.]

IV. INCREASING ABSORPTION

Gal[7] distinguishes two kinds of poles in the second quadrant, according to their location above or below the bisector. Bound states originate class (i) poles and resonances class (ii) poles as the potential becomes absorptive. The question arises if class (i) and class (ii) poles change their character as the absorption increases, i.e., if poles in the second quadrant cross the bisector as the imaginary part of the potential increases in absolute value. Class (ii) poles were found for the first time by Joffily[2] in the case of a complex square-well potential. The poles did not cross the bisector as the imaginary part of the potential tended to infinity. However, the real part of the potential considered in Ref. 2 was not strong enough to produce class (i) poles. That particular case of a square-well potential of constant real part and variable imaginary part was again discussed by Cassing, Stingl, and Weiguny,[3] but they did not consider either class (i) poles. In Fig. 5 we have represented the trajectories followed by s-wave poles of a square-well potential of constant real part $V=-25$, in units $\hbar^2/2\mu R^2$, and variable imaginary part W. The pole associated with one of the bound states, namely the most tightly bound one, remains a class (i) pole for any (negative) value of W. The other bound-state pole crosses the bisector but never approaches the real axis. The motion of the two poles coming from the third quadrant has also been represented: They never become class (i) poles.

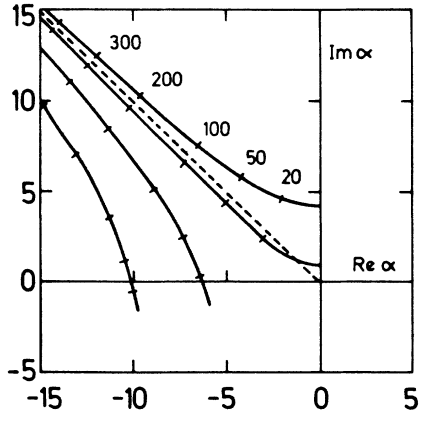

FIG. 5. S-wave poles of the S matrix for a complex square-well potential. The real part of the potential has been kept fixed, $V=-25$ in units $\hbar^2/2\mu R^2$. The imaginary part W has been varied from 0 to -400, in the same units. The numbers beside the first trajectory correspond to $|W|$. The bisector of the second quadrant has also been represented (dashed straight line).

The fact that all poles tend to infinity asymptotically to the bisector as W tends to infinity was proved in Ref. 2. The behavior of the poles for large values of W can be obtained by making use of approximate expressions of the reduced logarithmic derivatives of the Bessel functions. To the lowest order of approximation one obtains, in the case of angular momentum l,

$$\alpha_s\simeq(2\mu U)^{1/2}(R/\hbar)[1+(j_{\lambda,s})^2\hbar^2/4\mu UR^2]\ , \tag{4.1}$$

where $j_{\lambda,s}$ are the zeros of the Bessel function,[14] $J_\lambda(j_{\lambda,s})=0$, $\lambda=l+\frac{1}{2}$. The label s corresponds to the ordering, by increasing energy, of the bound states and resonances associated to the poles for $W=0$. By separating real and imaginary parts in the potential U, Eq. (4.1) can be written in the form

$$\alpha_s\simeq(2\mu iW)^{1/2}(R/\hbar)\times\{1-i[V+(j_{\lambda,s})^2\hbar^2/2\mu R^2]/2W\}\ , \tag{4.2}$$

showing explicitly that, as $W\to-\infty$, the bisector is approached from below by all poles except those with label s such that

$$V+(j_{\lambda,s})^2\hbar^2/2\mu R^2<0\ ,$$

which tend to the bisector from above.

V. CONCLUSIONS

We have suggested in the present work a definition of time delay for complex potentials as a plausible generalization of that for real ones. The definition can also be extended to negative energies. Applying our definition to the square-well potentials discussed by Morimatsu and Yazaki[5] we have obtained a time delay that presents a very narrow peak at the energy associated with a pole of the S matrix in the second quadrant. This result is in favor of the interpretation, given by Gal, Toker, and Alexander, of narrow Σ-hypernuclear states as being due to second quadrant poles of the S matrix for the optical potential, in spite of the fact that such poles do not produce any structure in the cross section.

It is important to notice that the time delay becomes negative for poles in the third quadrant just below the real axis. If we consider a potential slightly weaker than that studied in case c of Ref. 5, taking for instance -4.40 instead of -4.49 (cf. case c in Table I) for both the real and imaginary parts of the potential, the pole appears at $-2.20-i0.016$ and the time delay turns out to be nearly the opposite of that shown in Fig. 3. As is well known,[15] causality forbids one to have physical states with large negative values for the time delay. That sensitivity of UBS's to the intensity of the potential was already men-

tioned by Kawai and Iseri[16] in his paper on absorption of neutron partial waves by nuclear optical potentials, where a similar situation was found.

We have also shown in a simple example that, contrary to speculation by other authors, positive and negative-energy UBS's do not interchange their character as a consequence of increasing absorption. Instead, all of them become broader and broader, their lifetime tending to zero.

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