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Ab initio approach to the deuteron in the Skyrme-Witten model

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An *ab initio* calculation of the deuteron in the Skyrme-Witten model is presented. Without assuming any *a priori* symmetries of the configuration, two $B=1$ skyrmions are placed on a three-dimensional lattice and a numerical relaxation procedure is used to find the minimum energy configuration. Using no free parameters, the well depth, range, mass, and quadrupole moment of the configuration are found to be in reasonable agreement with deuteron properties. Although the product ansatz is used initially, it is discovered that the relaxed configuration no longer appears to be that of a product of chiral fields. Thus the calculation seems to justify the symmetry assumptions of Braaten and Carson in their derivation of the deuteron quantum numbers.

I. INTRODUCTION

Any fundamental theory of the strong interaction should be applicable to the realm of nuclear physics. Unfortunately, the quantum chromodynamic coupling constant becomes large at nuclear energy scales, thereby invalidating the usual perturbative expansion when investigating low-energy strong processes. Thus although QCD is generally accepted as the underlying theory describing the strong interaction, its predictive abilities for low-energy processes such as nuclear physics are severely limited.

In an attempt to find a suitable expansion parameter with which to implement perturbation theory, 't Hooft generalized quantum chromodynamics (QCD) from an SU(3) to an SU(N_c) gauge theory, where N_c is the number of colors.[1] Assuming confinement to be operative for any value of N_c, 't Hooft discovered that for a large number of colors, $1/\sqrt{N_c}$ emerges as a viable expansion parameter; QCD then reduces to a theory of mesons and glueballs at all energy scales. Expanding upon this approach, Witten showed that for large N_c, baryons emerge as solitons in a weakly coupled theory of mesons.[2]

This view of the strong interaction—and of baryons in particular—had already been advanced by Skryme[3] before the invention of QCD. Skyrme added a particular quartic term to the minimal SU(2)⊗SU(2) chirally invariant Lagrangian of the nonlinear sigma model in order to stabilize the solitonic solutions:

$$\mathcal{L}=\frac{F_\pi^2}{16}\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)-\frac{1}{32e^2}\mathrm{Tr}[(U^\dagger\partial_\mu U, U^\dagger\partial_\nu U)^2]\ . \tag{1}$$

Here F_π is the pion decay constant ($\simeq 186$ MeV) and e is a parameter introduced by Skyrme. The finite energy configurations $U(x)\epsilon$SU(2) fall naturally into topological sectors labelled by an integer, identified by Witten as the baryon number B.[4]

In addition to the large N_c motivation, the connection between the Skyrme model and QCD is reinforced by Witten's justification of the Wess-Zumino anomaly term in effective chiral Lagrangians.[4] Witten showed that just such a term is required in chiral Lagrangians in order that they possess only those symmetries respected by QCD. This turns out to be crucial, because although the anomaly vanishes in a two-flavor model, its addition to the three-flavor Skyrme Lagrangian allows the solitons to be quantized as fermions (which is essential if they are to be identified as baryons).[4]

The ability of the Skyrme model to reproduce low-energy phenomenology has been encouraging. Adkins, Nappi, and Witten used the spherically symmetric hedgehog ansatz,

$$U=e^{iF(r)\hat{r}\cdot\tau} \tag{2}$$

to derive properties of the nucleon and delta.[5] Adjusting F_π and e in order to correctly reproduce the nucleon and delta masses, they found results in $\simeq 30\%$ agreement with known values, certainly reasonable for a model with only two free parameters. More details of the Skyrme model are presented in the surveys listed in Ref. 6.

It is natural to consider extending these results to $B>1$ solutions. Indeed, if the Skyrme-Witten model encompasses low-energy strong-interaction phenomena, it should provide a new approach to nuclear physics. In particular, if such an approach has any validity, it must

at least reproduce the simplest of nuclear systems, the deuteron. Moreover, since F_π and e can be taken as fixed by the $B=1$ results,[5] we have the framework for an *ab initio* calculation of the deuteron.

The primary obstacle to describing a deuteron in the Skyrme-Witten model is mathematical. In particular, a way is needed to find the symmetry properties of the minimum energy static solutions for any sector with $B \geq 2$. Unfortunately the symmetry in the $B=1$ sector, though relatively simple, is not applicable to the $B=2$ sector. Specifically, the $B=1$ hedgehog ansatz has the very useful symmetry known as *equivariance:* It is invariant under combined spatial and isospin transformations,

$$U(r_i) \rightarrow A U(R_{ij} r_j) A^\dagger . \tag{3}$$

When quantized, this symmetry requires the spin and isospin quantum numbers to be equal; clearly, then, any equivariant solution cannot produce the deuteron, with $I=0$, $J=1$. One is therefore presented with a framework in which the deuteron is expected to emerge, while the configuration from which it springs is unknown.

Among the many attempts to find a description of the deuteron, the work of Jackson, Jackson, and Pasquier seems the most satisfactory.[7] Using the fact that the point-wise matrix product of two $B=1$ configurations has $B=2$, they constructed an approximate solution from the product of two $B=1$ skyrmions located a distance $R = |\mathbf{x}_1 - \mathbf{x}_2|$ apart:

$$U_{B=2} = A_1 U(\mathbf{r}-\mathbf{x}_1) A_1^\dagger A_2 U(\mathbf{r}-\mathbf{x}_2) A_2^\dagger . \tag{4}$$

They then identified the classical energy of this product ansatz, less two skyrmion masses, as the potential energy of the skyrmion-skyrmion interaction:

$$V(R,W) = E(R,W) - 2M_{B=1} , \tag{5}$$

where $W = A_1^\dagger A_2$ is the relative isospin orientation of the two skyrmions. In the absence of any relative rotation, the interaction was discovered to be everywhere *repulsive;* if one skyrmion were rotated by π about an axis perpendicular to the interskyrmion axis, however, an attractive potential resulted. The effect of such a rotation is shown in Fig. 1. Using these results, Braaten and Carson[8] analyzed the properties of this $B=2$ product configuration and obtained the correct quantum numbers for the deuteron.

There are both promising and disturbing aspects to this potential-energy approach. For large separation, $V(R,W)$ agrees with an interpretation in terms of single-pion exchange; in addition, the two skyrmions maintain their individual identities, since for large R the configuration of one skyrmion will not be greatly affected by the weak asymptotic field of the other. There are, however, problems with a product ansatz describing the deuteron. When there is a large overlap between the two skyrmions, they fuse into a single configuration and are not easily resolved into individual skyrmions with centers x_i and isospin orientation A_i. Indeed, for small separations, large distortions of the skyrmions are to be expected since the interaction energy may be comparable to the skyrmion rest mass. Thus the validity of the product ansatz breaks down for small R, and the concept of the two-skyrmion potential loses much of its relevance.[9] Moreover, the product ansatz is inherently ambiguous, since in general

$$U(x_1)U(x_2) \neq U(x_2)U(x_1) ,$$

and there seems no way to choose between the two.

Soon after the work of Jackson *et al.* an important contribution was made by Sommermann, Wyld, and Pethick,[10] who compared the effects of a product ansatz to a $B=2$ numerical solution. The baryon number distribution was discovered to be quite different in the two approaches. The correct deuteron solution can be distinguished, then, by its baryon number distribution, which should resemble the latter solution of Sommerman *et al.*

The present investigation was undertaken in order to discover what symmetries the exact deuteron configuration possesses. Assuming no underlying symmetries, two skyrmions are placed on a lattice and allowed to relax to the minimum value of the static energy.

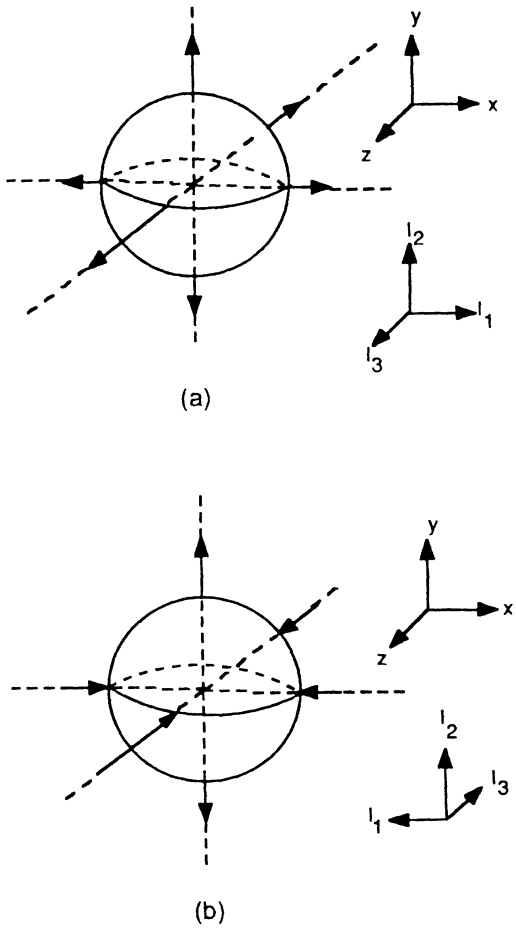
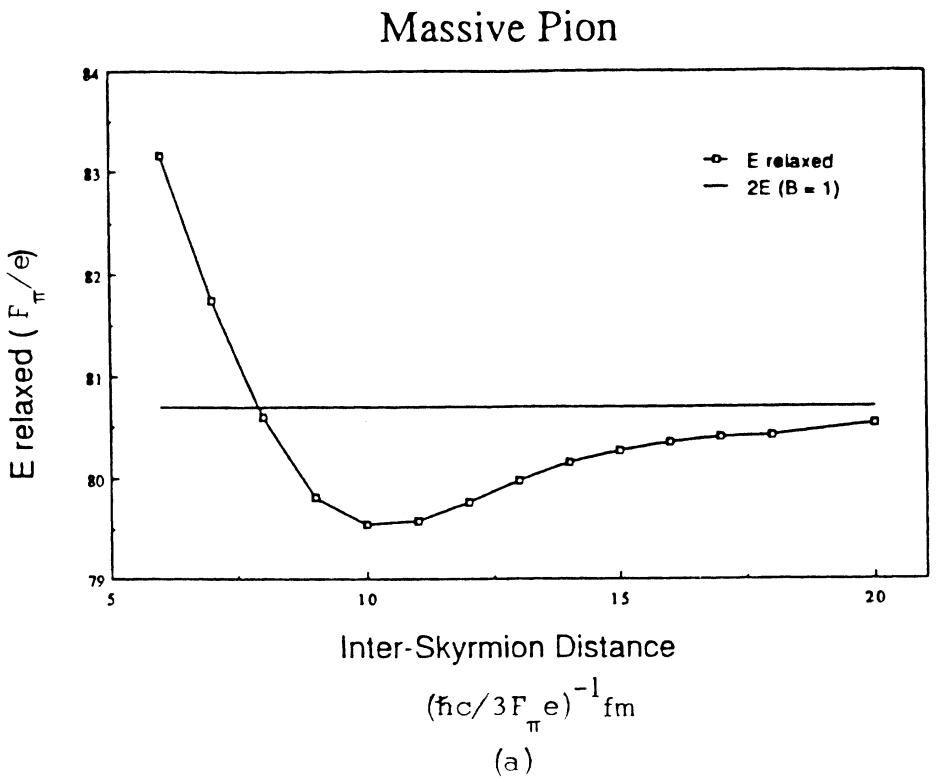


FIG. 1. Hedgehog configurations. The spheres represent contours of equal chiral angle, the arrows, the isospin direction. (a) Standard hedgehog; (b) Hedgehog with an isorotation of π about the I_2 axis.



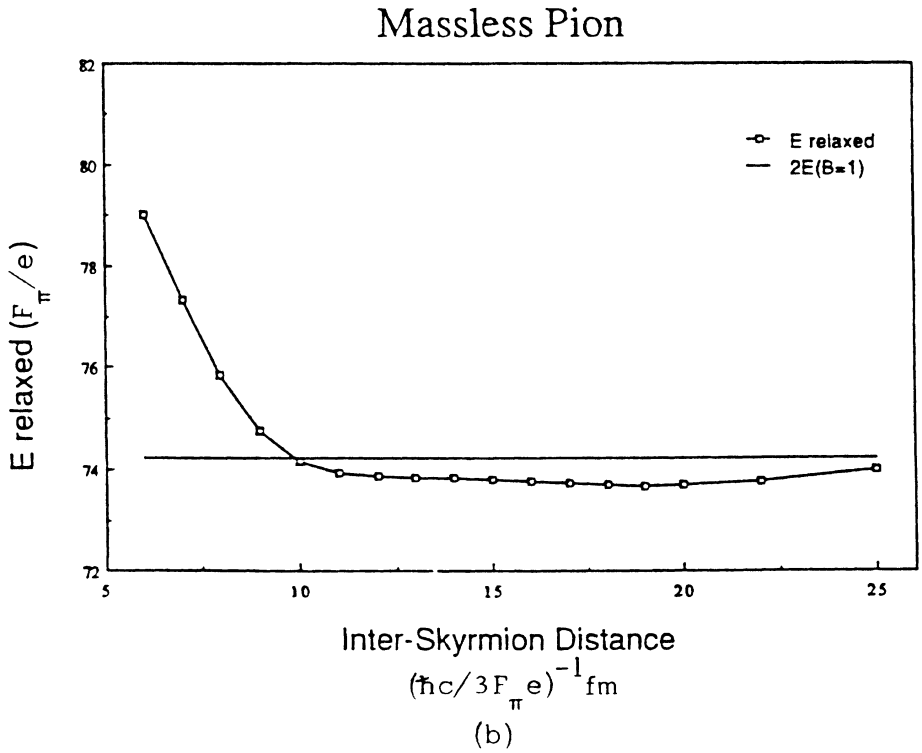


FIG. 2. Variation of the static biskyrmion mass with interskyrmion distance for (a) $m_\pi = 0$; (b) $m_\pi = 138$ MeV.

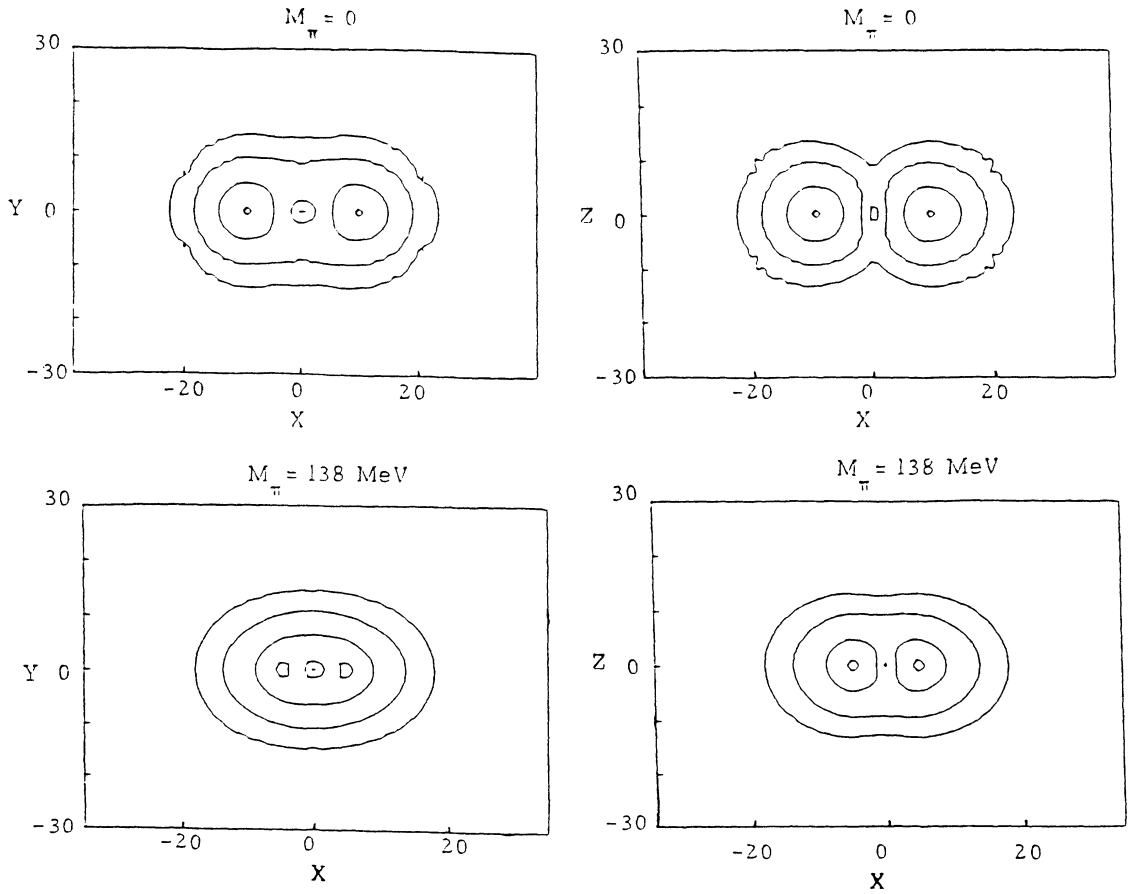


FIG. 3. Contours of constant baryon density for the relaxed configurations (a) $m_\pi = 0$ and (b) $m_\pi = 138$ MeV.

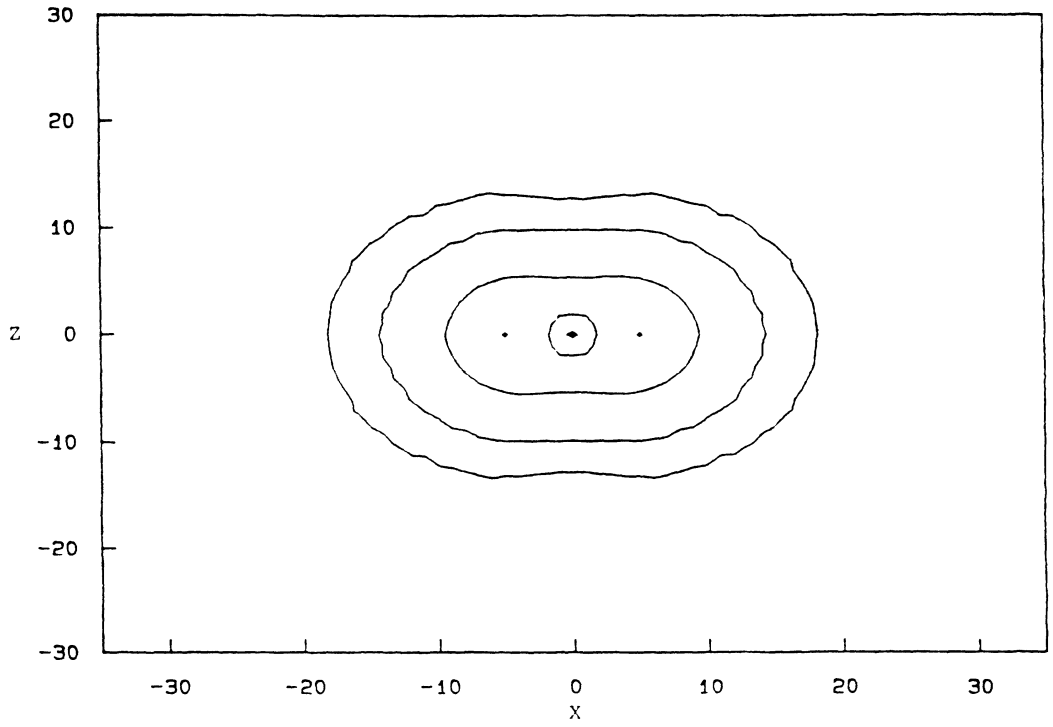


FIG. 4. Contour of constant baryon density for the initial $m_\pi = 138$ MeV configuration.

From the resulting $B=2$ configuration, the symmetries of this minimal energy solution are examined. In order to examine the effects of chiral symmetry breaking, two parallel tracks were undertaken, one following the approach of Adkins, Nappi, and Witten with a massless pion,[5] and one with a massive pion, as in the work of Adkins and Nappi.[11] By using values for F_π and e determined from these $B=1$ investigations, the present work becomes an *ab initio* calculation of the deuteron.

II. CALCULATIONAL METHOD

In order to solve the deuteron problem, two skyrmions are placed on a three-dimensional cubic lattice, and a numerical relaxation method is employed to solve the field equations. First one skyrmion, in the hedgehog configuration, is placed in a box and allowed to relax to minimum energy. This serves both as a source of data for the value of the field U on the lattice and as a check that the procedure yields results consistent with previous work on the $B=1$ hedgehog ansatz.[5,11] Using this newly generated data, two skyrmions are placed together and allowed to relax using the same relaxation procedure as for the single skyrmion. With malice aforethought, one of the skyrmions is rotated by π about an axis perpendicular to the line connecting the centers of the skyrmions in order to attain the most attractive skyrmion-skyrmion potential.[7] Despite the interpretative drawbacks, the product ansatz is used for the overlapping region of the two skyrmions; this ansatz seems the best starting configuration in light of the work of Braaten and Carson in determining the deuteron quantum numbers.[8] The interskyrmion separation is then taken to be the distance between the two lattice points where the baryon number density is a maximum (i.e., at the centers of the two skyrmions).

The numerical relaxation method used is a procedure adapted from the work of Klebanov.[12] For each iteration, the field U is updated according to

$$U'=(1+i\epsilon\cdot\tau)U \tag{6}$$

for some small parameter ϵ. The central element of the

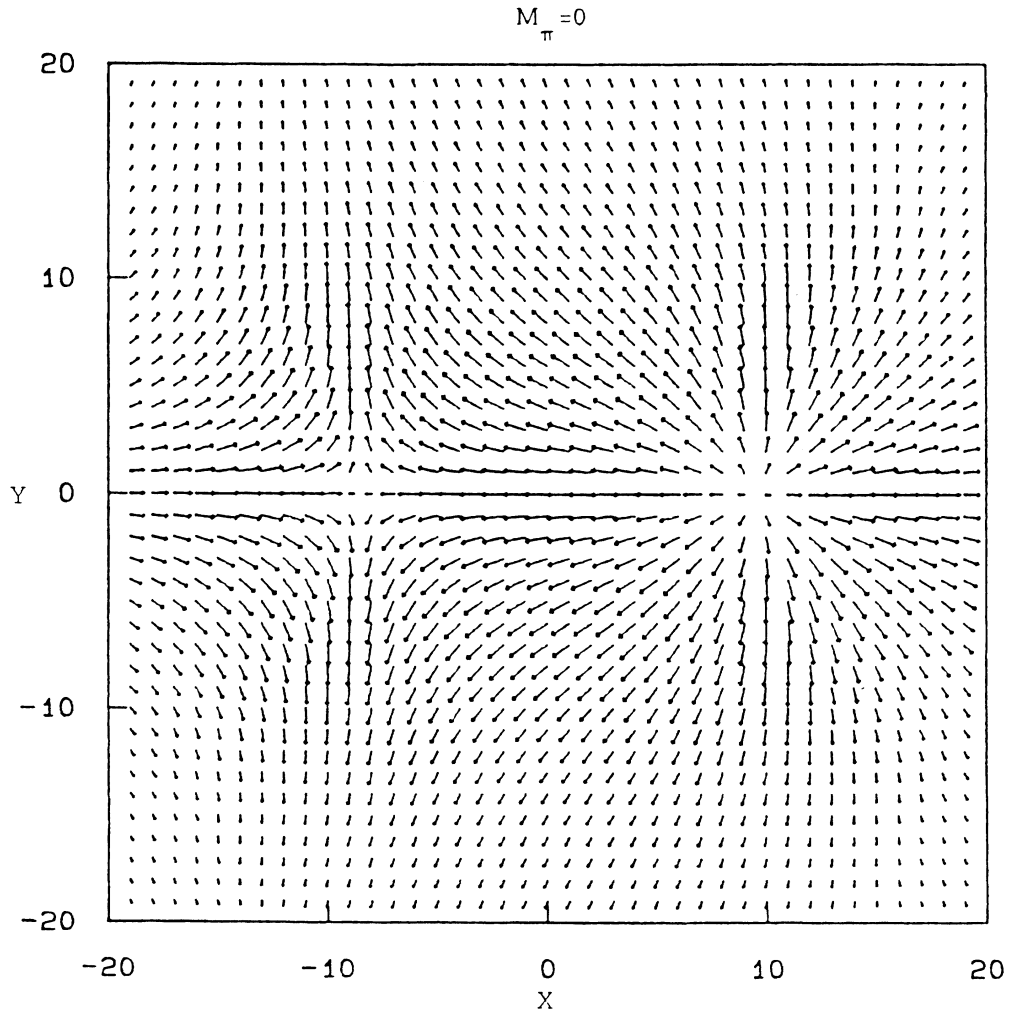


FIG. 5. Isovector pion field projected into the XY plane for the minimum-energy biskyrmion configuration with $m_\pi=0$. The magnitude of the projected field is proportional to the length of the arrow, and the direction in isospace is given by the direction of the arrow.

relaxation technique is the determination of the dependence of the updated energy upon this parameter. Allowing E_l to denote the contribution to the total energy at lattice site $l=(i,j,k)$, it can be shown that the change in E_l after one iteration takes the algebraic form

$$\delta E_l=\epsilon_l\cdot\mathbf{A}_l+O(\epsilon^2)\ , \tag{7}$$

where $\mathbf{A}_l$ is a function of U at (i,j,k) and the neighboring lattice sites. Once the algebraic form of A_l is known, it can be computed numerically for each iteration. Therefore, by defining ϵ according to

$$\epsilon_l\equiv -C\,\mathbf{A}_l \tag{8}$$

with C some positive constant, one finally obtains

$$\delta E_l=-C\,\mathbf{A}_l^2+O(\epsilon^2)\ . \tag{9}$$

For small C and thus small ϵ, the higher-order effects can be neglected; if so, the total energy necessarily decreases after every iteration. The relaxation is accomplished by sweeping through the lattice until the configuration converges to a minimum. Convergence is measured by determining the percentage change in the energy of the configuration after each iteration. For the present investigation, the lattice sweep was repeated until the convergence was within 0.005%.

The constant C is important not only to justify the discarding of the higher-order terms in (9), but also to ensure the conservation of baryon number. The baryon number is computed using a discrete analog of the usual continuum expression

$$B=\frac{1}{24\pi^2}\epsilon^{ijk}\int d^3x\ \mathrm{Tr}(U^\dagger\partial_i UU^\dagger\partial_j UU^\dagger\partial_k U)\ . \tag{10}$$

It is crucial that C be sufficiently small so that any given iteration will mimic a homotopic deformation of the continuum configuration U. Without this, the topology will change and baryon number will be lost with each iteration. As long as $B\simeq 1$, the homotopy is preserved and it is assumed that a good approximation to the continuum limit has been attained.

Ultimately, it is the value of the baryon number that gives a measure of the precision of the calculation; thus a lattice spacing that yields a good, stable baryon number is essential. Since the chiral angle $F(r)$ in Eq. (2) drops off very quickly, it was determined that the lattice need only

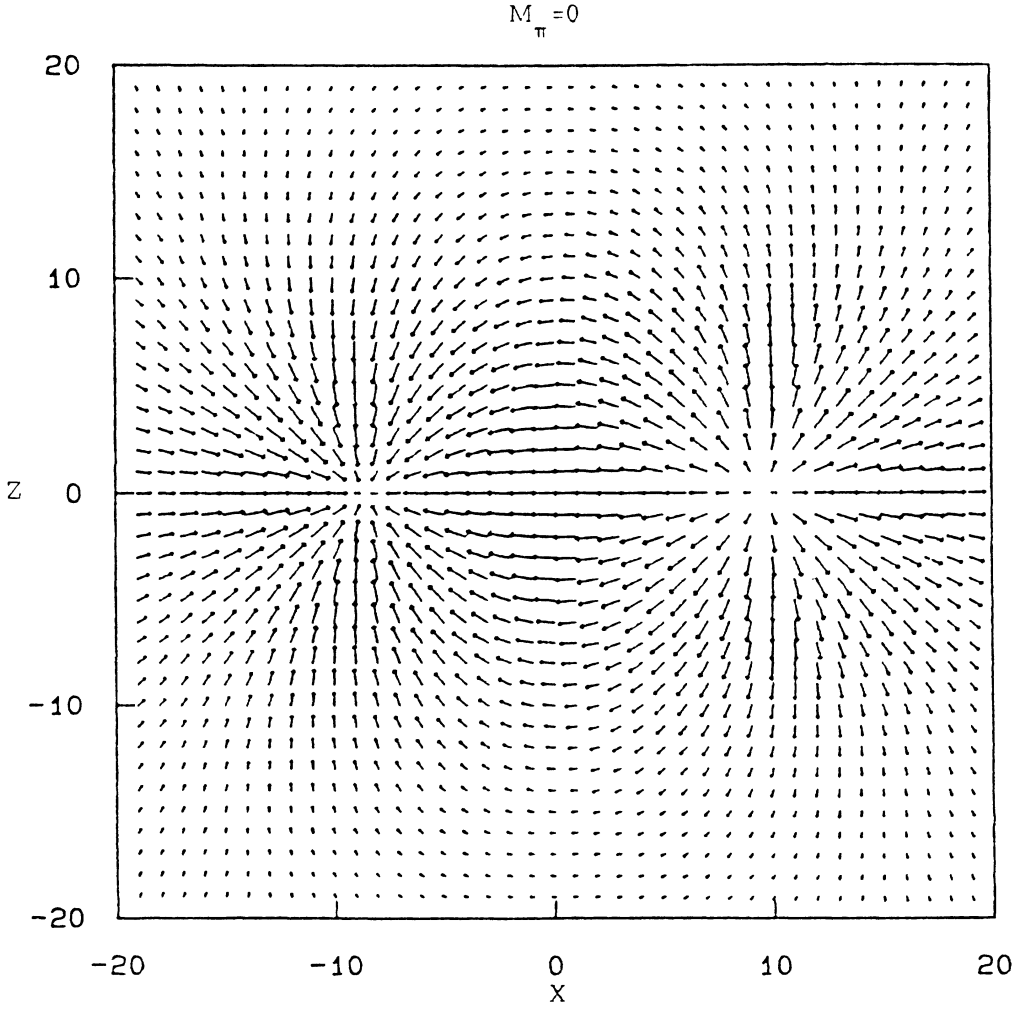


FIG. 6. Isovector pion field projected into the XZ plane for the minimum-energy biskyrmion configuration ($m_\pi=0$).

extend a distance $\rho \equiv eF_\pi r = 10$ from the origin. This distance was then divided into thirtieths, yielding a $60 \times 60 \times 60$ cubic lattice with a lattice spacing of $(\hbar c / 3F_\pi e)$ fm. This lattice was chosen in order to have a spacing which would optimize the baryon number while minimizing the memory requirements of the computer code; dividing each length of $eF_\pi r = 1$ into thirds produces a satisfactory compromise.

Since the $B = 1$ hedgehog configuration exhibits equivariance, only an octant—a $30 \times 30 \times 30$ lattice—is needed for the initial single-skyrmion computations. This being the case, the project was begun on a VAX 11/750; it was soon evident, however, that the VAX could not provide the memory and speed necessary for the full biskyrmion problem. As a result, the code was eventually adapted to exploit the capabilities of the FPS-164 attached array processor at the Triangle Universities Computation Center (TUCC), a machine designed for large array processing.

III. RESULTS

The single-skyrmion relaxation is measured against the standard results.[5,11] As already mentioned, the accuracy of the calculated baryon number determines the overall precision of the minimization. For a $30 \times 30 \times 30$ octant, the baryon number is initially found to be approximately 0.95 in both the massive and massless pion cases, and only decreases by about 1% over the course of the entire relaxation procedure. Although a baryon number closer to 1 would result from using a larger lattice, a 5% error is considered satisfactory. Moreover, the memory and time requirements of a larger lattice soon become overwhelming.

Once the single-skyrmion runs were completed, their relaxed configurations were used as input for the biskyrmion investigation. The goal, of course, is to find a bound state of the two-skyrmion system, signalled by a total energy less than twice the static energy of the $B = 1$ configuration. In order to generate a static potential and find the true energy minimum, several relaxation runs were required with different interskyrmion distances. Although the baryon number varies with the distance separating the two skyrmions, the relaxation retains an overall precision of approximately 5% to 8%.

In Fig. 2 are plots of the variation of static mass with interskyrmion distance for the massless and massive pion configurations, respectively. (For the latter, we take

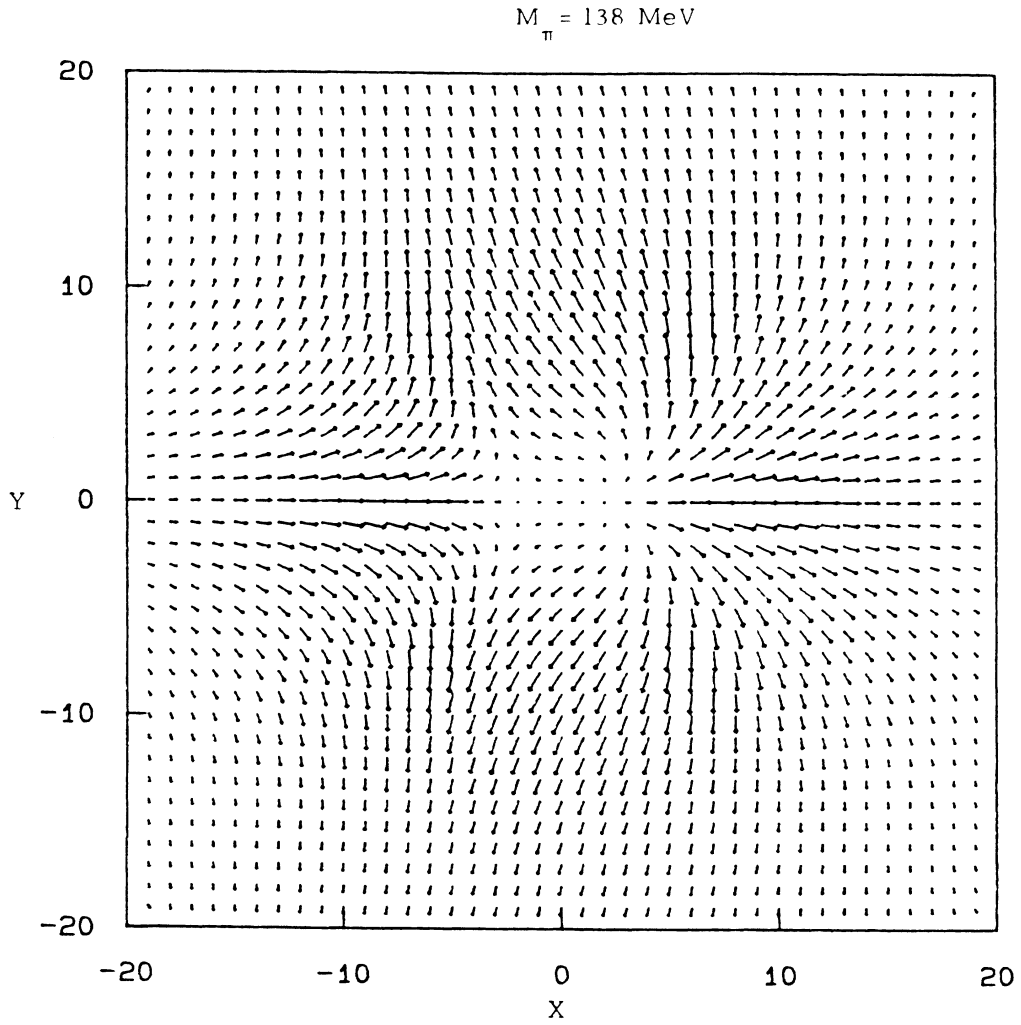


FIG. 7. Isovector pion field projected into the XY plane for the minimum-energy biskyrmion configuration ($m_\pi = 138$ MeV).

$m_\pi = 138$ MeV). A bound state occurs in both cases, but the massive pion shortens the range of the interaction, so that a bound state occurs at a smaller skyrmion separation. Using the appropriate values for F_π and e already determined,[11] the minimum occurs at

$$E_{B=2} - 2E_{B=1} = -1.17 F_\pi / e = -26 \text{ MeV} , \quad (11)$$

at a separation of $(10/3F_\pi e) = 1.26$ fm. The potential with a massless pion is, by comparison, considerably wider and shallower, with a minimum[5] of $0.56F_\pi/e = 13.2$ MeV at a distance of $(19/3F_\pi e) = 1.76$ fm. That the results using a massive pion should be better than for a massless pion is to be expected; indeed, additional improvement is anticipated with the inclusion of the ρ and ω vector mesons in the Lagrangian.

Several characteristics of the deuteron should emerge from these minimum-energy configurations.[13] At the classical level, the deuteron mass is the energy corresponding to the minimum of the curves in Fig. 2. Using the appropriate values of F_π and e, the classical biskyrmion mass is found to be 1744 MeV for a massless pion and 1775 MeV for the massive pion, only 5% to 7% from the deuteron mass of 1876 MeV. These results are most encouraging, for the quantum corrections are expected to improve the classical mass by only a few percent, as is the case for the single skyrmion quantization.[5,11]

One of the essential characteristics that must be reproduced by the biskyrmion configuration is the deuteron's quadrupole moment. Since the deuteron is an isoscalar, the electric charge density is equal to half the baryon number density; the electric quadrupole moment is thus a good measure of baryon number distribution. Indeed, the difference between the massless and massive pion configurations can easily be seen in the quadrupole moments. The classical quadrupole moment for the massless pion is found to be $Q_x \equiv \langle Q \rangle = 1.44$ fm^2, as compared with the actual deuteron quadrupole moment of $Q_d = 0.282$ fm^2. In contrast, however, is the classical quadrupole moment for the massive configuration, $Q_x = 0.492$ fm^2, only about 43% above the actual quadrupole moment of the deuteron. Though this appears to be a relatively large error, recall that the quantized hedgehog ansatz is only accurate at the 30% level; having effectively multiplied two hedgehogs together, a 43% error is consistent. Once again, proper quantization of the biskyrmion should improve the results.

Another deuteron characteristic which should emerge from this *ab initio* approach is the root-mean-square electromagnetic radius. This time, the massless pion configuration yields $(\langle r^2 \rangle)^{1/2} = 1.04$ fm, a slightly better result than does the massive pion configuration value of

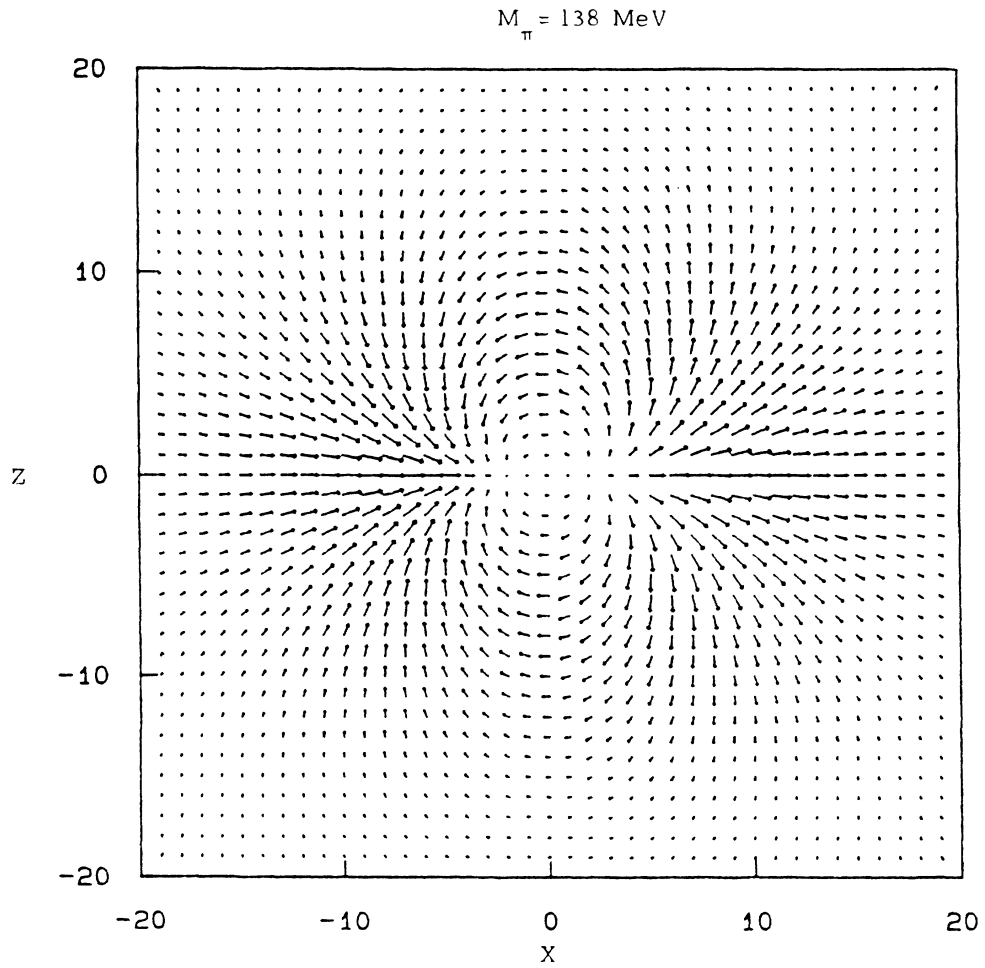


FIG. 8. Isovector pion field projected into the XY plane for the minimum-energy biskyrmion configuration ($m_\pi = 138$ MeV).

TABLE I. Static properties of $B=2$ relaxed configurations.

Quantity	Massless pion configuration	Massive pion configuration
Well depth	13.2 MeV	26 MeV
Range	1.78 fm	1.26 fm
Mass	1744 MeV	1775 MeV
$\langle Q \rangle$	1.44 fm^2	0.492 fm^2
$(\langle r^2 \rangle)^{1/2}$	1.04 fm	0.92 fm

0.92 fm. Comparison with the measured value for the deuteron $(\langle r^2 \rangle)^{1/2}=2.1$ fm, betrays a 50% error.

In order to get some measure of the inherent symmetries of the configurations, we consider the planar distribution of baryon number. Figure 3 contains contour plots of the baryon number for both the massless and massive pion configurations in perpendicular planes through the origin. (In these and the remaining diagrams, the axes are labelled in units of lattice points.) The baryon number distribution clearly indicates that despite the fact that the initial configuration uses the product ansatz (shown in Fig. 4 for a massive pion), the minimum energy configurations are no longer products of two hedgehog configurations. These identifications are consistent with the results of Sommermann, Wyld, and Pethick.[10] That the correct deuteron solution is not, in fact, a product configuration is encouraging, considering the difficulties of the product ansatz. Though this result is by no means conclusive, a larger lattice should yield greater precision and strengthen the identification.

If the deuteron solution is not a product, then the effect of relaxation on the product ansatz symmetries of the isospin fields must be examined as well. Figures 5–8 contain plots of the isovector pion fields projected into perpendicular planes through the origin for both pion configurations. In these figures, the magnitude of the projected field is proportional to the length of the arrow, and the direction in isospace is given by the direction of the arrow. Comparison with the depiction of the hedgehogs in Fig. 1 shows that in so far as isospin is concerned, the configuration is still that of two hedgehogs with a relative orientation of π about the Y axis. The key result is that the relaxed configuration maintains this initial setup, even though the baryon number distribution does not. This is important, as it implies that the relaxed, bound configuration does indeed display the symmetries assumed by Braaten and Carson,[8] even though it is no longer strictly a product of chiral fields.

IV. CONCLUSIONS

Without the imposition of *a priori* symmetry constraints, the product ansatz was used as an initial configuration in search for a proper deuteron configuration in the Skyrme model. In order to study the effects of chiral symmetry breaking, two different approaches were undertaken, one using a massless pion, the other with a massive pion. For both cases, most of the calculated properties of the presumed deuteron configuration are within the 30% limits of the usual single-skyrmion analysis; these results are reproduced in Table I.

Chiral symmetry breaking seems to play an important role in stabilizing the numerical analysis, as can be seen by comparing the contours in Fig. 3. Moreover, the presence of a pion mass term in the Lagrangian improves all but one of the static properties listed in Table I. Perhaps most importantly, however, is that a massive pion seems to reproduce a more realistic nucleon-nucleon potential, as seen by juxtaposing the curves in Fig. 2.

The symmetries of the relaxed configuration were found to be just those postulated by Braaten and Carson[8] for the product ansatz. Thus their analysis of the configuration and derivation of the deuteron quantum numbers apply to the energy configurations discovered in this investigation. Moreover, this would seem to be the case even though the relaxed biskyrmion no longer appears to be a product configuration. Thus the interpretative drawbacks of the product ansatz need not arise.

While this work was in progress, a preprint was discovered describing similar work by J. J. M. Verbaarschot.[14] Although different methods were employed, he is in general agreement with the results of the present investigation.

ACKNOWLEDGMENTS

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