Optical model for medium and high energy hadron-nucleus collisions

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The optical phase shift function is calculated to fifth order in a model in which center of mass correlations in the nucleus are taken into account. Differential cross sections are calculated for elastic scattering of protons by ^{4}He, ^{16}O, ^{58}Ni, and ^{208}Pb. The results are compared with exact Glauber calculations and with a simple $t_0\rho$-type approximation. The significance of the higher-order corrections is assessed.

I. INTRODUCTION

In recent years there have been a number of medium- and high-energy proton-nucleus elastic scattering measurements in which the differential cross section varies over many orders of magnitude, in some cases as many as 10.[1,2] In addition, the smallest cross sections measured are likely to be as many as some 12 orders of magnitude smaller than the nuclear cross section in the forward direction. These types of collisions are often analyzed by means of a microscopic optical potential which is used in the Schrödinger equation or, more recently, in the Dirac equation, which is then solved numerically. Usually the optical potential used is obtained from a first-order or $t\rho$-type approximation. On occasion some second-order correlation effects are estimated using a variety of methods. Another means of analysis is via Glauber-type optical calculations in which an optical phase shift function is calculated microscopically and used in an eikonal approximation. There, too, analyses generally employ a first-order or $t\rho$ approximation, and, on occasion, second-order correlation effects are estimated. The first-order results are not sensitive to correlation effects. Hence, corrections to the first order results are sometimes calculated or estimated solely from correlation effects. However, even with wave functions with no dynamical correlations, the first order or $t\rho$ results are still only approximations to the full optical potential or full optical phase shift function. Since the recent experiments have been able to measure such small cross sections and such large variations in the cross sections, it is perhaps worthwhile to reexamine the first order or $t\rho$ approximation and to ask how large corrections are to it, even in the absence of dynamical correlations, and how such corrections can be obtained. To this end we consider a model whose exact result can be calculated, namely Glauber theory. In Sec. II we develop a series for the optical phase shift function (and hence for the optical potential) and explicitly calculate the first-order through fifth-order terms. In Sec. III we calculate the differential cross sections obtained from the various approximations and compare them to the exact Glauber theory results for nuclei with mass numbers 4, 16, 58, and 208.

II. OPTICAL PHASE SHIFT FUNCTION

In Glauber theory the hadron-nucleus elastic scattering amplitude $F(q)$ is given[3] by the expectation value of a scattering amplitude operator in the ground state of the nucleus,

$$F(q)=\frac{ik}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}'}\left\{1-\prod_{j=1}^{A}[1-\Gamma_j(\mathbf{b}'-\mathbf{s}_j)]\right\}\times|\psi_A(\mathbf{r}_1,\ldots,\mathbf{r}_A)|^2\times\delta\left[\frac{1}{A}\sum_{j=1}^{A}\mathbf{r}_j\right]d\mathbf{r}_1\ldots d\mathbf{r}_A d^2b\ , \qquad (1)$$

where $\hbar q$ is the momentum transferred to the nucleus, $\mathbf{s}_j$ is the projection of the jth target nucleon coordinate $\mathbf{r}_j$ onto the impact parameter plane, $\mathbf{b}'$ is the impact parameter vector, Γ_j is the hadron-nucleon profile function, ψ_A is the nuclear wave function, A is the mass number of the nucleus, and $\hbar k$ is the incident momentum.

For simplicity, we assume that the nuclei can be described by an independent particle model for ψ_A,

$$|\psi_A|^2=\prod_{j=1}^{A}|\varphi(\mathbf{r}_j)|^2 \qquad (2a)$$

with a form factor given by

$$S_A(q)=e^{-R_A^2q^2/4}\ . \qquad (2b)$$

In this case we can evaluate the effect of the delta function and obtain

$$F(q)=\frac{ik}{2\pi}K(q)\int e^{i\mathbf{q}\cdot\mathbf{b}'}\left\langle\psi_A\left|\left\{1-\prod_{j=1}^{A}[1-\Gamma_j(\mathbf{b}'-\mathbf{s}_j)]\right\}\right|\psi_A\right\rangle d^2b \;, \tag{3}$$

where $K(q)$ is a center of mass correlation function given by

$$K(q)=e^{R_A^2q^2/4A} \;. \tag{4}$$

(This result also holds for the somewhat more general independent particle harmonic oscillator wave function.)

We define an optical phase shift function $\chi_{\rm opt}(b)$ by[4–7]

$$F(q)=\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}[1-e^{i\chi_{\rm opt}(b)}] \tag{5a}$$

$$=ik\int_0^{\infty}J_0(qb)[1-e^{i\chi_{\rm opt}(b)}]b\,db \;. \tag{5b}$$

From Eqs. (3) and (5) we obtain

$$e^{i\chi_{\rm opt}(b)}=(2\pi)^{-2}\int d^2b'd^2q e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\prod_{j=1}^{A}[1-\Gamma_j(\mathbf{b}'-\mathbf{s}_j)]\right|\psi_A\right\rangle \;. \tag{6}$$

We define a function $\varphi(b,\lambda)$ by

$$\varphi(b,\lambda)=\ln\left[(2\pi)^{-2}\int d^2b'd^2q e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\prod_{j=1}^{A}[1-\lambda\Gamma_j(\mathbf{b}'-\mathbf{s}_j)]\right|\psi_A\right\rangle\right] \;. \tag{7}$$

We then expand $\varphi(b,\lambda)$ in a power series about $\lambda=0$,

$$\varphi(b,\lambda)=\sum_{n=1}^{\infty}\frac{1}{n!}\varphi^{(n)}(b,0)\lambda^n \;, \tag{8}$$

where

$$\varphi^{(n)}(b,\lambda)\equiv\frac{\partial^n\varphi(b,\lambda)}{\partial\lambda^n} \;, \tag{9}$$

and where we have used $\varphi(b,0)=0$. From Eqs. (6), (7), and (8) we have

$$i\chi_{\rm opt}(b)=\varphi(b,1) \tag{10}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n!}\varphi^{(n)}(b,0) \tag{11}$$

$$\equiv\sum_{n=1}^{\infty}i\chi_n(b) \;, \tag{12}$$

where we have defined the nth order optical phase shift function $\chi_n(b)$ by Eq. (12). To evaluate the series (11) we note

$$\varphi^{(1)}(b,\lambda)=\frac{\partial\varphi(b,\lambda)}{\partial\lambda} \tag{13}$$

$$=-e^{-\varphi(b,\lambda)}(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\Gamma_j(\mathbf{b}'-\mathbf{s}_j)\prod_{k\neq j}[1-\lambda\Gamma_k(\mathbf{b}'-\mathbf{s}_k)]\right|\psi_A\right\rangle \;, \tag{14}$$

$$\varphi^{(2)}(b,\lambda)=-[\varphi^{(1)}(b,\lambda)]^2+e^{-\varphi(b,\lambda)}(2\pi)^{-2}$$
$$\times\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\Gamma_j(\mathbf{b}'-\mathbf{s}_j)\Gamma_k(\mathbf{b}'-\mathbf{s}_k)\prod_{l\neq j,k}[1-\lambda\Gamma_l(\mathbf{b}'-\mathbf{s}_l)]\right|\psi_A\right\rangle \;, \tag{15}$$

$$\varphi^{(3)}(b,\lambda)=-3\varphi^{(1)}\varphi^{(2)}-[\varphi^{(1)}]^3-e^{-\varphi(b,\lambda)}(2\pi)^{-2}$$
$$\times\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)$$
$$\times\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\Gamma_j(\mathbf{b}'-\mathbf{s}_j)\Gamma_k(\mathbf{b}'-\mathbf{s}_k)\Gamma_l(\mathbf{b}'-\mathbf{s}_l)\prod_{m\neq j,k,l}[1-\lambda\Gamma_m(\mathbf{b}'-\mathbf{s}_m)]\right|\psi_A\right\rangle \;, \tag{16}$$

$$\varphi^{(4)}(b,\lambda)=-4\varphi^{(1)}\varphi^{(3)}-3[\varphi^{(2)}]^2-6[\varphi^{(1)}]^2\varphi^{(2)}-[\varphi^{(1)}]^4+e^{-\varphi(b,\lambda)}(2\pi)^{-2}$$
$$\times\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\sum_{m\neq j,k,l}\Gamma_j\Gamma_k\Gamma_l\Gamma_m\prod_{n\neq j,k,l,m}(1-\lambda\Gamma_n)\right|\psi_A\right\rangle , \quad (17)$$

$$\varphi^{(5)}(b,\lambda)=-5\varphi^{(1)}\varphi^{(4)}-10\varphi^{(2)}\varphi^{(3)}-15\varphi^{(1)}[\varphi^{(2)}]^2-10[\varphi^{(1)}]^2\varphi^{(3)}-10[\varphi^{(1)}]^3\varphi^{(2)}-[\varphi^{(1)}]^5$$
$$-e^{-\varphi(b,\lambda)}(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)$$
$$\times\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\sum_{m\neq j,k,l}\sum_{n\neq j,k,l,m}\Gamma_j\Gamma_k\Gamma_l\Gamma_m\Gamma_n\prod_{\nu\neq j,k,l,m,n}(1-\lambda\Gamma_\nu)\right|\psi_A\right\rangle , \quad (18)$$

and so on. Consequently, with $\lambda=0$ we have

$$\varphi^{(1)}(b,0)=-(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\Gamma_j(\mathbf{b}-\mathbf{s}_j)\right|\psi_A\right\rangle \quad (19)$$

$$\varphi^{(2)}(b,0)=-[\varphi^{(1)}(b,0)]^2+(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\Gamma_j(\mathbf{b}'-\mathbf{s}_j)\Gamma_k(\mathbf{b}'-\mathbf{s}_k)\right|\psi_A\right\rangle , \quad (20)$$

$$\varphi^{(3)}(b,0)=-3\varphi^{(1)}(b,0)\varphi^{(2)}(b,0)-[\varphi^{(1)}(b,0)]^3-(2\pi)^{-2}$$
$$\times\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\Gamma_j(\mathbf{b}'-\mathbf{s}_j)\Gamma_k(\mathbf{b}'-\mathbf{s}_k)\Gamma_l(\mathbf{b}'-\mathbf{s}_l)\right|\psi_A\right\rangle , \quad (21)$$

$$\varphi^{(4)}(b,0)=-4\varphi^{(1)}(b,0)\varphi^{(3)}(b,0)-3[\varphi^{(2)}(b,0)]^2-6[\varphi^{(1)}(b,0)]^2\varphi^{(2)}(b,0)-[\varphi^{(1)}(b,0)]^4$$
$$+(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\sum_{m\neq j,k,l}\Gamma_j\Gamma_k\Gamma_l\Gamma_m\right|\psi_A\right\rangle , \quad (22)$$

$$\varphi^{(5)}(b,0)=-5\varphi^{(1)}(b,0)\varphi^{(4)}(b,0)-10\varphi^{(2)}(b,0)\varphi^{(3)}(b,0)-15\varphi^{(1)}(b,0)[\varphi^{(2)}(b,0)]^2$$
$$-10[\varphi^{(1)}(b,0)]^2\varphi^{(3)}(b,0)-10[\varphi^{(1)}(b,0)]^3\varphi^{(2)}(b,0)-[\varphi^{(1)}(b,0)]^5$$
$$-(2\pi)^{-2}\int d^2b'd^2q\, e^{i\mathbf{q}\cdot(\mathbf{b}'-\mathbf{b})}K(q)\left\langle\psi_A\left|\sum_{j=1}^{A}\sum_{k\neq j}\sum_{l\neq j,k}\sum_{m\neq j,k,l}\sum_{n\neq j,k,l,m}\Gamma_j\Gamma_k\Gamma_l\Gamma_m\Gamma_n\,\right|\psi_A\right\rangle , \quad (23)$$

and so on. We assume, for simplicity, that the hadron-nucleon profile functions Γ_j are identical, which is approximately true at high energies. (The generalization of our results to the case $\Gamma_n\neq\Gamma_p$ is straightforward, but tedious.) We define

$$C(b)=\langle\psi_A\,|\,\Gamma_j(\mathbf{b}-\mathbf{s}_j)\,|\,\psi_A\rangle . \quad (24)$$

Then the expectation value in Eq. (19) for $\varphi^{(1)}$ is written

$$\left\langle\psi_A\left|\sum_j\Gamma_j(\mathbf{b}'-\mathbf{s})\right|\psi_A\right\rangle=AC(b') . \quad (25)$$

The expectation values of the higher order terms involving products of profile functions may be expressed in terms of multiparticle densities, as in Ref. 6. For uncorrelated nuclear wave functions, ψ_A, all the expectation values in Eqs. (20)–(23) are easily expressed in terms of $C(b')$,

$$\left\langle\sum_j\sum_{k\neq j}\Gamma_j\Gamma_k\right\rangle=A(A-1)C^2 , \quad (26a)$$

$$\left\langle\sum\sum\sum\Gamma_j\Gamma_k\Gamma_l\right\rangle=A(A-1)(A-2)C^3 , \quad (26b)$$

$$\left\langle\sum\sum\sum\sum\Gamma_j\Gamma_k\Gamma_l\Gamma_m\right\rangle=A(A-1)(A-2)(A-3)C^4 , \quad (26c)$$

$$\left\langle\sum\sum\sum\sum\sum\Gamma_j\Gamma_k\Gamma_l\Gamma_m\Gamma_n\right\rangle=A(A-1)(A-2)(A-3)(A-4)C^5 , \quad (26d)$$

and so on.

In order to relate the optical phase shift function to the experimentally measured hadron-nucleon amplitudes, f_j, for hadron-nucleon scattering we have

$$f_j(\mathbf{q})=\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\Gamma_j(\mathbf{b}) , \quad (27)$$

which, upon Fourier inversion, leads to

$$\Gamma_j(\mathbf{b})=(2\pi ik)^{-1}\int d^2q\, e^{-i\mathbf{q}\cdot\mathbf{b}}f_j(\mathbf{q}) . \quad (28)$$

For the hadron-nucleon amplitudes $f_j(\mathbf{q})$ we shall take the usual high energy parametrization

$$f_j(q)=\frac{ik\sigma}{4\pi}(1-i\rho)e^{-aq^2/2} . \tag{29}$$

From Eqs. (2a), (2b), (24), (28), and (29) we obtain for $C(b)$ the result

$$C(b)=ye^{-b^2/(R_A^2+2a)} \tag{30}$$

with

$$y=\frac{\sigma(1-i\rho)}{2\pi(R_A^2+2a)} . \tag{31}$$

The functions $\varphi^{(i)}(b,0)$ given by Eqs. (19)–(23) may now be explicitly evaluated to yield

$$\frac{1}{n!}\varphi^{(n)}(b,0)=\frac{(-)^n A\binom{A}{n}y^n(R_A^2+2a)}{A(R_A^2+2a)-nR_A^2}\exp\left[-\frac{nAb^2}{A(R_A^2+2a)-nR_A^2}\right]-\frac{1}{n!}[\varphi^{(1)}(b,0)]^n(1-\delta_{n1})$$

$$-\frac{1}{(n-1)!}(1-\delta_{n1})(1-\delta_{n2})\varphi^{(1)}(b,0)\varphi^{(n-1)}(b,0)-\frac{D_n}{n!},\quad (n\leq A) \tag{32}$$

where

$$D_1=D_2=D_3=0 , \tag{33}$$

$$D_4=6[\varphi^{(1)}(b,0)]^2\varphi^{(2)}(b,0)+3[\varphi^{(2)}(b,0)]^2 , \tag{34}$$

$$D_5=10[\varphi^{(1)}(b,0)]^3\varphi^{(2)}(b,0)+10[\varphi^{(1)}(b,0)]^2\varphi^{(3)}(b,0)+15\varphi^{(1)}(b,0)[\varphi^{(2)}(b,0)]^2+10\varphi^{(2)}(b,0)\varphi^{(3)}(b,0) , \tag{35}$$

and so on, and δ_{ij} is the Kronecker delta function. Thus the optical phase shift function $i\chi_{\text{opt}}$ may be approximated by the explicit fifth order result

$$i\chi_{\text{opt}}(b)\approx\sum_{n=1}^{5}\frac{1}{n!}\varphi^{(n)}(b,0) \tag{36a}$$

$$=\sum_{n=1}^{5}i\chi_n(b) \tag{36b}$$

with $\varphi^{(n)}(b,0)/n!$ given by Eqs. (32)–(35). The higher order terms may be obtained, if necessary, by a straightforward continuation of our expansion.

The commonly used first-order result ($n=1$) is given by

$$i\chi_1(b)=-\frac{A\sigma(1-i\rho)}{2\pi[R_A^2(1-A^{-1})+2a]}\times\exp\left[-\frac{b^2}{R_A^2(1-A^{-1})+2a}\right] . \tag{37}$$

This is, in essence, the commonly used $t\rho$ approximation. It and its relativistic generalization are widely used in analyses of hadron-nucleus scattering. Hence, we will use the notation $\chi_{t\rho}$ interchangeably with χ_1,

$$\chi_{t\rho}(b)\equiv\chi_1(b) . \tag{38}$$

Often in these analyses the hadron-nucleon scattering amplitude is evaluated only in the forward direction and the result is again called the $t\rho$ approximation. We will call that approximation the $t_0\rho$ approximation. It is obtained here by setting $a=0$, yielding

$$i\chi_{t_0\rho}(b)=-\frac{A^2\sigma(1-i\rho)}{2\pi(A-1)R_A^2}\exp\left[-\frac{A}{A-1}\frac{b^2}{R_A^2}\right] \tag{39}$$

$$=-A\left[\frac{A}{A-1}\frac{1}{\pi R_A^2}e^{-(A/A-1)(b^2/R_A^2)}\right]\frac{2\pi f(0)}{ik} . \tag{40}$$

We should bear in mind that this $t_0\rho$ approximation is an approximation to $i\chi_1(b)\equiv i\chi_{t\rho}(b)$ which in turn is an approximation to $i\chi_{\text{opt}}(b)$.

We should caution that the results (32)–(35) hold only for $n\leq A$. For $n>A\geq 2$, the first term in Eq. (32) is absent. For $n>A+1\geq 3$, the second term is also absent, the coefficient of the third term is $(A+1)/n!$ instead of $1/(n-1)!$, and the integer coefficients in D_4 and D_5 are different from those shown in Eqs. (34) and (35). For example, if $A=3$, the fourth-order phase shift function $i\chi_4$ would be given by

$$24i\chi_4(b;A=3)=-[\varphi^{(1)}]^4-4\varphi^{(1)}\varphi^{(3)}-6[\varphi^{(1)}]^2\varphi^{(2)}-3[\varphi^{(2)}]^2 , \tag{41}$$

and the fifth-order phase shift function $i\chi_5$ would be given by

$$120i\chi_5(b;A=3)=-4\varphi^{(1)}\varphi^{(4)}-4[\varphi^{(1)}]^3\varphi^{(2)}-6[\varphi^{(1)}]^2\varphi^{(3)}-12\varphi^{(1)}[\varphi^{(2)}]^2-10\varphi^{(2)}\varphi^{(3)} . \tag{42}$$

The nth order result for the optical phase shift function will be denoted by

$$\chi^{(n)}(b)=\sum_{j=1}^{n}\chi_j(b) . \quad (43)$$

The various approximate results for χ_{opt} lead to corresponding approximate results for the scattering amplitude,

$$F^{(n)}(q)=ik\int_0^\infty J_0(qb)[1-e^{i\chi^{(n)}(b)}]b\,db \quad (44)$$

and

$$F^{(t_0\rho)}(q)=ik\int_0^\infty J_0(qb)[1-e^{i\chi_{t_0\rho}(b)}]b\,db . \quad (45)$$

The exact Glauber theory result is well known,

$$F^{(G)}(q)=\frac{ik\sigma(1-i\rho)}{4\pi}\sum_{n=1}^{A}\frac{(-)^{n-1}}{n}\binom{A}{n}\left[\frac{\sigma(1-i\rho)}{2\pi(R_A^2+2a)}\right]^{n-1}\exp\left[-\left(\frac{R_A^2+2a}{4n}-\frac{R_A^2}{4A}\right)q^2\right] . \quad (46)$$

The corresponding differential cross sections for the various approximations are obtained using

$$\frac{d\sigma}{dt}=\frac{\pi}{(\hbar k)^2}|F(q)|^2 \quad (47)$$

with $\hbar^2q^2=-t$ and with the corresponding approximate scattering amplitude for $F(q)$.

We might point out that the results (32)–(35) we have obtained with both the special forms of wave function and the hadron-nucleon scattering amplitude we have used could have been obtained more directly by Fourier transforming the exact result $F^{(G)}$ of Eq. (46) and then expanding the logarithm of unity diminished by $2\pi/ik$ times the Fourier transform. We chose not to do it this way since the method we have presented is more general and may be used with other kinds of wave functions or scattering amplitudes.

III. COMPARISON OF APPROXIMATE DIFFERENTIAL CROSS SECTIONS

In this section we calculate the differential cross section $d\sigma/dt$ for hadron-nucleus elastic scattering by four target nuclei; ^{4}He, ^{16}O, ^{58}Ni, and ^{208}Pb. For definiteness we use the proton-proton scattering parameters used in Ref. 6 for 2.1 GeV nucleons, namely $\sigma=42.7$ mb, $\rho=-0.28$, and $a=0.24142$ fm^2. The parameter R^2 in Eq. (2b) is obtained from

$$R_A^2=\tfrac{2}{3}[\langle r_A^2\rangle-\langle r_p^2\rangle]/(1-A^{-1}) .$$

We use $R_4^2=1.805$ fm^2, $R_{16}^2=4.671$ fm^2, $R_{58}^2=9.116$ fm^2, and $R_{208}^2=19.759$ fm^2, corresponding to the measured rms radii $\langle r_4^2\rangle^{1/2}=1.675$ fm, $\langle r_{16}^2\rangle^{1/2}=2.71$ fm, $\langle r_{58}^2\rangle^{1/2}=3.77$ fm, and $\langle r_{208}^2\rangle^{1/2}=5.502$ fm, with $\langle r_p^2\rangle^{1/2}=0.88$ fm. Since our aim is to compare various approximate theoretical results with the exact Glauber theory result, the precise values of the input parameters are of minor significance, provided they are not grossly different from the generally accepted values.

The optical phase shift method is expected to be most accurate for $A\gg 1$. Nevertheless χ_1 and the $t\rho$ and $t_0\rho$ approximations have often been used even for nuclei as light as ^{4}He. To compare the various approximations, we present in Fig. 1 the differential cross section $d\sigma/dt$ for p-^{4}He elastic scattering as a function of $-t$, using the various approximations ($t_0\rho$, $\chi^{(1)}=t\rho$, $\chi^{(2)}$, $\chi^{(3)}$, $\chi^{(4)}$, and $\chi^{(5)}$) and the exact Glauber theory result. The $t_0\rho$ approximation, Eq. (40), is seen to be a poor approximation for p-^{4}He collisions. The first order result ($\chi^{(1)}$, or the $t\rho$ approximation), given by Eq. (37), represents a significant improvement over the $t_0\rho$ approximation for the forward peak, but away from the forward peak it decreases too rapidly. The higher order results represent substantial improvements over the first order result. The fifth order ($\chi^{(5)}$) result is the best of the approximations shown. At the secondary maxima it is 1% and ~10% too low.

In Fig. 2 we show the differential cross sections for p-^{16}O elastic scattering as a function of $-t$. In the forward

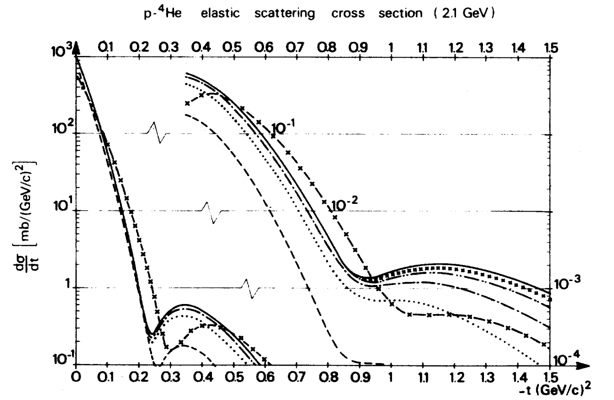

FIG. 1. Theoretical differential cross sections for p-^{4}He elastic scattering. The solid curve is the exact Glauber theory results. The dash-crossed curve is the $t_0\rho$ approximation described in the text. The dashed curve is the result obtained from the first order phase shift function, $\chi^{(1)}$. The dotted curve is the result obtained from $\chi^{(2)}$ which includes the second order phase shift function. The dashed-dotted curve is obtained from $\chi^{(3)}$. The dashed-double-dotted curve is obtained from $\chi^{(4)}$. The crossed curve is obtained from $\chi^{(5)}$.

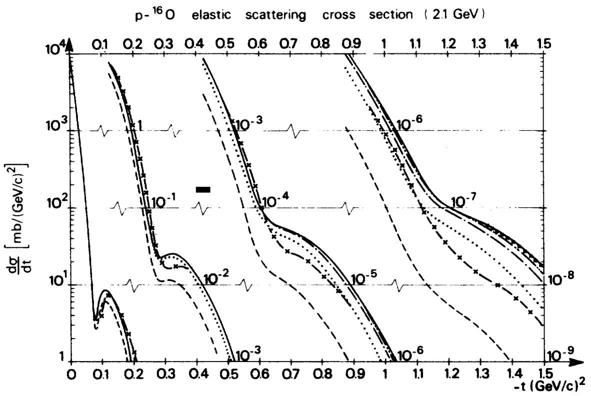

FIG. 2. Same as Fig. 1 for p-^{16}O elastic scattering.

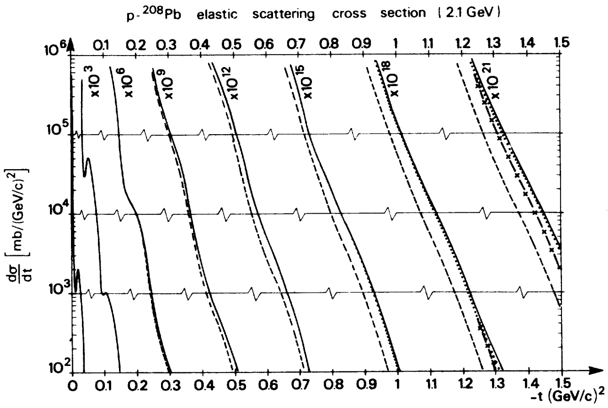

FIG. 4. Same as Fig. 1 for p-^{208}Pb elastic scattering.

diffraction peak the $t_0\rho$ approximation is the poorest one, being as much as a factor of 2 too large. The first order result differs from the exact Glauber results by only 2–8 % over the entire forward peak. The higher order results are even better. The relatively simple fifth-order result differs by $\lesssim 10\%$ as the cross section decreases by 14 orders of magnitude over the range $0 \le -t \le 1.8$ $(\mathrm{GeV}/c)^2$. We also note that since the $t_0\rho$ approximation generally yields larger cross sections than the $\chi^{(1)}$ or $t\rho$ approximation, except near $t=0$, and since the $\chi^{(1)}$ approximation is generally too low, numerically the $t_0\rho$ approximation is better than the $t\rho$ approximation for $-t \gtrsim 0.3$ $(\mathrm{GeV}/c)^2$, although it is not terribly good over most of the range shown. (An approximation to an approximation sometimes yields better numerical results than the original approximation.)

In Fig. 3 we show the differential cross sections for p-^{58}Ni elastic scattering, as a function of $-t$. In the forward diffraction peak the $t_0\rho$ approximation is the poorest one (although it is certainly not too bad), differing from the exact Glauber result by as much as 32% for $0 \le -t \le 0.03$ $(\mathrm{GeV}/c)^2$. On the other hand, the first order result differs from the exact Glauber result by no more than 9% for $0 \le -t \le 0.03$ $(\mathrm{GeV}/c)^2$. The higher order results are even better. The fifth order results differ by $\lesssim 0.3\%$ as the cross section decreases by 18 orders of magnitude over the range $0 \le -t \le 1.8$ $(\mathrm{GeV}/c)^2$.

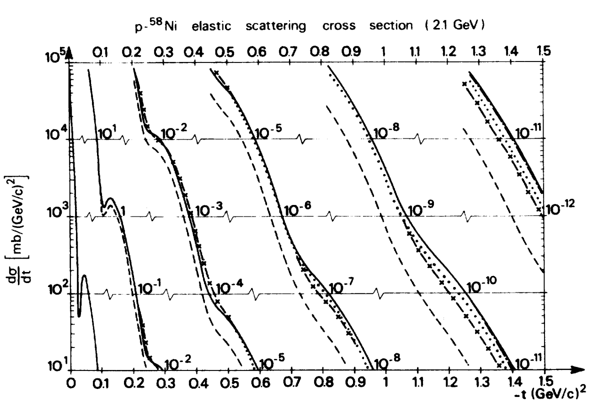

FIG. 3. Same as Fig. 1 for p-^{58}Ni elastic scattering.

In Fig. 4 we show the differential cross sections for p-^{208}Pb elastic scattering, as a function of $-t$. In the forward diffraction peak the $t_0\rho$ approximation is the poorest one, being 4% too low at $t=0$. On the other hand, the first order result ($\chi^{(1)}$) is only 1% too low at $t=0$. The higher order results are even better. The fifth order results differ by $<0.1\%$ as the cross section decreases by some 26 orders of magnitude over the range $0 \le -t \le 1.8$ $(\mathrm{GeV}/c)^2$.

A note of caution should be added to the consideration of the theoretical validity of the numerical results presented at the largest values of $-t$. It is difficult to establish the angular validity of Glauber theory, especially when employed in terms of the basic hadron-nucleon scattering amplitudes rather than in terms of potentials. Studies have shown the existence of cancellations of higher order contributions to the basic theory, extending the range of validity to angles larger (by unknown amounts) than that arrived at by simple order of magnitude estimates. In addition, applications of Glauber theory have often been successful at angles larger than would be considered justified. Nevertheless, we should point out that these simple order of magnitude estimates, using potential descriptions, would require the scattering angle to be less than $\sim 10°$ in our calculations; the largest angle for which we have shown cross sections in the figures is $\sim 24°$.

We have seen that for light nuclei, such as ^{4}He, the $t_0\rho$ approximation is poor everywhere, and the $t\rho$ (or χ_1) approximation is poor everywhere except for the forward diffraction peak. The higher order approximations improve the results substantially. The accuracy of the $t\rho$ (or $\chi^{(1)}$) approximation improves with increasing mass number. But even for ^{208}Pb it is a factor of 1.4 too small at scattering angles for which the cross section has decreased by 9 orders of magnitude relative to the cross section at 0°. (This occurs at a scattering angle of $\sim 10°$.) On the other hand, the second order results for ^{208}Pb are quite accurate, being only 1% too low when the cross section has decreased by 9 orders of magnitude. Our calculations indicate that higher order corrections to the optical phase shift function are necessary in order to obtain a reliable approximation in analyses of cross sections which decrease by several orders of magnitude.

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[1]See, for example, G. W. Hoffman *et al.*, Phys. Rev. C **21**, 1488 (1980).
[2]G. W. Hoffman *et al.*, Phys. Rev. Lett. **47**, 1436 (1981).
[3]See, for example, R. J. Glauber, in *High Energy Physics and Nuclear Structure,* edited by G. Alexander (Wiley, New York, 1967), p. 311.
[4]R. J. Glauber, in *Lectures in Theoretical Physics,* edited by W. E. Brittin and L. G. Dunham (Wiley-Interscience, New York, 1959), Vol. 1, p. 315.
[5]V. Franco and A. Tekou, Phys. Rev. C **16**, 658 (1977).
[6]V. Franco and G. K. Varma, Phys. Rev. C **18**, 349 (1978).
[7]The optical phase shift function $\chi_{\rm opt}(b)$ in the present work corresponds to the optical phase shift functions $\bar{\chi}(b)$ and $\bar{\chi}_{\rm opt}(b)$ in Refs. 5 and 6, respectively, containing, as they do, the effects of the center-of-mass correlation.