# Spurious components in the ideal boson states

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The nature of the appearance of the spurious states in the ideal boson basis states is investigated. It is shown that the occurrence of the spurious states can be mainly attributed to the orthogonality of the boson basis states. It is further observed that the matrix element of the pairing interaction between the ideal boson basis is identical to the corresponding fermion matrix element.

In recent years, the Dyson boson mapping (DBM) has been extensively employed[1-4] for mapping the bifermion operators onto a physical subspace of the boson space. The underlying aim of these studies is to relate the interacting boson model (IBM) parameters to the nuclear shell model.[5] In these investigations the major difficulty one encounters is in the mapping of the fermion basis states. These mapped states, referred to as the physical boson basis (PBB) states, have a very complex structure,[2,3] thereby nullifying any advantage in working in the boson representation. The various truncation schemes to these PBB states are still far from satisfactory.[6] An alternative approach is to employ the ordinary boson basis (BB) states.[7] Here one encounters the problem of the spurious (unphysical) states which appear due to the neglect of the Pauli principle.[4] In the present work the origin of these spurious states is investigated.

It is shown that the physical Hamiltonian does not connect the physical and unphysical states. This is exemplified by considering the pairing interaction between the identical nucleons. In the following we shall first briefly review the DBM.

The Dyson boson mapping is represented as

$$C_\alpha^\dagger C_\beta^\dagger \rightarrow \bar{b}^\dagger_{\alpha\beta} \equiv b^\dagger_{\alpha\beta} - \sum_{\gamma\delta} b^\dagger_{\alpha\gamma} b^\dagger_{\beta\delta} b_{\gamma\delta} \ , \tag{1a}$$

$$C_\beta C_\alpha \rightarrow \bar{b}_{\alpha\beta} \equiv b_{\alpha\beta} \ , \tag{1b}$$

$$C_\alpha^\dagger C_\gamma \rightarrow \sum_\gamma b^\dagger_{\alpha\gamma} b_{\beta\gamma} \ , \tag{1c}$$

with

$$\begin{aligned} b^\dagger_{\alpha\beta} &= -b^\dagger_{\beta\alpha} \ , \\ b_{\alpha\beta} &= (b^\dagger_{\alpha\beta})^\dagger \ , \end{aligned} \tag{2}$$

and $\alpha$ $(=nljm)$ labels the single particle shell model states. The operators $C_\alpha^\dagger(C_\beta)$ are the single particle fermion creation (annihilation) operators. The operators $b^\dagger_{\alpha\beta}$ $(b_{\gamma\delta})$ are the boson creation (annihilation) operators satisfying the following commutation relations:

$$\begin{aligned} [b_{\alpha\beta}, b^\dagger_{\gamma\beta}] &= \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\beta\gamma}\delta_{\alpha\delta} \ , \\ [b^\dagger_{\alpha\beta}, b^\dagger_{\gamma\delta}] &= [b_{\alpha\beta}, b_{\gamma\delta}] = 0 \ . \end{aligned} \tag{3}$$

The DBM Eqs. (1a)–(1c) in terms of the angular momentum operators is

$$A^\dagger_{JM}(ab) \rightarrow \bar{b}^\dagger_{JM}(ab) \equiv b^\dagger_{JM}(ab) - \sum_{\substack{J_1J_2J_3 \\ J_4cd}} \hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4 \begin{Bmatrix} j_a & j_c & J_1 \\ j_b & j_d & J_2 \\ J & J_3 & J_4 \end{Bmatrix} (-1)^{J_3+J_4+J} \{ [b^\dagger_{J_1}(ac) b^\dagger_{J_2}(bd)]_{J_4} \otimes \tilde{b}_{J_3}(cd) \}_{JM} \ , \tag{4a}$$

$$A_{JM}(ab) \rightarrow \bar{b}_{JM}(ab) \equiv b_{JM}(ab) \ , \tag{4b}$$

$$(C_a^\dagger \tilde{C}_b)_{JM} \rightarrow - \sum_{J_1J_2c} \hat{J}_1\hat{J}_2(-1)^{j_a+j_c+J+J_2} \begin{Bmatrix} j_a & j_c & J_1 \\ J_2 & J & j_b \end{Bmatrix} [b^\dagger_{J_1}(ac) \otimes \tilde{b}_{J_2}(bc)]_{JM} \ , \tag{4c}$$

where

$$\begin{aligned} \tilde{C}_a &= (-1)^{j_a-m_a} C_{-m_a} \ , \\ \hat{J}_1 &= (2J_1+1)^{1/2} \ , \\ A^\dagger_{JM}(ab) &= (C_a^\dagger C_b^\dagger)_{JM} \ , \\ b^\dagger_{JM}(ab) &= \sum_{m_a m_b} \begin{bmatrix} j_a & j_b & J \\ m_a & m_b & M \end{bmatrix} b^\dagger_{j_a m_a j_b m_b} \ . \end{aligned} \tag{5}$$

 

The commutation relations between the coupled bosons in Eqs. (4a)–(4c) are

$$[b_{JM}(ab),b^{\dagger}_{J'M'}(a'b')]=\delta_{JJ'}\delta_{mm'}[\delta_{aa'}\delta_{bb'}-(-1)^{j_a+j_b+J}\delta_{ab'}\delta_{ba'}] ,$$
$$[b^{\dagger}_{JM}(ab),b^{\dagger}_{J'M'}(a'b')]=[b_{JM}(ab),b_{J'M'}(a'b')]=0 . \tag{6}$$

It is to be noted that the DBM is (a) finite and (b) nonunitary. Due to (a) any fermion operator will have a finite number of terms in the boson space. For the evaluation of various matrix elements, one needs an appropriate set of basis states. Recently, the use of ideal boson basis states $|i)$, instead of the PBB states $|p)$, has been advocated[7] in favor of their constructional simplicity.

The relation between the two basis sets can be uniquely expressed as

$$|i)=|u)+|p) . \tag{7}$$

The presence of the unphysical states $|u)$ in $|i)$ drastically enlarges the space of diagonalization and also ensures the spurious components in the calculated eigenstates.

The above points can be exemplified by considering a simple case of four particles in a single $j=\frac{7}{2}$ shell. The fermion basis states for total angular momentum $J=0$ subspace are

$$[A^{\dagger}_0(77)A^{\dagger}_0(77)]_{00}[A^{\dagger}_2(77)A^{\dagger}_2(77)]_{00} ,$$
$$[A^{\dagger}_4(77)A^{\dagger}_4(77)]_{00}[A^{\dagger}_6(77)A^{\dagger}_6(77)]_{00} . \tag{8}$$

The single particle state $j=\frac{7}{2}$ is denoted by 7. The overlap integral between any two of the above basis states is

$$\langle 0|[A_J(77)A_J(77)]_0[A^{\dagger}_{J'}(77)A^{\dagger}_{J'}(77)]_0|0\rangle\neq 0 , \tag{9}$$

with $J,J'=0$, 2, 4, and 6. Therefore the basis states [Eq. (8)] are linearly dependent. The physical state is $|(\frac{7}{2})^4 v=0J=0\rangle$, which is, in fact, a linear combination of the above basis states [Eq. (8)], the expansion coefficients being the two particle coefficients of fractional parentage. Therefore, in the $J=0$ subspace there is only one eigenvalue (eigenstate).

The ideal boson basis states are obtained by replacing the bifermion operators in the fermion basis by the corresponding boson operators. These ideal boson states for Eq. (8) are

$$[b^{\dagger}_0(77)b^{\dagger}_0(77)]_{00},[b^{\dagger}_2(77)b^{\dagger}_2(77)]_{00} ,$$
$$[b^{\dagger}_4(77)b^{\dagger}_4(77)]_{00},[b^{\dagger}_6(77)b^{\dagger}_6(77)]_{00} . \tag{10}$$

The overlap integral between any of the two basis states [Eq. (10)] is

$$(0|[b^{\dagger}_J(77)b_J(77)]_0[b^{\dagger}_{J'}(77)b^{\dagger}_{J'}(77)]_0|0)=0 , \tag{11}$$

for $J,J'=0$, 2, 4, 6 and $J\neq J'$. From the overlap it is clear that the states [Eq. (10)] are linearly independent. Therefore, in the boson space for the present case, one has to diagonalize a 4×4 matrix yielding four eigenvalues (eigenvectors). As mentioned before, in the fermion space there is only one physical eigenstate, implying that out of the possible four states three are spurious. Thus, the occurrence of the spurious states can be attributed to the orthogonality of the boson basis states.

The spurious states can be identified by making use of the fact that the eigenvalue equation can be uniquely separated into two disjoint parts—one corresponding to the physical subspace and the other to the unphysical subspace. The spurious states in the calculated spectra are then separated from the physical subspace by requiring the invariance of the physical subspace under the action of the mapped operator $\hat{0}_B$, i.e.,

$$(u|\hat{0}_B|p)=0 . \tag{12}$$

In the following, these arguments are exemplified by considering the case of the pairing interaction between the identical nucleons.

The pairing interaction between the identical nucleons is written as

$$H_F=\sum_a \epsilon_a \hat{N}_a - G/2\sum_{ab}\sqrt{r_a r_b}\,A^{\dagger}_{00}(aa)A_{00}(bb) , \tag{13}$$

where

$$\hat{N}_a=\sum_{m_a} C^{\dagger}_{m_a}C_{m_a};\quad r_a=\frac{(2j_a+1)}{2}$$

and $\epsilon_a$ denotes the single particle energies, and $G$, possessing the dimensions of energy, is quoted as the strength of the pairing interaction. We restrict ourselves here to only two levels. For this case the algebraic expressions for the matrix are available.[8] The fermion basis states are

$$|m,n-m\rangle=M_m^{-1}(A^{\dagger}_{00}(aa))^m(A^{\dagger}_{00}(bb))^{n-m}|0\rangle . \tag{14}$$

Using the mapping [Eqs. (4a)–(4c)], we find that the pairing Hamiltonian Eq. (13) is given in the boson space by

$$H_B=\sum_a \epsilon_a b^{\dagger}_{00}(aa)b_{00}(bb)-G\sum_{ab}\sqrt{r_a r_b}\left[b^{\dagger}_{00}(aa)-\sum_{\substack{J_1J_2J_3\\cd}}\hat{J}_1\hat{J}_2\hat{J}_3\hat{j}_a^{-1}(-1)^{j_a+j_d+J_1+J_3}\begin{Bmatrix} j_c & J_1 & j_a\\ J_2 & j_d & J_3\end{Bmatrix}\right.$$
$$\left.\times[b^{\dagger}_{J_1}(ac)b^{\dagger}_{J_2}(ad)_{J_3}\otimes\tilde{b}_{J_3}(cd)]_{00}b_{00}(bb)\right] . \tag{15}$$

The ideal boson basis states corresponding to states Eq. (14) are

$$|\, m,n-m) = B_m^{-1}(b_{00}^{\dagger}(aa))^m (b_{00}^{\dagger}(bb))^{n-m}\,|\,0) \ , \qquad (16)$$

where

$$B_m^{-1} = \frac{1}{\sqrt{m!(n-m)!}} \ ,$$

and $|\,0)$ is the boson vacuum defined through

$$b_{JM}(ab)\,|\,0) = 0 \ .$$

The bra states corresponding to Eq. (16) are

$$(m',n-m'\,| = B_{m'}^{-1}(0\,|\,(b_{00}(aa))^{m'}(b_{00}(bb))^{n-m'} \ . \qquad (17)$$

The matrix of the first term of Eq. (15) between the states [Eqs. (16) and (17)] is

$$\delta_{mm'}(n-2m)(\epsilon_b - \epsilon_a) \ ,$$

which is identical to that given in Ref. 8. The matrix element of the second term has three components. These are

$$-G[m(r_a+1-m)+(n-m)(r_b+1-n+m)]\delta_{mm'} \ ,$$

denoted I,

$$-G\{[m(n-m+1)r_a r_b]^{1/2} - (n-m)[m(n-m+1)]^{1/2}\sqrt{r_a/r_b}\,\}\delta_{m',m-1} \ ,$$

denoted II,

$$-G\{[(m+1)(n-m)r_a r_b] - m[(m+1)(n-m)]^{1/2}\sqrt{r_a/r_b}\,\}\delta_{m',m+1} \ ,$$

denoted III. The diagonal term (I) is exactly the same as that given in Ref. 8. Due to the nonunitary character of the DBM, the matrix elements $H_{mm'} \neq H_{m'm}$ for $m \neq m'$. Using the hermitization prescription[2]

$$H_{mm'} = H_{m'm} = (H_{mm'}H_{m'm})^{1/2} \ ,$$

it is observed that the off diagonal terms (II) and (III) lead to similar results given in Ref. 8. Therefore, it turns out that the matrix element of the pairing interaction between the ideal boson basis states is equivalent to the corresponding fermion matrix element.

From the above analysis it follows that the physical subspace does not mix with the unphysical one. Thus, one need not worry about the spurious states while carrying out the calculations in the ideal boson basis states. The spurious states in the calculated spectra can be identified by making use of Eq. (12).

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