Algebraic solution of a general quadrupole Hamiltonian in the interacting boson model

S. Kuyucak and I. Morrison
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia
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A method based on an eigenmode condition and projection techniques is presented for solution of a general quadrupole Hamiltonian in the interacting boson model. It is shown that the intrinsic states obtained from the eigenmode condition provide a zeroth order solution to the diagonalization of the corresponding quadrupole Hamiltonian. The method is in effect a $1/N$ expansion, where N is the boson number, ideally suited for the deformed nuclei for which $N \approx 12-16$. Because of certain cancellations with the normalization, zeroth order solutions of the intrinsic states are sufficient to obtain matrix elements to order $O(1/N^2)$.

In recent years, the interacting boson model (IBM) has been almost a standard tool in phenomenological analysis of the low-lying nuclear data. Besides its simplicity, another appealing feature of the model has been the three limiting symmetries each of which can be associated with a well defined nuclear shape.[1] Although the dynamical symmetries are not strictly realized in nature, they allow analytical solution of the model problem which may be a convenient starting point. As long as deviations from the symmetry are small, these can be treated in perturbation theory with the benefit that analytical solutions often provide insight to the problem at hand which are hard to come by in a numerical solution.

Extensive numerical calculations of the deformed nuclei, which correspond closest to the SU(3) limit of the IBM, indicate that this limit is far from being realized.[2] Phenomenological values of the parameter χ in the quadrupole operator

$$Q_\mu = (s^\dagger \tilde{d} + d^\dagger s)^{(2)}_\mu + \chi (d^\dagger \tilde{d})^{(2)}_\mu \ , \tag{1}$$

range from[3] -0.4 to -0.6 to be compared with the SU(3) value of -1.32. Thus a perturbation treatment of the deformed nuclei based on the SU(3) limit is not practical, and one has to go beyond the group theoretical techniques for algebraic solution.

Another complication for the deformed nuclei arises from the need to include the g boson in addition to the usual s and d bosons. Microscopic calculations show that there are non-negligible admixtures of $J=4^\dagger$ (G) nucleon pairs in the low-lying collective states. Thus mapping of fermion pairs to bosons requires a certain amount of g boson which is the image of the G pair.[4] At the phenomenological level, inclusion of g boson is found to be essential for a detailed description of a complete set of spectra.[5] Empirical g-factor variations in the ground band is another phenomenon which is not explained in the sd IBM and requires inclusion of the g boson.[6]

Because the sdg IBM Hamiltonian contains many more parameters (32 compared to 9 of the sd model), selection and determination of a simple set of parameters through numerical analysis is an arduous task. Algebraic solutions of relevant model problems would certainly shed some light on this process.

In the following, we will indicate an algebraic solution of the Hamiltonian

$$H = -\kappa Q \cdot Q - \kappa' L \cdot L \ , \tag{2}$$

which has been successfully applied to the deformed region in the sd IBM numerically.[2,3] Since the $L \cdot L$ term is always diagonal, it has no effect on the wave functions and will be dropped henceforth. It can be easily restored by adding $-\kappa' L(L+1)$ to the final energy expressions.

We introduce the boson creation ($b^\dagger_{lm}$) and annihilation (b_{lm}) operators, where $l=0,2,4,\ldots,p$ correspond to $s,d,g,\ldots$ bosons, and m is the projection on an appropriate axis. The quadrupole operator, Eq. (1), can be generalized to arbitrary kinds of bosons by defining a parameter matrix q of order $1+p/2$

$$q = \begin{vmatrix} 0 & q_{02} & 0 & & \\ q_{20} & q_{22} & q_{24} & 0 & \vdots \\ 0 & q_{42} & q_{44} & q_{46} & 0 \\ & & \cdots & & \end{vmatrix} . \tag{3}$$

The matrix q is symmetric to ensure the hermiticity of the quadrupole operator which is now written as

$$Q_\mu = \sum_{j,l} q_{jl} [b^\dagger_j \tilde{b}_l]^{(2)}_\mu \ . \tag{4}$$

In the usual notation, $q_{02}=1$, $q_{22}=\chi$ in the (sd) case and $q_{02}=1$, $q_{22}=\beta$, $q_{24}=\gamma$, $q_{44}=\delta$ in the (sdq) case.

Intrinsic boson operators are given by

$$b^\dagger_m = \sum_l x_{lm} b^\dagger_{lm}, \quad \sum_l (x_{lm})^2 = \mathbf{x}_m \cdot \mathbf{x}_m = 1 \ , \tag{5}$$

where $m=0,1,\ldots,p$.

Applying the eigenmode condition[7]

$$[Q_0, b^\dagger_m] = \lambda_m b^\dagger_m \ , \tag{6}$$

to Eqs. (4) and (5) leads to the eigenvalue equation

$$\sum_l (-)^m \langle jml-m \mid 20 \rangle q_{jl} x_{lm} = \lambda_m x_{jm} \ , \tag{7}$$

which determines all the intrinsic boson operators in

terms of the quadrupole parameters q_{jl}. As an example, in the *sdg* IBM, Eq. (7) has order 3 for $m=0$, order 2 for $m=1,2$ and order 1 for $m=3,4$. Thus there are three solutions for $m=0$ denoted by $b_0^\dagger$, $b_0'^\dagger$, and $b_0''^\dagger$ (with eigenvalues $\lambda_0>\lambda_0'>\lambda_0''$) which correspond to the ground, β- and β'-band intrinsic operators.

Next we show that for the low-lying states, which are dominated by the ground-band intrinsic operator, $b_0^\dagger$, the leading order solutions obtained by varying the Hamiltonian, Eq. (2), are the same as Eq. (7). First, we treat the simple case of $m=0$ (from now on we drop the subscript if $m=0$, e.g., $b_0\equiv b$), and consider a trial intrinsic state with N bosons. Then

$$\langle Q\cdot Q\rangle_N=\frac{\langle -\mid b^N Q\cdot Q(b^\dagger)^N\mid -\rangle}{\langle -\mid b^N(b^\dagger)^N\mid -\rangle}\ . \tag{8}$$

Using standard boson commutation techniques, we obtain

$$\langle Q\cdot Q\rangle_N=N(N-1)\left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}}\right]^2+N\frac{C_0}{\mathbf{x}\cdot\mathbf{x}}\ , \tag{9}$$

where

$$\begin{aligned}A_m&=\sum_{jl}(-)^m\langle jml-m\mid 20\rangle q_{jl}x_{jm}x_{lm}\ ,\\ C_m&=\sum_{jl}\frac{5}{2l+1}(q_{jl}x_{lm})^2\ .\end{aligned} \tag{10}$$

Variation of the leading term in Eq. (9) is equivalent to varying A_0 subject to the condition $\mathbf{x}\cdot\mathbf{x}=1$ which (upon introducing the Lagrange multiplier λ) leads to Eq. (7) with $m=0$.

Evaluation of $\langle Q\cdot Q\rangle$ for a general intrinsic state

$$\mid N_0,N_1,\ldots,N_p\rangle=(b_0^\dagger)^{N_0}(b_1^\dagger)^{N_1}\cdots(b_p^\dagger)^{N_p}\mid -\rangle\ , \tag{11}$$

is somewhat more involved due to the presence of cross terms. The final result is

$$\langle Q\cdot Q\rangle_{N_0,N_1,\ldots,N_p}=\sum_m N_m\left[\frac{C_m}{\mathbf{x}_m\cdot\mathbf{x}_m}+(N_m-1)\left[\frac{A_m}{\mathbf{x}_m\cdot\mathbf{x}_m}\right]^2+2\sum_{m'\neq m}N_{m'}\left[\frac{Am}{\mathbf{x}_m\cdot\mathbf{x}_m}\cdot\frac{Am'}{\mathbf{x}_{m'}\cdot\mathbf{x}_{m'}}+\sum_{jl}\frac{U_{jmm'}}{\mathbf{x}_m\cdot\mathbf{x}_m}\frac{U_{lmm'}}{\mathbf{x}_{m'}\cdot\mathbf{x}_{m'}}x_{jm}x_{lm}\right]\right]\ , \tag{12}$$

where

$$U_{lmm'}=\sum_{j'}\langle lmj'-m'\mid 2m-m'\rangle q_{lj'}x_{j'm'}\ . \tag{13}$$

Variation of the leading terms in Eq. (12), subject to the conditions, $\mathbf{x}_m\cdot\mathbf{x}_m=1$, $m=0,1,\ldots,p$ gives

$$\sum_{m'}N_{m'}A_{m'}\sum_l(-)^m\langle jml-m\mid 20\rangle q_{jl}x_{lm}+\sum_{m'\neq m}N_{m'}U_{jmm'}\sum_l U_{lmm'}x_{lm}=\lambda_m x_{jm}\ . \tag{14}$$

Invoking the dominance of $b_0^\dagger$ (i.e., $N_0\gg N_m$, $m\neq 0$), it is easy to see that Eq. (14) reduces to Eq. (7) for $m=0$ and $m>2$. For $m=1,2$, the second term in Eq. (14) does not vanish, however comparison of the solutions in the *sdg* model shows that the two equations are equivalent in the SU(3) limit. Away from the SU(3) limit, especially for $m=2$ (which is more important because $b_2^\dagger$ generates the γ band) Eq. (7) remains an excellent approximation to Eq. (14).

Until now, we have discussed properties of intrinsic state solutions to Eq. (2). To each boson condensate, or excitations thereof, a band of states can be obtained by angular momentum projection. In variation before projection (VBP), the intrinsic state is fixed for all states in the band. In contrast, in variation after projection (VAP), the intrinsic state is varied independently for each L, and hence can produce stretching effects. In general, VAP incorporates band-mixing effects and is preferable to VBP when technically feasible.

In our model, we finally show that VAP changes only the next to leading order term in the VBP calculation, hence the two methods are equivalent to the leading order. The intrinsic state solutions are therefore stable, changes induced by rotation being of order $1/N$. Again, we first treat the simple case of $m=0$

$$\langle Q\cdot Q\rangle_{N,L}=\frac{\langle -\mid b^N Q\cdot QP_{00}^L(b^\dagger)^N\mid -\rangle}{\langle -\mid b^N P_{00}^L(b^\dagger)^N\mid -\rangle}\ , \tag{15}$$

where

$$P_{MK}^L=\frac{2L+1}{8\pi^2}\int D_{MK}^{L*}(\Omega)R(\Omega)d\Omega\ .$$

is the projection operator. Writing the rotation operator explicitly, Eq. (15) becomes

$$\langle Q\cdot Q\rangle_{N,L}=\frac{\int d\beta\sin\beta d_{00}^L(\beta)\langle -\mid b^N Q\cdot Qe^{-i\beta J_y}(b^\dagger)^N\mid -\rangle}{\int d\beta\sin\beta d_{00}^L(\beta)\langle -\mid b^N e^{-i\beta J_y}(b^\dagger)^N\mid -\rangle}\ , \tag{16}$$

where $d_{mm'(\beta)}^L$ are the reduced rotation matrices. Defining the rotated intrinsic operator as

$$b_R^\dagger=e^{-i\beta J_y}b^\dagger e^{i\beta J_y}\ , \tag{17}$$

the matrix elements in Eq. (16) can be calculated as

$$\langle - \mid b^N e^{-i\beta J_y}(b^\dagger)^N \mid - \rangle = N! \left[\frac{\partial b_R^\dagger}{\partial b^\dagger} \right]^N = N! \left[\sum_l x_l^2 d_{00}^l(\beta) \right]^N \equiv N![Z(\beta)]^N , \tag{18}$$

$$\begin{aligned}
\langle - \mid b^N Q\cdot Q e^{-i\beta J_y}(b^\dagger)^N \mid - \rangle &= NN![Z(\beta)]^{N-2} \left\langle - \left| \left[\frac{N-1}{4} \left[\frac{\partial}{\partial b^\dagger} \right]^2 \left[\frac{\partial}{\partial b_R} \right]^2 + Z(\beta) \frac{\partial}{\partial b^\dagger} \frac{\partial}{\partial b_R} \right] Q\cdot Q \right| - \right\rangle \\
&= NN![Z(\beta)]^{N-2} \left[(N-1) \sum_\mu \left[\sum_{jl} \langle j0l\mu \mid 2\mu \rangle q_{jl} x_j x_l d_{\mu 0}^l \right]^2 + Z(\beta) \sum_{jl} \frac{5}{2l+1} (q_{jl}x_l)^2 d_{00}^l \right] \\
&= 5NN![Z(\beta)]^{N-2} \left[(N-1) \sum_{\substack{Kll' \\ jj'}} \langle j0j'0 \mid K0 \rangle \langle l0l'0 \mid K0 \rangle \begin{Bmatrix} j' & j & K \\ l & l' & 2 \end{Bmatrix} q_{jl} q_{j'l'} x_j x_{j'} x_l x_{l'} d_{00}^K \right. \\
&\quad \left. + Z(\beta) \sum_{jl} \frac{1}{2l+1} (q_{jl}x_l)^2 d_{00}^l \right] ,
\end{aligned} \tag{19}$$

where in the last step, we have used angular momentum algebra to combine the d matrices. In order to evaluate the β integrals, we approximate $[Z(\beta)]^N$ with a Gaussian,[8] which is valid for large N

$$[Z(\beta)]^N \simeq (\mathbf{x}\cdot\mathbf{x})^N e^{-\beta^2/\Gamma}, \quad \Gamma = \frac{2}{Ny} \tag{20}$$

with

$$y = \tfrac{1}{2} \sum_l l(l+1) x_l^2 / \mathbf{x}\cdot\mathbf{x} .$$

Extending the β integration to ∞, and using the integral formula[8]

$$\int_0^\infty d\beta \sin\beta P_L(\cos\beta) e^{-\beta^2/\Gamma} = \frac{\Gamma}{2} \left[1 - \frac{\Gamma}{4}(\mathcal{L} + \tfrac{2}{3}) + \frac{\Gamma^2}{2!4^2}(\mathcal{L}^2 + 2\mathcal{L} + \tfrac{8}{15}) + \cdots \right] , \tag{21}$$

where $\mathcal{L} \equiv L(L+1)$, we obtain after some algebra

$$\langle Q\cdot Q \rangle_{N,L} = N^2 \left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}} \right]^2 + N \left[\left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}} \right]^2 - \frac{1}{2y} \frac{A_0 B_0}{(\mathbf{x}\cdot\mathbf{x})^2} + \frac{C_0}{\mathbf{x}\cdot\mathbf{x}} \right] + \frac{\mathcal{L}}{y} \left[- \left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}} \right]^2 + \frac{1}{4y} \frac{A_0 B_0}{(\mathbf{x}\cdot\mathbf{x})^2} \right] + \cdots . \tag{22}$$

Here A_0 and C_0 are defined as in Eq. (10) and B_0 as

$$B_0 = \sum_{jl} (2l^2 + 2l - 3) \langle j0l0 \mid 20 \rangle q_{jl} x_j x_l . \tag{23}$$

Since VAP is equivalent to the Schrödinger equation in the full boson space (and this equivalence is carried to the leading order in the case of a special subspace) this completes the proof that the eigenmode condition provides a zeroth order solution to the $m=0$ intrinsic states.

Before proceeding further, we have a few comments on Eq. (22). First, it correctly reproduces the energy eigenvalues in the SU(3) limit. (Corrections coming from the variation of Eq. (22) vanish in that limit.) Secondly, the general expression for the expansion has the form

$$\langle Q\cdot Q \rangle_{N,L} = N^2 \sum_{n,m=0}^{\infty} \left[\frac{\mathcal{L}}{N^2} \right]^n \frac{C_{nm}}{N^m} , \tag{24}$$

where C_{00}, C_{01}, and C_{10} correspond to the three terms in Eq. (22). The coefficient C_{20}/N^2 determines the deviation from the $L(L+1)$ rule, which is usually obtained through band-mixing calculations in the geometrical model. Here, Eq. (24) automatically includes mixing of all the $K=0$ bands [subsequent to our choice of $K=0$ subspace in Eq. (15)], and no further parameter is needed other than those of the quadrupole operator. Finally, variation of Eq. (24), with respect to x_l, leads to solutions of a similar form

$$x_l = \sum_{n,m=0}^{\infty} \left[\frac{\mathcal{L}}{N^2} \right]^n \frac{y_{nm}}{N^m} , \tag{25}$$

where y_{nm} are the coefficients obtained from the variational equations. Equation (25) shows that, in general, the structure of the intrinsic state depends on L, i.e., the boson system stretches in response to the rotation. [The (sd) case and the SU(3) limit are exceptions where no stretching occurs[6].]

Rather than going into the general case, which is technically complicated, we will repeat the above calculation for the β band which should also illustrate the general case. Denoting the intrinsic operator for the β band by $b'^\dagger$, the expectation value of $Q\cdot Q$ is given by

$$\langle Q\cdot Q \rangle_{\beta,L} = \frac{\langle - \mid b^{N-1} b' Q\cdot Q P_{00}^L (b^\dagger)^{N-1} b'^\dagger \mid - \rangle}{\langle - \mid b^{N-1} b' P_{00}^L (b^\dagger)^{N-1} b'^\dagger \mid - \rangle} . \tag{26}$$

Following steps similar to Eqs. (16)–(22), we obtain

$$\langle Q\cdot Q\rangle_{\beta,L}=N^2\left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}}\right]^2+N\left[-\left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}}\right]^2+2\frac{A_0}{\mathbf{x}\cdot\mathbf{x}}\frac{A_0'}{\mathbf{x}\cdot\mathbf{x}'}-\frac{1}{2y}\frac{A_0B_0}{(\mathbf{x}\cdot\mathbf{x})^2}+\frac{C_0}{\mathbf{x}\cdot\mathbf{x}}\right]+\frac{\mathcal{L}}{y}\left[-\left[\frac{A_0}{\mathbf{x}\cdot\mathbf{x}}\right]^2+\frac{1}{4y}\frac{A_0B_0}{(\mathbf{x}\cdot\mathbf{x})^2}\right]+\cdots, \tag{27}$$

where primes denote the β band and $A_0'=A_0(x')$. Equation (27) shows that the leading term for the β band is left intact after projection. Similar calculations can be repeated for the other intrinsic states.

In the SU(3) limit, Eq. (27) reproduces the energy eigenvalues obtained from the Casimir operator. This might come as a surprise at first thought because the form of the intrinsic state

$$|\beta\rangle=\left[(b^\dagger)^{N-1}b'^\dagger+(b^\dagger)^{N-2}\sum_{m\neq 0}a_m b_m^\dagger b_{-m}^\dagger+\cdots\right]|-\rangle\ , \tag{28}$$

suggests corrections to Eq. (27) at the $O(1/N)$ level due to $m\neq 0$ intrinsic operators. However, evaluation of $\langle Q\cdot Q\rangle$ for the above state gives to leading order for each term

$$\langle Q\cdot Q\rangle_{\beta,L}=\frac{N^2A_0^2+NA_0^2\sum_m a_m^2+\cdots}{(\mathbf{x}\cdot\mathbf{x})^2\left[1+\frac{1}{N}\sum_m a_m^2+\cdots\right]}\ , \tag{29}$$

which shows that, due to cancellation with the normalization, such corrections only enter at the $O(1/N)^2$ level.

Another consequence of the dominance of the ground-band intrinsic operator is that the moment of inertia of different bands remain the same to the leading order [see the C_{10} terms in Eqs. (22) and (27)]. In general, the C_{11} terms will be different and the moment of inertia of neighboring bands will differ at the $1/N$ level.

The present method is very economical in the sense that it is purpose specific, i.e., in order to calculate a nuclear property in a given band, one need not diagonalize the Hamiltonian in the full space. On the other hand it is general enough to allow study of the influence of higher bosons ($g,i,\ldots$). The method enables prediction of nuclear properties in terms of the quadrupole parameters g_{jl} [three in the (sdg) case]. It has already been used to indicate g-boson and stretching effects[6] in the rare earth region after proof that a standard VBP approach could not suffice.[9] In principle, it would be possible to correlate algebraically most measurables involving ground β and γ bands in terms of the quadrupole matrix, as a guide to trend analysis or to assist more accurate (but tedious) numerical calculations. Application of the method to calculation of other properties (e.g., electromagnetic transition rates) will be given in a future publication.

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