Invariant light front perturbation theory

Michael G. Fuda
Department of Physics and Astronomy, State University of New York at Buffalo, Buffalo, New York 14260
(Received 6 April 1987)

It is shown that with the introduction of an external lightlike vector, a light front perturbation theory for the S matrix in quantum field theory can be developed in which the individual terms in the series are invariant functions of the particle variables and the external vector. No limiting processes are involved. The dependence of the results on the external lightlike vector is discussed.

I. INTRODUCTION

It is common knowledge that a change of variables often simplifies the solution of a problem and can dramatically alter the appearance of a theory. One of the most striking examples of this occurs in the implementation of the principles of special relativity, when the standard space-time coordinates (t,x,y,z) are replaced by the so-called light front coordinates (τ,x,y,ζ), where $\tau=2^{-1/2}(t+z)$ and $\zeta=2^{-1/2}(t-z)$. This has the effect of changing the metric tensor from the standard one whose nonzero elements are $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$ to one whose nonzero elements are $g^{03}=g^{30}=-g^{11}=-g^{22}=1$. In the usual tensor notation, the commutation relations for the generators of the Poincaré group look the same with either choice of the coordinates, however, the change from the standard to the light-front metric has a disproportionate effect on the implementation and solution of the commutation relations.

It was Dirac[1] who first emphasized this point and stressed the corresponding lack of uniqueness in the formulation of Poincaré invariant Hamiltonian theories. Essentially, the various dynamical schemes correspond to different choices for the hypersurface on which the state of a system is specified. In the standard and light front formulations, the hypersurfaces are $t=0$ and $\tau=0$, respectively, and all of the surfaces related by Poincaré transformations. It is known[2] that there are five inequivalent classes of such surfaces with reasonable symmetry properties. The generators that induce transformations from one point to another on one of these hypersurfaces are kinematical, while those that produce transformations from one hypersurface to another are dynamical and are called Hamiltonians. Of the five possible dynamical schemes, the light front approach has the smallest number of Hamiltonians, namely three. This has important practical consequences and makes light front dynamics quite appealing.

Interest in the light front approach was greatly stimulated by Weinberg's analysis[3] of the infinite momentum limit of time ordered or old fashioned perturbation theory. In a Lorentz frame moving at a high velocity in a direction opposite to a system's total three momentum $\mathbf{P}$, all particles move with large velocities more or less in the direction of $\mathbf{P}$. In the limit of infinite $|\mathbf{P}|$, many of the diagrams of time ordered perturbation theory (TOPT) vanish, and the energy denominators are replaced by invariant s denominators, where s is the square of the total four momentum.

Susskind[4] demonstrated that Weinberg's infinite momentum limit is equivalent to changing variables from (t,x,y,z) to (τ,x,y,ζ). Susskind used the light front variables to draw attention to an isomorphism between a subgroup of the Poincaré group and the Galilean symmetry group of nonrelativistic quantum mechanics in two dimensions. This isomorphism results in a nonrelativistic structure for quantum mechanics in the light front scheme.

This nonrelativistic analogy has provided much of the inspiration for attempts to construct relativistic Hamiltonian theories for few particle systems within the light front framework. Bardacki and Halpern[5] have succeeded in constructing potential models for two-particle systems which contain ten generators that satisfy the commutation relations for the Poincaré group. Leutwyler and Stern[2] have analyzed very systematically one of the most desirable features of light front dynamics in this context, namely they have shown that, in general, the inner variables that describe the structure of a system uncouple from the motion of the system as a whole. This is, of course, analogous to the separation of the center of mass variables in nonrelativistic quantum mechanics.

Besides these attempts to solve the Poincaré group commutation relations directly, there has been some effort to derive few particle equations by summing subsets of light front perturbation theory (LFPT) diagrams. In fact, in his original paper on the infinite momentum frame, Weinberg[3] presented a light front ladder approximation for the relativistic two-body problem, and for this reason such equations are often called Weinberg equations.

A shortcoming of the original Weinberg equation[3] is that it is not Lorentz invariant. This is a direct consequence of the fact that the LFPT developed by Weinberg is a limiting case of TOPT, which gives series whose individual terms are not Lorentz invariant. Namyslowski and co-workers[6] have overcome this difficulty to some extent by introducing a variation of the original Weinberg variables which involves invariant projections of four vectors on tetrads, where a tetrad is a set of four mutually orthonormal four vectors.

The lack of invariance for the individual terms in LFPT can also be overcome by first reformulating TOPT so that its individual terms are Lorentz invariant. Such a reformulation was developed several years ago by Kadyshevsky.[7] In his approach an invariant time direction is established by introducing a timelike, unit four vector λ whose components $\lambda^\mu=(\lambda^0,\boldsymbol{\lambda})$ satisfy $\lambda^2=(\lambda^0)^2-\boldsymbol{\lambda}^2=1$. In the diagrams which arise in connection with the perturbation theory he has developed for the S matrix, all of the particles are on the mass shell and the total four momentum is conserved at each vertex. Besides the particles described by the underlying quantum fields, there appear in the diagram so-called quasiparticles or spurions which make it possible for intermediate particles to be on the mass shell while conserving the total four momentum. The appearance of the quasiparticles makes Kadyshevsky's diagrams more complicated than those of Feynman, TOPT or LFPT.

Recently it has been shown[8] that an invariant version of TOPT can be formulated with an external vector λ, without introducing quasiparticles. This invariant perturbation theory leads to graphical rules which are the same as those of TOPT except for the replacement of three-momentum conserving δ functions by invariant three-dimensional δ functions and the use of invariant denominators rather than energy denominators. The invariants that occur in the new denominators are of the form $\lambda\cdot P$, where P is the total four momentum of a state. This invariant time ordered perturbation theory (ITOPT) becomes identical to TOPT in a set of Lorentz frames called λ frames. A λ frame is one in which $\lambda=(1,\mathbf{0})$. If λ is chosen parallel to the total four momentum of the system of interest, a λ frame is the same as a c.m. frame.

An invariant version of LFPT has been obtained[9] by taking the infinite λ limit of ITOPT. This limit is determined by letting the components of λ become infinite subject to the constraint $\lambda^2=\lambda_0^2-\boldsymbol{\lambda}^2=1$. That this limit should lead to a version of LFPT can be seen quite simply. In TOPT the spacelike surfaces $t=$const play a privileged role, while in ITOPT these surfaces are replaced by the spacelike surfaces $\lambda\cdot x=\tau$, where τ is an invariant time parameter. When the infinite λ limit is taken, these surfaces become light fronts. The graphical rules for this invariant version of LFPT can be stated in a form identical to that given by Weinberg.[3] The difference lies in the definition of the variables that are used to label the particle lines. For an on-the-mass shell particle with four momentum p in a state (initial, final, or intermediate) with total four momentum P, the original Weinberg variables are $\eta=(p^0+p^3)/(P^0+P^3)$ and $\mathbf{q}=\mathbf{p}_\perp=(p^1,p^2)$ while the new definitions are $\eta=\xi\cdot p/\xi\cdot P$ and $\mathbf{q}=\mathbf{p}_\perp-\eta\mathbf{P}_\perp$. Here ξ is a lightlike vector that arises from λ in the infinite λ limit. Clearly the η's are invariants, and it can be shown[9] that dot products formed from the two vectors $\mathbf{q}$ are also.

The use of the new Weinberg variables has also been considered by Namyslowski,[10] and he has stressed the fact that they provide a manifestly invariant form for the cluster decomposition property. This property plays an important role in systems with three or more particles, and amounts to the reasonable requirement that a subsystem should behave as if it were isolated when its interactions with the rest of the system are negligible.

The invariant LFPT (ILFPT) has been used to derive[9] a set of three-particle integral equations within the context of a model field theory which describes the interaction of the quanta ψ of a charged scalar field with the quanta ϕ of a neutral scalar field according to the virtual process $\psi\rightleftarrows\psi+\phi$. The equations for ϕ-ψ scattering were obtained by summing all ILFPT diagrams with $|\psi\rangle$, $|\psi,\phi\rangle$, and $|\psi,2\phi\rangle$ intermediate states. The analysis showed that the amplitudes for $\phi+\psi\to\phi+\psi$ and $\phi+\psi\to2\phi+\psi$ can be obtained by solving a manifestly invariant, linear, three-dimensional integral equation which satisfies the cluster property. Moreover, the amplitudes satisfy two and three particle unitarity. The fact that the equations satisfy the cluster property without introducing a spurious singularity in the three-particle s variable[11] is consistent with Namyslowski's analysis.[10] It is known[12] that this type of spurious singularity can lead to spurious bound state solutions of three-particle integral equations, so avoiding its introduction while maintaining the cluster property is of some practical value.

Since the three-particle integral equations obtained with the help of invariant LFPT have so many desirable features, it is natural to try to extend the analysis to a physical system such as the pion-nucleon system. Before this can be done, however, it is necessary to develop ILFPT in a broader context than that considered in Ref. 9. There the development was carried out in the framework of the model field theory described above. In order to construct a model for pion-nucleon scattering, it is necessary to include the spin and isospin degrees of freedom. It has been known for some time[13,14,10] that the infinite momentum limit of a theory with fermions present has additional features and complications that do not arise in the spinless case. Therefore it is not surprising that taking the infinite λ limit of ITOPT when fermions are present is awkward, to say the least. This raises the question as to whether it is possible to develop ILFPT without resorting to limiting processes. Here we will see that this question can be answered in the affirmative.

The approach that will be followed here in developing ILFPT without limiting processes is similar to that used to obtain ITOPT.[8] The main difference is that it will be assumed here that the underlying quantum field theory is formulated within the framework of light front dynamics. In particular, it will be assumed that the components of the four-momentum operators are light front components, which means that P^0 P^1 P^2, and P^3 generate translations in ζ, x, y, and τ, respectively. The construction of the Poincare' generators within the context of quantum field theory has been carefully discussed by Yan *et al.*[15] Also, the role played by the *timelike* vector λ in the development of ITOPT, will be taken over here by a *lightlike* vector ξ. Instead of a λ frame, which is one in which the ordinary components of λ are $\lambda^\mu=(1,\mathbf{0})$, we will deal with a ξ frame, which is one in which the *light front* components of ξ are $\xi_\mu=(1,0,0,0)$. In the ξ frame we will develop a perturbation theory which is analogous to TOPT, but with the role of the time and energy taken over by τ and P^3, respectively. We will show that when individual terms in the expansion of the S matrix are written in an arbitrary frame of reference, they are invariant functions of the particles' variables and ξ.

A question arises as to whether or not the results obtained with the invariant perturbation theories developed in Refs. 8 and 9 and here depend on the arbitrary external vectors λ and ξ, respectively. The answer to the question is in principle, no; in practice, maybe. Assuming that the underlying quantum field theory is covariant, the S-matrix elements should not depend on an arbitrary four vector. A formal proof of this is given in Ref. 8, where it is shown that all of the ITOPT contributions to an S-matrix element in a particular order of the coupling constant must combine to give a result which is independent of λ. It is not difficult to see by example that combining the contributions from ITOPT diagrams that differ only in the ordering of the vertices gives the same result as the corresponding Feynman diagram, which of course is independent of λ. It should be emphasized that in general this is only true on shell, i.e., when the initial and final total four momenta in the amplitudes are the same, as in S-matrix elements. It can be argued, in a somewhat cavalier fashion, that since the invariant version of LFPT can be obtained as the infinite λ limit of ITOPT, the above comments also apply with regard to the ξ dependence of ILFPT contributions to the S-matrix. In Sec. IV, where the formalism developed in the previous section is applied to the model field theory of Refs. 8 and 9, we will see by means of an example how the ξ dependence drops out when the ordered contributions are combined on shell.

The outline of the paper is as follows. In Sec. II we briefly comment on the light front variables and summarize how the state vectors, field operators, and four-momentum operator transform from one Lorentz frame to another. The formal perturbation theory is developed in Sec. III, and it is verified that individual contributions to the S matrix are invariant functions of the particle variables and the external, lightlike vector ξ. In Sec. IV the formalism is illustrated by applying it to the model field theory used in Refs. 8 and 9, and the ξ dependence of some S-matrix elements is examined. A brief discussion and suggestions for future work is presented in Sec. V.

II. LIGHT FRONT VARIABLES

We denote the coordinates of a space-time point in the ordinary coordinate system by

$$\hat{x}^{\mu}=(\hat{x}^{0},\hat{x}^{1},\hat{x}^{2},\hat{x}^{3})=(t,x,y,z)=(t,\mathbf{x}) , \tag{1}$$

and the light front coordinates by

$$x^{\mu}=(x^{0},x^{1},x^{2},x^{3})=\left[\frac{t+z}{\sqrt{2}},x,y,\frac{t-z}{\sqrt{2}}\right] . \tag{2}$$

We use the usual covariant tensor notation for quantities in the light front coordinate system, e.g.,

$$x\cdot x=x_{\mu}x^{\mu} , \tag{3}$$

with

$$x_{\mu}=g_{\mu\nu}x^{\nu} , \tag{4a}$$

$$x^{\mu}=g^{\mu\nu}x_{\nu} , \tag{4b}$$

where the metric and its inverse are given by

$$g_{\mu\nu}=g^{\mu\nu}=\begin{bmatrix}0&0&0&1\\0&-1&0&0\\0&0&-1&0\\1&0&0&0\end{bmatrix} . \tag{5}$$

Just as with the usual variables given by (1), a Lorentz transformation and its inverse are given in terms of the light front components by

$$x^{\mu'}=a^{\mu}{}_{\nu}x^{\nu} , \tag{6a}$$

$$x^{\mu}=(a^{-1})^{\mu}{}_{\nu}x^{\nu'}=x^{\nu'}a_{\nu}{}^{\mu} , \tag{6b}$$

with the orthogonality relation

$$a_{\mu\lambda}a^{\nu\lambda}=a_{\lambda\mu}a^{\lambda\nu}=\delta^{\nu}_{\mu} . \tag{7}$$

Corresponding to (6a), the state vectors of a system transform according to[16,17]

$$|\psi'\rangle=U(a)|\psi\rangle , \tag{8}$$

where $U(a)$ is a unitary operator which can be written in terms of the light front components of the angular momentum tensor. The unitary operators satisfy the well known group properties[17]

$$U(a')U(a)=U(a'a) , \tag{9a}$$

$$U(a^{-1})=U(a)^{-1} , \tag{9b}$$

and can be used to transform field operators just as in the conventional formalism, i.e.,

$$U(a)\phi_{r}(x)U(a)^{-1}=S^{-1}_{rs}(a)\phi_{s}(x') . \tag{10}$$

The *light front components*, P^{μ}, of the four-momentum operator transform according to[17]

$$U(a)P^{\mu}U(a)^{-1}=(a^{-1})^{\mu}{}_{\nu}P^{\nu}=P^{\nu}a_{\nu}{}^{\mu} . \tag{11}$$

III. PERTURBATION THEORY

We begin in an arbitrary frame of reference where the coordinates are given by (1) or (2), and introduce a set of orthonormal three-vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, in terms of which we define a set of four vectors, whose ordinary components are given by

$$\hat{\xi}^{\mu}=\frac{1}{\sqrt{2}}(1,-\mathbf{u}) , \tag{12a}$$

$$\hat{\rho}^{\mu}=(0,-\mathbf{v}) , \tag{12b}$$

$$\hat{\lambda}^{\mu}=(0,-\mathbf{w}) , \tag{12c}$$

$$\hat{\zeta}^{\mu}=\frac{1}{\sqrt{2}}(1,\mathbf{u}) . \tag{12d}$$

Clearly, ξ and ζ are lightlike vectors, while ρ and λ are spacelike. We introduce a Lorentz transformation to a special frame of reference by

$$x^{\mu}_{\xi}=d^{\mu}{}_{\nu}x^{\nu} , \tag{13}$$

where

$$(d^0{}_\nu, d^1{}_\nu, d^2{}_\nu, d^3{}_\nu) = (\xi_\nu, \rho_\nu, \lambda_\nu, \zeta_\nu) . \tag{14}$$

We call this special frame the ξ frame and indicate components in this frame by a subscript or superscript ξ, as in (13).

From (12)–(14), it follows immediately that

$$x_\xi^\mu = (\xi\cdot x, \rho\cdot x, \lambda\cdot x, \zeta\cdot x) \tag{15a}$$

$$= \left[\frac{x_+}{\sqrt{2}}, \mathbf{x}_\perp, \frac{x_-}{\sqrt{2}}\right] , \tag{15b}$$

where

$$x_\pm = \hat{x}^0 \pm \mathbf{u}\cdot\mathbf{x} , \tag{16a}$$

$$\mathbf{x}_\perp = (\mathbf{v}\cdot\mathbf{x})\mathbf{v} + (\mathbf{w}\cdot\mathbf{x})\mathbf{w} = \mathbf{x} - (\mathbf{u}\cdot\mathbf{x})\mathbf{u} . \tag{16b}$$

In terms of these quantities, the four dot product is given by

$$x\cdot y = \tfrac{1}{2}(x_+ y_- + x_- y_+) - \mathbf{x}_\perp\cdot\mathbf{y}_\perp . \tag{17}$$

The vectors given by (12) have particularly simple light front components in the ξ frame, i.e.,

$$\xi_\xi^\mu = (0,0,0,1) , \tag{18a}$$

$$\rho_\xi^\mu = (0,-1,0,0) , \tag{18b}$$

$$\lambda_\xi^\mu = (0,0,-1,0) , \tag{18c}$$

$$\zeta_\xi^\mu = (1,0,0,0) . \tag{18d}$$

The ξ frame plays a role analogous to the λ frame introduced in the invariant formulation of time ordered perturbation theory[8] based on the use of a timelike external four vector λ [not to be confused with the vector given by (12c) and (18c)]. A λ frame is one in which the ordinary spatial components of λ vanish.

We now formulate the light front dynamics of the system in the ξ frame. We take as our evolution parameter

$$\tau = x_\xi^0 = \xi\cdot x , \tag{19}$$

and denote, as in (11), the components of the light front four-momentum operator by

$$P^\mu = (P^0, P^1, P^2, P^3) = (P_3, -P_1, -P_2, P_0) . \tag{20}$$

With our conventions,[14] P^0, P^1, and P^2 do not contain the interaction and are therefore kinematical, while P^3, which contains the interaction, gives the dynamics.[15] Since P^3 induces translations in τ, it plays the role of a Hamiltonian and so we write

$$H = P^3 = P_0 . \tag{21}$$

If $|\psi\rangle$ is the state vector of a system in the Heisenberg picture, then the state vector in the Schrödinger picture is given by

$$|\psi_S(\tau)\rangle = e^{-iH\tau}|\psi\rangle , \tag{22}$$

and satisfies

$$i\frac{d}{d\tau}|\psi_S(\tau)\rangle = H|\psi_S(\tau)\rangle . \tag{23}$$

The interaction picture is given by the relations

$$H = H_0 + H_1 , \tag{24}$$

$$|\psi_I(\tau)\rangle = e^{iH_0\tau}|\psi_S(\tau)\rangle , \tag{25}$$

$$i\frac{d}{d\tau}|\psi_I(\tau)\rangle = H_I(\tau)|\psi_I(\tau)\rangle , \tag{26}$$

$$H_I(\tau) = e^{iH_0\tau}H_1 e^{-iH_0\tau} , \tag{27}$$

where H_0 is the free Hamiltonian and H_1 gives the interaction. Just as in the usual formulation,[18] a perturbation series for the S operator can be obtained by converting (26) to an integral equation and iterating. The result for the S operator in the ξ frame is

$$S_\xi = 1 + \sum_{n=1}^{\infty} (-i)^n \int d\tau_1 d\tau_2 \ldots d\tau_n H_I(\tau_1) \times \theta(\tau_1 - \tau_2) H_I(\tau_2) \ldots H_I(\tau_n) . \tag{28}$$

The S matrix is constructed by evaluating matrix elements of (28) with the free states $|K_\xi\rangle$ which satisfy

$$P_{(0)}^\mu |K_\xi\rangle = K_\xi^\mu |K_\xi\rangle , \tag{29}$$

where

$$P_{(0)}^\mu = (P^0, P^1, P^2, H_0) . \tag{30}$$

The subscript 0 is used here and subsequently to denote free operators. If we let (28) act on a free state, it is straightforward to show that[8,18]

$$(S_\xi - 1)|K_\xi\rangle = -2\pi i \delta(H_0 - K_\xi^3)\left[H_1 + H_1 \frac{1}{K_\xi^3 + i\epsilon - H_0} H_1 + \cdots \right]|K_\xi\rangle . \tag{31}$$

By transforming (31) back to our original arbitrary frame, we will be able to write each term of the series in an invariant form.

According to (8), (13), (11), and (29) we have

$$|K_\xi\rangle = U_0(d)|K\rangle , \tag{32}$$

where

$$P_{(0)}^\mu |K\rangle = K^\mu |K\rangle , \tag{33}$$

with

$$K^\mu = (d^{-1})^\mu{}_\nu K_\xi^\nu = K_\xi^\nu d_\nu{}^\mu . \tag{34}$$

It is worth noting that the generator for the unitary transformation in (32) is the *free-field*, light front angular momentum tensor. It follows from (30), (11), (9b), and (14) that

$$U_0(d)^{-1} H_0 U_0(d) = \zeta\cdot P_{(0)} . \tag{35}$$

If we define

$$V(\xi)=U_0(d)^{-1}H_1U_0(d)\ , \tag{36}$$

and let

$$S=U_0(d)^{-1}S_\xi U_0(d)\ , \tag{37}$$

we obtain from (31), with the help of (32), (35), (15a), and (36), the basic result

$$(S-1)\,|\,K\,\rangle=-2\pi i\delta(\zeta\cdot P_{(0)}-\zeta\cdot K)\left[V(\xi)+V(\xi)\frac{1}{\zeta\cdot K+i\epsilon-\zeta\cdot P_{(0)}}V(\xi)+\cdots\right]|\,K\,\rangle\ . \tag{38}$$

It should be noted that according to (36) and (14), V should depend on all four of the vectors ξ, ρ, λ, and ζ. It turns out that in general the dependence on ρ, λ, and ζ in (36) is illusory, so the notation $V(\xi)$ actually makes sense. We will see an example of this in Sec. IV.

We now show that the matrix elements of each term in the above series are invariant functions of ξ and the particle variables. We consider a Lorentz transformation from a frame in which the coordinates are x^μ to one in which they are $x^{\mu'}$, as in (6a). According to (13) the transformation from the prime frame to the ξ-frame is given by

$$\begin{aligned} x^\mu_\xi &= d'^\mu{}_\nu x^{\nu'} \\ &= d'^\mu{}_\alpha a^\alpha{}_\nu x^\nu \\ &= d^\mu{}_\alpha x^\alpha\ , \end{aligned} \tag{39}$$

so we see that

$$d=d'a\ . \tag{40}$$

Combining this with (14), we got the obvious result that the elements of d' are given by the components of ξ, ρ, λ, and ζ in the prime frame.

In the prime frame, the interaction term is given by (36) with ξ and d replaced by ξ' and d', respectively. Using this, as well as (9) and (40), we find that

$$V(\xi')=U_0(a)V(\xi)U_0(a)^{-1}\ , \tag{41}$$

while from (11) we obtain

$$\zeta'\cdot P_{(0)}=U_0(a)\zeta\cdot P_{(0)}U_0(a)^{-1}\ . \tag{42}$$

According to (8), the free states in the two frames are related by

$$|\,K'\,\rangle=U_0(a)\,|\,K\,\rangle\ , \tag{43}$$

which when used in conjunction with (41) and (42) shows that

$$\begin{aligned} &\langle K'_\beta\,|\,V(\xi')\frac{1}{\zeta'\cdot K'+i\epsilon-\zeta'\cdot P_{(0)}}V(\xi')\ldots V(\xi')\,|\,K'_\alpha\,\rangle \\ &\quad=\langle K_\beta\,|\,V(\xi)\frac{1}{\zeta\cdot K+i\epsilon-\zeta\cdot P_{(0)}}V(\xi)\ldots V(\xi)\,|\,K_\alpha\,\rangle\ , \end{aligned} \tag{44}$$

where α and β refer to the initial and final state, respectively. This verifies that the matrix elements of the terms in (38) are invariant functions of the various variables involved.

Two aspects of this result on invariance are worth noting. First of all the proof shows that the matrix elements, are invariant off-shell, i.e., there need not be any special relation among K_α, K_β, and K in order for (44) to be valid. Secondly, the invariance holds if only parts of V are considered, or, going back to (36), if only part of H_1 are retained. By using projection operators to limit the action of H_1 to some subspace of the free states, the set of intermediate states that come into play can be truncated without destroying the invariance of the S-matrix elements. This amounts to an invariant formulation of the Tamm-Dancoff method.

In applying the perturbation theory based on (38), it is important to realize that there are three conserved quantities. For a Poincare invariant theory, the components of the light-front total momentum operator must commute with each other,[14] i.e.,

$$[P^\mu,P^\nu]=0\ . \tag{45}$$

As pointed out above P^0, P^1, and P^2 do not contain the interaction, so we have

$$[P^\mu_{(0)},H_0]=[P^\mu_{(0)},H_1]=0\quad(\mu=0,1,2)\ . \tag{46}$$

By sandwiching (46) between $U_0(d)^{-1}$ and $U_0(d)$, and using (11), (9b), (14), (12), (2), and (36), it follows that

$$[\xi\cdot P_{(0)},V(\xi)]=0\ , \tag{47a}$$

$$[\mathbf{P}_{0\perp},V(\xi)]=0\ , \tag{47b}$$

where

$$\begin{aligned} \mathbf{P}_{0\perp} &= (\mathbf{v}\cdot\mathbf{P}_0)\mathbf{v}+(\mathbf{w}\cdot\mathbf{P}_0)\mathbf{w} \\ &= \mathbf{P}_0-(\mathbf{u}\cdot\mathbf{P}_0)\mathbf{u}\ . \end{aligned} \tag{48}$$

Here $\mathbf{P}_0$ is the usual free, three-momentum operator.

From (47) it follows that the initial, final, and intermediate states that occur in the S-matrix perturbation theory based on (38) all have the same eigenvalues of $\xi\cdot P_{(0)}$ and $\mathbf{P}_{0\perp}$, which we write as $\xi\cdot K$ and $\mathbf{K}_\perp$. Since the particles which occur in the various states are all on the mass shell, we can specify the four momentum of each particle by the three parameters[8,10]

$$\eta=\xi\cdot k/\xi\cdot K\ , \tag{49}$$

and

$$\mathbf{q}=\mathbf{k}_\perp-\eta K_\perp\ , \tag{50}$$

where k is the particle's four momentum. Clearly η is a Lorentz scalar. According to (15), the plus component of the four vector $k-\eta K$ vanishes. Using this observation in conjunction with (17) leads to the conclusion that dot products of the form $\mathbf{q}\cdot\mathbf{q}'$ are Lorentz scalars. Since $\xi\cdot K$ and $\mathbf{K}_\perp$ are the same in all states (initial, intermediate, and final), it follows from (49) and (50) that

$$\sum_n \eta_n = 1 , \tag{51}$$

$$\sum_n \mathbf{q}_n = 0 , \tag{52}$$

where the sums are over all the particle in a state.

The existence of the conserved quantities $\xi\cdot P_{(0)}$ and $\mathbf{P}_{0\perp}$ makes it possible to replace the denominator factors in (38) with an alternate form. According to (11), (9b), and (7)

$$P^2_{(0)} = U_0(a)^{-1} P^2_{(0)} U_0(a) . \tag{53}$$

By choosing $a=d$ and using (3), (4), (11), (14), (15), and (48), it is straightforward to show that

$$P^2_{(0)} = 2\xi\cdot P_{(0)}\zeta\cdot P_{(0)} - \mathbf{P}^2_{0\perp} , \tag{54}$$

while from (15)–(17), it follows that

$$K^2 = 2\xi\cdot K\zeta\cdot K - \mathbf{K}^2_\perp . \tag{55}$$

Those last two relations allow us to make the replacement

$$\frac{1}{\zeta\cdot K + i\epsilon - \zeta\cdot P_{(0)}} \to \frac{2\xi\cdot K}{K^2 + i\epsilon - P^2_{(0)}} \tag{56}$$

in (38). This makes the denominators s denominators where s is the square of the total four momentum. It is not difficult to show[9,10] that for any state (initial, intermediate, or final) s is given by

$$s = \sum_n \frac{\mathbf{q}_n^2 + m_n^2}{\eta_n} , \tag{57}$$

where m_n is the mass of the nth particle, and the sum is over all the particles in the state.

When the perturbation theory given by (38) is applied to quantum field theories, the essential features of the diagrammatic rules that emerge are quite simple. The sums of the η's and $\mathbf{q}$'s are conserved at each vertex, and s denominators, which can be calculated from (57), are associated with the intermediate states. Taking into account (49), (50), (16b), and (12a), it is clear that the only dependence on the external vectors ξ, ρ, λ, and ζ that survives in the S-matrix elements is associated with ξ, or more precisely, the arbitrary direction of the unit vector $\mathbf{u}$. In Sec. IV we will study this dependence in the context of a simple field theory.

IV. AN EXAMPLE

We consider a field theory for which the interaction is given in the interaction picture by

$$H_I(\tau) = \int d^4x_\xi \mathcal{H}_I(x_\xi)\delta(x^0_\xi - \tau) , \tag{58}$$

where $\mathcal{H}_I$ is a Lorentz invariant Hamiltonian density. It should be noted that in general the interaction can be written in the above form only for scalar fields.[19] It has been shown that for coupled scalar-Dirac fields an additional noncovariant term comes in, whose effect is canceled by a noncovariant term in the fermion propagator when the Feynman-Dyson-Wick expansion is used to obtain the S-matrix. We will not consider such complications here. According to (10) and (13) the transformation of the density from the ξ frame to an arbitrary frame is given by

$$U_0(d)^{-1}\mathcal{H}_I(x_\xi)U_0(d) = \mathcal{H}_I(x) . \tag{59}$$

Using this in conjunction with (36), (27), and (19), we find

$$V(\xi) = \int d^4x \mathcal{H}_I(x)\delta(\xi\cdot x) , \tag{60}$$

which justifies our notation, as this shows that the only external vector that V depends upon is ξ.

We take for the Hamiltonian density

$$\mathcal{H}_I(x) = g{:}\psi^\dagger(x)\phi(x)\psi(x){:} , \tag{61}$$

where $\phi(x)$ and $\psi(x)$ are Hermitian and non-Hermitian scalar field operators, respectively. This interaction corresponds to the elementary virtual process

$$\psi \rightleftarrows \psi + \phi , \tag{62}$$

where ϕ is the neutral scalar particle and ψ is the charged scalar particle. The field operators describe free fields, since we are in the interaction picture. Accordingly we can write

$$\phi(x) = \int \frac{d^3k\theta(k^0)}{(2\pi)^{3/2}2k^0}[A(k)e^{-ik\cdot x} + A^\dagger(k)e^{ik\cdot x}] ,$$
$$k^2=\mu^2,\quad d^3k = dk^0dk^1dk^2 \tag{63}$$

and

$$\psi(x) = \int \frac{d^3p\theta(p^0)}{(2\pi)^{3/2}2p^0}[B(p)e^{-ip\cdot x} + D^\dagger(p)e^{ip\cdot x}] ,$$
$$p^2 = m^2,\quad d^3p = dp^0dp^1dp^2 . \tag{64}$$

Here $A^\dagger$, $B^\dagger$, and $D^\dagger$ create a ϕ, ψ, and anti-ψ particle, respectively. The creation and annihilation operators used here are Lorentz scalars, so that, for example,

$$U_0(a)A^\dagger(k)U_0(a)^{-1} = A^\dagger(ak) , \tag{65}$$

which is consistent with (10). Free states constructed by letting such creation operators operate on the vacuum state obey (43), which in turn implies that the free states have an invariant norm. By using standard results,[16] it is straightforward to show that

$$[A(k), A^\dagger(k')] = 2k^0\delta(k^0 - k^{0'})\delta(k^1 - k^{1'})\delta(k^2 - k^{2'}) , \tag{66}$$

as well as similar relations for the other nonzero commutators.

If we put (61) into (60) and use the expansions (63) and (64), we encounter the invariant function

$$\delta_\xi(\Delta p) = \frac{1}{(2\pi)^3}\int d^4x e^{ix\cdot\Delta p}\delta(\xi\cdot x) , \tag{67}$$

where Δp can be interpreted as the change in four-momentum at a vertex. Evaluating the integral in the ξ frame, and using (15), leads to

$$\delta_\xi(\Delta p)=\delta(\Delta\xi\cdot p)\delta^2(\Delta\mathbf{p}_\perp) \ . \tag{68}$$

The presence of such δ functions is, of course, related to (47). Since $\xi\cdot K$ and $\mathbf{K}_\perp$ do not change from state to state, (49) and (50) allows us to write

$$\delta_\xi(\Delta p)=(\xi\cdot K)^{-1}\delta(\Delta\eta)\delta^2(\Delta\mathbf{q}) \ , \tag{69}$$

which verifies that the sums of the η's and $\mathbf{q}$'s are conserved at each vertex.

By working out a few examples with the help of the relations given in this section, it is not difficult to show that (38), (60), and (61) lead to a perturbation expansion for the S matrix given by

$$\langle K_\beta\,|\,S-1\,|\,K_\alpha\rangle=-\frac{(2\pi)^4}{(2\pi)^{3N/2}}i\delta^4(K_\beta-K_\alpha)T_{\beta\alpha} \ , \tag{70}$$

where N is the number of particles in the initial state plus the number of final state particles, and the T-matrix elements $T_{\beta\alpha}$ are given by the graphical rules of Ref. 9.

In order to illustrate the dependence of the T-matrix elements on the external vector ξ, we consider the diagrams shown in Fig. 1. These diagrams are the lowest-order contributions to ϕ-ψ scattering, with the dashed lines, ϕ particles, and the solid lines, ψ particles. The corresponding contributions to $T_{\beta\alpha}$ are

$$B_a(\eta',\mathbf{q}';\eta,\mathbf{q};s)=\frac{g^2\theta(1-\eta'-\eta)}{(1-\eta'-\eta)[s-s_a(\eta',\mathbf{q}';\eta,\mathbf{q})]} \tag{71}$$

and

$$B_b(\eta',\mathbf{q}';\eta,\mathbf{q};s)=\frac{g^2\theta(\eta'+\eta-1)}{(\eta'+\eta-1)[s-s_b(\eta',\mathbf{q}';\eta,\mathbf{q})]} \ . \tag{72}$$

Here s_a and s_b are the s values for the intermediate states indicated in Fig. 1 and can be calculated from (57). The

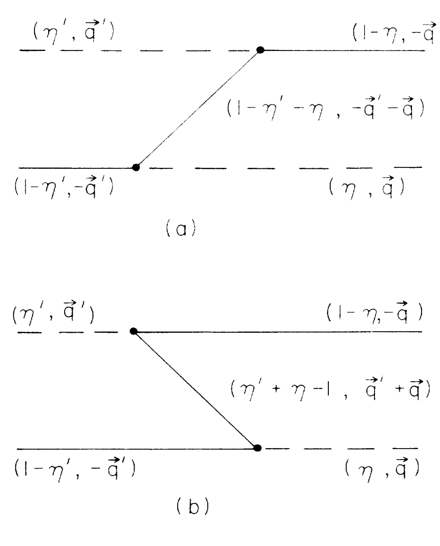

FIG. 1. Lowest order contributions to ϕ-ψ scattering.

expressions (71) and (72) are off shell in that the initial and final s values are not necessarily equal to each other or the s appearing in the denominators.

We can rewrite (71) and (72) so as to give them a more familiar form and make the ξ dependence more transparent. According to (15) and (17) the ξ frame, light front components of any four-momentum p are given by

$$p_\xi^\mu=\left[\xi\cdot p,\mathbf{p}_\perp,\frac{\mathbf{p}_\perp^2+p^2}{2\xi\cdot p}\right] \ . \tag{73}$$

When this is combined with the fact that $\xi\cdot K$ and $\mathbf{K}_\perp$ are the same in every state, we obtain

$$s-s_a=2\xi\cdot K(K_\xi^3-K_{a\xi}^3) \ . \tag{74}$$

Using the conservation laws for the vertices given by (68), we can write

$$K_{a\xi}^3=k_\xi'^3+k_\xi^3+\frac{(\mathbf{K}_\perp-\mathbf{k}_\perp'-\mathbf{k}_\perp)^2+m^2}{2\xi\cdot(K-k'-k)} \ , \tag{75}$$

which when put into (74) leads to, with the help of (49),

$$(1-\eta'-\eta)(s-s_a)=(K-k'-k)^2-m^2 \ . \tag{76}$$

Here k and k' are the initial and final four momenta of the ϕ particles, while below p and p' are those of the ψ particles. Similarly it can be shown that

$$(\eta'+\eta-1)(s-s_b)=(K-p'-p)^2-m^2 \ , \tag{77}$$

so we see that the denominators in (71) and (72) do not depend on the external vector ξ. Only the step functions depend on ξ, as a result of (49). If we consider the on-shell limits of (71) and (72), then, of course, the total four momentum is conserved, i.e.,

$$K=k+p=k'+p' \ , \tag{78}$$

and the denominators become identical, which in turn implies

$$B_a+B_b=\frac{g^2}{(K-k'-k)^2-m^2}\quad\text{(on shell)} \ . \tag{79}$$

This is the familiar Feynman diagram result. As pointed out in the Introduction, this is what happens in general, i.e., when the contributions from the various ordered diagrams which correspond to a single Feynman diagram are combined on shell, the dependence on the external vector ξ disappears. Since LFPT can be thought of as TOPT in the infinite momentum frame, and it is known that combining time ordered diagrams leads to Feynman diagrams, this result is not too surprising.

It is not anticipated that in practical applications of the formalism developed here and in Ref. 9, all of the ordered diagrams corresponding to a Feynman diagram will be retained. For example, the three-particle equations developed in Ref. 9 ignore the contribution of Fig. 1(b) and retain Fig. 1(a). This seems reasonable since in (72) we have $s_b\geq(3m)^2$ while in (71) we have $s_a\geq(2\mu+m)^2$, where μ and m can be thought of as the pion mass and nucleon mass, respectively. As a result of this type of approximation, the equations developed in Ref. 9 are invariant but not truly covariant, since there is some residual dependence on the unit vector $\mathbf{u}$ that appears in ξ. To the extent that the approximations made are good, this dependence should be weak.

V. DISCUSSION

The formalism developed here provides a practical basis for the development of relativistic equations for few particle systems. Since the particles that occur in initial, final, and intermediate states are on the mass shell, the integral equations developed from ILFPT have fewer variables than those obtained from Feynman diagrams. In general, two-body equations will be three-dimensional, while three-body equations will be six dimensional. As the analysis of Ref. 9 shows, it is possible to develop *three-dimensional*, three-particle equations which should be accurate for a system such as the pion-nucleon system at low and intermediate energies. Since we have proven in Sec. III that the individual terms in ILFPT are Lorentz invariant functions of the variables involved, we are guaranteed that integral equations obtained from this formalism will be manifestly invariant.

In the diagrams of ILFPT the internal lines are associated with intermediate states characterized by the number of on-mass shell particles present. This has two related advantages. First of all, it makes it easier to justify the omission of particular diagrams, and second, unitarity becomes quite transparent. As the analysis of Ref. 9 indicates, the derivation of discontinuity and unitarity relations is quite straightforward. In fact, the formal procedures are identical to those of nonrelativistic quantum mechanics.[20] The only difference worth noting is the shift in emphasis from energy variables to invariant s variables.

The analysis of Namyslowski[10] and the equations developed in Ref. 9, make it clear that use of the variables given by (49) and (50) makes it possible, within the framework of ILFPT, to satisfy the cluster decomposition property in a manifestly invariant way. The additive character of the right-hand side of (57) provides the algebraic basis for proving this. There have been relativistic three-particle equations developed that satisfy the cluster property,[21] but they do so at the expense of introducing spurious singularities. The cluster property is crucial for the description of the scattering of composite systems and makes it possible to explain many-body systems in terms of two, three, etc. particle interactions.

Summing sets of diagrams to obtain few particle equations can become quite complicated, especially in three or more particle systems. Fortunately, the one-to-one correspondence between the internal lines of ILFPT diagrams and intermediate states makes it possible to use projection operator techniques to keep track of things. Previous work[22] on a static model for the pion-nucleon system indicates that the use of these techniques will make it possible to develop relativistic equations for a system such as this, which go beyond the isobar model.

One of the practical problems that remains in the application of the formalism developed here is the optimal way to choose the external vector ξ, or more specifically the unit vector $\mathbf{u}$. This point has been considered by Namyslowski[10] and he has made some practical suggestions. Karmanov[23] has also considered this issue in an analysis of state vectors defined on light fronts. His results are interesting in that they show that the appearance of the external vector ξ can be turned to an advantage. In light front dynamics the angular momentum operators depend on the interaction, i.e., they are not kinematical generators as in the conventional approach.[1,2,4–6,10,13,14] Karmonov has shown how to reformulate the light front theory of angular momentum in terms of rotations of the ξ vector, so as to make the construction of states with definite angular momentum a purely kinematic problem.

An analysis of the theory of angular momentum within the light front framework and the application of the formalism developed here to the pion-nucleon system are presently underway.

[1]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[2]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).

[3]Steven Weinberg, Phys. Rev. **150**, 1313 (1966).

[4]L. Susskind, Phys. Rev. **165**, 1535 (1968); *Mathematical Methods in Theoretical Physics,* Vol. XI-D of *Lectures in Theoretical Physics,* edited by K. T. Mahanthappa and W. E. Brittin (Gordon and Breach, New York, 1969).

[5]K. Bardakci and M. B. Halpern, Phys. Rev. **176**, 1686 (1968).

[6]J. M. Namyslowski, Phys. Rev. D **18**, 3676 (1978); P. M. Fishbane and J. M. Namyslowski, *ibid.* **21**, 2406 (1980); J. M. Namyslowski and H. J. Weber, Z. Phys. A **295**, 219 (1980).

[7]V. G. Kadyshevsky, Zh. Eksp. Teor. Fiz. **46**, 654 (1964) [Sov. Phys.—JETP **19**, 443 (1964)]; **46**, 872 (1964) [**19**, 597 (1964)]; Dokl. Acad. Nauk. SSR **160**, 573 (1965) [Sov. Phys. Dokl. **10**, 46 (1965)].

[8]M. G. Fuda, Phys. Rev. C **33**, 996 (1986).

[9]M. G. Fuda, Phys. Rev. C **35**, 226 (1987).

[10]J. M. Namyslowski, Prog. Part. Nucl. Phys. **14**, 49 (1984).

[11]J. L. Basdevant and R. L. Omnes, Phys. Rev. Lett. **17**, 775 (1966).

[12]H. Garcilazo and L. Mathelitsch, Phys. Rev. C **28**, 1272 (1983); L. Mathelitsch and H. Garcilazo, *ibid.* **32**, 1635 (1985).

[13]S.-J. Chang and S.-K. Ma, Phys. Rev. **180**, 1506 (1969); D. Flory, Phys, Rev. D **1**, 2795 (1970).

[14]J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).

[15]S.-J. Chang, R. G. Root, and T. M. Yan, Phys. Rev. D **7**, 1113 (1973).

[16]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1964).

[17]S. G. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).

[18]David Lurie', *Particles and Fields* (Interscience, New York, 1968).

[19]S.-J. Chang and T.-M. Yan, Phys. Rev. D **7**, 1147 (1973).

[20]K. L. Kowalski, Phys. Rev. **188**, 2235 (1969); Phys. Rev. D **2**, 812 (1970); M. G. Fuda, Phys. Rev. C **3**, 485 (1971).

[21]R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. **174**, 2022 (1968).

[22]M. G. Fuda, Phys. Rev. C **31**, 1365 (1985); 2024 (1985).

[23]V. A. Karmanov, Zh. Eksp. Teor. Fiz. **71** 399 (1976) [Sov. Phys.—JETP **44**, 210 (1976)]; Nucl. Phys. **B166**, 93 (1980); Zh. Eksp. Teor. Fiz. **83**, 3 (1982) [Sov. Phys.—JETP **56**, 1 (1982)].