Kinetic equation of nuclear gas

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The kinetic equation on nuclear gas is derived by means of the Bogoliubov approach. It is an improved Boltzmann-Uehling-Uhlenbeck equation including correctional binary collisions with many-body effects.

The study of collisions between heavy ions at medium energy is a subject which is still poorly understood. The time-dependent Hartree-Fock (TDHF) equation has been used at low energy.[1] The TDHF equation requires that binary collisions be neglected in comparison to the mean field generated by the nucleons. However, at medium energy binary collisions between the nucleons are important. In this case, the Boltzmann-Uehling-Uhlenbeck (BUU) equation has become a starting point for microscopic studies.[2] The first derivation of this equation was given by Uehling-Uhlenbeck in 1933.[3] They derived it from physical arguments. The collision integral on the right-hand side differs from the classical Boltzmann equation by the Pauli blocking factor. Later attempts have been made to derive this equation from the Liouville–von Neumann equation of quantum statistics.[4] In these derivations an important condition is that the gas is dilute. Whether the BUU equation can be used in the nuclear gas of heavy ion collisions is an open question. Recently, the modified Boltzmann equation has been derived by the self-consistent Brueckner-Bethe-Goldstone method.[5] The equation is expressed by the Brueckner G matrix, which is very difficult to solve.

In this paper an improved Boltzmann-Uehling-Uhlenbeck equation is derived by means of the Wigner distribution function and a Bogoliubov approach. The equation considers both modified mean field interactions and correctional binary collisions by the effect of many-body effects. This is of advantage for applications.

The time evolution of s particle density matrices ρ_s is determined by the quantum BBGKY hierarchy[5] as follows:

$$i\hbar\frac{\partial\rho_s}{\partial t}=\sum_{j=1}^{s}[K_j,\rho_s]+\sum_{i<j}[V_{ij},\rho_s]+\mathrm{Tr}\sum_{j=1}^{s}[V_{js+1},\rho_{s+1}]\ . \tag{1}$$

It is convenient to introduce directly the symmetry requirements on the function ρ_s by means of

$$\rho_s=A_sF_s\ , \tag{2}$$

where A_s is an antisymmetrization operator defined by

$$A_s=\prod_{j=2}^{s}\left[1-\sum_{k=1}^{j-1}P_{jk}\right]\ . \tag{3}$$

Here, P_{jk} denotes the permutation operator. Since A_s satisfies the relation

$$A_{s+1}=A_s\left[1-\sum_{j=1}^{s}P_{js+1}\right] \tag{4}$$

and commutes with the operators K_i and V_{ij}, one may substitute Eq. (2) into Eq. (1) to obtain the equation

$$i\hbar\frac{\partial F_s}{\partial t}=\sum_{j=1}^{s}[K_j,F_s]+\sum_{i<j}[V_{ij},F_s]+\mathop{\mathrm{Tr}}_{s+1}\sum_{j=1}^{s}[V_{js+1},F_{s+1}]-\mathop{\mathrm{Tr}}_{s+1}\sum_{j=1}^{s}\left[V_{js+1},\sum_{j=1}^{s}P_{js+1}F_{s+1}\right]\ . \tag{5}$$

It is convenient to introduce the Wigner distribution functions

$$f_s(q^sp^st)=\frac{1}{(2\pi\hbar)^{3s}}\int F_s(q'^sq''^st)\times\exp(-p^s\cdot\gamma^s/i\hbar)d\gamma^s\ . \tag{6}$$

and

$$F_s(q'^sq''^st)=\int f_s(q^sp^st)\exp(p^s\cdot\gamma^s/i\hbar)dp^s\ . \tag{7}$$

One may substitute Eq. (7) into Eq. (5) to obtain the quantum BBGKY hierarchy of the Wigner distribution functions f_s. When $s=1$ and 2, one finds

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1}{2\mathbf{q}_1}+\frac{1}{\hbar v}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})f_2-\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})P_{12}f_2=0\ , \tag{8}$$

$$\frac{\partial f_2}{\partial t}+\sum_{j=1}^{2}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial f_2}{\partial\mathbf{q}_j}+\frac{i}{\hbar}(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})f_2+\frac{i}{\hbar v}\int d\mathbf{x}_3\sum_{j=1}^{2}(e^{(i\hbar/2)\theta_{j3}}-e^{-(i\hbar/2)\theta_{j3}})f_3-\frac{i}{\hbar}\sum_{j=1}^{2}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta_{j3}}-e^{(i\hbar/2)\theta_{j3}})P_{j3}f_3=0\ , \tag{9}$$

where v is the mean occupied volume of every nucleon, x_s are all of variable q_s and p_s. The operators θ_{12} and θ_{j3} are expressed as

$$\theta_{12}=\frac{\partial V_{12}(q_1-q_2)}{\partial \mathbf{q}_1}\cdot\left[\frac{\partial}{\partial \mathbf{p}_1}-\frac{\partial}{\partial \mathbf{p}_2}\right], \quad \theta_{j3}=\frac{\partial V_{13}(q_j-q_3)}{\partial \mathbf{q}_j}\cdot\frac{\partial}{\partial \mathbf{p}_j} . \tag{10}$$

Equations (8) and (9) are exact. Since f_2 depends on f_3, accurate solutions of the hierarchy are impossible. It is necessary that the approximated approach will be applied to solve.

According to the Bogoliubov hypothesis,[6] provided the average time between collisions is much longer than the collision time, it is possible to find a kinetic state. In this case

$$f_s(\mathbf{x}_1,\ldots,\mathbf{x}_s t)=f_s(\mathbf{x}_1,\ldots,\mathbf{x}_s\,|\,f_1), \quad \frac{\partial f_1}{\partial t}=A(\mathbf{x}f_1) . \tag{11}$$

The average separation between collisions is d. When the length is measured in d the mean volume v is a small quantity. One can write

$$\frac{\partial f_1}{\partial t}=A^0(\mathbf{x}f_1)+vA^1(\mathbf{x}f_1)+\cdots , \tag{12}$$

$$f_s=f_s^0+vf_s^1+v^2f_s^2+\cdots . \tag{13}$$

In a first order approximation we set $f_2=f_1(x_1)f_1(x_2)$ and find

$$\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{q}_1}+\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)-\frac{iv}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})P_{12}f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)=0 . \tag{14}$$

This is a self-consistent equation called the quantum Vlasov equation. In a second order approximation we write a formal solution

$$f_s'(\mathbf{x}_1,\ldots,\mathbf{x}_3\,|\,f_1)=\sum_{i<j\le s} g(\mathbf{x}_i\mathbf{x}_j)\prod_{\gamma\ne i\ne j} f_1(\mathbf{x}_\gamma) , \tag{15}$$

where

$$g(\mathbf{x}_i\mathbf{x}_j)=f_2^1(\mathbf{x}_i\mathbf{x}_j\,|\,f_1) .$$

$g(\mathbf{x}_i\mathbf{x}_j)$ is the two-body correlative function, whose boundary condition is

$$\lim_{\tau\to\infty} \mathcal{S}^{(2)}_{-\tau}g(\mathbf{x}_i\mathbf{x}_j)\to 0 . \tag{16}$$

Equation (15) means that s-body effects are correlated by two-body effects. We may write

$$\frac{\partial f_2}{\partial t}=\frac{\partial f_2}{\partial f_1}\frac{\partial f_1}{\partial t}=\left[\frac{\partial f_2^0}{\partial f_1}+v\frac{\partial f_2^1}{\partial f_1}\right][A^0(\mathbf{x}F_1)+vA^1(\mathbf{x}f_1)]\approx\mathcal{D}_0f_2^0+v[\mathcal{D}_0g(\mathbf{x}_1\mathbf{x}_2)+\mathcal{D}_1f_2^0] . \tag{17}$$

We use Eq. (15) to obtain

$$\begin{aligned}\mathcal{D}_ig(\mathbf{x}_1\mathbf{x}_2)&+\sum_{j=1}^{2}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial}{\partial \mathbf{q}_j}g(\mathbf{x}_1\mathbf{x}_2)+\frac{i}{\hbar}\sum_{j=1}^{2}(e^{(i\hbar/2)\eta_j}-e^{-(i\hbar/2)\eta_j})g(\mathbf{x}_1\mathbf{x}_2)\\
&=-\frac{i}{\hbar}(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{13}}-e^{-(i\hbar/2)\theta'_{13}})g(\mathbf{x}_2\mathbf{x}_3)f_1(\mathbf{x}_5)\\
&\quad-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{23}}-e^{-(i\hbar/2)\theta'_{23}})f_1(\mathbf{x}_2)g(\mathbf{x}_1\mathbf{x}_3)+\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{13}}-e^{-(i\hbar/2)\theta'_{13}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)\\
&\quad+\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{23}}-e^{-(i\hbar/2)\theta'_{23}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3) .\end{aligned} \tag{18}$$

Once one knows $g(\mathbf{x}_1,\mathbf{x}_2)$, we shall be able to obtain the two-order-approximated equation of f_1,

$$\begin{aligned}\frac{\partial f_1(\mathbf{x}_1)}{\partial t}&+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{2\mathbf{q}_1}+\frac{i}{\hbar}(e^{(i\hbar/2)\eta_1}-e^{-(i\hbar/2)\eta_i})f_1(\mathbf{x}_1)+\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})g(\mathbf{x}_1\mathbf{x}_2)\\
&-\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)=0 ,\end{aligned} \tag{19}$$

where

$$\theta'_{12}=\theta_{12}/v , \quad \eta_1=\frac{\partial U_1(q)}{\partial \mathbf{q}}-\frac{\partial}{\partial \mathbf{p}_1} , \quad U_1(q_1)=\frac{1}{v}\int V_{12}(q_1-q_2)f_1(\mathbf{x}_2)d\mathbf{x}_2 . \tag{20}$$

$U_1(q_1)$ is the mean field potential. In general, solving the simultaneous equations (18) and (19) is very difficult. In the following paragraph it is shown how Eq. (18) may be solved in quasihomogeneous systems.

The conditions of quasihomogeneous systems are

$$g(\mathbf{x}_1\mathbf{x}_2)=g(\mathbf{q}_1-\mathbf{q}_2\mathbf{p}_1\mathbf{p}_2)\ . \tag{21}$$

This shows that the correlative function only depends on the relative coordinate. In this case one may obtain

$$\begin{aligned}g(\mathbf{q}\mathbf{p}_1\mathbf{p}_2)=\int_0^\infty dt\Bigg[\frac{i}{\hbar}\Bigg\{&\exp\left[\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\left[\frac{\partial}{\partial\mathbf{p}_1}-\frac{\partial}{\partial\mathbf{p}_2}\right]\right]-\exp\left[-\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\left[\frac{\partial}{\partial\mathbf{p}_1}-\frac{\partial}{\partial\mathbf{p}_2}\right]\right]\Bigg\}V\left[\left|\mathbf{q}-\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]t\right|\right]f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)\\
&+\frac{i}{\hbar}\int d\mathbf{q}'d\mathbf{p}_3\Bigg\{\left[\exp\left[\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{p}_1}\right]-\exp\left[-\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{p}_1}\right]\right]V\left[\left|\mathbf{q}-\mathbf{q}'-\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]t\right|\right]\\
&\qquad\times[g(-\mathbf{q}'\mathbf{p}_2\mathbf{p}_3)f_1(\mathbf{x}_1)-f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)]\\
&-\left[\exp\left[\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{p}_2}\right]-\exp\left[-\frac{i\hbar}{2}\frac{\partial}{\partial\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{p}_2}\right]\right]V\left[\left|\mathbf{q}-\mathbf{q}'-\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]t\right|\right]\\
&\qquad\times[g(\mathbf{q}'\mathbf{p}_1\mathbf{p}_3)f_1(\mathbf{x}_2)-f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)]\Bigg\}\Bigg]\ .\end{aligned} \tag{22}$$

It is convenient to introduce the Fourier transform to solve Eqs. (19) and (22). We set

$$\begin{aligned}\tilde{g}(\mathbf{k},\mathbf{p}_1\mathbf{p}_2)&\equiv\int d\mathbf{q}\,g(\mathbf{q}\mathbf{p}_1\mathbf{p}_2)e^{-i\mathbf{k}\cdot\mathbf{q}}\ ,\\ \tilde{V}(\mathbf{k})&\equiv\int d\mathbf{q}\,V(\mathbf{q})e^{-i\mathbf{k}\cdot\mathbf{q}}\ .\end{aligned} \tag{23}$$

We may substitute Eq. (23) into Eq. (19) to obtain

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1}{\partial\mathbf{q}_1}+\mathcal{F}\cdot\frac{\partial f_1}{\partial\mathbf{p}_1}=\frac{-i}{(2\pi)^3\hbar}\int d\mathbf{k}\left(e^{(\hbar\mathbf{k}/2)\cdot\partial/\partial\mathbf{p}_1}-e^{-(\hbar\mathbf{k}/2)\cdot\partial/\partial\mathbf{p}_1}\right)\tilde{V}_{12}h(\mathbf{k},\mathbf{p})\ , \tag{24}$$

where

$$\begin{aligned}h(kp_1)&=\int dp_2\tilde{g}(kp_1p_2)\ ,\\ \mathcal{F}\cdot\frac{\partial f_1}{\partial\mathbf{p}_1}&=\frac{i}{\hbar}\left(e^{(i\hbar/2)\eta_1}-e^{-(i\hbar/2)\eta_1}\right)f_1(\mathbf{x}_1)-\frac{i}{\hbar}\int d\mathbf{x}_2\left(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}}\right)f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)\ .\end{aligned} \tag{25}$$

Performing the Fourier transform of Eq. (22), one may find, after some manipulation,

$$\mathrm{Im}h(\mathbf{k}\mathbf{p}_1)=\int\frac{\pi\tilde{V}_{12}(\mathbf{k})}{\hbar k\,|1+(1/\hbar)\tilde{V}_{23}\Psi|^2}\delta\left[\mathbf{k}\cdot\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]\right][f_1^+(\mathbf{x}_1)f_1^-(\mathbf{x}_2)-f_1(\mathbf{x}_1)f_1^+(\mathbf{x}_2)]\ , \tag{26}$$

where

$$\begin{aligned}f^{\pm}&=f\left[\mathbf{p}\pm\frac{\hbar\mathbf{k}}{2}\right]\left[1-f\left[\mathbf{p}\mp\frac{\hbar\mathbf{k}}{2}\right]\right]\ ,\\ f\left[\mathbf{p}\pm\frac{\hbar\mathbf{k}}{2}\right]&=e^{\pm(\hbar\mathbf{k}/2)\cdot\partial/\partial\mathbf{p}}f(\mathbf{p})\ ,\\ \Psi&=\int_{-\infty}^{\infty}\frac{d\mathbf{p}_3}{\mathbf{k}\cdot(\mathbf{p}_2/m-\mathbf{p}_3/m)-i\epsilon}\left[f\left[\mathbf{p}_3+\frac{\hbar\mathbf{k}}{2}\right]-f\left[\mathbf{p}_3-\frac{\hbar\mathbf{k}}{2}\right]\right]\ .\end{aligned} \tag{27}$$

Substituting Eq. (26) into Eq. (24), one finds

$$\begin{aligned}\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial\mathbf{q}_1}+\mathcal{F}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial\mathbf{p}_1}&=\frac{\pi}{(2\pi)^3\hbar}\int d\mathbf{k}\left(e^{(\hbar\mathbf{k}/2)\cdot\partial/\partial\mathbf{p}_1}-e^{-(\hbar\mathbf{k}/2)\cdot\partial/\partial\mathbf{p}_1}\right)\\ &\quad\times\int d\mathbf{p}_2\delta\left[\mathbf{k}\cdot\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]\right]\frac{\tilde{V}_{12}^2(\mathbf{k})}{|1+(1/\hbar)\tilde{V}_{23}\Psi|^2}\\ &\quad\times[f_1^+(\mathbf{x}_1)f_1^-(\mathbf{x}_2)-f_1^-(\mathbf{x}_1)f_1^+(\mathbf{x}_2)]\ .\end{aligned} \tag{28}$$

Equation (28) is the kinetic equation of a nuclear gas in the quasihomogeneous case. This is an improved BUU equation. It is reduced to the usual BUU equation provided to neglect many-body effects and to take the first approximation of the term $\mathcal{F}$:

$$\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{q}_1}-\nabla U_1\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{p}_1}=\frac{\pi}{\hbar(2\pi)^9}\int d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' \,|\langle \mathbf{p}_1\mathbf{p}_2 | V_{12} | \mathbf{p}_1'\mathbf{p}_2'\rangle|^2$$
$$\times\{f_1(\mathbf{x}_1')f_1(\mathbf{x}_2')[1-f_1(\mathbf{x}_1)][1-f_1(\mathbf{x}_2)]$$
$$-f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)[1-f_1(\mathbf{x}_1')][1-f_1(\mathbf{x}_2')]\}\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2') . \tag{29}$$

In summary, the BUU equation can be applied by restricting the systems to be dilute: that is, the density of the particle should be low enough so that each nucleon essentially is independent of every other nucleon in the system. However, this condition is not satisfied in heavy ion collisions. Since Eq. (28) includes the influence of many-body effects, it is inevitable that Eq. (28) is better than Eq. (29) for heavy ion collisions.

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