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Quark cluster model for nuclear matter

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A model of nuclear matter is given in which nonrelativistic quarks, carrying color, moving in one dimension, interact through a contact interaction. Solutions for the ground-state energy as a function of density may be found exactly using a method based on the Bethe ansatz, which is explained in detail. The model system shows characteristic behavior of a Fermi gas of nucleons (bound clusters of quarks) at low density, and of a Fermi gas of free quarks at high density, with a smooth transition between these limits.

I. INTRODUCTION

In this paper we present theoretical methods for finding the ground-state properties of a model of nuclear matter considered as a system of interacting quarks. The model is meant to represent some of the features of nuclear systems that are expected to emerge from any theory based on quark degrees of freedom, such as quantum chromodynamics (QCD). An outline of this model has been presented in an earlier publication,[1] with some of the results. In the present paper we explain the theory of the ground state, derive in detail its properties, and give some further results. Earlier methods, based on a variational approach, are explained in Ref. 2.

The motivation for the present quark model of nuclear matter is to be able to explore the consequences to nuclear physics of the basic idea that nucleons and nuclear systems alike consist of interacting quarks, as given by the theory of QCD. Since the evidence of low energy nuclear physics indicates that interacting nucleons provide the dominant degrees of freedom for low energy excitations, one expects to find quarks in nuclei to be organized more or less into nucleons as correlated objects. However, QCD leads us to expect that under compression to higher than normal density, quarks may be less localized than in isolated nucleons. At sufficiently high densities (or at high temperature) nucleon systems may transform to a quark-gluon plasma phase, with totally unlocalized quarks.

The model treated in this paper consists of a system of nonrelativistic fermions (quarks) of three colors, moving in one spatial dimension, interacting through a contact potential with color exchange. For the present we do not consider spin or flavor degrees of freedom. The potential represents only part of the interaction among quarks; it can bind quarks into clusters (nucleons) but it will not confine them against all excitation. Color confinement is represented by boundary conditions on the system as a whole, i.e., quarks are confined to the nuclear volume.

This simple many-body model has the interesting property that it exhibits clustering behavior analogous to that discussed above in connection with QCD; namely, at low density the ground state behaves like a Fermi gas of nucleons, while at high density the behavior is closer to that of a Fermi gas of quarks. The situation representing real nuclei lies somewhere in between, presumably closer to the low-density (nucleonic) end.

The model is one of a class of many-body systems for which there exist exact constructive techniques for eigenstates and energies that go under the title of the Bethe ansatz. This allows us to explore the nature of the ground state in some detail, and to exhibit the degree of clustering as a function of density. Although the basic exact techniques have been around for more than twenty years, they have not been much applied to problems of clustering in Fermi systems. Therefore some of the methods in this paper are new, while many others follow earlier results. We have attempted to make the presentation self-contained, since the theoretical subject is not well known to nuclear physicists. It is possible to use the techniques of this paper to explore other clustering phenomena in nuclear physics, e.g., the old problem of α clusters in nuclei.

The present paper is devoted to the construction of the ground-state wave function, and derivation of the ground-state energy. In Sec. II we explain the model and introduce the Bethe ansatz form of the wave function. The existence of bound clusters and their properties is discussed in Sec. III. In Secs. IV and V we find algebraic equations which determine the ground state and its energy, for an N-particle system. These are converted into an integral equation for the infinite system, in Sec. VI. Finally, properties of the ground-state energy are obtained in Sec. VII. This is followed by a discussion section and conclusions.

II. ONE-DIMENSIONAL MODEL AND THE BETHE ANSATZ

The model system consists of N nonrelativistic identical particles (quarks) obeying Fermi statistics, and moving in one space dimension. The only explicit internal degree of freedom is SU(3) color. (Spin and flavor can be included, but at the cost of considerable complication, and are not considered in this paper. One can consider the present model to have spin and flavor both frozen into "up" states, thus having no effect on the dynamics.)

The interaction is a two-body contact potential,

$$V_{ij}=\kappa\lambda_i\cdot\lambda_j\delta(x_i-x_j)\ , \tag{2.1}$$

where $\lambda_i\cdot\lambda_j$ gives the SU(3) color exchange form of one-gluon exchange, although the short range form of (2.1) has nothing to do with the usual perturbative gluon potential. For our purposes (2.1) is an effective interaction between quarks, which can produce clusters (nucleons), but not color confinement. Since we have suppressed other degrees of freedom, the antisymmetry of the quarks in space and color can be used to eliminate the color-exchange operator. The contact interaction (2.1) only acts on space-symmetric quark pairs, which are therefore color antisymmetric. The operator $\lambda_i\cdot\lambda_j$ then always takes on the numerical value $\langle\lambda_i\cdot\lambda_j\rangle=-(\frac{8}{3})$, so that we may simply introduce a new coupling constant $g=-(\frac{8}{3})\kappa$, and write the Hamiltonian in the form

$$H=-\frac{1}{2M}\sum_i^N\partial^2/\partial x_i^2-g\sum_{i<j}\delta(x_i-x_j)\ , \tag{2.2}$$

with $g>0$ for an attractive quark-quark interaction. For the present model, the Hamiltonian is therefore effectively color independent. However, when spin and flavor degrees of freedom are included, the color-exchange form (2.1) limits the nucleon cluster size to three quarks, which (2.2) does not (see Ref. 2, Sec. V B).

For the many-body system, we impose periodic boundary conditions in length L, containing N quarks. In Sec. VI, we shall find the ground state in the limit of an infinite system: $N,L\to\infty$ with fixed particle density $\rho=N/L$.

The Schrödinger equation with this Hamiltonian has solutions in the form (Bethe ansatz)

$$\psi(x_1,\ldots,x_N)=\sum_P a(Q,P)\exp\left[i\sum_j k(P_j)x(Q_j)\right]\ , \tag{2.3}$$

where $k(j)$, $j=1,\ldots,N$, are N different quantum numbers (momenta), P are the permutations of the indices $\{j\}$ of the $k(j)$, and Q are permutations of the indices $\{j\}$ of the positions $x(j)=x_j$, to put them in the order

$$0\le x(Q_1)\le x(Q_2)\le\cdots\le x(Q_N)\le L\ . \tag{2.4}$$

The notable property of (2.3) is that the wave functions can be expressed everywhere as linear combinations of plane-wave functions, with the *same* set $\{k(j)\}$ of momenta. The configuration space is divided into domains by the ordering (2.4) of the particle positions, which we label by the permutations Q. In each domain, each particle Q_j may carry any of the N momenta $k(P_j)$, according to the permutations P. The particles interact only at the boundaries of the domain (2.4), i.e., where $x_i=x_j$ for some i,j. [The form for $N=2$ is shown in (5.3), for $x_2>x_1$.]

For an open system (no boundary conditions imposed at $x_j=0,L$) a solution of the form of (2.3) exists for every set of *unequal* momenta $\{k(j)\}$. The amplitudes are determined by the interaction, for a given set of $\{k(j)\}$, as we see in Sec. IV. The imposition of periodic boundary conditions introduces a set of constraints on the values of the $k(j)$. The dynamical problem reduces to satisfying the constraints. The solution (2.3) has the energy

$$E=(2M)^{-1}\sum_j k^2(j)\ , \tag{2.5}$$

equal to the kinetic energy in any domain (2.4) away from the boundaries $x_i=x_j$.

A wave function of the type of (2.3) was first introduced by Bethe[3] as the solution of a one-dimensional chain of interacting (magnetic) spins; the term Bethe ansatz now refers to a variety of related constructions.[4] The use of such forms for the Hamiltonian of (2.2) was discovered by Lieb and Liniger[5] in the context of a Bose gas with a repulsive interaction and by McGuire,[6] for a general (finite) system of N particles. General proofs of the consistency of solutions of the form (2.3) for the Hamiltonian (2.2) were given by Brezin and Zinn-Justin[7] for an open system of any symmetry, and by Yang,[8] who extended them to periodic boundary conditions.

Gaudin[9] gave a method of solution for a spin-$\frac{1}{2}$ Fermi gas with the attractive interaction (2.2), leading to an integral equation for the ground state, analogous to that of Lieb and Liniger for the Bose gas with $g<0$. Yang[8] gave a different (equivalent) method for the spin-$\frac{1}{2}$ Fermi system with repulsion ($g<0$); Takahashi[10] showed the connection of Gaudin's and Yang's solutions for the ground state. The spin-$\frac{1}{2}$ Fermi system is equivalent to a two-color system, analogous to the three-color system of this paper. Sutherland[11] extended Yang's method to N colors, with repulsive interactions. Yang[12] gave some further properties of bound clusters for the Hamiltonian (2.2).

The method used in the present paper draws on the work of Refs. 5–12, but does not follow specifically the constructions of Gaudin and Yang. Since the immediate interest is in the ground state, our derivation is somewhat more direct and simpler than those of Refs. 8–11, and closer in spirit to that of Ref. 5. We will need first some properties of bound clusters, to which we turn next.

III. BOUND CLUSTERS

N particles acting under the Hamiltonian (2.2) may form a bound state (cluster) which is space symmetric, for a set of momenta for the wave function (2.3) (see Refs. 6 and 12):

$$\begin{aligned}&k(j)=K+imc\ ,\\ &m=\left[\frac{N-1}{2}\right],\left[\frac{N-1}{2}\right]-1,\ldots,-\left[\frac{N-1}{2}\right]\ .\end{aligned} \tag{3.1}$$

Here we have defined $c=Mg$, and have not (yet) included the constraints of Fermi statistics, or imposed periodic boundary conditions. The set of momenta (3.1) have the same real part K but different imaginary parts which sum to zero. Therefore, the c.m. momentum of the cluster is given by NK. In the c.m. the energy is [from (2.5)]

$$E_N=-Mg^2N(N^2-1)/24\ . \tag{3.2a}$$

For the present three-color model antisymmetry limits the cluster size to $N=2$ and $N=3$, with energies

$$E_2=-Mg^2/4\ , \tag{3.2b}$$

$$E_3=-Mg^2\ . \tag{3.2c}$$

The $N=3$ cluster is color neutral, and serves as our model nucleon. The $N=2$ cluster carries color, but is less well bound than the nucleon: from (3.2b) and (3.2c) we see that $E_3/3<E_2/2$, and thus will play no role in the many-body ground state. For $N=3$, (3.1) gives momenta

$$k_+=K+ic,\quad k_0=K,\quad k_-=K-ic\ . \tag{3.3}$$

The cluster function, for total momentum zero, takes the form (not normalized)

$$\psi_c(x_1,x_2,x_3)=\exp[-c(x_3-x_1)]\ , \tag{3.4}$$

for

$$x_1<x_2<x_3\ ,$$

or

$$\psi_c=\exp\left[-\frac{c}{2}\sum_{i<j}^{3}|x_i-x_j|\right]\ , \tag{3.5}$$

for any ordering of the x_i. Comparing (3.4) and (3.5) to (2.3), one sees that only one permutation (P) of momenta (3.3) has a nonzero amplitude $a(Q,P)$ for each Q. This follows from the usual requirement of bounded solutions for any $x\to\pm\infty$.

The cluster wave function (3.5) is evidently symmetric under permutation of the x_i. The color wave function is totally antisymmetric:

$$\chi_{123}=(\alpha\beta\gamma+\beta\gamma\alpha+\gamma\alpha\beta-\beta\alpha\gamma-\alpha\gamma\beta-\gamma\beta\alpha)\ , \tag{3.6}$$

where α,β,γ are the three color states, and particles are ordered 123 from the left. A cluster with total momentum ($3K$), given by (3.3), has the total wave function

$$\Psi_K(123)=\psi_c(x_1,x_2,x_3)\chi_{123}\exp[iK(x_1+x_2+x_3)]\ . \tag{3.7}$$

The imposition of periodic boundary conditions in $0<x<L$ restricts the values of K, which property will be used in the following sections. The boundary conditions do also introduce corrections to the imaginary parts (3.1) and therefore to the wave functions (3.5), which, however, can be shown to be small for $cL\gg1$, and disappear in the limit $L\to\infty$. Therefore, the boundary conditions only affect the real momentum K, in the large L limit, in which we are interested.

IV. GROUND STATE

For a color-neutral state we take $N=3n$ quarks, n of each color. Since the interaction (2.2) is attractive, the ground-state wave function should have maximal space symmetry consistent with the Pauli principle: i.e., $[3^n]$ in space (three equal columns, in a Young's tableau) and $[n^3]$ in color (three equal rows). These are one-dimensional permutation representations, so the wave function factors

$$\Psi(1,\ldots,N)=\psi(x_1,\ldots,x_N)\chi_{1,\ldots,N}\ . \tag{4.1}$$

With no boundary conditions imposed, the ground state consists of n bound clusters (of three quarks), each of the form (3.5), at large separation from each other, so there is vanishing cluster-cluster energy. Since there are no bound states with $N>3$ in this system, this arrangement gives the most binding energy. More appropriate to the Bethe form (2.3), the clusters can be said to have form (3.7), with vanishing momenta K. We expect the form of the ground state to remain similar, even with periodic boundary conditions, as long as $cL\gg n$, so that the clusters rarely overlap. [Note that the cluster size is of order c^{-1}, e.g., from (3.5).]

With these considerations, we construct the ground-state wave function as follows. Starting with the Bethe form (2.3), we assign the N momenta $k(j)$, $j=1,\ldots,N$, in triplets, k_+,k_0,k_-, as in (3.3), with a common real part K for each triplet. We label these real momenta by $m=1,\ldots,n$: $K(m)$. With the $K(m)$ taken to be unequal real quantities, we also satisfy the requirement that the $k(j)$ be unequal, which is sufficient for (2.3) to be a solution of the Schrödinger equation with Hamiltonian (2.2), for an open system. The imposition of periodic boundary conditions will only constrain the real momenta $K(m)$, since the imaginary parts are unaffected as long as $cL\gg1$ [see following (3.7)]. This construction gives the true ground state in the limit of $L\to\infty$ for finite N, as we see later. We assume the same form for all values of N/L; this will be justified by the analytic properties of the ground state which are discussed in Sec. VII.

To determine the values of the momenta $K(m)$, we first consider the N-body wave function (4.1) constrained to have three particles at each of n points y_m, $m=1,\ldots,n$; e.g.,

$$\begin{aligned} &y_1=x_1=x_2=x_3\ ,\\ &y_2=x_4=x_5=x_6\ ,\\ &\vdots\\ &y_n=x_{N-2}=x_{N-1}=x_N\ . \end{aligned} \tag{4.2}$$

Next, consider the constrained wave function

$$\Psi=\phi(y_1,\ldots,y_n)\chi_{1,\ldots,N}\ , \tag{4.3}$$

at well separated points, i.e., $|y_i-y_j|\gg c^{-1}$. Under this condition, the original Bethe form (2.3) can be seen to take a *reduced* Bethe ansatz form

$$\phi(y_1,\ldots,y_N)=\sum_p\alpha(q,p)\exp\left[i\sum_j^n 3K(p_j)y(q_j)\right]\ , \tag{4.4}$$

where the p are the permutations of the indices m of the $K(m)$ and the q are permutations of the indices i of the $y(i)=y_i$. The point is that for well-separated coordi-

nates y_i, each triplet in (4.2) forms a bound cluster, with momentum $3K(m)$. The internal wave functions drop out: $\psi_c=1$, (3.5). All other components of (2.3) with different distributions of the $k(j)$ other than those included in (4.4) become vanishingly small at large separations, due to the damping of plane waves with complex momenta (3.3).

We may consider (4.4) to be a cluster wave function with cluster (c.m.) coordinates y_i, and cluster momenta $K(m)$. It represents only the one assignment of quarks to clusters given in (4.2); other assignments can be obtained from the permutation symmetry of (4.1). It also represents the actual ground state only for nonoverlapping clusters. However, if we can determine the amplitudes $\alpha(q,p)$ and the momenta $K(j)$ of (4.4), we actually can determine all the amplitudes $a(Q,P)$ of (2.3) [and of course the momenta $k(j)$], and thus construct the wave function (4.1) at all points.

Now for a fixed L ($cL\gg1$) we use the periodic boundary condition on the reduced wave function (4.4), i.e.,

$$\phi(0,y_2,\ldots,y_n)=\phi(L,y_2,\ldots,y_n)\ , \tag{4.5}$$

and similarly for all y_i. This boundary condition is sufficient to determine the allowed momentum $K(j)$, by methods similar to those used originally by Lieb and Liniger,[5] as follows.

First, the antisymmetry of the cluster wave function (4.4) in the y_j allows us to restrict the discussion to one permutation q of the coordinates, say $q=I$: $q_j=j$. The amplitudes in all other domains q' are related to those for $q=I$ by

$$\alpha(q',p)=(-1)^s\alpha(p)\ , \tag{4.6}$$

where s is the signature (odd or even) of the permutation $I\to q'$, and $\alpha(p)=\alpha(I,p)$.

The boundary condition (4.5) can be rewritten, using antisymmetry

$$\phi(0,y_2,\ldots,y_n)=(-1)^{n-1}\phi(y_2,\ldots,L)\ . \tag{4.7}$$

This implies a relation in $q=I$:

$$(-1)^{n-1}e^{3iK(p_1)L}\alpha[p(n)]=\alpha[p(1)]\ , \tag{4.8}$$

where $p(1)=p_1,p_2,\ldots,p_n$; $p(n)=p_2,\ldots,p_n,p_1$.

Next, we consider the cyclic permutation from $p(1)$ to $p(n)$ as decomposed into a series of pair permutations $p_1\leftrightarrow p_2, p_1\leftrightarrow p_3,\ldots,p_1\leftrightarrow p_n$, each defining a permutation $p(j)=p_2,\ldots,p_j,p_1,p_{j+1},\ldots,p_n$.

We shall find (Sec. V) a relation between amplitudes whose permutations differ by any one of these pair permutations:

$$\alpha[p(j)]=A(p_1,p_{j+1})\alpha[p(j+1)]\ , \tag{4.9}$$

with $A(i,j)$ a complex amplitude of unit modulus:

$$A(i,j)=-\left[\frac{ix+c}{ix-c}\right]\left[\frac{ix+2c}{ix-2c}\right]\ , \tag{4.10}$$

$$x=K(p_i)-K(p_j)\ . \tag{4.11}$$

Therefore

$$\alpha[p(1)]=\prod_{j=2}^{n}A(p_1,p_j)\alpha[p(n)]\ . \tag{4.12}$$

From (4.8) we obtain

$$(-1)^{n-1}e^{3iK(p_1)L}=\prod_{j=2}^{n}A(p_1,p_j)\ , \tag{4.13}$$

which is a set of n algebraic equations [with (4.10) and (4.11)] relating the n momenta $K(j)$.

The ground state is given by the solution of these equations which gives the lowest energy. In Sec. V we derive (4.9)–(4.11), returning in Sec. VI to find the equations for the ground state.

V. AMPLITUDES

The wave function (4.1) for N quarks, now including the color labels explicitly, may be written

$$\psi_{c_1,\ldots,c_N}(x_1,\ldots,x_N)=\langle c_1,\ldots,c_N\mid\Psi(1,\ldots,N)\rangle\ , \tag{5.1}$$

where c_j is the color state of the jth quark. The coefficients $a(Q,P)$ in the expansion (2.3) of (5.1) will also carry the labels $c_1,\ldots,c_N$. Antisymmetry of the Ψ in space and color allows us to work in one region of Q, e.g., $Q=I$;

$$0\le x_1\le x_2\cdots\le x_N\le L\ . \tag{5.2}$$

With this restriction we drop the label Q: $a(I,P)=a(P)$.

Concentrate on a boundary, e.g., $x=x'$, with $x=x_j$, $x'=x_{j+1}$, and the dependence of (5.1) on x,x' and colors. Single out the following terms with color labels $c_j=a$, $c_{j+1}=b$,

$$a_{ab}(12)\exp[i(k_1x+k_2x')]+a_{ab}(21)\exp[i(k_2x+k_1x')]\ , \tag{5.3}$$

for two momenta k_1,k_2; the other factors are suppressed.

Using the continuity of (5.1) across the boundary and the discontinuity of the derivative, one can establish the following condition[8] relating the coefficients in (5.3) for colors ab and ba (Appendix A):

$$\begin{bmatrix}a_{ab}(12)\\a_{ba}(12)\end{bmatrix}=M_{12}\begin{bmatrix}a_{ab}(21)\\a_{ba}(21)\end{bmatrix}\ , \tag{5.4a}$$

where M_{12} is an operator on color indices

$$M_{12}=\frac{-iZ_{12}P_{12}+c}{iZ_{12}-c}\ , \tag{5.4b}$$

with $Z_{12}=k_1-k_2$, and P_{12} interchanges color indices: $P_{12}a_{ab}=a_{ba}$.

Now we apply these relations to amplitudes involving clusters. As noted for (3.5), a bound cluster has only one permutation of the momenta (3.3), namely k_-, k_0, k_+, for increasing x_j. The color state is given in (3.6).

First consider a four-quark system, consisting asymptotically of one cluster plus one quark of momentum k_1 and color α—this is the asymptotic description when $x_1 \ll x_2,x_3,x_4$ or $x_1,x_2,x_3 \ll x_4$. (This can be considered the state for scattering of a quark of momentum k_1 from a cluster of momentum K; see Ref. 11.)

Define the following permutations P_i of the momenta: $1=k_1$, $2=k_-$, $3=k_0$, $4=k_+$;

$$\begin{aligned} P_1&=1\,2\,3\,4\ , \\ P_2&=2\,1\,3\,4\ , \\ P_3&=2\,3\,1\,4\ , \\ P_4&=2\,3\,4\,1\ . \end{aligned} \tag{5.5}$$

The color states corresponding to P_1 and P_4 must be

$$\chi(P_1)=\alpha\chi_{234}\ , \tag{5.6a}$$

$$\chi(P_4)=\chi_{123}\alpha\ , \tag{5.6b}$$

with χ_{abc} given by (3.6), to give the appropriate asymptotic behavior of particle plus cluster. The coefficients of these states are therefore given by the following linear combinations of the amplitudes $a_{abcd}(P)$:

$$\begin{aligned} B(P_1)=(6)^{-1/2}&(a_{\alpha\alpha\beta\gamma}+a_{\alpha\beta\gamma\alpha}+a_{\alpha\gamma\alpha\beta} \\ &-a_{\alpha\beta\alpha\gamma}-a_{\alpha\alpha\gamma\beta}-a_{\alpha\gamma\beta\alpha})\ , \end{aligned} \tag{5.7a}$$

$$\begin{aligned} B(P_4)=(6)^{-1/2}&(a_{\alpha\beta\gamma\alpha}+a_{\beta\gamma\alpha\alpha}+a_{\gamma\alpha\beta\alpha} \\ &-a_{\beta\alpha\gamma\alpha}-a_{\alpha\gamma\beta\alpha}-a_{\gamma\beta\alpha\alpha})\ , \end{aligned} \tag{5.7b}$$

with P_1,P_4 understood for each $a(P)$ in (5.7a) and (5.7b).

Now the amplitudes $a(P)$ for different permutations P_i of (5.5) are related by operators M_{1i}, (5.4). For example,

$$M_{14}a_{\alpha\beta\gamma\alpha}(P_4)=\frac{-iZ_{14}a_{\alpha\beta\alpha\gamma}(P_3)+ca_{\alpha\beta\gamma\alpha}(P_3)}{iZ_{14}-c}\ . \tag{5.8a}$$

Then, if we consider the coefficients $a_{abcd}(P_i)$ to define vectors on which the M_{1i} operate, we may relate the two coefficients (5.7) by a numerical factor,

$$B(P_1)=\phi(k_1-k_0)B(P_4)\ , \tag{5.8b}$$

where the factor ϕ is the matrix element in this vector space obtained by repeated operation of (5.4a) on (5.7b):

$$\phi(k_1-k_0)=\langle B(P_1)\,|\,M_{12}M_{13}M_{14}\,|\,B(P_4)\rangle\ . \tag{5.8c}$$

Straightforward algebra (Appendix B) yields

$$\phi(k_1-k_0)=-\left[\frac{i(k_1-k_0)+c}{i(k_1-k_0)-c}\right]\ . \tag{5.9}$$

The result (5.9) can be seen to be unitary (for real k_1,k_0) and independent of the odd particle color (α), as expected.

Next, consider six quarks, arranged so that asymptotically they form two clusters of momenta $3K_1,3K_2$; the momentum labels are taken to be

$$\begin{aligned} &1=K_1-ic,\quad 2=K_1,\quad 3=K_1+ic\ , \\ &4=K_2-ic,\quad 5=K_2,\quad 6=K_2+ic\ , \end{aligned} \tag{5.10}$$

and we denote $p(1)$ to be the permutation order 123456, and $p(2)=456123$ to be the order with the clusters interchanged. The amplitudes for these permutations are $\alpha(p)$, corresponding to (4.6).

We find easily by applying (5.8a) and (5.9) three times that

$$\alpha[p(1)]=A(1,2)\alpha[p(2)]\ , \tag{5.11a}$$

$$A(1,2)=\phi(K_1-ic-K_2)\phi(K_1-K_2)\phi(K_1+ic-K_2)\ , \tag{5.11b}$$

$$A(1,2)=-\left[\frac{ix+c}{ix-c}\right]\left[\frac{ix+2c}{ix-2c}\right]\ , \tag{5.12}$$

with $x=K_1-K_2$. This leads directly to the results quoted in (4.9)–(4.11), for an n-cluster system.

VI. INTEGRAL EQUATION FOR THE GROUND STATE

We return to finding the ground state from (4.13) following Lieb and Liniger.[5] We expect the $K(j)$ to be bounded symmetrically: $-K\le K(j)\le K$ with smallest possible spacings. Define

$$\Delta_j=K(j+1)-K(j)\ . \tag{6.1}$$

First setting $K(p_1)=K(j)$, then $K(p_1)=K(j+1)$ in (4.13), and taking the logarithm, obtain

$$3i\Delta_jL=\ln\prod_{l=1}^{n}A(j+1,1)/A(j,1)+2\pi m_j\ , \tag{6.2}$$

with integer m_j chosen to minimize the Δ_j. For $cL\gg1$, expand (6.2) in $\Delta_j/c\sim(cL)^{-1}$, using (4.10):

$$\begin{aligned} 3\Delta_jL=2c\Delta_j\sum_l&\left\{\frac{1}{[k(j)-k(1)]^2+c^2}\right. \\ &\left.+\frac{2}{[k(j)-k(1)]^2+(2c)^2}\right\} \\ &+2\pi m_j+O(\Delta^2/c^2)\ . \end{aligned} \tag{6.3}$$

Define the density of momentum states per unit k:

$$f(k_j)=(\Delta_jL)^{-1}\ . \tag{6.4}$$

Now take the limit of (6.3) as $L\to\infty$:

$$3f^{-1}(k)=2\pi m+2f(k)^{-1}\int_{-K}^{K}dq\,f(q)G(q-k)\ , \tag{6.5a}$$

or, alternatively:

$$2m\pi f(k)=3-2\int_{-K}^{K}dq\,f(q)G(q-k)\ , \tag{6.5b}$$

with

$$G(q)=\frac{c}{c^2+q^2}+\frac{2c}{(2c)^2+q^2} . \tag{6.6}$$

The integer m is independent of k, and can be determined by taking the limit $c\to 0^+$, for which

$$\lim G(q-k)=2\pi\delta(q-k) , \tag{6.7}$$

and

$$2m\pi f(k)=3-4\pi f(k) ,$$

or

$$f(k)=\frac{3}{(2m+4)\pi} . \tag{6.8}$$

Since in this limit of no interaction, the density of momentum in the ground state must have the free quark (Fermi) gas value $f(k)=(2\pi)^{-1}$; we find $m=1$. The integral equation for the ground state is then

$$2\pi f(k)=3-2\int_{-K}^{K} dq\, f(q)G(q-k) . \tag{6.9}$$

Since the number of particles (quarks) is given by

$$N=3\sum_{j=1}^{n} 1 ,$$

the density $\rho=N/L$ in the $L\to\infty$ limit is given by

$$\rho=3\int_{-K}^{K} dk\, f(k) . \tag{6.10}$$

The ground-state energy per quark is given by one-third of the binding energy of a nucleon cluster (3.2c) plus the average kinetic energy

$$\frac{E}{N}=\frac{1}{2M}\left[\frac{-2c^2}{3}+\frac{3}{N}\sum_{j}^{n} k(j)^2\right] ,$$

which becomes, in the limit

$$\frac{E}{N}=\frac{1}{2M}\left[\frac{-2c^2}{3}+\frac{3}{\rho}\int_{-K}^{K} dk\, k^2 f(k)\right] . \tag{6.11}$$

These equations were given in Koltun and Tosa.[1] An alternative derivation of these equations may be obtained starting with Sutherland's treatment[11] of the three-color system with a repulsive potential.[13,14]

VII. PROPERTIES OF THE GROUND-STATE ENERGY

The solution $f(k)$ of the integral equation (6.9) for a given value of K determines both the ground state density of the system, through (6.10), and the ground-state energy per quark, from (6.11). It is simpler, both for formal properties and for numerical solution, to rescale the momentum variables and constants as follows:

$$z=k/K, \quad v=c/K , \tag{7.1}$$

so that (6.9) may be rewritten in dimensionless from using (7.1)

$$2\pi f_v(z)=3-2v\int_{-1}^{1} dx\, f_v(x)g_v(z-x) , \tag{7.2a}$$

$$g_v(x)=\frac{1}{x^2+v^2}+\frac{2}{x^2+(2v)^2} . \tag{7.2b}$$

Then (6.10) and (6.11) become

$$r=\rho/c=\frac{3}{v}\int_{-1}^{1} dx\, f_v(x) , \tag{7.3}$$

$$e=\left[\frac{3M}{c^2}\right]\frac{E}{N}=\left[-1+\frac{9}{2rv^3}\int_{-1}^{1} dx\, x^2 f_v(x)\right] , \tag{7.4}$$

where r and e are the dimensionless density and energy. The notation emphasizes the dependence of the solution $f_v(z)$ (and therefore r,e) on only one parameter, v. It is clear from (7.4) that the density enters only as $r=\rho/c$, so that the ground state behavior for $\rho\to\infty$ is given by $c\to 0$, and for $\rho\to 0$ by $c\to\infty$.

Physical solutions require non-negative $f_v(z)$, as is seen from (6.4). Some further properties of $f_v(z)$ for $(-1\le z\le 1)$ are easily obtained directly from the integral equation (7.2), and this restriction:

$$f_v(-z)=f_v(z), \quad f_v'(z)\ge 0 \quad \text{for } z\ge 0 , \tag{7.5a}$$

$$0\le f_v(0)\le f_v(z)\le 3/2\pi . \tag{7.5b}$$

Two interesting limiting cases are given by

$$v\to\infty, \quad vg(x)\to 0 ,$$
$$\lim f(z)=3/2\pi , \tag{7.6a}$$
$$r\to 0, \quad e\to -1 .$$

$$v\to 0, \quad vg(x)\to 2\pi\delta(x) ,$$
$$\lim f(z)=1/2\pi , \tag{7.6b}$$
$$r\to\infty, \quad e\to\pi^2 r^2/18 .$$

For fixed c, these give the $\rho\to 0$, $\rho\to\infty$ limits. Since $\rho\to\infty$ is equivalent to $c\to 0$, the system in this limit is a Fermi gas of quarks, with a density of states (per color)

$$f(k)=1/2\pi, \quad -K\le k\le K \tag{7.7a}$$

where K is the quark Fermi momentum and

$$\rho\to 3K/\pi, \quad \frac{E}{N}\to\frac{K^2}{6M}=\frac{\pi^2\rho^2}{54M} , \tag{7.7b}$$

as $K=c/v\to\infty$.

The other limit gives a dilute gas of nucleons (clusters) with energy per nucleon

$$\frac{E}{n}=3\frac{E}{N}=-Mg^2 , \tag{7.8}$$

equal to the cluster energy (3.2c). For $0<\rho\ll c$, this can be seen to behave like a Fermi gas of nucleons, as follows. Expanding all quantities in $K=c/v$ and keeping lowest order terms in (7.2)–(7.4), we find

$$f(k)\simeq 3/2\pi, \quad -K<k<K ,$$
$$\rho\simeq 9K/\pi , \tag{7.9a}$$
$$\frac{E}{N}\simeq-\frac{Mg^2}{3}+\frac{K^2}{6M}=-\frac{Mg^2}{3}+\frac{\pi^2\rho^2}{9^2 6M} .$$

This expression was also obtained by a variational method, independent of the Bethe ansatz, in Ref. 2. These quantities refer to the quarks: momenta k, densi-

ty ρ, energy E/N. The nucleons have momenta $P=3k$, so the nucleon Fermi momentum is $P_F=3K$, and the quantities in (7.9a) may be rescaled for nucleons:

$$f_N(P)=\frac{f(k)}{3}\simeq\frac{1}{2\pi},\quad -P_F<P<P_F\ ,$$
$$\rho_N=\rho/3\simeq P_F/\pi\ , \tag{7.9b}$$
$$\frac{E}{n}\simeq -Mg^2+\frac{P_F^2}{6(3M)}=-Mg^2+\frac{\pi^2\rho_N^2}{6(3M)}\ ,$$

where $f_N(P)$ is the density of states in nucleon momentum, and ρ_N the nucleon density. Comparing (7.9b) with (7.7), the differences between the Fermi gases at the two extremes of density can be seen to come from the different masses: M for the quarks and $(3M)$ for the nucleons; and, from the color, three states for the free quarks and one (neutral) state for the nucleons.

The nucleons at low density actually do have a mutual interaction, which is expressed in the amplitudes $A(i,j)$ of (4.10). (The interpretation of these terms as cluster-cluster scattering amplitudes has been discussed by McGuire[6] and Yang.[11]) The phase of (4.10) is always equivalent to a repulsive interaction. In the low density limit, $x\to 0$, (4.11), and $A\to -1$, the amplitude for a repulsive "core" (effective) potential

$$V_{NN}(y_i-y_j)=\lim_{G\to\infty} G\delta(y_i-y_j)\ . \tag{7.10}$$

However, the interaction (7.10) has no effect on the energy in this limit, since the nucleons (clusters) are Pauli excluded from $y_i=y_j$ by antisymmetry. Thus, the energy at low density given by (7.9b) has only internal nucleon energy plus kinetic energy.

At higher density, the interaction will appear in the ground-state energy as a repulsion relative to the nucleon Fermi gas, or as an attraction relative to the quark Fermi gas. It is in fact easy to establish upper and lower bounds for E/N at any density ρ. First,

$$\frac{E}{N}(\rho)\le\frac{\pi^2\rho^2}{54M}-\frac{1}{3}g\rho \tag{7.11}$$

follows variationally, since the interaction is attractive; the quark Fermi gas kinetic energy is an upper bound. The linear term is the Hartree-Fock contribution of the potential of (2.2). (See Lieb and DeLlano,[15] for the two-color case.)

A lower bound may be obtained from the inequality

$$\int_{-1}^{1} dx\, x^2 f_v(x)\ge\tfrac{1}{3}\int_{-1}^{1} dx\, f_v(x)\ , \tag{7.12a}$$

which follows from the concavity of $f(x)$, (7.5a). Using this with (7.3) and (7.4) gives

$$e\ge -1+(2v^2)^{-1}\ . \tag{7.12b}$$

Further,

$$\rho\le\frac{9c}{\pi v} \tag{7.12c}$$

follows from (7.3) with $f(x)\le 3/2\pi$. Then combining (7.12a), (7.12b), and (7.12c) we obtain the lower bound

$$\frac{E}{N}(\rho)\ge -\frac{Mg^2}{3}+\frac{\pi^2\rho^2}{9^2 6M}\ , \tag{7.13}$$

which is the nucleon Fermi gas result of (7.9a).

The functional dependence of E/N on ρ is that of a monotonically rising function bounded by (7.11) and (7.13). This is related to the fact that $\rho\to 0$ gives the true ground state for no boundary condition (see discussion below).

The solutions of the integral equation may be found numerically: sample results are shown in Figs. 1 and 2. In Fig. 1, we see the smooth change of the density of states $f_v(x)$ with varying v, passing between the two limits of (7.6). Similarly, Fig. 2 shows the smooth variation of energy with density, approaching the lower bound (7.12b) at low density, and the upper bound (7.11) at high density. A simple approximation of assuming $f_v(x)$ to be a constant $\bar{f}_v$ for all v is surprisingly accurate for the energy; this is shown by the solid line of Fig. 2. The resulting equations are obtained by setting $f_v(0)=\bar{f}_v$ in (7.2):

$$\bar{f}_v=3\left[2\pi+4\tan^{-1}\left[\frac{1}{v}\right]+4\tan^{-1}\left[\frac{1}{2v}\right]\right]^{-1}\ ,$$
$$r=6\bar{f}/v\ , \tag{7.14}$$
$$e=\left[-1+\frac{1}{2v^2}\right]\ .$$

This clearly gives exactly both the limiting cases of (7.6), since $f_v(z)=\bar{f}_v$ in these limits.

We come to the question of whether the solution of (7.2) represents the true ground state at the density by (7.4). First, with no restriction of the density (zero pressure) the true ground state is given by isolated $(N=3)$ clusters, with energy (7.8). This follows from the completeness of the Bethe ansatz wave functions (see, e.g., Li *et al.*[16]) and the limitation of clusters to $N\le 3$ for the present model [see (3.1) and (3.2)]: no other bound states exist.

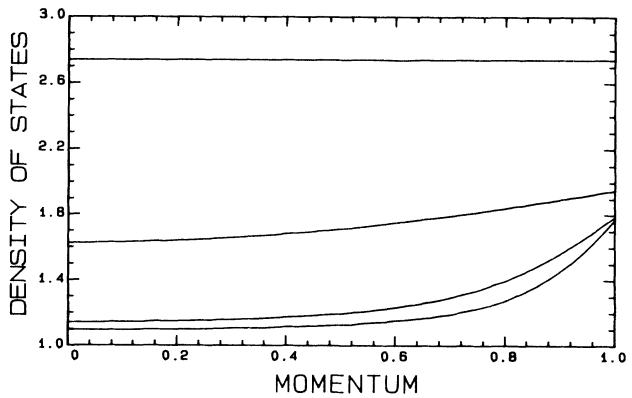

FIG. 1. The density of states $2\pi f_v(x)$ is plotted vs the dimensionless momentum $x=k/K$, for several values of $v=c/K$, (7.1); from the top: $v=10$, 1, 0.25, and 0.166, from numerical calculations.

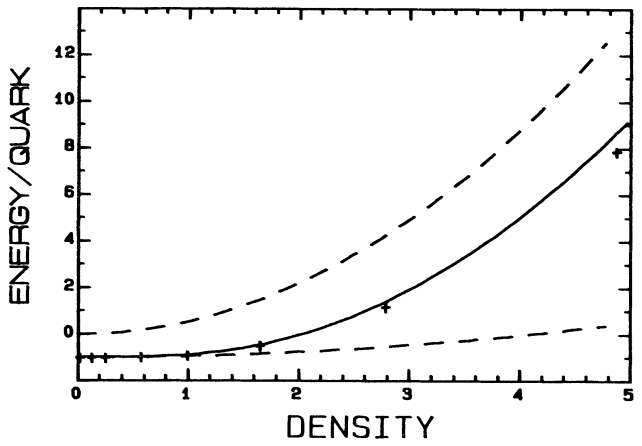

FIG. 2. Plot of energy per quark vs quark density in dimensionless form: e [Eq. (7.4)] vs r [Eq. (7.3)]. The dashed lines give the quadratic upper and lower bounds, corresponding to (7.11) (with linear term neglected) and (7.12). The crosses are calculated from numerical solutions of the integral equation (7.2), the solid line from the approximation (7.14).

Second, Eqs. (7.2)–(7.4) can be seen to have smooth variation with parameter c, except near $c=0$. For the related (Gaudin) problem of a two color system, Takahashi has proved the analyticity of the ground-state energy (6.11) or (7.4) for real c ($c\neq 0$) of either sign. This obtains also for the present model, giving analytic behavior of $f_v(x)$, r, and e in the parameter $v=c/k$. It follows that $E/N(\rho)$ is analytic in ρ for all finite ρ, which means that there are no "level crossings." It also follows from the variational principle that at each density the energy can be lowered by decrease of density, when gives the monotonic behavior of $E/N(\rho)$ (see Ref. 15).

Strictly speaking, this means that solution of (7.2) gives the lowest state of the same permutation symmetry: we have restricted ourselves to $[3^n]$ in the construction, beginning in Sec. IV. Therefore, states of different symmetry could conceivably become the ground state at some density. This could only be eliminated by solution of the Bethe ansatz equations for the appropriate symmetry. It is clear, however, that for sufficiently low density the $N=3$ cluster must be the true ground state: e.g., for $E/N<0$, only the $N=2$ or $N=3$ clusters compete.

VIII. DISCUSSION

We have found in the previous section that the ground-state energy of our model system approaches that of a Fermi gas of nucleons at low density, and that of a Fermi gas of quarks at high density, with a smooth transition between those limits (see Fig. 2). What happens in the wave functions? For very low density the construction followed in Sec. IV effectively assigned every particle to a nucleon cluster, the clusters forming an interacting gas, as in (4.3) and (4.4), with the permutation symmetry of the entire system preserved. In this limit, the nucleon momenta are small, $3K \ll c$, so that on the average, the clusters are separated by distances large compared to the internal size of a cluster as given by (3.5), i.e., $c\,|y_i-y_j| \gg 1$, for which (4.4) is a valid representation of (4.1). The clusters are anticorrelated, as discussed in connection with (7.10), so that the cluster wave function (4.4) has nodes at $y_i=y_j$, in the low density limit. Thus the wave function exhibits the nucleon Fermi gas behavior seen in the energy, in this limit. The anticorrelation can equally well be considered to be a consequence of the repulsive effective interaction (7.10) between clusters. For the present model, this does not affect the energy, because the wave function (4.4) is *also* antisymmetric. However, for closely related models with different cluster symmetry, the repulsive interaction at low density is apparent, e.g., for the "two-color" fermion model of Gaudin, for which the clusters are Bose-like: in the low density limit, the energy is that of a Fermi gas of clusters, because of the nodes imposed by the repulsion. (The repulsion is clearly a Pauli exclusion effect of the quarks.)

At higher densities, the form (4.1) of the ground-state wave function is unchanged, with the momenta assigned in triplets corresponding to clusters. However, the clusters are generally overlapping, and the wave function (4.1) is not well represented by (4.4) everywhere; there are other significant terms not of that form, for $c\,|y_i-y_j| \lesssim 1$. [This does not affect the method of Secs. IV–VI for the determination of the $K(m)$, since (4.4) can be used for any *pair* of clusters at large separation, independent of the positions of the other particles.] In the regions of overlap, the spatial forms of the clusters are changed from that of (3.5), and indeed the identities of the clusters are mixed. This regime of partially overlapping nucleons is of interest in the context of looking for quark effects in low energy nuclear systems. For the present model, the quark effects appear in the energy as a repulsive interaction between nucleons. But since the nucleons in this model are modified when they overlap, it would be interesting to study effects specific to these changes, such as the response to excitation by external probes, e.g., in scattering of electrons. It would also be interesting to follow the ground-state properties (other than energy) as a function of density.

Here the availability of exact constructive methods for the eigenstates of the model are of some use. For example, it is easy to see that the lowest-energy excitations of the system will involve changes of single-nucleon momenta, rather than break up of nucleon clusters. However, the present technology of the Bethe ansatz does not lead directly to interesting quantities like the two-body correlation function, or the response function, in a manner analogous to the ground-state energy. These are a subject for further study. For some purposes, it may be more profitable to study finite systems at fixed L, rather than the infinite-system limit.

The present model obviously has many shortcomings as a representation of quark structure in nuclear physics. Some of these may be obviated by improvements in the model; others are clearly beyond its scope. One obvious extension is to include other internal degrees of freedom

of the quarks, such as spin and flavor. This is currently under investigation (see also Ref. 2, Sec. V B and a recent paper by Kebukawa[17]).

The restriction to one space dimension is a severe limitation to making the model more realistic. The methods used in this paper (Bethe ansatz) cannot be extended to more dimensions, but the variational approach of Ref. 2 is possible for a generalized three-dimensional model. However, other elements in a more complete theory are also missing: e.g., relativistic quarks, confining interactions, gluonic degrees of freedom, etc. These are clearly beyond the scope and the goal of the present approach.

IX. CONCLUSIONS

The main goal of this paper has been to obtain the ground-state energy of our quark model of nuclear matter, and demonstrate the transformation from nucleonic to quark matter, with increasing density. This has been accomplished, using a construction based on the Bethe ansatz, which reduces the problem to the solution of a linear integral equation. The transition of the energy with density can then be followed between the nucleonic and quark-gas limits. Further study of this model system and extensions of the model are under way.

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APPENDIX A: DERIVATION OF EQ. (5.4)

Starting with the wave function (5.1), consider two quark variables only: x,x' with colors a,b; all other variables are held fixed. We study the neighborhood of the boundary $x=x'$. Antisymmetry gives

$$\psi_{ab}(x,x')=-\psi_{ba}(x',x)\ . \tag{A1}$$

Continuity of ψ at the boundary and (A1) give the condition

$$\psi_{ab}(x,x)=-\psi_{ba}(x,x)\ . \tag{A2}$$

The Schrödinger equation for ψ with the Hamiltonian given in (2.2) may be integrated across the boundary, to obtain

$$\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]\psi_{ab}(x,x')\Bigg|_{x=x'-\epsilon}^{x=x'+\epsilon}=-2c\,\psi_{ab}(x,x)\ , \tag{A3}$$

or using (A1) and (A2)

$$\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right][\psi_{ab}(x-\epsilon,x)-\psi_{ba}(x-\epsilon,x)] =c\,[\psi_{ab}(x,x)-\psi_{ba}(x,x)]\ . \tag{A4}$$

Equations (A2) and (A4) are boundary conditions on the color-antisymmetric wave functions; the color-symmetric wave functions have a node at the boundary.

If we apply these conditions to the Bethe form (2.3), now expressed as in (5.3), Eq. (A2) yields

$$a_{ab}(12)+a_{ab}(21)+a_{ba}(12)+a_{ba}(21)=0\ , \tag{A5}$$

while (A4) leads to

$$iZ_{12}[a_{ab}(12)-a_{ab}(21)-a_{ba}(12)+a_{ba}(12)] =c\,[a_{ab}(12)+a_{ab}(21)-a_{ba}(12)-a_{ba}(21)]\ , \tag{A6}$$

with $Z_{12}=k_1-k_2$. These can be rearranged in the matrix form given in (5.4).

APPENDIX B: EQUATION (5.9)

Write the operator of (5.8c) as $T=M_{12}M_{13}M_{14}$, with M_{1i} of (5.4) expressed in the form

$$M_{1i}=A_iP_{1i}+B_i,\quad A_i+B_i=-1\ ,\qquad A_i=\frac{-iZ_{1i}}{iZ_{1i}-c},\quad B_i=\frac{c}{iZ_{1i}-c}\ . \tag{B1}$$

Then operating on the six components of the coefficient $B(P_4)$ of (5.7b) (we indicate only the color indices),

$$\begin{aligned}T(\alpha\beta\gamma\alpha-\alpha\gamma\beta\alpha)&=B_2B_3B_4(\alpha\beta\gamma\alpha-\alpha\gamma\beta\alpha)\\&\quad+B_2B_3A_4(\alpha\beta\alpha\gamma-\alpha\gamma\alpha\beta)\\&\quad-A_3A_4(\alpha\alpha\beta\gamma-\alpha\alpha\gamma\beta)\\&\quad+B_2A_3B_4(\alpha\gamma\beta\alpha-\alpha\beta\gamma\alpha)+\cdots\ ,\end{aligned} \tag{B2a}$$

$$\begin{aligned}T(\gamma\alpha\beta\alpha-\beta\alpha\gamma\alpha)&=-A_2A_4(\alpha\gamma\alpha\beta-\alpha\beta\alpha\gamma)\\&\quad+A_2B_3B_4(\alpha\gamma\beta\alpha-\alpha\beta\gamma\alpha)+\cdots\ ,\end{aligned} \tag{B2b}$$

$$T(\beta\gamma\alpha\alpha-\gamma\beta\alpha\alpha)=-A_2A_3(\alpha\beta\gamma\alpha-\alpha\gamma\beta\alpha)\ , \tag{B2c}$$

where we have kept only the components which actually contribute to the vector $B(P_i)$, (5.7a); those components orthogonal to (5.7a) are denoted by $\cdots$. The amplitude for each component of $B(P_i)$ is given by the (normalized) sum of coefficients of that component in Eq. (B2), e.g.,

$$\langle \alpha\alpha\beta\gamma \mid T \mid B(P_4)\rangle = -6^{-1/2} A_3 A_4$$
$$= +6^{-1/2} \frac{(k_1-k_0)(k_1-k_0-ic)}{[i(k_1-k_0)-c][i(k_1-k_0)]}$$
$$= -6^{-1/2} \frac{i(k_1-k_0)+c}{i(k_1-k_0)-c} . \tag{B3}$$

Similarly, for each component of P_1 one obtains by straightforward algebra the same coefficient as for (B3), which then leads directly to (5.9).

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