PHYSICAL REVIEW C VOLUME 36, NUMBER 4 OCTOBER 1987

Poincaré invariant potential model

Michael G. Fuda
Department of Physics and Astronomy, State University of New York at Buffalo, Buffalo, New York 14260
(Received 8 June 1987)

A general Poincaré invariant potential model for two-particle systems is constructed within the framework of light front dynamics. The model specifies all ten of the generators that are necessary to satisfy the Poincaré algebra. Formal expressions for the scattering states are constructed, and it is verified explicitly that they transform correctly under Lorentz transformations. The S-matrix elements are shown to be Lorentz invariant functions of the initial and final four-momenta of a scattering process. The T matrix is found to satisfy an integral equation that has the mathematical simplicity of its nonrelativistic counterpart.

I. INTRODUCTION

Many years ago Dirac[1] established the necessary conditions that a Hamiltonian formulation of relativistic dynamics must satisfy. In quantum mechanics these conditions arise from the requirement that to each inhomogeneous Lorentz transformation there corresponds a unitary operator that transforms the state vectors of the system. For those transformations that can be built up from infinitesimal transformations, these unitary operators can be expressed in terms of ten Hermitian generators P_μ and $J_{\mu\nu}$, where P_μ is the four-momentum operator and $J_{\mu\nu}$ is the angular momentum tensor. From the fact that the inhomogeneous Lorentz transformations form a group, it follows that these generators must satisfy a set of commutation relations which define the so-called Poincaré algebra. These commutation relations are the necessary conditions that a relativistic dynamics must satisfy. A dynamics that does so is said to be Poincaré invariant.

In constructing the ten generators, P_μ and $J_{\mu\nu}$, of the Poincaré group it is necessary to decide on the dynamically independent variables. Causality suggests the variables associated with a hypersurface Σ in Minkowski space that does not contain timelike directions.[2] In the usual form of dynamics, Σ is taken to be the spacelike surface $t=0$, or any other surface related to it by a Poincaré transformation. As is well known, the generators associated with the transformations which map this Σ onto itself are the three-momentum (P^1,P^2,P^3) and the angular momentum (J_{23},J_{31},J_{12}). These generators are the simple or kinematical ones in that they can be taken to be of the same form as those of a noninteracting system, while the remaining ones $(P^0,J_{10},J_{20},J_{30})$ depend on the interaction and are said to be dynamical. Dirac[1] calls all of the dynamical generators Hamiltonians.

It was Dirac[1] who first emphasized that it is possible to work with hypersurfaces Σ other than $t=0$. Other choices of Σ lead to different ways of splitting the ten Poincaré generators into kinematical ones and Hamiltonians. In principle, Σ can be any surface which does not contain timelike directions, but in order to get a practical theory Σ should be associated with some subgroup of the Poincaré group, so that it has some symmetries. This subgroup is called the stability group of Σ. The generators of the stability group are the kinematical ones, while the remaining ones are the Hamiltonians.

It can be shown[2] that if it is required that it be possible to map every point of Σ into any other point of Σ by an element of the stability group, then there are only five inequivalent classes of surfaces. Inequivalent means that the surfaces are not related by Poincaré transformations. Representatives of these classes are the following: (i) the time instant $t=0$, (ii) one sheet of a hyperboloid $t^2-x^2-y^2-z^2=a^2$, $t>0$; (iii) the null plane $t+z=0$; one sheet of the hyperboloids (iv) $t^2-x^2-y^2=a^2$, $t>0$, and (v) $t^2-z^2=a^2$, $t>0$. The number of generators for the stability groups of the surfaces (i)–(v) is 6, 6, 7, 4, and 4, respectively.

Dynamics based on the last two cases appear to be impractical, since they each involve six Hamiltonians. Dynamics based on the first three classes are called[1] (i) the instant form, (ii) the point form, and (iii) the front or light front form. These three forms have all been used to construct Hamiltonian quantum theories for systems with a finite number of degrees of freedom. As might be expected, the instant form has the longest history and has been the most thoroughly investigated.[3] Interest in the point form[4] and the front form[2,5–7] has developed somewhat more recently.

Here we will be mainly concerned with the light front formulation of the quantum mechanics of two-particle systems. We will not be concerned with the light front approach to quantum field theory, although it must be stated that quantum field theory has provided much of the impetus for studying the light front approach.[8] This aspect of the subject has been thoroughly documented by Namyslowski.[9]

For our purposes the most important papers are those of Bardakci and Halpern[5] and Leutwyler and Stern.[2,7] It appears that Bardakci and Halpern[5] were the first to construct Poincaré invariant potential models for two particle systems using the light front approach. As their work makes clear, the light front approach actually has a more thoroughgoing nonrelativistic analogy than the instant form approach. This makes it easier to construct models which satisfy the separable condition, which is the rather obvious requirement that a theory reduces to a free particle theory when interactions are negligible.

The most thorough and systematic analysis of the light front approach to Hamiltonian quantum theories of finitely many degrees of freedom has been carried out by Leutwyler and Stern.[2] They have stressed the analogy with nonrelativistic quantum mechanics. In particular, they have shown that, just as in nonrelativistic quantum mechanics, the variables that describe the internal structure of a system can always be uncoupled from those that describe the motion of the system as a whole. To be precise, they have shown that the dynamical content of a theory resides in a mass operator M and spin operator $\mathcal{J}$ that act in the space of the internal degrees of freedom. Since the ten generators of the Poincaré group are either purely kinematical or involve combinations of kinematical operators with M and $\mathcal{J}$, constructing a Poincaré invariant theory amounts to finding models for M and $\mathcal{J}$. The only constraints on these operators are as follows: M must commute with $\mathcal{J}$, the components of $\mathcal{J}$ must satisfy angular momentum commutation rules, and M and $\mathcal{J}$ must commute with all the generators of the stability group except J_3. The angular momentum J_3 is associated with spatial rotations in the null plane and is simply related to $\mathcal{J}_3$. Because of its relation to a stability group generator, $\mathcal{J}_3$ is essentially kinematical although it is a necessary part of the U(2) algebra associated with the dynamics. The interactions only occur in M, $\mathcal{J}_1$, and $\mathcal{J}_2$.

Here we will develop a rather general Poincaré invariant potential model for two particle systems. This model is characterized by the assumption that the internal spin operator $\mathcal{J}$ is the same as that for two free particles. The mass operator M is given by $M^2=M_0^2+U$, where M_0 is the free particle mass operator and U is the interaction. If it is assumed that U is such that scattering states exist, then the resulting S matrix turns out to be an invariant function of the initial and final state four-momenta. We will see that the T matrix that such a U gives rise to satisfies an integral equation that is very similar to, and just as simple as, its nonrelativistic counterpart.

The outline of the paper is as follows. In Sec. II we outline the results of Leutwyler and Stern[2] that are necessary for establishing the Poincaré invariance of our potential model. The model is developed in Sec. III and its Poincaré invariance is verified. The scattering theory for the model is presented in Sec. IV and the Lorentz invariance of the S matrix is demonstrated. The integral equation for the T matrix is also obtained. Section V contains a summary of the results, as well as some comments on requirements that might be imposed on relativistic systems, other than Poincaré invariance.

II. THE LIGHT FRONT APPROACH

An inhomogeneous Lorentz transformation is of the form

$$x'^{\mu}=a^{\mu}{}_{\nu}x^{\nu}+b^{\mu} , \tag{1}$$

corresponding to a homogeneous transformation, followed by a displacement. We assume that there exists a unitary transformation $U(a,b)$ associated with (1), which transforms the x-frame state $|\Psi\rangle$ to the x'-frame state $|\Psi'\rangle$ according to

$$|\Psi'\rangle=U(a,b)|\Psi\rangle . \tag{2}$$

Two successive Lorentz transformations with parameters (a,b) and (a',b') are equivalent to a single transformation with parameters (a'',b'') given by

$$a''=a'a , \quad b''=a'b+b' . \tag{3}$$

This implies that the unitary operators which transform states should satisfy the multiplication law

$$U(a'',b'')=U(a',b')U(a,b) . \tag{4}$$

For the infinitesimal transformations

$$b_{\mu}=\epsilon_{\mu} , \quad a_{\mu\nu}=g_{\mu\nu}+\epsilon_{\mu\nu}, \quad \epsilon_{\mu\nu}=-\epsilon_{\mu\nu} , \tag{5}$$

where $g_{\mu\nu}$ is the metric tensor, the corresponding unitary operators can be written in the form

$$U(a,b)=1+i\epsilon_{\mu}P^{\mu}-\frac{i}{2}\epsilon_{\mu\nu}J^{\mu\nu} . \tag{6}$$

The multiplication law (4) implies that the ten generators, P^{μ} and $J^{\mu\nu}$, satisfy the commutation rules[10]

$$[P_{\mu},P_{\nu}]=0 , \tag{7}$$

$$[J_{\mu\nu},P_{\rho}]=i(g_{\nu\rho}P_{\mu}-g_{\mu\rho}P_{\nu}) , \tag{8}$$

$$[J_{\mu\nu},J_{\rho\lambda}]=i(g_{\mu\lambda}J_{\nu\rho}+g_{\nu\rho}J_{\mu\lambda}-g_{\mu\rho}J_{\nu\lambda}-g_{\nu\lambda}J_{\mu\rho}) , \tag{9}$$

where indices have been lowered using the rule

$$x_{\mu}=g_{\mu\nu}x^{\nu} . \tag{10}$$

A model or theory is said to be Poincaré invariant if it contains a set of ten generators which satisfy the Poincaré algebra given by (7)–(9). In general, the unitary operators that transform the state of a system described by such a model will satisfy the multiplication law given by (3) and (4).

The Poincaré algebra defined by (7)–(9) can be expressed in terms of any metric. Here we will use the metric tensor implied by the use of the light-front components.[11]

$$x^{\mu}=(x^0,x^1,x^2,x^3)=\left[\frac{t+z}{\sqrt{2}}, \mathbf{x}, \frac{t-z}{\sqrt{2}}\right] , \tag{11}$$

where $\mathbf{x}$ is a two-vector transverse to the arbitrarily chosen z direction. In order that

$$x^2=x\cdot x=x_{\mu}x^{\mu} , \tag{12}$$

we must have

$$(g_{\mu\nu})=(g^{\mu\nu})=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} . \tag{13}$$

The use of this nondiagonal metric has important consequences, e.g., the light-front components of the four-momentum of an on-mass-shell particle of mass m are given by

$$p^{\mu}=\left[p^0,\mathbf{p},\frac{\mathbf{p}^2+m^2}{2p^0}\right] . \tag{14}$$

It is interesting and important to note that p^3 has the structure of a nonrelativistic kinetic energy in two dimensions with p^0 analogous to the mass.

Here we will adopt the notation of Kogut and Soper[11] for the components of the four-momentum and angular momentum operators, i.e.,

$$P^{\mu}=(P^0,\mathbf{P},H) , \tag{15}$$

$$J_{\mu\nu}=\begin{bmatrix}0 & -S_1 & -S_2 & K_3\\ S_1 & 0 & J_3 & B_1\\ S_2 & -J_3 & 0 & B_2\\ -K_3 & -B_1 & -B_2 & 0\end{bmatrix} . \tag{16}$$

Since (6) implies that P^3 generates translations in $x_3=x^0$, denoting P^3 by H draws attention to the fact that we are choosing x^0 to play a role analogous to time, so that P^3 plays the role of a Hamiltonian. In (16) J_3 represents the z component of angular momentum, while K_3 generates boosts along z. The transverse components of the angular momentum $\mathbf{J}$ and boost operator $\mathbf{K}$ are related to the light-front components $J_{\mu\nu}$ by

$$J_1=(S_2-B_2)/\sqrt{2},\quad J_2=(B_1-S_1)/\sqrt{2} , \tag{17}$$

$$K_1=(S_1+B_1)/\sqrt{2},\quad K_2=(S_2+B_2)/\sqrt{2} . \tag{18}$$

It is easy to check, with the help of (9), that the components of $\mathbf{J}$ and $\mathbf{K}$ satisfy the well known commutation rules for angular momentum and boost operators.

It is not difficult to find generators that describe systems of free particles. The problem is how to determine generators for systems with interaction. Leutwyler and Stern[2] have developed a systematic procedure for solving this problem in the framework of light-front dynamics. In particular, they have shown that the generators of the stability group of the null plane, which are

$$(P^0,\mathbf{P},\mathbf{B},K_3,J_3)\text{-stability group generators} , \tag{19}$$

can be taken to be the same as the free particle generators, while H and $\mathbf{S}$ are modified by interactions. With our conventions, the equation for the null plane is

$$x^0=0 \text{ (null plane)} , \tag{20}$$

and the stability group is the subgroup of the Poincaré group that maps this hypersurface onto itself. Using (1), (5), and (6), it is easy to verify that the generators of the stability group are given by (19); that is, these generators correspond to Lorentz transformations for which $x'^0=x^0=0$.

Representations for the stability group generators can be obtained by introducing the states defined by

$$|(p^0,\mathbf{p})n\rangle=U(\beta)|(\hat{p}^0,\mathbf{0})n\rangle , \tag{21}$$

$$U(\beta)=e^{-i\boldsymbol{\beta}\cdot\mathbf{B}}e^{-i\beta_3K_3} , \tag{22a}$$

$$\boldsymbol{\beta}=\mathbf{p}/p^0,\quad \beta_3=\ln(p^0/\hat{p}^0) , \tag{22b}$$

where $|(\hat{p}^0,\mathbf{0})n\rangle$ is an eigenstate of P^0 and $\mathbf{P}$ with eigenvalues $\hat{p}^0$ and $\mathbf{0}$, respectively, and n refers to any other quantum numbers of interest. By using the well known result[10]

$$U(a)P_{\mu}U(a)^{-1}=P_{\nu}a^{\nu}{}_{\mu} , \tag{23}$$

it is easy to verify that the unitary transformation $U(\beta)$ corresponds to the homogeneous Lorentz transformation given by

$$x^{\mu}=a^{\mu}{}_{\nu}(\beta)\hat{x}^{\nu} , \tag{24}$$

$$[a^{\mu}{}_{\nu}(\beta)]=\begin{bmatrix}e^{\beta_3} & 0 & 0 & 0\\ e^{\beta_3}\beta_1 & 1 & 0 & 0\\ e^{\beta_3}\beta_2 & 0 & 1 & 0\\ e^{\beta_3}\dfrac{\beta_1^2+\beta_2^2}{2} & \beta_1 & \beta_2 & e^{-\beta_3}\end{bmatrix} . \tag{25}$$

From this transformation it follows that

$$\bar{P}|\bar{p}n\rangle=\bar{p}|\bar{p}n\rangle , \tag{26}$$

where here, and from now on, the overbar indicates either the triad of operators $(P^0,\mathbf{P})$ or the triad of eigenvalues $(p^0,\mathbf{p})$. Using (9), (13), (16), (21), and (22), it is straightforward to show that

$$e^{i\omega B_r}|(p^0,\mathbf{p})n\rangle=|(p^0,p^r-\omega p^0,p^s)n\rangle \quad (r\neq s) , \tag{27}$$

$$e^{i\omega K_3}|(p^0,\mathbf{p})n\rangle=|(e^{-\omega}p^0,\mathbf{p})n\rangle , \tag{28}$$

which, in turn, leads to, upon letting ω become infinitesimal,

$$\langle\bar{p}n|B_r=-ip^0\frac{\partial}{\partial p^r}\langle\bar{p}n| , \tag{29}$$

$$\langle\bar{p}n|K_3=-ip^0\frac{\partial}{\partial p^0}\langle\bar{p}n| . \tag{30}$$

Thus the states defined by (21) and (22) fix the representation of all the stability group generators except J_3 in a relatively simple way. The case of J_3 is only slightly more complicated.

Leutwyler and Stern[2] have shown that the operator $\mathcal{J}_3$ defined by

$$\mathcal{J}_3=\frac{B_1}{P^0}P^2-\frac{B_2}{P^0}P^1+J_3 \tag{31}$$

commutes with all of the generators of the stability group, and is therefore a Casimir operator for this group. Clearly the action of $\mathcal{J}_3$ on the states $|(\hat{p}^0,\mathbf{0})n\rangle$

is the same as J_3, and since J_3 leaves $(\hat{p}^0,\mathbf{0})$ invariant, it can be diagonalized on this set of states. Each eigenvector of J_3 with eigenvalue h gives rise to a separate representation of the stability group. Since $\mathcal{J}_3$ commutes with $\mathbf{B}$ and K_3, (21) and (22) allow us to write for such an eigenvector

$$\mathcal{J}_3 \,|\, \bar{p}n \rangle = h \,|\, \bar{p}n \rangle \ , \tag{32}$$

where the helicity quantum number h is now one of the n. From (31), (32), (26), and (29), we find

$$\langle \bar{p}n \,|\, J_3 = \left[\left[i\frac{\partial}{\partial p^1}p^2 - i\frac{\partial}{\partial p^2}p^1 \right] + h \right] \langle \bar{p}n \,| \ , \tag{33}$$

which gives the representation of J_3.

In treating the interaction dependent operators H and $\mathbf{S}$, it is convenient to introduce the mass operator M and two spin operators $\mathcal{J}_r$ according to

$$M^2 = P^\mu P_\mu \ , \tag{34}$$

$$M\mathcal{J}_r = \mathcal{J}_r M = \epsilon_{rs}(P^0 S_s - P^s K_3 - HB_s) - P^r \mathcal{J}_3$$
$$= \epsilon_{rs}(S_s P^0 - K_3 P^s - B_s H) - \mathcal{J}_3 P^r, \quad r = 1,2 \tag{35}$$

where ϵ_{rs} is a Levi-Civitá symbol. From (34), (10), and (13), we find

$$H = (\mathbf{P}^2 + M^2)/2P^0 \ . \tag{36}$$

It can be shown[2] that the three spin operators $\mathcal{J}_i$ and the mass operator M satisfy the commutation rule

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k \ , \tag{37}$$

$$[M, \mathcal{J}_i] = 0 \ , \tag{38}$$

and commute with all of the stability group generators except J_3.

Since we already have a representation for the stability group generators, the problem of constructing a Poincaré invariant theory amounts to finding forms for H and $\mathbf{S}$, or equivalently, M, $\mathcal{J}_1$, and $\mathcal{J}_2$. Leutwyler and Stern[2] have shown that it is sufficient for M and $\mathcal{J}$ to satisfy the commutation rules (37) and (38), and commute with all of the stability group generators, except J_3, to guarantee that the stability group generators and the H and $\mathbf{S}$ given by (35) and (36) satisfy the Poincaré algebra. Thus the problem of constructing a Poincaré invariant theory amounts to finding an M, $\mathcal{J}_1$, and $\mathcal{J}_2$ that satisfy the just mentioned conditions. In the next section we will construct a potential model for a two-particle system that satisfies these requirements.

III. A POTENTIAL MODEL

We begin by introducing a "momentum space" basis $|\, \bar{p}_1 \bar{p}_2 \rangle$ for the two particle system where here, and subsequently, the subscripts 1 and 2 on the "momenta" and the various operators will distinguish the two particles. We take for the stability group generators

$$\bar{P} = \bar{P}_1 + \bar{P}_2 \rightarrow \bar{p}_1 + \bar{p}_2 \ , \tag{39a}$$

$$B_r = B_{r1} + B_{r2} \rightarrow -ip_1^0 \frac{\partial}{\partial p_1^r} - ip_2^0 \frac{\partial}{\partial p_2^r} \ , \tag{39b}$$

$$K_3 = K_{31} + K_{32} \rightarrow -ip_1^0 \frac{\partial}{\partial p_1^0} - ip_2^0 \frac{\partial}{\partial p_2^0} \ , \tag{39c}$$

$$J_3 = J_{31} + J_{32} \rightarrow \left[i\frac{\partial}{\partial p^1}p^2 - i\frac{\partial}{\partial p^2}p^1 \right]_1 + \left[i\frac{\partial}{\partial p^1}p^2 - i\frac{\partial}{\partial p^2}p^1 \right]_2 \ , \tag{39d}$$

where the expressions after the arrows are the momentum space representations of the operators. We are assuming that the particles are spinless. Following Leutwyler and Stern,[2] we introduce the relative momentum operators

$$K^0 = (P_1^0 - P_2^0)/2P^0 \ , \tag{40a}$$

$$\mathbf{K} = (P_2^0 \mathbf{P}_1 - P_1^0 \mathbf{P}_2)/P^0 \ , \tag{40b}$$

which commute with P^0, $\mathbf{P}$, $\mathbf{B}$, and K_3 and satisfy

$$[\mathcal{J}_3, K^0] = 0 \ , \tag{41a}$$

$$[\mathcal{J}_3, K^r] = i\epsilon_{rs}K^s \ , \tag{41b}$$

where $\mathcal{J}_3$ is defined by (31) and (39d). We rewrite our momentum space basis vectors $|\, \bar{p}_1 \bar{p}_2 \rangle$ as $|\, \bar{p}\,\bar{k} \rangle$, which are eigenvectors of $\bar{P} = (P^0, \mathbf{P})$ and $\bar{K} = (K^0, \mathbf{K})$ with eigenvalues

$$\bar{p} = \bar{p}_1 + \bar{p}_2 \ , \tag{42}$$

$$k^0 = (p_1^0 - p_2^0)/2p^0 \equiv \eta - \tfrac{1}{2} \ , \tag{43a}$$

$$\mathbf{k} = (p_2^0 \mathbf{p}_1 - p_1^0 \mathbf{p}_2)/p^0 \ . \tag{43b}$$

In (43a) we have introduced a parameter η which will turn out to be convenient. It is trivial to rewrite the representations given by (39) in terms of the total variables $\bar{p}$ and relative variables $\bar{k}$, and verify that they agree with (29) and (30), and that (31) leads to

$$\langle \bar{p}\,\bar{k} \,|\, \mathcal{J}_3 = \mathcal{J}_3(\bar{k}) \langle \bar{p}\,\bar{k} \,| \ , \tag{44}$$

with

$$\mathcal{J}_3(\bar{k}) = i\frac{\partial}{\partial k^1}k^2 - i\frac{\partial}{\partial k^2}k^1 \ . \tag{45}$$

We now have all of our stability group generators and therefore turn our attention to the Hamiltonians H and $\mathbf{S}$, or, equivalently, M, $\mathcal{J}_1$, and $\mathcal{J}_2$. For two noninteracting particles of mass m_1 and m_2, H and $\mathbf{S}$ are given by

$$H^0 = \sum_{\alpha=1}^{2} \frac{\mathbf{P}_\alpha^2 + m_\alpha^2}{2P_\alpha^0} \ , \tag{46}$$

$$S_r^0 = \sum_{\alpha=1}^{2} \frac{1}{P_\alpha^0} \left[P_\alpha^r K_{3\alpha} + \frac{\mathbf{P}_\alpha^2 + m_\alpha^2}{2P_\alpha^0} B_{r\alpha} \right] \ , \tag{47}$$

where these expressions have been obtained by using (35) and (36) for each particle, with the understanding that for each particle $\mathcal{J}_\alpha=0$. Recall that we have assumed that the particles are spinless. By transforming to total and relative variables, and again using (35) and (36), it is straightforward to show that

$$H^0=(\mathbf{P}^2+M_0^2)/2P^0\ , \tag{48}$$

where

$$M_0^2\,|\bar{p}\,\bar{k}\rangle=s(\eta,\mathbf{k})\,|\bar{p}\bar{k}\rangle\ , \tag{49}$$

with

$$s(\eta,\mathbf{k})=W^2(\eta,\mathbf{k})=\frac{\mathbf{k}^2+m_1^2}{\eta}+\frac{\mathbf{k}^2+m_2^2}{1-\eta}=(p_1+p_2)^2 \tag{50}$$

and

$$\langle\bar{p}\,\bar{k}\,|\,\mathcal{J}_r^0=\mathcal{J}_r^0(\bar{k})\langle\bar{p}\,\bar{k}\,|\ , \tag{51}$$

with

$$\mathcal{J}_r^0(\bar{k})=\frac{\epsilon_{rs}}{W(\eta,\mathbf{k})}\left\{-\eta(1-\eta)k^s i\frac{\partial}{\partial\eta}\frac{1}{\eta(1-\eta)}+i\frac{\partial}{\partial k^s}\left[\frac{2\eta-1}{2}s(\eta,\mathbf{k})-\frac{m_1^2-m_2^2}{2}\right]\right\}. \tag{52}$$

Here $\mathcal{J}_r^0$ is a component of the spin operator for the free, two-particle system. This expression for $\mathcal{J}_r^0$ does not look like an angular momentum, but nevertheless it is, as can be shown by verifying (37).

It is convenient at this point to introduce an alternative to the variables η and k^0 defined by (43a). In what follows it should be kept in mind that the particles are on the mass shell, so the light-front components of the four-momenta p_1 and p_2 are related as in (14). If in (22b) $\hat{p}^0$ is taken to be $W(\eta,\mathbf{k})/\sqrt{2}$, then the Lorentz transformation given by (25) becomes the transformation from the c.m. frame of the two particles to the frame in which their four-momenta are p_1 and p_2. Using this, it is very easy to verify that the c.m. light-front components of the two particles' momenta are

$$(p_1^\mu)_{\text{c.m.}}=\left[\frac{\eta W(\eta,\mathbf{k})}{\sqrt{2}},\ \mathbf{k},\ \frac{\sqrt{2}}{W(\eta,\mathbf{k})}\frac{\mathbf{k}^2+m_1^2}{2\eta}\right], \tag{53a}$$

$$(p_2^\mu)_{\text{c.m.}}=\left[\frac{(1-\eta)W(\eta,\mathbf{k})}{\sqrt{2}},\ -\mathbf{k},\ \frac{\sqrt{2}}{W(\eta,\mathbf{k})}\frac{\mathbf{k}^2+m_2^2}{2(1-\eta)}\right]. \tag{53b}$$

From the relation between light-front components and ordinary components as given by (11), it follows that the z component of the momentum of particle 1 in the c.m. frame is

$$q_z=\frac{2\eta-1}{2}W(\eta,\mathbf{k})-\frac{m_1^2-m_2^2}{2W(\eta,\mathbf{k})}\ , \tag{54}$$

which we take as the variable to replace η and k^0. The transverse components of $\mathbf{q}$ are given by

$$\mathbf{q}_\perp=\mathbf{k}\ . \tag{55}$$

It is also convenient to introduce two other quantities ω_q and ϵ_q which are the c.m. energies of particles 1 and 2, respectively. These, of course, are given by

$$\omega_q=(\mathbf{q}^2+m_1^2)^{1/2}\ , \tag{56a}$$

$$\epsilon_q=(\mathbf{q}^2+m_2^2)^{1/2}\ , \tag{56b}$$

which, in turn, allow us to write

$$W(\eta,\mathbf{k})=W_q=\omega_q+\epsilon_q=s_q^{1/2}\ . \tag{57}$$

According to (43a) and (25), we have

$$\eta=\left[\frac{p_1^0}{p^0}\right]_{\text{c.m.}}=\frac{\omega_q+q_z}{W_q}\ , \tag{58}$$

which, in effect, inverts (54). It is now straightforward to show that the angular momentum given by (45) and (52) is also given by

$$\mathcal{J}^0(\mathbf{q})=i\nabla_q\times\mathbf{q}\ , \tag{59}$$

which, of course, is recognized as an angular momentum.

We are now in a position to state our model. We write the Hamiltonian in the form

$$H=H^0+\frac{U}{2P^0}=H^0+V\ , \tag{60}$$

which by (34) and (48) leads to the mass operator

$$M^2=M_0^2+U\ . \tag{61}$$

For the internal angular momentum operator, we take the free operator analyzed above, i.e.,

$$\mathcal{J}=\mathcal{J}^0\ , \tag{62}$$

which, of course, satisfies (37) and commutes with all of the stability group generators, except J_3. In order to have a Poincaré invariant model, we must choose U so that M commutes with these generators and satisfies (38). It is not difficult to show that M satisfies these requirements if and only if they are satisfied by M^2.

Rewriting our momentum space basis vectors $|\bar{p}\,\bar{k}\rangle$ as $|\bar{p}\mathbf{q}\rangle$, we choose a normalization so that the completeness relation for these vectors is

$$\int|\bar{p}\mathbf{q}\rangle\frac{dp^0\theta(p^0)d\mathbf{p}}{(2\pi)^3 2p^0}\frac{W_q d\mathbf{q}}{(2\pi)^3 2\omega_q\epsilon_q}\langle\bar{p}\mathbf{q}\,|=1\ . \tag{63}$$

The volume element in (63) is Lorentz invariant as it is proportional to a transformed version of $d\bar{p}_1\theta(p_1^0)d\bar{p}_2\theta(p_2^0)/(p_1^0p_2^0)$, which, in turn, is a product of two invariant single particle volume elements. For the interaction we take

$$U=\int|\bar{p}\mathbf{q}\rangle\frac{d\bar{p}\,\theta(p^0)}{(2\pi)^3 2p^0}\frac{W_q d\mathbf{q}}{(2\pi)^3 2\omega_q\epsilon_q}\times U(\mathbf{q},\mathbf{q}')\frac{W_{q'}d\mathbf{q}'}{(2\pi)^3 2\omega_{q'}\epsilon_{q'}}\langle\bar{p}\mathbf{q}'\,|\ , \tag{64}$$

with

$$U(\mathbf{q},\mathbf{q}')=\sum_{j,\lambda} Y_j^\lambda(\hat{\mathbf{q}})U_j(q,q')Y_j^{\lambda *}(\hat{\mathbf{q}}') , \quad (65)$$

where $Y_j^\lambda(\hat{\mathbf{q}})$ is a spherical harmonic depending on the angles specifying the direction of $\mathbf{q}$.

It is obvious that U commutes with P^0 and $\mathbf{P}$. In order to see that U commutes with $\mathbf{B}$ and K_3, it is easiest to subject U to a unitary transformation $\exp(i\omega X)$ with $X=\mathbf{B},K_3$, and use (27), (28), and the invariance of the $\bar{p}$ volume element under such a transformation. From (62) and (59), it follows that

$$e^{-i\theta\hat{\mathbf{u}}\cdot\mathcal{J}}|\bar{p}\mathbf{q}\rangle=|\bar{p}\mathbf{q}'\rangle , \quad (66)$$

where $\mathbf{q}'$ is obtained from $\mathbf{q}$ by rotating about the direction $\hat{\mathbf{u}}$ through the angle θ. Subjecting U to the unitary transformation in (66) immediately leads to the conclusion that U is invariant under such a transformation and hence commutes with $\mathcal{J}$. This completes the verification of the Poincaré invariance of the model. In the next section we will show that the S-matrix elements for the two-particle scattering processes that arise in this model are Lorentz invariant functions of the initial and final momenta.

IV. THE INVARIANT S MATRIX

We begin by assuming that the interaction given by (64) and (65) is such that the two particles are asymptotically free, so that we can construct scattering states. We are interested in simultaneous eigenstates of the components of the four-momentum operator, i.e., we want

$$P^\mu|(p^0,\mathbf{p})\mathbf{q}\rangle^{(\pm)}=p^\mu|(p^0,\mathbf{p})\mathbf{q}\rangle^{(\pm)} , \quad (67)$$

where

$$p^\mu=\left[p^0,\mathbf{p},\frac{\mathbf{p}^2+W_q^2}{2p^0}\right] . \quad (68)$$

The plus and minus superscripts indicate in and out scattering states,[12] respectively. Assuming the validity of the standard results of scattering theory,[12] we can write, with the help of (60), the equations

$$|\bar{p}\mathbf{q}\rangle^{(\pm)}=|\bar{p}\mathbf{q}\rangle+\frac{1}{p^3\pm i\epsilon-H^0}V|\bar{p}\mathbf{q}\rangle^{(\pm)} \quad (69)$$

$$=\left[1+\frac{1}{p^3\pm i\epsilon-H}V\right]|\bar{p}\mathbf{q}\rangle , \quad (70)$$

where the states with no superscripts are the free, two-particle states introduced in Sec. III. From (60), (48), (68), (61), and the fact that P^0 and $\mathbf{P}$ commute with U, it follows that (69) and (70) can be rewritten as

$$|\bar{p}\mathbf{q}\rangle^{(\pm)}=|\bar{p}\mathbf{q}\rangle+G_0(s_q\pm i\epsilon)U|\bar{p}\mathbf{q}\rangle^{(\pm)} \quad (71)$$

$$=[1+G(s_q\pm i\epsilon)U]|\bar{p}\mathbf{q}\rangle , \quad (72)$$

where we have introduced the resolvents

$$G_0(s)=(s-M_0^2)^{-1} , \quad (73)$$

$$G(s)=(s-M^2)^{-1} . \quad (74)$$

It is not difficult to verify that the states defined by (72) transform properly under a homogeneous Lorentz transformation. The operators $\mathbf{B}$, K_3, and J_3 commute with M^2 and U, so we have

$$e^{i\omega X}|\bar{p}\mathbf{q}\rangle^{(\pm)}=[1+G(s_q\pm i\epsilon)U]e^{i\omega X}|\bar{p}\mathbf{q}\rangle , \quad X=\mathbf{B},K_3,J_3 . \quad (75)$$

In order to see what happens with the transformations generated by $\mathbf{S}$, we solve (35) for S_r and let the resulting expression act on $|\bar{p}\mathbf{q}\rangle^{(\pm)}$, which allows us to replace the operators P^μ with the eigenvalues p^μ. Using this, the commutivity of $\mathbf{B}$, K_3, and $\mathcal{J}$ with M^2 and U, and the fact that $\mathcal{J}$ is given by the free particle expression, we find

$$e^{i\omega S_r}|\bar{p}\mathbf{q}\rangle^{(\pm)}=[1+G(s_q\pm i\epsilon)U]e^{i\omega S_r^0}|\bar{p}\mathbf{q}\rangle . \quad (76)$$

Given the fact that the free states transform properly under the homogeneous Lorentz transformations, we can now conclude that

$$U(a)|\bar{p}\mathbf{q}\rangle^{(\pm)}=U(a)|p_1,p_2\rangle^{(\pm)}=|ap_1,ap_2\rangle^{(\pm)} , \quad (77)$$

where p_1 and p_2 are the on mass shell four-momenta that determine $\bar{p}$ and $\mathbf{q}$ through (42), (43), (50), (54), and (55).

As is well known,[12] the S-matrix elements are determined by the overlap of the in states and out states, and since it follows from (77) that

$${}^{(-)}\langle p_{1f},p_{2f}|p_{1i},p_{2i}\rangle^{(+)}={}^{(-)}\langle ap_{1f},ap_{2f}|ap_{1i},ap_{2i}\rangle^{(+)} , \quad (78)$$

we see that the S-matrix elements are Lorentz invariant functions of the initial and final four-momenta.

We now introduce a T operator, in the standard way,[12] by

$$T(s)=U+UG(s)U , \quad (79)$$

which when combined with the well known resolvent identities,

$$G(s)=G_0(s)+G_0(s)UG(s) \quad (80a)$$

$$=G_0+G(s)UG_0(s) , \quad (80b)$$

leads to

$$T(s)=U+UG_0(s)T(s) \quad (81a)$$

$$=U+T(s)G_0U . \quad (81b)$$

From (72), (74), (79), and (68), it follows that the S-matrix elements are related to the T-matrix elements by

$${}^{(-)}\langle p_{1f},p_{2f}|p_{1i},p_{2i}\rangle^{(\pm)}=(2\pi)^6 2p_{1i}^0\delta^{(3)}(\bar{p}_{1f}-\bar{p}_{1i})2p_{2i}^0\delta^{(3)}(\bar{p}_{2f}-\bar{p}_{2i})-(2\pi)^4 i\delta^{(4)}(p_f-p_i)T(\mathbf{q}_f,\mathbf{q}_i;p_i^2+i\epsilon) , \quad (82)$$

where we have defined the T-matrix elements by

$$\langle \bar{p}\mathbf{q} \mid T(s) \mid \bar{p}'\mathbf{q}' \rangle = (2\pi)^3 2p^0 \delta^{(3)}(\bar{p} - \bar{p}')T(\mathbf{q},\mathbf{q}';s) \ . \quad (83)$$

Using (81), (64), (63), (73), (49), and (57) it is straightforward to show that the integral equation for the T-matrix elements is

$$T(\mathbf{q},\mathbf{q}';s) = U(\mathbf{q},\mathbf{q}') + \int \frac{d\mathbf{q}'' W_{q''}}{(2\pi)^3 2\omega_{q''}\epsilon_{q''}} \frac{U(\mathbf{q},\mathbf{q}'')}{s - s_{q''}} T(\mathbf{q}'',\mathbf{q}';s) \ . \quad (84)$$

The equation above has a form similar to the Lippman-Schwinger equation, familiar from nonrelativistic potential scattering; however, there is an important difference in the interpretation of the momentum variables $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{q}''$. In the nonrelativistic theory, the total three-momentum is conserved throughout a scattering process, so there is a c.m. frame common to initial, final, and intermediate states. As a result of this the variables that play the roles of $\mathbf{q}'$, $\mathbf{q}$, and $\mathbf{q}''$ in the nonrelativistic theory can be interpreted as the momenta of one of the particles in the common c.m. frame. In light-front dynamics the conserved quantities are P^0 and $\mathbf{P}$, so the values of p^0 and $\mathbf{p}$ are the same in initial, final, and intermediate states. The light-front components of the total four-momentum in the final state, for example, are given by

$$p^\mu = \left[p^0, \mathbf{p}, \frac{\mathbf{p}^2 + W_q^2}{2p^0} \right] , \quad (85)$$

where W_q is given by (57) and (56). As a consequence of this, there is a different c.m. frame in the final state for each value of $\mathbf{q}^2$. Similar remarks apply to the initial and intermediate states. Accordingly, the momenta $\mathbf{q}'$, $\mathbf{q}$, and $\mathbf{q}''$ in (84) can be interpreted as the momenta of particle 1 in c.m. frames determined by $W_{q'}$, W_q, and $W_{q''}$, respectively.

This raises the question of whether or not the model presented here has a reasonable nonrelativistic limit. By putting the c's back in, it is easy to show from (43a) that

$$\eta \underset{c \to \infty}{\longrightarrow} \frac{m_1}{m_1 + m_2} + \frac{\rho_z}{(m_1 + m_2)c} + \cdots \ , \quad (86)$$

where ρ_z is the z component of the nonrelativistic relative momentum defined by

$$\boldsymbol{\rho} = \frac{m_2 \boldsymbol{\rho}_1 - m_1 \boldsymbol{\rho}_2}{m_1 + m_2} \ . \quad (87)$$

Here, $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are the three-momenta of the particles. Using this result, it then follows from (43b), (54), and (55) that

$$\mathbf{q} \underset{c \to \infty}{\longrightarrow} \boldsymbol{\rho} \ , \quad (88)$$

which makes it obvious that (84) goes over to its nonrelativistic counterpart when $c \to \infty$.

V. SUMMARY AND DISCUSSION

We have succeeded in constructing a rather general Poincaré invariant potential model for two-particle systems within the framework of light front dynamics. By Poincaré invariant, we mean that the model specifies ten generators, P_μ and $J_{\mu\nu}$, which satisfy the commutation relations (7)–(9) which define the Lie algebra for the Poincaré group. This, in turn, implies the existence of a group of unitary operators $U(a,b)$ that transform the quantum mechanical state vectors from one frame to another in correspondence with the underlying inhomogeneous Lorentz transformations [see (1) and (2)].

The representations for the various generators have been given in a "momentum" space whose basis vectors are $|\bar{p}_1\bar{p}_2\rangle = |\bar{p}\,\bar{k}\rangle = |\bar{p}\mathbf{q}\rangle$, where, for example, $\bar{p}$ stands for the triad $(p^0,\mathbf{p})$. The stability group generators $\bar{P}$, $\mathbf{B}$, K_3, and J_3 are defined by the simple kinematical expressions given in (39). When these are rewritten in terms of the external variables $\bar{p}$ and internal variables $\bar{k}$ defined by (42) and (43), respectively, the general representations given by (26), (29), (30), and (33) are recovered. The generic quantity n becomes identified with $\bar{k}$ or $\mathbf{q}$. In (33) h can be replaced by $\mathcal{J}_3(\bar{k})$ defined by (45). The only stability group generator that depends on the internal variables is J_3, but as (45) shows the dependence is kinematical.

The three dynamical or Hamiltonian generators H and $\mathbf{S}$ are given in terms of the internal operators M and $\boldsymbol{\mathcal{J}}$ by (36) and (35), respectively. The mass operator M is specified by (61), (49), (50), (64), and (65), while the spin operator $\boldsymbol{\mathcal{J}}$ is given simply by (62) and (59).

The scattering states of the model can be labeled with the on-shell four-momenta p_1 and p_2, which can be interpreted as the initial or final momenta of an in or out state, respectively. Under a homogeneous Lorentz transformation $x' = ax$ a state labeled with p_1 and p_2 is simply transformed to a state labeled with $p_1' = ap_1$ and $p_2' = ap_2$, which, in turn, implies that the S matrix is a Lorentz invariant function of the initial and final four-momenta.

The equation for the T matrix given by (84) is quite appealing in that it has the mathematical simplicity of its nonrelativistic counterpart. A reduction of this three-dimensional equation to a set of uncoupled one-dimensional equations can be carried out by inserting the partial wave expansion (65) along with a similar expansion for $T(\mathbf{q},\mathbf{q}';s)$.

It should not be difficult to extend the model here to particles with spin and other internal symmetries. Leutwyler and Stern[2] have already laid the foundation for treating spin-$\frac{1}{2}$ particles. With this extension it will be possible to construct separable potential models similar to those that already exist for the pion-nucleon[13–15] and kaon-nucleon[16] systems, but which are truly Poincaré invariant. In this connection it will be interesting to see if the light front approach can be easily extended to the treatment of coupled channels, as this is necessary for the just mentioned systems.[15,16]

In connection with the type of model considered here, it is of interest to determine whether or not there exists a covariant bilocal wave function $\Phi(x_1,x_2)$ that describes the two-particle system. Here, x_1 and x_2 are the space-time four-vectors of the two particles. Leutwyler and Stern[2] have shown that for spinless particles the existence of such a wave function is related to the existence of a set of covariant vectors $|x_1,x_2\rangle$ that transform under the action of the Poincaré group according to

$$U(a,b)\,|x_1,x_2\rangle = |x_1',x_2'\rangle = |ax_1+b,\, ax_2+b\rangle \ . \tag{89}$$

With these vectors a covariant wave function can be defined by

$$\Phi(x_1,x_2)=\langle x_1,x_2\,|\,\phi\rangle \ , \tag{90}$$

where $|\phi\rangle$ is a state vector for the composite system. The transformation law for such a wave function is evidently

$$\Phi(x_1,x_2)=\langle x_1,x_2\,|\,U^{-1}U\,|\,\phi\rangle = \langle x_1',x_2'\,|\,\phi'\rangle = \Phi'(x_1',x_2') \ . \tag{91}$$

Leutwyler and Stern[2] have shown that, in general, something more than Poincaré invariance is needed to ensure the existence of a wave function that transforms according to (91). That something is what they call the "angular condition," which is an algebraic relation that involves M and $\mathcal{J}$. It turns out that for states of spin zero, this angular condition is automatically satisfied, so for the model presented here it is possible to obtain covariant bilocal wave functions corresponding to the $j=0$ partial wave states.

As Leutwyler and Stern[2] have discussed, the existence of a covariant bilocal wave function facilitates the construction of the interaction between the system and a local field such as the photon field. In other words, its existence makes it easier to construct a current operator that transforms like a vector field under the Poincaré group. Also, a model which possesses the sort of consistent space-time description implied by (91) will presumably not suffer from the peculiar Lorentz contraction effects recently found[17] in some Poincaré invariant instant form models.

At present, work is under way to extend the model presented here to include internal symmetries, so as to construct Poincaré invariant separable potential models for the pion-nucleon and kaon-nucleon systems. Also, an attempt is being made to establish systematic procedures for developing potential models that are not only Poincaré invariant, but also possess covariant bilocal wave functions as well as sensible current operators.

---

[1]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[2]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978).

[3]B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953); L. L. Foldy, *ibid.* **122**, 275 (1961); F. Coester, Helv. Phys. Acta **38**, 7 (1965); F. Coester, S. C. Pieper, and F. J. D. Serduke, Phys. Rev. C **11**, 1 (1975); F. Coester and P. Havas, Phys. Rev. D **14**, 2556 (1976); R. Fong and J. Sucher, J. Math. Phys. **5**, 456 (1964); T. F. Jordan, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. B **133**, 487 (1964); L. B. Redei, J. Math. Phys. **6**, 487 (1965); *The Theory of Action-at-a-Distance in Particle Dynamics,* edited by E. H. Kerner (Gordon and Breach, New York, 1973); L. L. Foldy and R. A. Krajcik, Phys. Rev. D **10**, 1777 (1974); **12**, 1700 (1975).

[4]S. Fubini, A. J. Hanson, and R. Jackiw, Phys. Rev. D **7**, 1732 (1973); C. M. Sommerfield, Ann. Phys. (N.Y.) **84**, 285 (1974); D. Gromes, H. J. Rothe, and B. Stech, Nucl. Phys. **B75**, 313 (1974); James V. Lindesay, Alexander J. Markevich, H. Pierre Noyes, and George Pastrana, Phys. Rev. D **33**, 2339 (1986); Alexander J. Markevich, *ibid.* **33**, 2350 (1986).

[5]K. Bardakci and M. B. Halpern, Phys. Rev. **176**, 1686 (1968).

[6]J. M. Namyslowski, Phys. Rev. D **18**, 3676 (1978); Paul M. Fishbane and J. M. Namyslowski, *ibid.* **21**, 2406 (1980).

[7]H. Leutwyler and J. Stern, Nucl. Phys. **B133**, 115 (1978).

[8]S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[9]J. M. Namyslowski, Prog. Part. Nucl. Phys. **14**, 49 (1984).

[10]S. G. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).

[11]J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).

[12]R. G. Newton, *Scattering Theory of Waves and Particles,* 2nd ed. (Springer-Verlag, New York, 1982).

[13]Charles Coronis and Rubin H. Landau, Phys. Rev. C **24**, 605 (1981).

[14]R. J. McLeod and I. R. Afnan, Phys. Rev. C **32**, 222 (1985).

[15]B. Blankleider and G. E. Walker, Phys. Lett. **152B**, 297 (1985).

[16]Rubin H. Landau, Phys. Rev. C **27**, 2191 (1983); **28**, 1324 (1983).

[17]W. Glöckle and Y. Nogami, Phys. Rev. D **35**, 3840 (1987).