Fermion dynamical symmetry model of nuclei: Basis, Hamiltonian, and symmetries

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A fermion dynamical symmetry model is presented. The model has a rich variety of dynamical symmetries, with fully microscopic connections between these dynamical symmetries and the underlying shell structure. In the low angular momentum region, without explicit introduction of bosons, *all* the dynamical symmetries contained in the phenomenological interacting boson model are recovered. Furthermore, the model predicts several new dynamical symmetries, one of which has recently been empirically verified, and has limits appropriate to the study of high-spin physics.

I. INTRODUCTION

Dynamical symmetry is an important concept in physics.[1] Mathematically, dynamical symmetry allows one to obtain simple analytical solutions for the quantum many-body problem. Physically, dynamical symmetries depict various types of motion for such systems. Therefore, it is always desirable to separate the full Hamiltonian into a part possessing the symmetry which is responsible for a particular mode, and a symmetry-breaking part which represents deviations away from the "benchmark" symmetry. If this separation is successful, it is possible to greatly simplify the many-body problem and to gain a deeper understanding of collective motion.

In nuclear physics, there exists a rich variety of collective motion. The application of dynamical symmetries to the study of one of these modes (rotational) may be traced to the SU_3 work of Elliott,[2] which was motivated by the observation of strong deformation in light nuclei ($A \lesssim 24$). Unfortunately, the Elliott model can only be applied to a rather limited region of the periodic table.

There has been an effort in the last two decades to extend the Elliott model to investigate low-energy collective states in regions other than the *s-d* shell.[3–5] This was pioneered by Arima, Harvey, and Shimizu[6] and simultaneously but separately by Hecht and Adler[7] by introducing the concept of pseudospin $\frac{1}{2}$ and the pseudo-SU_3 model. By analogy with the Elliott model, the pseudo-SU_3 model explores the SU_3 symmetry in the (pseudo-) orbital part of the wave functions.

Another development in the use of dynamical symmetry in nuclear physics which has particular relevance to the work presented in this paper is the phenomenological interacting boson model (IBM), which was proposed by Arima and Iachello in 1974.[8–11] Motivated by the assumption that there are strongly correlated valence nucleon pairs with angular momenta zero (*S* pairs) and two (*D* pairs), the *s* and *d* bosons were introduced as the basic building blocks of low energy collective motion. With this assumption, the model has the unitary group U_6 as its highest symmetry and three dynamical symmetry chains: O_6, U_5, and SU_3. The IBM differs considerably from the pseudo-SU_3 model in the usage of dynamical symmetry. The pseudo-SU_3 model, just as the Elliott model, uses a *mono*-chain (SU_3) to construct a Hilbert space where the rotational aspect of the collective motion is emphasized. On the other hand, the IBM is a *multi*-chain model where each chain provides both a complete mathematical basis and a collective mode (O_6, γ soft; U_5, vibrational; SU_3, rotational) which may physically be realized in various areas of the periodic table.[12] Therefore, the IBM suggests that a dynamical symmetry group chain may represent the algebraic manifestation of a particular collective mode.

At the phenomenological level, the IBM has only a tenuous link to the fundamental shell structure (through the boson number). Therefore, although it provides a unified algebraic view for various collective modes, it is beyond the scope of the model to provide a deeper understanding of these collective modes and the related dynamical symmetries at the fermion level. There has been considerable effort[13] to provide a better way of mapping a fermion system to a boson system so that a microscopic understanding of the IBM could be

achieved. *However, any dynamical symmetries relevant to nuclear structure should manifest themselves directly from the fermion degrees of freedom without explicitly introducing bosons.* This is the basic theme of this paper.

An important milestone for fermion dynamical symmetry in nuclear physics occurred when Ginocchio proposed a fermionic model and investigated its possible dynamical symmetries.[14,15] In this model, the single-nucleon angular momentum **j** is separated into a pseudo-orbital angular momentum **k** and pseudospin **i**, where

$$\mathbf{j}=\mathbf{k}+\mathbf{i} \ . \tag{1.1a}$$

The single nucleon creation operator $b^{\dagger}_{km_k im_i}$ in the k-i scheme is related to the fermion creation operator $a^{\dagger}_{jm}$ by a Clebsch-Gordan coefficient,

$$a^{\dagger}_{jm}=\sum_{m_k m_i}\langle km_k im_i \mid jm\rangle b^{\dagger}_{km_k jm_i} \ . \tag{1.1b}$$

Ginocchio pointed out that in this scheme there are two alternatives to construct the S and D (fermion) pairs as well as the multipole operators P^r (with $r\leq 3$). The first is to take $k=1$ and couple **i** to zero for the fermion pair. This we call the *k-active coupling scheme.* The other is to take $i=\frac{3}{2}$ and couple k to zero for the pair. This we call the *i-active coupling scheme.* These couplings are illustrated in Fig. 1. Ginocchio demonstrated that the k-active scheme gives rise to an Sp_6 Lie algebra, while the i-active scheme gives an SO_8 algebra. By assuming that the Hamiltonian is a scalar (rotational invariant) constructed out of the generators of Sp_6 or SO_8, this model

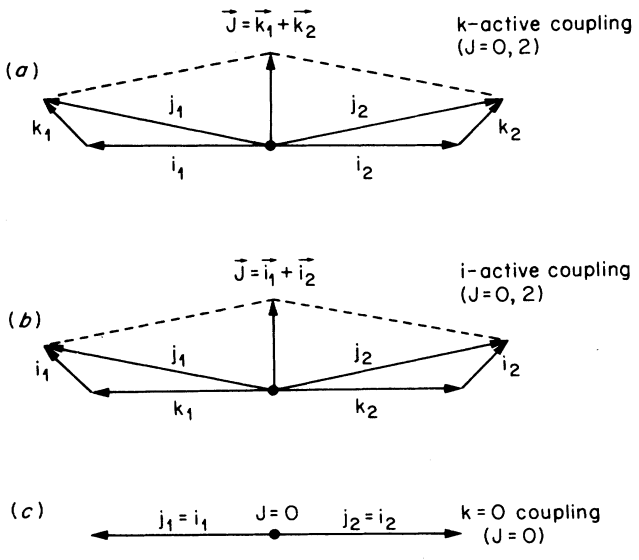

FIG. 1. The coupling schemes in the FDSM. The i-active and k-active schemes are identical to those of the Ginocchio model, and are appropriate for normal-parity orbitals. These couplings lead to SO_8 and Sp_6 symmetries, respectively, for the S- and D-pair condensate in the normal-parity orbitals. The $k=0$ coupling is relevant for abnormal-parity orbitals and is associated with an $\mathcal{SU}_2$ (quasispin) symmetry for the $\mathcal{S}$-pair condensate.

has the property that the (S,D) subspace is decoupled from the rest of the space. Ginocchio found the following (multichain) dynamical symmetries:

$$\begin{array}{lccccccc} & & & & SO_5\times SU_2 & & & \\ & & \nearrow & & & \searrow & & \\ i \text{ active} & SO_8 & \longrightarrow & SO_6 & \longrightarrow & SO_5 & \longrightarrow & SO_3 \ , \\ & & \searrow & & & \nearrow & & \\ & & & & SO_7 & & & \end{array} \tag{1.2}$$

$$\begin{array}{lccccc} & & & SU_3 & & \\ & & \nearrow & & \searrow & \\ k \text{ active} & Sp_6 & & & & SO_3 \ . \\ & & \searrow & & \nearrow & \\ & & & SU_2\times SO_3 & & \end{array} \tag{1.3}$$

The spectra of the dynamical symmetries $SO_5\times SU_2$ and $SU_2\times SO_3$ are formally identical to those of the IBM's U_5 vibrational limit and SO_6 is formally identical to the IBM's O_6 γ-soft limit, while the SO_7 limit does not occur in the IBM. The Sp_6 symmetry contains the SU_3 (rotational) chain but suffers from what has been viewed as a serious problem:[4,15] namely when the nucleon pair number $N>\Omega/3$, where Ω $[=\Sigma(2j+1)/2]$ is the total pair degeneracy, the Pauli principle forbids the occurrence of the highest SU_3 representation $(\lambda,\mu)=(2N,0)$, which is normally associated with ground state rotational band. Similar Pauli restrictions occur for other representations. Thus, the lowest energy SU_3 representations do not always exist in the fermion model. In fact, Ginocchio estimated that due to the Pauli restriction, his fermion SU_3 contained only about 70% of the states of the boson (IBM) SU_3 model. As a consequence, Ginocchio abandoned the Sp_6 model, along with its important rotational limit. Furthermore, in the Ginocchio model the relationship between the pseudoorbit and pseudospin angular momenta and the shell structure was not explored. Therefore, it was not obvious that this model could be applied to real nuclei.

Despite the apparently serious difficulty associated with the SU_3 chain, and the ill-defined relation to the shell model, the Ginocchio effort appears to be *the first serious attempt* in nuclear structure physics to seek a variety of dynamical symmetries (representing various collective motions) from a unified fermion point of view.

Recently we have proposed a fermion dynamical symmetry model (FDSM).[16] This model, which may be regarded as a further development of the Ginocchio model, is intimately related to the shell structure of nuclei and has a multitude of dynamical symmetries. In addition, because of the relation to the shell structure, one finds that the SU_3 limit of the Ginocchio model is resurrected in the FDSM. It has also been shown[17] that there is a one-to-one correspondence between the generators, irreducible basis, and the building blocks (i.e., the s,d bosons versus the S and D fermion pairs) of the subgroups SO_6, U_5, and SU_3 contained in the dynamical group chains of the IBM and those of the FDSM, and that the FDSM can describe phenomena in high-spin physics normally considered to be the exclusive domain

of the geometrical model.[23] Thus this model appears capable of providing a unified shell-model basis for both boson and geometrical pictures.

In this series of papers we give a comprehensive introduction to the FDSM. To illustrate the basic idea of this model, we shall concentrate in this paper on the simplest situation: identical S and D fermion pairs moving in one major shell. In the forthcoming papers of this series, we shall discuss (a) the situation where the neutrons and protons are distinguished and the effect of n-p interaction is included; and (b) the case with broken pairs or unpaired particles, which is relevant for the study of odd-even and odd-odd systems and high spin structures.

In Sec. II, we shall discuss the (k-i) basis of the FDSM and, in Sec. III, the general Hamiltonian of the FDSM will be derived directly from the nuclear shell model. The primary symmetry limits $Sp_6 \times \mathcal{SU}_2$ and $SO_8 \times \mathcal{SU}_2$ of the FDSM are described in Sec. IV. This is followed in Sec. V by a discussion of the rotational limit in the Sp_6 symmetry and the γ-soft limit in the SO_8 symmetry. In Sec. VI, the three "vibrational-like" subchains, two of SO_8 symmetry and one of Sp_6 symmetry, are compared. In Sec. VII, we argue that the FDSM is a model of effective interactions in nuclei and a comparison with the pairing plus quadrupole model is discussed. Finally, in Sec. VIII, the paper is summarized, and the differences between the FDSM and IBM, the effects of the n-p interaction and other symmetry breaking terms, and open questions requiring further study are discussed.

In order to emphasize the physical aspect of the FDSM, we shall omit most of the group theoretical derivations in this paper. Interested readers are directed to Ref. 17 for the mathematical details.

II. THE k-i BASIS OF THE FDSM

It is well known that two identical nucleons participating in a low-lying collective mode preferentially couple their single particle (s.p.) angular momenta $\mathbf{j}_1$ and $\mathbf{j}_2$ to zero (S pair) and two (D pair).[18] To implement this property of the nuclear many-body problem at the fermion level, it is desirable to have a basis which is tailored to decouple the coherent S-D subspace from the entire shell model space. The single-particle (s.p.) pseudoorbit and pseudospin basis (k-i basis), introduced by Ginocchio with the modification presented here to take into account the abnormal-parity level in each major shell, has precisely this property.

To understand the physical meaning of the k-i basis, it is useful to introduce the concepts of the *active* part and the *inert* part of a single-particle angular momentum $\mathbf{j}$:

$$\mathbf{j}=\mathbf{j}_{\text{active}}+\mathbf{j}_{\text{inert}} . \tag{2.1}$$

The active part $\mathbf{j}_{\text{active}}$ is defined as the part of the single-particle angular momentum which for two identical nucleons can *only* couple to angular momentum $J=0$ and 2, while the inert part $\mathbf{j}_{\text{inert}}$ is the remaining single particle angular momentum. The dominance of the S-D pairs in the low-lying collective modes suggest that, for each identical nucleon pair, the inert parts of the angular momenta tend to couple to zero and become inactive in the low-energy region. This is the reason we call this part of angular momentum "inert." *The k-i basis is a basis with the single-particle angular momentum* $\mathbf{j}$ *decomposed into an active and inert parts* instead of conventional decomposition of orbital and spin parts. As shown in Fig. 1, there are two alternatives as to which part of the angular momentum is active in the k-i decomposition $\mathbf{j}=\mathbf{k}+\mathbf{i}$, where $\mathbf{k}$ is an integer called pseudoorbital angular momentum and $\mathbf{i}$ is a half-integer called pseudospin. In Fig. 1(a) $\mathbf{k}_1$ and $\mathbf{k}_2$ are active whereas $\mathbf{i}_1$ and $\mathbf{i}_2$ are inert, and the S and D pairs are

$$|(k_1k_2)K(i_1i_2)0;KM\rangle=[b^\dagger_{k_1i_1}b^\dagger_{k_2i_2}]^{K0}_{M0}|0\rangle \qquad (k \text{ active}, K=0,2) . \tag{2.1a}$$

In Fig. 1(b), $\mathbf{i}_1$ and $\mathbf{i}_2$ are active while $\mathbf{k}_1$ and $\mathbf{k}_2$ are inert. The S and D pairs are

$$|(k_1k_2)0(i_1i_2)I;IM\rangle=[b^\dagger_{k_1i_1}b^\dagger_{k_2i_2}]^{0I}_{0M}|0\rangle \qquad (i \text{ active}, I=0,2) . \tag{2.1b}$$

Figure 1(c) corresponds to the situation that the entire angular momentum $\mathbf{j}$ is inert. In this case there are only S pairs. From Table I it can be seen that, for the k-active scheme, k must be 1 and $\mathbf{k}_1=\mathbf{k}_2=\mathbf{k}$, since if $k=0$, then only S pairs can be formed, and if $k>1$, then pairs with $J>2$ could occur. Likewise for the i-active scheme, i must be $\frac{3}{2}$ and $\mathbf{i}_1=\mathbf{i}_2=\mathbf{i}$ (as illustrated in Table I).

It is important to emphasize that such S and D pairs are actually highly coherent, i.e., of collective nature. This can be seen after transforming the k-i basis back into the shell model basis by using the normalized 9-j recoupling coefficient:

$$|(k_1k_2)K(i_1i_2)I;r\mu\rangle=[b^\dagger_{k_1i_1}b^\dagger_{k_2i_2}]^{(KI)r}_{\mu}|0\rangle=\sum_{j_1j_2}\begin{bmatrix}k_1 & i_1 & j_1\\ k_2 & i_2 & j_2\\ K & I & r\end{bmatrix}[a^\dagger_{j_1}a^\dagger_{j_2}]^r_\mu|0\rangle . \tag{2.2}$$

It is easily seen that the S pair, in either the k- or i-active scheme, is precisely the Cooper pair in the pairing condensate:

$$|(kk)0(ii)0;00\rangle=\sum_j\sqrt{\Omega_j/\Omega_{ki}}[a^\dagger_ja^\dagger_j]^0_0|0\rangle , \tag{2.3}$$

where the pair degeneracies Ω_{ki} and Ω_j are defined as

$$\Omega_{ki}=(2k+1)(2i+1)/2, \quad \Omega_j=(2j+1)/2 . \tag{2.4}$$

Thus we see that a highly coherent pair, which has very strong configuration mixing in the shell model basis, can

TABLE I. Allowed values of the total angular momentum J for different choices of k and i. The absence of odd values of J when $k_1=k_2$ follows from Pauli principle for identical nucleons.

k-active scheme			i-active scheme		
k_1	k_2	$K=J$	i_1	i_2	$I=J$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
0	1	1	$\frac{1}{2}$	$\frac{3}{2}$	1,2
1	1	0,2	$\frac{3}{2}$	$\frac{3}{2}$	0,2
1	2	1,2,3	$\frac{3}{2}$	$\frac{5}{2}$	1,2,3,4
2	2	0,2,4	$\frac{5}{2}$	$\frac{5}{2}$	0,2,4

have a very simple, i.e., a pure configuration, structure in the k-i basis. This is precisely the characteristics one desires for a basis designed to describe the microscopic structure of collective excitations.

To build a complete basis for a many-body system, additional degrees of freedom associated with activating the inert parts of the angular momenta of the pairs are required. For example, in the k-active case [Eq. (2.1a) and Fig. 1(a)], the restriction $\mathbf{i}_1+\mathbf{i}_2=\mathbf{0}$ may be relaxed while maintaining the coupling $\mathbf{k}_1+\mathbf{k}_2=\mathbf{0},\mathbf{2}$. Likewise for the i-active case the restriction $\mathbf{k}_1+\mathbf{k}_2=\mathbf{0}$ may be relaxed while maintaining $\mathbf{i}_1+\mathbf{i}_2=\mathbf{0},\mathbf{2}$. This will physically correspond to "breaking" S (D) pairs and allowing the inert parts of their angular momenta to add vectorially to that generated by the S-D condensate. Notice that we use the terminology "broken" even for pairs of angular momentum 0 and 2, if that angular momentum comes from the inert parts of the single-particle angular momentum. By "activating" the inert angular momenta of nucleons, i.e., by considering all possible broken pairs, the k-i basis becomes a complete many-body basis. Defining a quantum number u, which we shall call "heritage,"

$u \equiv$ the number of particles which do not form coherent S and D pairs

[the coherent S and D pairs are defined in Eq. (2.1)], the shell model space can be classified according to the heritage number as shown in Table II. The advantage of using the k-i basis is that the S-D subspace can be easily carved out from the entire shell model space by setting the heritage number $u=0$. By including all possible u's, one would span the entire shell model space as illustrated in Fig. 2.

The correspondence between the shell model basis $|\, n(l\,l/2)jm\,\rangle$ $(=a^\dagger_{jm}\,|\,0\rangle)$ and the k-i basis $|\, km_k im_i\,\rangle$ $(=b^\dagger_{km_k im_i}\,|\,0\rangle)$ is generally many-to-one, i.e., in Eq. (1.1b) the choices of k and i are not unique since they are only restricted by $\mathbf{k}+\mathbf{i}=\mathbf{j}$. For the k-i basis to be useful, it is prerequisite that each j in a major shell should be decomposed uniquely into a definite k and i, and all j's within a major shell should be reproduced, no more and no less, from $\mathbf{k}+\mathbf{i}=\mathbf{j}$, which implies that (i) $\sum_{ki}\Omega_{ki}=\sum_j \Omega_j$; (ii) $\Delta i=3$ for the k-active scheme or $\Delta k=4$ for the i-active scheme, where Δi (Δk) is the difference between two neighboring i's (k's). Requirement (ii) is necessary to guarantee that each j within one major shell generated by $\mathbf{k}+\mathbf{i}=\mathbf{j}$ occurs only once.

It can be shown that two conditions ensure a unique decomposition: (a) the normal-parity levels in a major shell must be either totally k active ($k=1$) or i active ($i=\frac{3}{2}$); (b) the abnormal-parity level must be assigned $k=0$, i.e., $b^\dagger_{0j}=a^\dagger_j$.

Point (a) can be checked by trial and error, as illustrated in Table III for shells 6 and 7. Point (b) constitutes a modification of the Ginocchio scheme and its proof is trivial: for the abnormal level, the choice of any k other than zero will give rise to more than one abnormal-parity level, $j=|\,k-i\,|,\ldots,k+i$, in contradiction to the fact that in a major shell there is only one such level. Therefore, the only possibility is $k=0$, $i=j_0$ as illustrated in Table III.

TABLE II. The many-particle k-i basis. A broken pair means a pair of nucleons with inert parts of the angular momentum activated.

System	u	Building blocks
Even	0	S and D pairs
	2	S and D pairs plus 1 broken pair[a]
	⋮	
Odd	1	S and D pairs plus 1 odd nucleon
	3	S and D pairs plus 1 broken pair plus 1 odd nucleon
	⋮	

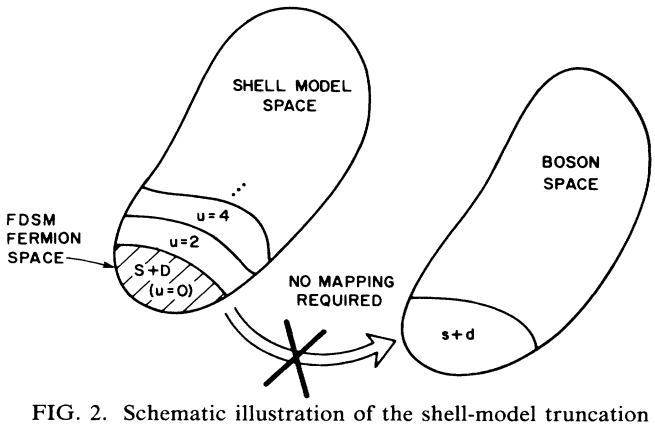

FIG. 2. Schematic illustration of the shell-model truncation implicit in the FDSM. For no broken pairs (heritage $u=0$), the coherent $(S,D,\mathcal{S})$ subspace is decoupled from the remainder of the shell-model space. By breaking pairs ($u\neq 0$), a richer basis of states may be constructed. Since the FDSM is formulated entirely in the *fermion* space, it requires no fermion→boson mapping procedure.

TABLE III. (Upper) An example (shell No. 6) of k-i decomposition. (Lower) An example (shell No. 7) of k-i decomposition. The numbers in each parentheses correspond to the possible values (ki) in either scheme. The (ki) values marked with an asterisk are the only choice of (ki) where $k+i=j$ which can reproduce all the normal parity levels. For the abnormal-parity level, the only possible k-i decomposition is $(0\ \frac{11}{2})$ for the upper portion and $(0\ \frac{13}{2})$ for the lower portion of the table.

lj	k-active scheme	i-active scheme
$s_{1/2}$	$(1\ \frac{1}{2})\ (1\ \frac{3}{2})$	$(1\ \frac{3}{2})\ (2\ \frac{3}{2})^*$
$d_{3/2}$	$(1\ \frac{1}{2})\ (1\ \frac{3}{2})(1\ \frac{5}{2})$	$(0\ \frac{3}{2})(1\ \frac{3}{2})(2\ \frac{3}{2})^*(3\ \frac{3}{2})$
$d_{5/2}$	$(1\ \frac{3}{2})(1\ \frac{5}{2})(1\ \frac{7}{2})$	$(1\ \frac{3}{2})(2\ \frac{3}{2})^*(3\ \frac{3}{2})\ (4\ \frac{3}{2})$
$g_{7/2}$	$(1\ \frac{5}{2})(1\ \frac{7}{2})\ (1\ \frac{9}{2})$	$(2\ \frac{3}{2})^*(3\ \frac{3}{2})\ (4\ \frac{3}{2})(5\ \frac{3}{2})$
$h_{11/2}$	$(1\ \frac{9}{2})(1\ \frac{11}{2})(1\ \frac{13}{2})\ \ (0\ \frac{11}{2})^*$	$(4\ \frac{3}{2})(5\ \frac{3}{2})(6\ \frac{3}{2})(7\ \frac{3}{2})$
$p_{1/2}$	$(1\ \frac{1}{2})^*(1\ \frac{3}{2})$	$(1\ \frac{3}{2})(2\ \frac{3}{2})$
$p_{3/2}$	$(1\ \frac{1}{2})^*(1\ \frac{3}{2})(1\ \frac{5}{2})$	$(0\ \frac{3}{2})(1\ \frac{3}{2})(2\ \frac{3}{2})\ (3\ \frac{3}{2})$
$f_{5/2}$	$(1\ \frac{3}{2})(1\ \frac{5}{2})(1\ \frac{7}{2})^*$	$(1\ \frac{3}{2})(2\ \frac{3}{2})\ (3\ \frac{3}{2})\ (4\ \frac{3}{2})$
$f_{7/2}$	$(1\ \frac{5}{2})(1\ \frac{7}{2})^*(1\ \frac{9}{2})$	$(2\ \frac{3}{2})\ (3\ \frac{3}{2})\ (4\ \frac{3}{2})(5\ \frac{3}{2})$
$h_{9/2}$	$(1\ \frac{7}{2})^*(1\ \frac{9}{2})(1\ \frac{11}{2})$	$(3\ \frac{3}{2})\ (4\ \frac{3}{2})(5\ \frac{3}{2})(6\ \frac{3}{2})$
$i_{13/2}$	$(1\ \frac{11}{2})(1\ \frac{13}{2})(1\ \frac{15}{2})(0\ \frac{13}{2})^*$	$(5\ \frac{3}{2})(6\ \frac{3}{2})(7\ \frac{3}{2})(8\ \frac{3}{2})$

The *one-to-one correspondence* between the k-i basis and the shell model basis is shown in Table IV. It is particularly intriguing that the different major shells are found to demand a different coupling scheme. *Physically this means that the possible collective modes at low energy are intimately related to the shell structure.* We shall discuss this in more detail in the following sections.

Finally, we should note that the coherent nature of the S-D subspace in the k-i basis implies that it is most suitable for describing collective modes, or the coupling of single-particle modes to collective modes, rather than the pure single particle degrees of freedom. For the latter, it is better to use the conventional shell model basis. One should not expect that our D-pair wave functions would have large overlap with the wave functions of nuclei which have two nucleons outside closed shells, since they are mainly of noncollective nature. On the other hand, the conventional shell model basis has difficulty in describing collective motion. Thus the k-i basis used in FDSM and the conventional shell model basis, which tends to emphasize the single-particle degrees of freedom, are complementary.

III. THE HAMILTONIAN OF THE FDSM

In this paper, we will restrict ourselves to the case of identical particles interacting by two-body residual interactions in one major shell. The effective nuclear Hamiltonian can be generally written as

$$H=\sum_j e_j a_j^\dagger a_j + V\ , \tag{3.1a}$$

$$V=V_p+V_Q\ . \tag{3.1b}$$

In (3.1), V_p and V_Q are the pairing and multipole interactions, respectively.

TABLE IV. Reclassification of shell model single-particle levels. The column No. labels the shell ordering; n, k, i label the principle, pseudoorbit, and pseudospin quantum numbers; Ω_0 and Ω_1 are the pair degeneracies of the abnormal-parity and normal-parity levels for each shell. The number n means the maximum allowable nucleon number up to and including that particular major shell. The symbols $\mathbf{G}_6$, $\mathbf{G}_8$, and $\mathbf{G}_3$ are shorthand notation for the symmetries: $\mathbf{G}_6=(\mathrm{Sp}_6^k\times\mathrm{SO}_3^i)\times(\mathcal{SU}_2\times\mathcal{SO}_3)$ (k active); $\mathbf{G}_8=(\mathrm{SO}_8^i\times\mathrm{SO}_3^k)\times(\mathcal{SU}_2\times\mathcal{SO}_3)$ (i active); $\mathbf{G}_3=(\mathrm{SU}_3^k\times\mathrm{SO}_6^i)\times(\mathcal{SU}_2\times\mathcal{SO}_3)$ (k/i active). For details of these symmetries, see the discussions in text. The script symbols in the definition of $\mathbf{G}_6$, $\mathbf{G}_8$, and $\mathbf{G}_3$ imply the symmetry for the abnormal parity level in each shell. For the s-d shell, there is no $\mathcal{SU}_2\times\mathcal{SO}_3$ groups due to the absence of an abnormal-parity level in the shell.

No.	1	2	3	4	5		6		7			8		
n	0	1	2	3	3	4	4	5	5	5	6	6	6	7
k	0	1	1	0	1	0	2	0	1	1	0	1	1	0
i	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{11}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	$\frac{13}{2}$	$\frac{3}{2}$	$\frac{9}{2}$	$\frac{15}{2}$
Con.	$s_{1/2}$	$p_{1/2}$	$s_{1/2}$	$f_{7/2}$	$p_{1/2}$	$g_{9/2}$	$s_{1/2}$	$h_{11/2}$	$p_{1/2}$	$f_{5/2}$	$i_{13/2}$	$s_{1/2}$	$g_{7/2}$	$j_{15/2}$
		$p_{3/2}$	$d_{3/2}$		$p_{3/2}$		$d_{3/2}$		$p_{3/2}$	$f_{7/2}$		$d_{3/2}$	$g_{9/2}$	
			$d_{5/2}$		$f_{5/2}$		$d_{5/2}$			$h_{9/2}$		$d_{5/2}$	$i_{11/2}$	
							$g_{7/2}$							
Sym.				$\mathbf{G}_6\mathbf{G}_8\mathbf{G}_3$	$\mathbf{G}_6\mathbf{G}_8\mathbf{G}_3$		$\mathbf{G}_8$		$\mathbf{G}_6$			$\mathbf{G}_6$		
Ω_0	0	0	0	0	5		6		7			8		
Ω_1	1	3	6	4	6		10		15			21		
n	2	8	20	28	50		82		126			184		

$$V_p=\tfrac{1}{4}\sum_{\lambda j_1 j_2 j_1' j_2'}\langle j_1 j_1' \lambda \mid V_p \mid j_2 j_2' \lambda\rangle \times[a_{j_1}^{\dagger}a_{j_1'}^{\dagger}]^{\lambda}\cdot[\tilde{a}_{j_2}\tilde{a}_{j_2'}]^{\lambda}\ , \tag{3.2a}$$

$$V_Q=\tfrac{1}{4}\sum_{r j_1 j_2 j_1' j_2'}\langle j_1 \bar{j}_2 r \mid V_Q \mid \bar{j}_1' j_2' r\rangle \times[a_{j_1}^{\dagger}\tilde{a}_{j_2}]^{r}\cdot[a_{j_1'}^{\dagger}\tilde{a}_{j_2'}]^{r}\ . \tag{3.2b}$$

In (3.2b), the usual time-reversed notation ($\sim$) is used for the annihilation operator, $\tilde{a}_{jm}=(-)^{j-m}a_{j-m}$. The particle-hole type matrix element of (3.2b) is defined in terms of the regular two-body matrix elements by Eq. (3.5). It should be noted that there is a single-particle energy term embedded in V_Q, since V_Q can be rewritten as

$$V_Q=\sum_j \epsilon_j^Q a_j^{\dagger}a_j + \tfrac{1}{4}\sum_{\lambda j_1 j_2 j_1' j_2'}\langle j_1 j_1' \lambda \mid V_Q \mid j_2 j_2' \lambda\rangle \times[a_{j_1}^{\dagger}a_{j_1'}^{\dagger}]^{\lambda}\cdot[\tilde{a}_{j_2}\tilde{a}_{j_2'}]^{\lambda}\ , \tag{3.3}$$

where

$$\epsilon_j^Q=\sum_{\lambda j'}\frac{1}{4}\langle jj'\lambda \mid V_Q \mid jj'\lambda\rangle\frac{2\lambda+1}{2j+1} \tag{3.4}$$

and the two-body matrix elements of V_Q are

$$\langle j_1 j_1' \lambda \mid V_Q \mid j_2 j_2' \lambda\rangle=\sum_r \langle j_1 \bar{j}_2 r \mid V_Q \mid \bar{j}_1' j_2' r\rangle\sqrt{(2r+1)/(2\lambda+1)}\begin{bmatrix} j_1 & j_2 & r \\ j_1' & j_2' & r \\ \lambda & \lambda & 0\end{bmatrix}\ . \tag{3.5}$$

Therefore the single particle energies ϵ_j should be understood as the sum of e_j and ϵ_j^Q:

$$\epsilon_j=e_j+\epsilon_j^Q\ . \tag{3.6}$$

It should also be noted that one term (either V_Q or V_p) in Eq. (3.1b) is sufficient, since any two-body interaction can be expressed as either a V_p-type or a V_Q-type expansion. In practice, however, the V_p-type expansion is more suitable for the description of short-range correlations. To describe long-range correlations with V_p, one needs to include high-rank (λ) terms, while the V_Q-type expansion has precisely the opposite property. Since practical calculations require truncation of the multipole series, it is usually advantageous to expand the short and long range parts of V in V_p- and V_Q-type expansions, respectively.

One cannot go further without simplifying V in Eq. (3.1). One such simplification is the pairing-plus-quadrupole ($P+Q$) model where only $\lambda=0$ and $r=2$ are chosen. The relationship between the FDSM and $P+Q$ model will be discussed in Sec. VII. In the present model, we shall proceed differently. We introduce three assumptions to simplify the full shell model Hamiltonian (3.1) to the Hamiltonian of the FDSM. They are the following.

Assumption 1. The residual pairing interaction V_p is dominated by the monopole ($\lambda=0$) and quadrupole ($\lambda=2$) terms. The implications of this assumption are best manifested in the *k-i* basis which we have discussed in Sec. II. In this basis, V_p of Eq. (3.2a) can be expressed as

$$V_p=\tfrac{1}{4}\sum\langle (k_1k_2)K(i_1i_2)I;\lambda \mid V_p \mid (k_1'k_2')K'(i_1'i_2')I';\lambda\rangle[b_{k_1i_1}^{\dagger}b_{k_2i_2}^{\dagger}]^{(KI)\lambda}\cdot[\tilde{b}_{k_1'i_1'}\tilde{b}_{k_2'i_2'}]^{(K'I')\lambda}\ , \tag{3.7}$$

where the summation is over all the indices. There are two possibilities which ensure that only $\lambda=0$ and 2 contribute to (3.7): either $I=0$ and $K=0,2$ or $K=0$ and $I=0,2$. Any other choice of K and I will result in λ having values other than 0 and 2.

According to Table I, one can see that for the $K=0,2$ case, in the *normal-parity orbitals* we must have $k_1=k_2=1$ and $i_1=i_2$ since $I=0$. This is the *k*-active scheme. Similarly, for the $I=0,2$ case, we must have $i_1=i_2=\frac{3}{2}$ and $k_1=k_2$, since $K=0$, which is the *i*-active scheme. In the abnormal-parity orbitals, we have shown in Sec. II that a unique reclassification of the shell model in the *k-i* basis requires the assignment $k_1=k_2=0$, $i_1=i_2=j_0$. Thus, assumption (1) is equivalent to requiring the quantum numbers in the pairing matrix elements to satisfy the following conditions. For all cases,

$$k_1=k_2\equiv k,\quad i_1=i_2\equiv i\ ,\qquad k_1'=k_2'\equiv k',\quad i_1'=i_2'\equiv i'\ . \tag{3.8a}$$

(1) For the *k*-active normal levels,

$$k(k')=1,\quad K(K')=\lambda,\quad I(I')=0\ . \tag{3.8b}$$

(2) For the *i*-active normal levels,

$$i(i')=\tfrac{3}{2},\quad K(K')=0,\quad I(I')=\lambda\ . \tag{3.8c}$$

(3) For the abnormal level,

$$k(k')=0\quad K(K')=0,\quad I(I')=0,\lambda=0\ . \tag{3.8d}$$

As mentioned before, the appropriate coupling scheme depends on which major shell one is considering. For example, from Table IV, shells 7 and 8 are *k*-active

($k=1$), and shell 6 is i-active ($i=\frac{3}{2}$). For shells 3 and 5 where both k and i could be active (i.e., $k=1$ and $i=\frac{3}{2}$), the situation is more complicated, since there now also exist matrix elements with $K\neq K'$ and $I\neq I'$ (e.g., $\langle (11)2(\frac{3}{2}\frac{3}{2})0;2 \mid V_p \mid (11)0(\frac{3}{2}\frac{3}{2})2;2\rangle$). This coupling will be denoted as the k/i active scheme.

Assumption 2. The two-body pairing matrix elements are parametrized by assuming that they are proportional to the degeneracy of the levels participating in the pairing correlations. In the present model, we take this to be the degeneracy of a major shell,

$$\langle (kk)K(ii)I;\lambda \mid V_p \mid (k'k')K'(i'i')I';\lambda\rangle = 2\sqrt{\Omega_{ki}\Omega_{k'i'}}G_{\lambda}^{\phi\phi'} . \quad (3.9)$$

The superscript ϕ (ϕ') defines whether a pair belongs to the abnormal level ($\phi=a$) or normal levels ($\phi=n_k,n_i$), while the subscripts k and i denote whether it is k active or i active.

Inserting Eqs. (3.9) into Eq. (3.7), V_p becomes

$$V_p=\sum_{\phi\phi'}[G_0^{\phi\phi'}S^{\dagger}(\phi)S(\phi')+G_2^{\phi\phi'}D^{\dagger}(\phi)\cdot D(\phi')] . \quad (3.10)$$

The operators $S^{\dagger}(\phi)[S(\phi')]$ and $D^{\dagger}(\phi)$ $[D(\phi')]$ denote the S and D pair creation (annihilation) operators, respectively:

(1) For the k-active normal levels,

$$S^{\dagger}(\phi=n_k)\equiv S^{\dagger}=\sum_i \sqrt{\Omega_{1i}/2}[b_{1i}^{\dagger}b_{1i}^{\dagger}]_{00}^{00} , \quad (3.11a)$$

$$D_{\mu}^{\dagger}(\phi=n_k)\equiv D_{\mu}^{\dagger}=\sum_i \sqrt{\Omega_{1i}/2}[b_{1i}^{\dagger}b_{1i}^{\dagger}]_{\mu 0}^{20} . \quad (3.11b)$$

(2) For the i-active normal levels,

$$S^{\dagger}(\phi=n_i)\equiv S^{\dagger}=\sqrt{\Omega_{k3/2}/2}[b_{k3/2}^{\dagger}b_{k3/2}^{\dagger}]_{00}^{00} , \quad (3.12a)$$

$$D_{\mu}^{\dagger}(\phi=n_i)\equiv D_{\mu}^{\dagger}=\sqrt{\Omega_{k3/2}/2}[b_{k3/2}^{\dagger}b_{k3/2}^{\dagger}]_{0\mu}^{02} . \quad (3.12b)$$

(3) For abnormal parity level, there are only S pairs [and no D pairs, see Eqs. (3.8)]

$$S^{\dagger}(\phi=a)=\mathcal{S}^{\dagger}=\sqrt{\Omega_{j_0}/2}[b_{0j_0}^{\dagger}b_{0j_0}^{\dagger}]_{00}^{00} , \quad (3.13)$$

where j_0 is the angular momentum of the abnormal-parity orbital. In general, the right-hand side of Eq. (3.12) should be summed over k. However, as is seen from Table IV, there is only one k value for each shell in the known nuclei. Only for the shells higher than 8 could more than one k value occur. For example, shell 9 will have $k=1$ and 5.

To simplify notation, throughout the paper we shall use script letters to denote the quantities associated with the abnormal level, as shown in Eq. (3.13). Furthermore, we shall suppress the indices $\phi=n_k$ and $\phi=n_i$ and simply use $S^{\dagger}(S)$ and $D^{\dagger}(D)$ to denote the $S(D)$ operators in normal levels. This will not induce confusion unless we consider the k/i active case ($k=1$ and $i=\frac{3}{2}$), for which we have two kinds of S and D pairs. As one can see from Table IV, this situation happens only for shell 3 and shell 5. All the higher shells (corresponding to medium and heavier nuclei) are either k active or i active and there is only one kind of normal-parity $S(D)$ pair. Only the latter cases are considered in this paper, and Eq. (3.10) is thus simplified as

$$V_p=\sum_{\phi\phi'}G_0^{\phi\phi'}S^{\dagger}(\phi)S(\phi')+G_2D^{\dagger}\cdot D , \quad (3.14)$$

where ϕ stands either for a or n to distinguishing abnormal and normal-parity S pairs. More explicitly,

$$\sum_{\phi\phi'}G_0^{\phi\phi'}S^{\dagger}(\phi)S(\phi')=G_0^{nn}S^{\dagger}S+G_0^{na}(S^{\dagger}\mathcal{S}+\mathcal{S}^{\dagger}S)+G_0^{aa}\mathcal{S}^{\dagger}\mathcal{S} . \quad (3.15)$$

By assuming that $G_0^{nn}=G_0^{na}=G_0^{aa}$, and denoting it by G_0, the monopole pairing term can be written simply as $G_0S^{T\dagger}\mathbf{S}$, which is identical to the pairing part of the pairing plus quadrupole model (see Sec. VII) with

$$S^{T\dagger}\equiv S^{\dagger}+\mathcal{S}^{\dagger}=\sum_j \sqrt{\Omega_j/2}[a_j^{\dagger}a_j^{\dagger}]_0^0 . \quad (3.16)$$

Assumption 3. The terms involving the s.p. energies and multipole interactions are approximated so that the Hamiltonian is a simple function of the generators of a tractable Lie algebra. This assumption is based on the expectation that a dynamical symmetry of the Hamiltonian corresponds to a particular nuclear collective mode.

In order for the Hamiltonian to possess dynamical symmetries, the single particle energy term [the first term of Eq. (3.1a)] must be simplified. It can be rewritten as

$$\sum_j e_j a_j^{\dagger}a_j=e_0n_0+\sum_i e_{ki}n_{ki} + \sum_{\lambda\neq 0}\sum_i e_{ki}^{\lambda}\sqrt{2\Omega_{ki}}[b_{ki}^{\dagger}\tilde{b}_{ki}]_0^{(\lambda\lambda)0} , \quad (3.17a)$$

where e_0 is the s.p. energy of the abnormal-parity level, n_{ki} is the number operator of the particles occupying the k-i orbit:

$$n_{ki}=\sqrt{2\Omega_{ki}}[b_{ki}^{\dagger}\tilde{b}_{ki}]_{00}^{00}=\sum_{m_km_i} b_{km_kim_i}^{\dagger}\tilde{b}_{km_kim_i} , \quad (3.17b)$$

$$e_{ki}^{\lambda}=\sum_{j\in i} e_j \begin{bmatrix} k & i & j \\ k & i & j \\ \lambda & \lambda & 0 \end{bmatrix}\sqrt{\Omega_j/\Omega_{ki}} , \quad (3.17c)$$

and e_{ki} ($=e_{ki}^{\lambda=0}$) can be regarded as the s.p. energy of the normal-parity k-i basis. For most of the shells, e_{ki} is a constant since there is only one i value for the normal-parity levels in each major shell except for shells 7 and 8 which have two i's (see Table IV). The last term on the right-hand side of Eq. (3.17a) (termed the s.p. symmetry-breaking term) does not close under commutations with the generators in V_p [Eq. (3.14)] and therefore is neglected according to our assumptions. In fact, its matrix elements within the $(S,D,\mathcal{S})$ subspace (i.e., heritage $u=0$) are always zero. Thus, under the $u=0$ approximation, the s.p. energy term in the Hamiltonian can be written as

$$\sum_j e_j a_j^{\dagger}a_j\cong e_0n_0+\sum_i e_{ki}n_{ki} . \quad (3.18)$$

We see that, in the zero-heritage representations of the k-i basis, the normal-parity levels reduce to just one or two degenerate orbits in spite of the fact that they are not degenerate in the shell model s.p. basis.

It was shown recently by Kirson and Leviatan[19] that introducing nondegeneracies of the single particle energies has only a small effect on the dynamical symmetry of $SO_8 \supset SO_5 \times SU_2$, i.e., the seniority v is still approximately a good quantum number. This effect can be incorporated via renormalization of the interaction strength of the Hamiltonian without removing the s.p. energy degeneracies. For other dynamical symmetry cases, the importance of the s.p. symmetry-breaking term remains to be studied. For the situation where $u \neq 0$ configurations are important (e.g., for odd nuclei or high-spin states) the s.p. symmetry-breaking term may not be negligible. In any case the symmetric case is a good starting point and symmetry-breaking terms can always be taken into account numerically if necessary.

In order to simplify the multipole interaction, we need to first transform V_Q [Eq. (3.2b)] into the k-i basis,

$$V_Q = \tfrac{1}{4} \sum \langle (k_1 \bar{k}_2) K (i_1 \bar{i}_2) I; r \mid V_Q \mid (\bar{k}'_1 k'_2) K' (\bar{i}'_1 i'_2) I'; r \rangle \times [b^\dagger_{k_1 i_1} \tilde{b}_{k_2 i_2}]^{(KI)} \cdot [b^\dagger_{k'_1 i'_1} \tilde{b}_{k'_2 i'_2}]^{(K'I')r} , \tag{3.19}$$

where the sum is over all indices. In order that the Hamiltonian be a function of the generators of a tractable Lie algebra, we truncate V_Q by only considering terms which satisfy the following conditions: (1) $k_1 = k_2$, $i_1 = i_2$, $k'_1 = k'_2$, $i'_1 = i'_2$; (2) either $K=0$ then $r=I$, or $I=0$ then $r=K$; (3) if the inert part of the angular momentum is activated ($K \neq 0$ for the i-active case and $I \neq 0$ for the k-active or abnormal case), only odd-rank r is considered. Furthermore, we impose for the V_Q an assumption analogous to Eq. (3.9) for the pairing matrix elements:

$$\langle (k\bar{k}) K (i\bar{i}) I; r \mid V_Q \mid (\bar{k}'k') K (\bar{i}'i') I; r \rangle = 2\sqrt{\Omega_{ki} \Omega_{k'i'}} B_r^{\alpha\alpha'} . \tag{3.20}$$

By inserting (3.20) into (3.19), the multipole interaction V_Q becomes

$$V_Q = \sum_{\alpha\alpha' r} B_r^{\alpha\alpha'} P^r(\alpha) \cdot P^r(\alpha') , \tag{3.21}$$

where the index $\alpha(\alpha')$ takes three possible values: k, i, and a. The index k means that the total angular momentum **r** comes from the **k** (pseudoorbital) parts of the normal levels, i.e., $r=K$ and $I=0$. The index i indicates that **r** comes from the **i** (pseudospin) parts of normal levels, i.e., $r=I$ and $K=0$, while the index a is used to specify the abnormal level. The multipole indices r range over the following values:

(i) When $\alpha(\alpha')=k$,

$$r = \begin{cases} 0,1,2 & \text{for } k\text{-active normal levels } (k=1) \\ 1,3,\ldots,2k-1 & \text{for } k \text{ activated in the } i\text{-active scheme } (i=\frac{3}{2}) . \end{cases}$$

(ii) When $\alpha(\alpha')=i$,

$$r = \begin{cases} 0,1,2,3 & \text{for } i\text{-active normal levels } (i=\frac{3}{2}) \\ 1,3,\ldots,2i & \text{for } i \text{ activated in the } k\text{-active scheme } (k=1) . \end{cases}$$

(iii) When $\alpha(\alpha')=a$,

$r=0$ or odd for the abnormal-parity level $(k=0)$.

The $P^r(\alpha)$'s are defined as

$$P^r_\mu(k) = \sum_i \sqrt{\Omega_{ki}/2} [b^\dagger_{ki} \tilde{b}_{ki}]^{r0}_{\mu 0} , \tag{3.22a}$$

$$P^r_\mu(i) = \sqrt{\Omega_{ki}/2} [b^\dagger_{ki} \tilde{b}_{ki}]^{0r}_{0\mu} , \tag{3.22b}$$

$$P^r_\mu(a) \equiv \mathcal{P}^r_\mu = \sqrt{\Omega_{j_0}/2} [a^\dagger_{j_0} \tilde{a}_{j_0}]^r_\mu . \tag{3.22c}$$

(Notice that the P^r_μ defined here is one half of that defined by Ginocchio.[15]) No sum over k is required in Eq. (3.22b) as noted in connection with Eq. (3.12). Thus, with our three assumptions, the most general FDSM Hamiltonian for either the k-active or i-active scheme is

$$H_{\text{FDSM}} = e_0 n_0 + \sum_i e_{ki} n_{ki} + \sum_{\phi,\phi'} G_0^{\phi\phi'} S^\dagger(\phi) \cdot S(\phi') + G_2 D^\dagger \cdot D + \sum_{r,\alpha,\alpha'} B_r^{\alpha\alpha'} P^r(\alpha) \cdot P^r(\alpha') . \tag{3.23a}$$

As noted before, the s.p. energy e_{ki} in the k-i basis has only one value except for shells 7 and 8, where there are two variables of i. Therefore we shall further simplify it by assuming that $e_{ki} = e_1$ (a constant), and Eq. (3.23a) becomes

$$H_{\text{FDSM}} = e_0 n_0 + e_1 n_1 + \sum_{\phi,\phi'} G_0^{\phi\phi'} S^\dagger(\phi) \cdot S(\phi') + G_2 D^\dagger \cdot D + \sum_{r,\alpha,\alpha'} B_r^{\alpha\alpha'} P^r(\alpha) \cdot P^r(\alpha') , \tag{3.23b}$$

where the multipole operators are defined by (3.22) and the pairing operators are defined by either (3.11) or (3.12), depending on whether the shell being considered is k or i active. In this FDSM Hamiltonian there are two classes of multipole operators. The first class consists of the multipole operators coupled from the active parts of the angular momenta ("active" multipole operators); the other consists of those coupled from the inert parts of the angular momentum ("inert" multipole operators). Explicitly,

	Active multipole operators	Inert multipole operators	
For k-active normal	$P^r_\mu(k)$ $r=0,1,2$	$P^r_\mu(i)$ $r=1,3,\ldots,2i$	(3.24a)
For i-active normal	$P^r_\mu(i)$ $r=0,1,2,3$	$P^r_\mu(k)$ $r=1,3,\ldots,2k-1$	(3.24b)
For the abnormal level	$\mathcal{P}^0$ $r=0$	$\mathcal{P}^r_\mu$ $r=1,3,\ldots,2j_0$	(3.24c)

The active multipole and pair operators for the normal-parity levels and those for the abnormal-parity level are separately closed under commutation i.e.,

(a) $k=1$: The 21 generators $\{S^\dagger,S,D^\dagger_\mu,D_\mu,P^r_\mu,r=0,1,2\}$ form an Sp_6 algebra, (3.25a)

(b) $i=\frac{3}{2}$: The 28 generators $\{S^\dagger,S,D^\dagger_\mu,D_\mu,P^r_\mu,r=0,1,2,3\}$ form an SO_8 algebra, (3.25b)

(c) $\mathcal{S}^\dagger$, $\mathcal{S}$, and $\mathcal{P}^0$ form an $\mathcal{SU}_2$ algebra. (3.25c)

Thus, neglecting the inert multipole operators, the Hamiltonian has $Sp_6\times\mathcal{SU}_2$ (for the k-active case) or $SO_8\times\mathcal{SU}_2$ (for the i-active case) symmetry. Including inert multipole operators, it is easy to see the following.

(1) For the k-active normal levels, the multipole operators $P^r_\mu(i)$ with $r=1,3,\ldots,2i$ form an Sp_{2i+1} algebra, which is commutative with Sp_6. Therefore, the dynamical symmetry group for the normal levels is enlarged to $Sp_6\times Sp_{2i+1}$.

(2) For the i-active normal levels, the multipole operators $P^r_\mu(k)$ with $r=1,3,\ldots,2k-1$ form an SO_{2k+1} algebra, which is commutative with SO_8. Therefore the dynamical symmetry group for the normal levels is enlarged to $SO_8\times SO_{2k+1}$.

(3) For the abnormal level, the multipole operators $\mathcal{P}^r_\mu$ with $r=1,3,\ldots,2j_0$ form the algebra of $\mathcal{S}\!\!\!/_{2j_0+1}$ which is commutative with $\mathcal{SU}_2$, and the dynamical symmetry group of the abnormal level is enlarged to $\mathcal{SU}_2\times\mathcal{S}\!\!\!/_{2j_0+1}$.

The inert multipole operators are effective only in the $u\neq0$ subspace. If r is restricted to 1, then Sp_{2i+1}, SO_{2k+1}, and $\mathcal{S}\!\!\!/_{2j_0+1}$ are reduced to SO^i_3, SO^k_3, and $\mathcal{SO}_3$, respectively. These groups are associated with the angular momenta of the unpaired particles. Since the inert multipole operators commute with S, D, and $\mathcal{S}$ pair operators, they will not mix the $S,D,\mathcal{S}$ subspace (the $u=0$ space) with the $u\neq0$ space. Thus the FDSM Hamiltonian has a very useful property that the $(S,D,\mathcal{S})$ subspace (heritage zero subspace) is completely decoupled from the remaining fermion space. If we only consider the $(S,D,\mathcal{S})$ subspace (this could be a reasonable approximation for the low lying collective states of an even-even system), the inert multipole operators will play no role since their action on any state in the zero-heritage space will always result in a null vector. In this case, the Hamiltonian (3.23b) can be reduced to $H^0_{\rm FDSM}$

$$H^0_{\rm FDSM}=e_0n_0+e_1n_1+\sum_{\phi,\phi'}G_0^{\phi\phi'}S^\dagger(\phi)\cdot S(\phi')+G_2D^\dagger\cdot D+\sum_{r,\phi,\phi'}B_r^{\phi\phi'}P^r(\phi)\cdot P^r(\phi')\ , \tag{3.26}$$

where $\phi(\phi')=a$ and n, and

$$P^r_\mu(a)=\mathcal{P}^0\delta_{r0},\quad P^r_\mu(n)\equiv P^r_\mu=\begin{cases}P^r_\mu(k) & r=0,1,2\ \text{(for } k \text{ active)}\\ P^r_\mu(i) & r=0,1,2,3\ \text{(for } i \text{ active)}\ .\end{cases} \tag{3.27}$$

For high-spin or odd-nucleon systems, where the broken pair or an unpaired particle become important, we have to use the Hamiltonian (3.23b) and take the inert multipole operators into account. In this manner, the FDSM can naturally depict the coupling of unpaired particles to the "core" constructed out of the S and D pairs. Of course, the s.p. symmetry-breaking term, as well as some symmetry-breaking terms in the multipole interaction, which are not included in the Hamiltonian (3.23b), may have to be considered in realistic calculations. This will be discussed in more detail in subsequent papers.

A summary of the generators and the associated algebras of the FDSM for the k-active and i-active schemes is given in Table V. For the scheme where k and i can both be active ($k=1$ and $i=\frac{3}{2}$), the situation is more involved and we must consider the algebra SO_{24}. This group requires further study as to its implications in nuclear physics and will not be discussed here.

IV. THE $Sp_6\times\mathcal{SU}_2$ AND $SO_8\times\mathcal{SU}_2$ SYMMETRIES OF THE FDSM

In this section, the zero-heritage situation, i.e., no broken pairs existing in either the normal-parity or the abnormal-parity levels, is discussed. The FDSM Hamiltonian $H_{\rm FDSM}$ of Eq. (3.26) can be rewritten as

$$H^0_{\rm FDSM}=e_0n_0+e_1n_1+\mathcal{H}_a+H_n+H'\ , \tag{4.1a}$$

$$\mathcal{H}_a=\mathcal{G}_0\mathcal{S}^\dagger\cdot\mathcal{S}+(\mathcal{B}_0/4)n_0^2\ , \tag{4.1b}$$

$$H_n=G_0S^\dagger\cdot S+G_2D^\dagger\cdot D+\sum_r B_rP^r\cdot P^r\ , \tag{4.1c}$$

$$H'=g_0(S^\dagger\cdot\mathcal{S}+\mathcal{S}^\dagger\cdot S)+(b_0/2)n_0n_1\ . \tag{4.1d}$$

The operators $\mathcal{S}^\dagger(\mathcal{S})$, $S^\dagger(S)$, $D^\dagger(D)$, and P^r_μ are defined in Eqs. (3.11)–(3.13), (3.22), and (3.27), and the number operators are

$$n_1=2P^0 \text{ and } n_0=2\mathcal{P}^0\ . \tag{4.2}$$

TABLE V. Generators of the $k(i)$ active groups in the FDSM. $A_\mu^{[r}(KI)=\sqrt{\Omega_{ki}/2}[b_{ki}^\dagger b_{ki}^\dagger]_\mu^{(KI)r}$; $P_\mu^r(KI)=\sqrt{\Omega_{ki}/2}[b_{ki}^\dagger \tilde{b}_{ki}]_\mu^{(KI)r}$ $(K\neq 0, I\neq 0)$.

Scheme	Generators	Group
k active	$S\ S^\dagger\ D_\mu(k)\ D_\mu^\dagger(k)\ P_\mu^r(k)$ $(r=0,1,2)$	Sp_6
	$P_\mu^r(i)$ $(r=\text{odd})$	Sp_{2i+1}
	$[A_\mu^{[r}(KI),\ I\neq 0], P_\mu^r(KI), [P_\mu^r(i)\ r=\text{even}]$ are neglected	
i active	$S\ S^\dagger\ D_\mu(i)\ D_\mu^\dagger(i)\ P_\mu^r(i)$ $(r=0,1,2,3)$	SO_8
	$P_\mu^r(k)$ $(r=\text{odd})$	SO_{2k+1}
	$[A_\mu^{[r}(KI),\ K\neq 0], P_\mu^r(KI), [P_\mu^r(k)\ r=\text{even}]$ are neglected	
abnormal	$\mathcal{S}^\dagger\ \mathcal{S}\ \mathcal{P}^0$	$\mathcal{SU}_2$
	$\mathcal{P}_\mu^r$ $(r=\text{odd})$	$\mathcal{SR}_{2i+1}$

In Eq. (4.1), $\mathcal{H}_a$ is the Hamiltonian of the $\mathcal{S}$ pair in the abnormal parity level, and H_n is the Hamiltonian of the S and D pairs in the normal parity levels. The cross term H' represents scattering of pairs between the normal and abnormal parity levels. The parameters in H^0_{FDSM} are related to the parameters of Eq. (3.26) as follows:

$$G_0=G_0^{nn},\quad g_0=G_0^{na},\quad \mathcal{G}_0=G_0^{aa},\quad G_2=G_2^{nn},$$
$$b_r=B_r^{na},\quad \mathcal{B}_0=B_0^{aa},\quad B_r=B_r^{nn}, \tag{4.3}$$

with $G_\lambda^{na}=G_\lambda^{an}$ and $B_r^{na}=B_r^{an}$. The commutators for the operators in the Hamiltonian are as follows:

$$[\mathcal{S}^\dagger, P_\mu^r]=[\mathcal{S}^\dagger, S]=[\mathcal{S}^\dagger, D]=0, \tag{4.4a}$$

$$[\mathcal{S}, \mathcal{S}^\dagger]=-2\mathcal{S}_0,\quad \mathcal{S}_0=(\mathcal{N}_0-\Omega_0)/2, \tag{4.4b}$$

$$[S, S^\dagger]=-2S_0,\quad S_0=(n_1-\Omega_1)/2, \tag{4.4c}$$

$$[A_\mu^r, A_\nu^{s\dagger}]=\Omega_1\delta_{r,s}\delta_{\mu,\nu} -2\sum_t K^{t\sigma}_{r-\mu,s\nu}(-1)^\mu P^t_\sigma, \tag{4.4d}$$

$$[P_\mu^r, A_\nu^{s\dagger}]=\sum_t K^{t\sigma}_{r\mu,s\nu}A^{t\dagger}_\sigma, \tag{4.4e}$$

$$[P_\mu^r, P_\nu^s]=\tfrac{1}{2}\sum_t[(-1)^t-(-1)^{r+s}]K^{t\sigma}_{r\mu,s\nu}P^t_\sigma, \tag{4.4f}$$

where

$$K^{t\sigma}_{r\mu,s\nu}=\begin{cases}\sqrt{3}\hat{r}\,\hat{s}\langle r\mu s\nu\,|\,t\sigma\rangle \begin{Bmatrix} r & s & t\\ 1 & 1 & 1\end{Bmatrix}, & \text{for } k \text{ active}\\ -2\hat{r}\,\hat{s}\langle r\mu s\nu\,|\,t\sigma\rangle \begin{Bmatrix} r & s & t\\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}, & \text{for } i \text{ active}\end{cases} \tag{4.5}$$

and $A_0^{0\dagger}=S^\dagger$ and $A_\mu^{2\dagger}=D_\mu^\dagger$. For the i-active scheme, the commutation relationships of Eqs. (4.4) are identical to those given by Eqs. (4.6a)–(4.6f) of Ref. 15.

According to Eq. (4.4b), $\mathcal{H}_a$ has $\mathcal{SU}_2$ dynamical symmetry. The term H_n in (4.1c) is essentially the Ginocchio Hamiltonian [see Eqs. (6.4) and (6.6) as well as the comment below Eq. (6.6) in Ref. 15], and exhibits Sp_6 or SO_8 dynamical symmetries, depending on whether the k-active or i-active scheme is in use. Summarizing, the model Hamiltonian (4.1) has $Sp_6\times\mathcal{SU}_2$ dynamical symmetry for the k-active scheme and $SO_8\times\mathcal{SU}_2$ dynamical symmetry for the i-active scheme.

In order to explore the dynamical symmetries of this Hamiltonian, it is expedient to rewrite it in terms of the independent Casimir operators of the relevant subgroups. The group chain decompositions leaving the total angular momentum invariant are shown in Fig. 3. The Casimir operators are summarized in Table VI. In addition, using the second-quantized form of angular momentum, one can show that

$$P^1=\sqrt{3/[4t(t+1)]}\,\mathbf{L}, \tag{4.6a}$$

where $\mathbf{L}$ is the angular momentum operator associated with the angular momentum t, and $t=k$, i, or j_0. Letting $t=k=1$ or $t=i=\frac{3}{2}$, we get

$$P^1=\begin{cases}\sqrt{3/8}\,\mathbf{L} & \text{for } Sp_6\\ \sqrt{1/5}\,\mathbf{L} & \text{for } SO_8.\end{cases} \tag{4.6b}$$

Using Table VI and Eqs. (4.6), the Hamiltonian (4.1) can be recast as follows.

(1) The $Sp_6\times\mathcal{SU}_2$ symmetry (k-active scheme):

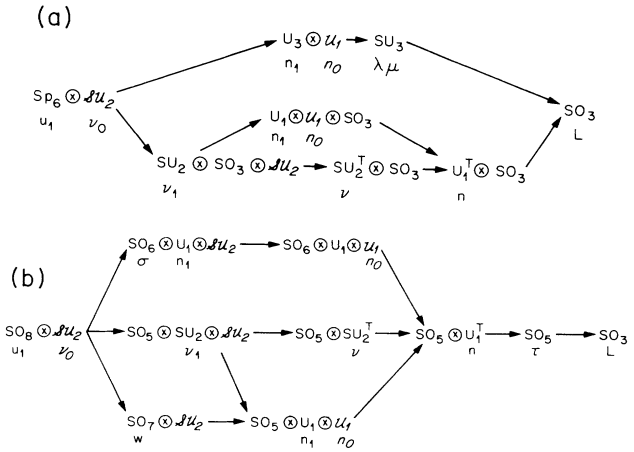

FIG. 3. (a) Group chains for the $Sp_6\times\mathcal{SU}_2$ dynamical symmetries. Quantum numbers are shown below the group responsible for them. Groups with italic symbols originate from the abnormal parity level. The symbol SU_2^T denote the group generated by $(S^T, S^{T\dagger}, P^{T0})$. (b) Same as for (a), but for the $SO_8\times\mathcal{SU}_2$ dynamical symmetries.

TABLE VI. Casimir operators, eigenvalues, and quantum numbers of the groups in the FDSM.

$\chi(\sigma_1,\sigma,\sigma_3)=\frac{1}{2}\sigma_1\sigma_3+\frac{3}{4}\sum_{i=1,3}[\sigma_i(\sigma_i+4)+\sigma\sigma_i]$; $\zeta(\theta_2,\theta_3)=\theta_2(\theta_2+3)+\frac{1}{2}\theta_3(\theta_3+4)+\theta_2\theta_3$;

$\phi(\rho_1,\rho_2,\rho_3)=\frac{1}{2}(\rho_1^2+\rho_2^2)+\frac{1}{4}(\rho_1+\rho_3)(\rho_1+\rho_3+4\rho_2+12)+\rho_2(\rho_2+4)$; $\Psi(a_1,a_2)=\frac{1}{2}(a_1^2+a_2^2+a_1a_2+3a_1+3a_2)$.

For the $u_1=0$ case, all the above functions vanish. $(\lambda\mu)$, $(\tau\omega)$, $(\sigma_1,\sigma,\sigma_3)$, $\{\theta_i\}$, $\{\rho_i\}$, and $\{a_i\}$ are the Dynkin labels (Ref. 17) for the irreps of SU_3, SO_5, SO_6, SO_7, SO_8, and Sp_6, respectively. The heritage number is defined by $u=u_1+\nu_0$, and thus u_1 is called SO_8 (Sp_6) heritage. The SO_7 quantum number w is the number of particles which do not form D pairs.

Group	Casimir operators	Eigenvalues	Quantum numbers
SU_2	$S^\dagger S+S_0(S_0-1)$	$\frac{1}{4}(\Omega_1-\nu_1)(\Omega_1-\nu_1+2)$	ν_1
$\mathcal{SU}_2$	$\mathcal{S}^\dagger\mathcal{S}+\mathcal{S}_0(\mathcal{S}_0-1)$	$\frac{1}{4}(\Omega_0-\nu_0)(\Omega_0-\nu_0+2)$	ν_0
SU_2^T	$S^{T\dagger}S^T+S_0^T(S_0^T-1)$	$\frac{1}{4}(\Omega-\nu)(\Omega-\nu+2)$	ν
SU_3	$\sum_{r=1,2}P^r\cdot P^r$	$\frac{1}{2}(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)$	(λ,μ)
SO_5	$\sum_{r=1,3}P^r\cdot P^r$	$\tau(\tau+3)+\frac{1}{2}\omega(\omega+4)+\tau\omega$	(τ,ω)
SO_6	$\sum_{r=1,2,3}P^r\cdot P^r$	$\sigma(\sigma+4)+\chi(\sigma_1,\sigma,\sigma_3)$	$(\sigma_1,\sigma,\sigma_3)$
SO_7	$D^\dagger\cdot D+S_0(S_0-5)+\sum_{r=1,3}P^r\cdot P^r$	$\frac{1}{4}(\Omega_1-w)(\Omega_1-w+10)+\zeta(\theta_2,\theta_3)$	$(\theta_1\theta_2\theta_3)$ $\theta_1=\frac{\Omega_1-w}{2}-\theta_2-\frac{\theta_3}{2}$
SO_8	$S^\dagger\cdot S+D^\dagger\cdot D+S_0(S_0-6)+\sum_{r=1,2,3}P^r\cdot P^r$	$\frac{1}{4}(\Omega_1-u_1)(\Omega_1-u_1+12)+\phi(\rho_1,\rho_2,\rho_3)$	$(\rho_1,\rho_2,\rho_3,\rho_4)$ $\rho_4=\frac{\Omega_1-u_1}{2}-\frac{1}{2}(\rho_1+2\rho_2+\rho_3)$
Sp_6	$S^\dagger\cdot S+D^\dagger\cdot D+S_0(S_0-6)+\sum_{r=1,2}P^r\cdot P^r$	$\frac{1}{4}(\Omega_1-u_1)(\Omega_1-u_1+12)+\Psi(a_1,a_2)$	(a_1,a_2,a_3) $a_3=\frac{\Omega_1-u_1}{3}-\frac{1}{3}(a_1+2a_2)$

$$H_6=H_0+v_0\mathcal{C}_{\mathcal{SU}_2}+v_1C_{SU_2}+g_0C_{SU_2^T}+s_3C_{SU_3}+G_2C_{Sp_6}+s\mathbf{L}^2 , \quad (4.7a)$$

$$H_0=h_0+\epsilon_0 n_0+\epsilon_1 n_1+\eta_0 n_0(n_0-1)/2+\eta_1 n_1(n_1-1)/2+\eta n(n-1)/2 . \quad (4.7b)$$

(2) The $SO_8\times\mathcal{SU}_2$ symmetry (i-active scheme):

$$H_8=H_0+v_0\mathcal{C}_{\mathcal{SU}_2}+v_1C_{SU_2}+g_0C_{SU_2^T}+g_5C_{SO_5}+g_6C_{SO_6}+G_2C_{SO_8}+g\mathbf{L}^2 , \quad (4.8)$$

where H_0 is the same as that defined in Eq. (4.7). The generators of SU_2^T are $\{S^{T\dagger},S^T,P^{T0}\}$, where $S^{T\dagger}(S^T)$ is defined in Eq. (3.16) and $P^{T0}=P^0+\mathcal{P}^0$. The relationship between the parameters in Eqs. (4.7) and (4.8) and the parameters of the Hamiltonian in (4.1) are listed in Table VII.

Note that

$$C_{SO_6}=C_{SO_8}-C_{SO_7}+C_{SO_5}-S^\dagger S+S_0 . \quad (4.9)$$

Therefore, Eq. (4.8) can also be expressed in terms of the Casimir of SO_7,

$$H_8=H_0'+v_0\mathcal{C}_{\mathcal{SU}_2}+v_1'C_{SU_2}+g_0C_{SU_2^T}+g_5'C_{SO_5}+g_7'C_{SO_7}+B_2C_{SO_8}+g\mathbf{L}^2 , \quad (4.10a)$$

$$H_0'=h_0'+\epsilon_0 n_0+\epsilon_1' n_1+\eta_0 n_0(n_0-1)/2+\eta_1' n_1(n_1-1)/2+\eta n(n-1)/2 . \quad (4.10b)$$

The primed quantities in Eq. (4.10) are defined in Table VIII, while the others are the same as given in Table VII.

From Eqs. (4.7) and (4.8) and (4.10), it is seen that H^0_{FDSM} has the following limiting cases.

(1) For the k-active coupling scheme, $v_1=g_0=0$, the $SU_3\times\mathcal{SU}_2$ limit; $s_3=0$, the $SU_2\times SO_3\times\mathcal{SU}_2$ limit.

(2) For the i-active coupling scheme, $v_1=g_0=0$, the $SO_6\times\mathcal{SU}_2$ limit; $g_6=0$, the $SO_5\times SU_2\times\mathcal{SU}_2$ limit; $v_1'=g_0=0$, the $SO_7\times\mathcal{SU}_2$ limit.

These five limiting cases and their corresponding group chains are shown in Figs. 3(a) and 3(b). The overall group structure of the FDSM is shown in Fig. 4, which is discussed in Ref. 16. For a full major shell the pairing and multipole operators generate an $SO_{4\Omega}$ algebra. Specific choices of the Ginocchio quantum numbers k and i then generate the Sp_6 and SO_8 group chains discussed here, and the SO_{24} chain (k/i active) which is neglected in this paper.

The physical conditions for the various symmetry limits to occur in the model can be understood as follows: For simplicity, we shall assume the following FDSM Hamiltonian:

TABLE VII. Transformation between the parameters in Eqs. (4.7), (4.8), and (4.1).

$$\epsilon_0=e_0+\frac{\mathcal{G}_0}{4}(2\Omega_0+1)+\frac{g_0}{2}\Omega_1+\tfrac{1}{4}\mathcal{B}_0 \qquad v_1=(G_0-g_0-G_2)$$

$$\epsilon_1=e_1+\frac{G_0}{4}(2\Omega_1+1)+\frac{g_0}{2}\Omega_0+\tfrac{1}{4}B_0+\tfrac{5}{2}G_2 \qquad s_3=B_2-G_2$$

$$\eta_0=\tfrac{1}{2}(\mathcal{B}_0-b_0-\mathcal{G}_0+g_0) \qquad s=\tfrac{3}{8}(B_1-B_2)$$

$$\eta_1=\tfrac{1}{2}(B_0-b_0-G_0+g_0) \qquad g_5=B_3-B_2$$

$$\eta=\tfrac{1}{2}(b_0-g_0) \qquad g_6=B_2-G_2$$

$$v_0=(\mathcal{G}_0-g_0) \qquad g=\tfrac{1}{5}(B_1-B_3)$$

$$h_0=-\tfrac{1}{4}g_0\Omega(\Omega+2)-\tfrac{5}{2}\Omega_1G_2-\frac{(\mathcal{G}_0-g_0)}{4}\Omega_0(\Omega_0+2)-\frac{(G_0-g_0)}{4}\Omega_1(\Omega_1+2)$$

$$H^0_{\mathrm{FDSM}}=G_0S^{T\dagger}S^T+B_2P^2\cdot P^2\ . \tag{4.11}$$

This Hamiltonian is very similar to the pairing plus quadrupole model which we discuss in Sec. VII, except that the quadrupole operator P^2_μ used here is different. In this case,

$$H_6=\epsilon n-G_0n(n-1)/4+G_0[C_{\mathrm{SU}_2^T}-\Omega(\Omega+2)/4]+B_2C_{\mathrm{SU}_3}-(3B_2/8)\mathbf{L}^2\ , \tag{4.12}$$

$$H_8=\epsilon n-G_0n(n-1)/4+G_0[C_{\mathrm{SU}_2^T}-\Omega(\Omega+2)/4]+B_2C_{\mathrm{SO}_6}-B_2C_{\mathrm{SO}_5}\ , \tag{4.13}$$

where $\epsilon=G_0(2\Omega+1)/4$. We see clearly from the above two equations that the dominance of monopole pairing (obtained formally by setting $B_2=0$) will lead to the SU_2^T limiting symmetry which is known to have a vibrational spectrum. When the quadrupole-quadrupole interaction dominates (i.e., $G_0=0$), we obtain the SU_3 limit and SO_6 for the k-active and i-active schemes, respectively. The SU_3 limit is known to have the spectrum of an axially symmetric rotor, while the SO_6 limit corresponds to γ-soft rotational nuclei.[15,8]

Notice that the parameters G_λ and B_r are residual interaction strengths. If the model is to be self-consistent, these parameters may only have small systematic varia-

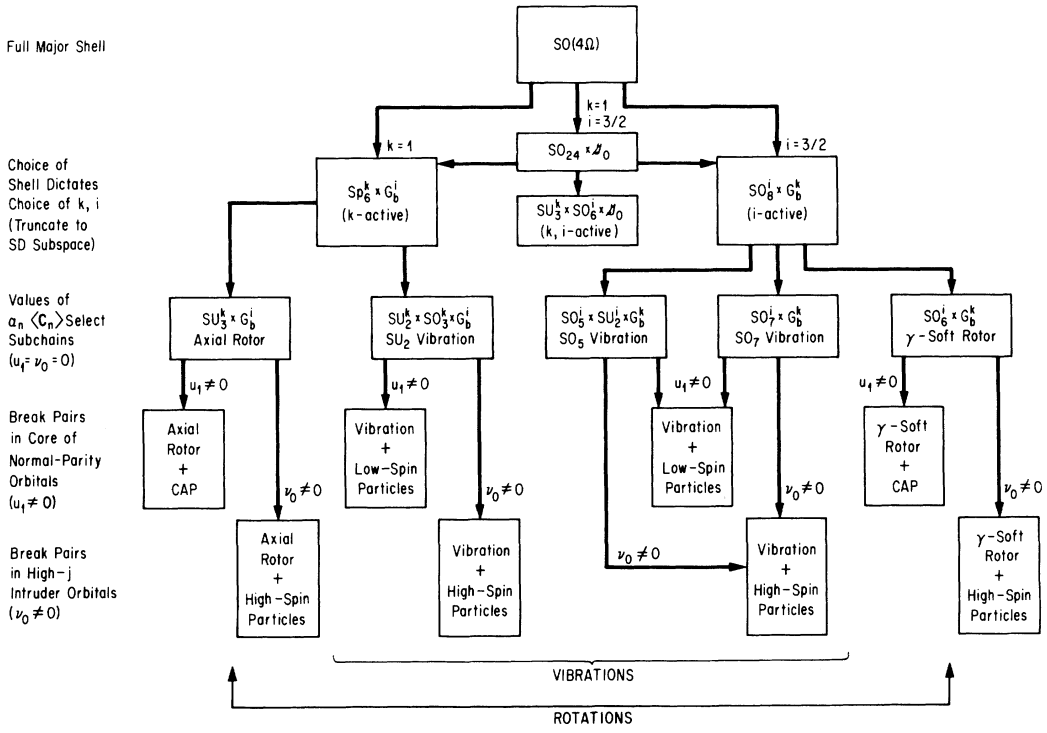

FIG. 4. The overall group structure of the FDSM. The symbols G^k_b and G^i_b denote the group structure associated with the abnormal orbitals ($\mathcal{SO}_3\times\mathcal{SU}_2$) and the angular momentum of decoupled particles (SO^k_3 or SO^i_3): $G^k_b=\mathrm{SO}^k_3\times\mathcal{SO}_3\times\mathcal{SU}_2$, $G^i_b=\mathrm{SO}^i_3\times\mathcal{SO}_3\times\mathcal{SU}_2$, and $\mathcal{G}_0=\mathcal{SO}_3\times\mathcal{SU}_2$. The designation CAP means Coriolis antipairing. The expectation values of Casimir operators are denoted by $\langle C_n\rangle$. α_n denotes the corresponding strength.

TABLE VIII. Additional parameters used in Eq. (4.10).

$\epsilon_1'=\epsilon_1+\frac{1}{4}(G_2-B_2)(2\Omega_1-1)$	$v_1'=(G_0-g_0-B_2)$
$\eta_1'=\eta_1+\frac{1}{2}(B_2-G_2)$	$g_5'=B_3-G_2$
$h_0'=h_0-\frac{G_2-B_2}{4}\Omega_1^2$	$g_7'=G_2-B_2$

tion over an entire series of nuclei; in other words, they must be nearly constants. Therefore the condition $G_0=0$ is actually a shorthand notation for the requirement

$$[G_0\langle S^{T\dagger}S^T\rangle]\ll[B_2\langle C_{\mathrm{SU}_3}\rangle] \tag{4.14}$$

or

$$[G_0\langle S^{T\dagger}S^T\rangle]\ll[B_2\langle C_{\mathrm{SO}_6}\rangle]\ . \tag{4.15}$$

Both of these equations imply the dominance of the long range quadrupole (Q-Q) interaction over the monopole pairing interaction. Physically, this is just the well-known condition for the occurrence of rotational spectra. Similarly, the condition $B_2=0$ implies a requirement just the opposite to that of (4.14) and (4.15), namely the dominance of the pairing interaction over the Q-Q interaction, which is the well-known condition for the appearance of vibrational spectra.

We now discuss the conditions for the occurrence of the SO_7 limit. From Eq. (4.10) and Tables VII and VIII, one sees that if $g_0=0$ and $G_0=B_2$, so that $v_1'=0$, the Hamiltonian becomes

$$H_{\mathrm{SO}_7}=H_0''+(B_3-G_2)C_{\mathrm{SO}_5}+(G_2-B_2)C_{\mathrm{SO}_7}+(B_1-B_3)/5\mathbf{L}^2\ , \tag{4.16}$$

where $H_0''=H_0'+\eta_2 C_{\mathrm{SO}_8}+g_0 C_{\mathcal{SU}_2}$.

From Table VI:

$$C_{\mathrm{SO}_7}=D^\dagger\cdot D+C_{\mathrm{SO}_5}+S_0(S_0-5)\ . \tag{4.17}$$

Using (4.17), Eq. (4.16) can be recast as

$$H_{\mathrm{SO}_7}=H_0'''+G_2'D^\dagger\cdot D+\sum_{r=1,3}B_r'P^r\cdot P^r\ , \tag{4.18}$$

where $H_0'''=H_0''+(G_2-B_2)S_0(S_0-5)$, $G_2'=G_2-B_2$, and $B_r'=B_r-B_2$. The term H_0''', in the absence of broken pairs, depends only on the particle number and therefore has no effect on the spectrum for a fixed number of valence particles. Thus the SO_7 mode corresponds to the excitation of the quadrupole pairs in the system.[14,15] The condition for this to occur is that the monopole pairing interaction between the normal and abnormal levels is weak ($g_0\langle S^\dagger\mathcal{S}+\mathcal{S}^\dagger S\rangle=0$), and that the monopole pairing strength for the normal-parity levels is equal to the quadrupole-quadrupole strength ($G_0=B_2$).

The zero heritage $\mathrm{SO}_8\times\mathcal{SU}_2$ symmetry limit of the FDSM is very similar to the SO_8 model which has been discussed extensively by Ginocchio.[15] In the subsequent sections, we shall mainly concentrate on the discussion of the rotational limit of the $\mathrm{Sp}_6\times\mathcal{SU}_2$ symmetry. Also, we shall compare the three "vibrational-like" limits, $\mathrm{SO}_7\times\mathcal{SU}_2$ and $\mathrm{SO}_5\times\mathrm{SU}_2\times\mathcal{SU}_2$ (from $\mathrm{SO}_8\times\mathcal{SU}_2$), and $\mathrm{SU}_2\times\mathrm{SO}_3\times\mathcal{SU}_2$ (from $\mathrm{Sp}_6\times\mathcal{SU}_2$). For the SO_6 limit of the $\mathrm{SO}_8\times\mathcal{SU}_2$ symmetry we shall only give a brief summary, since it is many ways analogous to the SU_3 limit of the $\mathrm{Sp}_6\times\mathcal{SU}_2$ symmetry, and has been discussed by Ginocchio.[15]

V. THE ROTATIONAL AND γ-SOFT LIMITS OF THE FDSM

A. The rotational limit

For heritage number $u=0$, the general FDSM Hamiltonian in the $\mathrm{SU}_3\times\mathcal{SU}_2$ limit (SU_3 for short from now on) which describes collective axially-symmetric rotational motion, is obtained formally from the Hamiltonian (4.7) by setting $g_0=0$ and $v_1=0$ (i.e., $G_0=G_2$, see Table VII):

$$H_{\mathrm{SU}_3}=H^0_{\mathrm{SU}_3}+\tfrac{3}{8}(B_1-B_2)\mathbf{L}^2-(G_2-B_2)C_{\mathrm{SU}_3}\ , \tag{5.1}$$

where

$$H^0_{\mathrm{SU}_3}=H_0+g_0C_{\mathcal{SU}_2}+G_2C_{\mathrm{Sp}_6} \tag{5.2}$$

and H_0 is defined in Eq. (4.7b).

This Hamiltonian is associated with the SU_3 dynamical chain of Fig. 3(a). We use

$$|N_0N_1(\lambda,\mu)KLM\rangle \tag{5.3}$$

to denote the irreducible basis of this chain. The meanings of the quantum numbers are as follows: $N_0=n_0/2$ and $N_1=n_1/2$ are the number of pairs in the abnormal-parity level ($k=0$) and the normal-parity levels ($k=1$), respectively, K is the Vergados quantum number;[20] and L,M are angular momentum quantum numbers. Using Eqs. (4.7b), (5.1), and (5.2) and Tables VI and VII, the eigenvalues of H_{SU_3} are

$$E[N_1(\lambda\mu)KL]=E_0(N_1)+\alpha L(L+1)-\beta C(\lambda,\mu)\ , \tag{5.4}$$

where

$$\alpha=(\tfrac{3}{8})(B_1-B_2),\quad \beta=\tfrac{1}{2}(G_2-B_2)\ , \tag{5.5}$$

$$C(\lambda,\mu)=2C_{\mathrm{SU}_3}(\lambda,\mu)=\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu\ , \tag{5.6}$$

$$E_0(N_1)=A_0N_1^2-2B_0N_1+C_0\ , \tag{5.7}$$

and

$$A_0=2(\eta_0+\eta_1)\ , \tag{5.8a}$$

$$B_0=\Delta\epsilon-(\eta_0-\eta_1)/2+2\eta_0N\ , \tag{5.8b}$$

$$C_0=\epsilon_0n+(\eta+\eta_1)n(n-1)/2\ , \tag{5.8c}$$

where $\Delta\epsilon=\epsilon_0-\epsilon_1$, and $N=N_0+N_1=n/2$. In Eq. (5.4) as well as for subsequent discussions in this section, the notation $C_g(\kappa)$ means the eigenvalue with quantum number (or numbers) κ of the Casimir operator of the group g (see Table VI). The parameters ϵ_0, ϵ_1, η_0, η_1, and η are defined in Table VII.

For each N_1, the allowed values of (λ,μ) in (5.4) are

$$(\lambda,\mu)=(2N_1,0),(2N_1-4,2),\ldots,(0,N_1)\ \ [\text{or}\ (2,N_1-1)]\ ,$$
$$(2N_1-6,0),(2N_1-10,2),\ldots,(0,N_1-3)\ \ [\text{or}\ (2,N_1-4)]\ ,$$
$$(2N_1-12,0),(2N_1-16,2),\ldots,(0,N_1-6)\ \ [\text{or}\ (2,N_1-7)],\ \ldots\ . \tag{5.9}$$

Let

$$K_0=\min(\lambda,\mu),\quad K_m=\lambda+\mu\ , \tag{5.10a}$$
$$K=K_0,K_0-2,K_0-4,\ldots,0\ \text{or}\ 1\ . \tag{5.10b}$$

The energy levels are grouped into rotational bands

$$L=\begin{cases}K_m,K_m-2,K_m-4,..,0\ \text{or}\ 1 & \text{if } K=0\ ,\\ K,K+1,K+2,\ldots,K_m+1-K, & \text{if } K\neq 0\ .\end{cases} \tag{5.11}$$

The ground state band of a strongly deformed nuclei belongs to the irrep$(2N_1,0)$. On the other hand, the β and γ bands belong to the $(2N_1-4,2)$ representation. This is the familiar situation for the IBM SU_3 limit.

In fact, Eq. (5.4) is formally identical to the energy formula for the interacting boson model (IBM) in the rotational limit [cf. Eq. (2.8) of Ref. 10]. It can be shown[17] that the γ-transition rates in the SU_3 limit are also identical to those of the IBM. However, there exist some differences in details. In the IBM, the boson number N is assumed to be one-half of all the valence nucleons, whereas in the FDSM the SU_3 representation is determined solely by N_1, the pair number in the normal parity levels. Furthermore, the parameters α and β in the FDSM are now directly related to the parameters of the effective nucleon-nucleon interaction. We expect that usually the quadrupole-quadrupole interaction is attractive ($B_2<0$), and larger in magnitude than the dipole-dipole and quadrupole pairing strengths ($|B_2|>|B_1|$, $|B_2|>|G_2|$). Thus from Eq. (5.5) we can immediately obtain the correct sign for α and β, i.e., α and $\beta>0$. As a purely phenomenological model, the IBM is not able, *a priori,* to determine the signs of these parameters.

It should be pointed out that, due to the Pauli restriction, the reduction rule of Eqs. (5.9)–(5.11) is valid only for $N_1\leq\Omega_1/3$. When $N_1>\Omega_1/3$, the irrep(λ,μ) with $\lambda+\mu>2\Omega_1/3$ is forbidden. This is different from the IBM and has been termed the "fatal-flaw" of the Ginocchio Sp_6 model. It has been discussed in Ref. 15 and reiterated in Ref. 4, and which caused Ginocchio to reject the Sp_6 model as unphysical.

We shall discuss why the "fatal flaw" is *not* a problem in the $Sp_6\times SU_2$ symmetry of the FDSM. For the Ginocchio Sp_6 model, the range of $(\lambda\mu)$ is also given by (5.9) but with the significant difference that N_1 is replaced by N (total number of valence pairs). When $N>\Omega/3$, the ground band SU_3 irrep$(2N,0)$ is forbidden! This is seemingly at odds with the empirical fact that many strongly-deformed nuclei are at or near midshell ($N\sim\Omega/2$). In the $Sp_6\times\mathcal{SU}_2$ symmetry of the FDSM, for a nucleus with N_1 (N_0) valence nucleon pairs in the normal (abnormal) parity orbits, the highest SU_3 irrep$(2N_1,0)$ is also not allowed when $N_1>\Omega_1/3$ (instead of $\Omega/3$). On the other hand, since $N=N_0+N_1$, even when $N\sim\Omega/2$, it is still possible that $N_1\leq\Omega_1/3$, thus realizing a $(2N_1,0)$ ground band. Physically, this means that the abnormal parity orbitals can serve as a "sink" to absorb the nucleons driven by the Pauli principle from the normal orbitals. Hence we see that it is the $\mathcal{SU}_2$ of the abnormal-parity level in the FDSM which plays the crucial role in reviving the Ginocchio Sp_6 model.

The values of N_1 and N_0, i.e., the distribution of the valence nucleon pairs in the normal and abnormal levels in the ground state of a nucleus, can be roughly estimated from the deformation of the nucleus and the Nilsson diagrams. Such an empirical analysis has been undertaken[21] and the result shows that for most deformed nuclei, N_1 is indeed less than or equal to $\Omega_1/3$ (see Table IX).

The value of N_1 can be estimated theoretically by minimizing the ground energy $E_{\text{g.s.}}(N_1)$ of Eq. (5.4) as

$$\frac{\partial}{\partial N_1}E_{\text{g.s.}}(N_1)\bigg|_{N_1=N_{1g}}=0 \tag{5.12a}$$

and

$$E_{\text{g.s.}}(N_1)=E_0(N_1)-\beta C(2N_1,0)\ . \tag{5.12b}$$

Thus we obtain

$$N_{1g}=\frac{B_0+3\beta}{A_0-4\beta}\ . \tag{5.13}$$

Inserting Eq. (5.8) into Eq. (5.13), we obtain

$$N_{1g}=a+bN\ , \tag{5.14}$$

where a and b are constants depending on interactions and the separation between normal and abnormal level single-particle energies

$$a=\frac{2\Delta\epsilon+\eta_1-\eta_0+6\beta}{4(\eta_1+\eta_0)-8\beta},\quad b=\frac{2\eta_0}{2(\eta_1+\eta_0)-4\beta}\ . \tag{5.15}$$

The excitation energy ΔE of the states with $N_1=N_{1g}\pm1$ (with $N=N_0+N_1$ fixed) can be calculated from (5.4) and (5.13) to be

$$\Delta E=A_0-4\beta=2(\eta_1+\eta_0)-4\beta\ . \tag{5.16}$$

Such an excitation corresponds physically to a redistribution of pairs between normal and abnormal parity orbitals.

B. The γ-soft limit

The $SO_6\times\mathcal{SU}_2$ limit (SO_6 for short) of the FDSM which describes the γ-soft behavior of nuclei is obtained from the $SO_8\times\mathcal{SU}_2$ Hamiltonian (4.8). The physical conditions for the occurrence of this symmetry are similar to those of the SU_3 symmetry, except that it can only appear in SO_8 shells (see Table IV). The SO_6 energy formula for the $u=0$ case is

TABLE IX. Estimated distributions. The symbols N_0^π (N_0^ν) and N_1^π (N_1^ν) are the numbers of proton (neutron) pairs which occupy the abnormal and normal levels, respectively, and n^π (n^ν) is the total number of protons (neutrons). For the actinides, the protons and neutrons occupy shells no. 7 and 8, respectively, while for the rare earths, neutrons are in shell 7. The estimate of occupancy is based on Nilsson levels with deformation $\epsilon \sim 0.3$.

Estimated proton distribution of the actinide nuclei

n^π	84	86	88	90	92	94	96	98	100
N_0^π	0	1	1	2	2	3	3	3	4
N_1^π	1	1	2	2	3	3	4	5	5
$\Omega_1^\pi/3$	5	5	5	5	5	5	5	5	5

Estimated neutron distribution of the actinide nuclei

n^ν	128	130	132	134	136	138	140	142	144	146	148
N_0^ν	0	0	0	1	2	2	3	3	4	4	4
N_1^ν	1	2	3	3	3	4	4	5	5	6	7
$\Omega_1^\nu/3$	7	7	7	7	7	7	7	7	7	7	7

Estimated neutron distribution of rare-earth nuclei

n^ν	84	86	88	90	92	94	96	98	100
N_0^ν	0	0	1	1	2	3	3	3	4
N_1^ν	1	2	2	3	3	3	4	5	5
$\Omega_1^\nu/3$	5	5	5	5	5	5	5	5	5

$$E[N_1(\sigma\tau)n_\Delta L]=E_0(N_1)-A\sigma(\sigma+4)+B\tau(\tau+3)+CL(L+1)\ , \tag{5.17a}$$

where

$$A=G_2-B_2,\quad B=B_3-B_2,\quad C=(B_1-B_3)/5\ . \tag{5.17b}$$

In Eq. (5.17a), σ and τ are quantum numbers to denote the irreps of the $SO_6 \supset SO_5$ chain [see Table VI and Fig. 3(b)], while n_Δ is an additional quantum number. For given N_1 ($\leq \Omega_1/2$) the allowed values of σ, τ, n_Δ, and L are

$$\sigma=N_1,N_1-2,N_1-4,\ldots,0 \text{ or } 1\ , \tag{5.18a}$$

$$\tau=3n_\Delta+\lambda=\sigma,\sigma-1,\sigma-2,\ldots,0\ , \tag{5.18b}$$

$$L=\lambda,\lambda+1,\ldots,2\lambda-2,2\lambda\ . \tag{5.18c}$$

The N_1 value can also be determined by minimizing the ground-state energy [Eq. (5.12a)] and can be expressed using Eq. (5.14) with

$$E_{\text{g.s.}}(N_1)=E_0(N_1)-AN_1(N_1+4) \tag{5.19}$$

and

$$a=\frac{2\Delta\epsilon+\eta_1-\eta_0+4A}{4(\eta_1+\eta_0)-2A},\quad b=\frac{2\eta_0}{2(\eta_1+\eta_0)-A}\ . \tag{5.20}$$

Just as for the SU_3 case, both the spectra and γ-transition rates are identical to those of the IBM's O_6 symmetry,[17] if the boson number is taken as N_1. The geometrical analog of the SO_6 limit is a γ-unstable particle-rotor model.

It is interesting to note that, according to Eq. (5.17b), the parameters A and B in the SO_6 energy formula should have roughly the same magnitude and positive sign since the quadrupole-quadrupole interaction is expected to be dominant and attractive (i.e., $|B_2| \gg |B_1|$, $|B_2| \gg |G_2|$, and $B_2<0$). The condition $A=B$ is well known in IBM phenomenology. The FDSM provides an immediate microscopic justification.[22]

C. The high-spin limit

The S and D pairs alone are inadequate to study high-spin phenomena since broken pairs ($u_1 \neq 0$ and/or $v_0 \neq 0$) play an important role. By including nonzero heritage states, however, the FDSM can take into account such effects. This has been discussed in Ref. 23, and will be considered in detail in the next paper of this series. Here we present only a brief discussion of high-spin physics in the SU_3 limit of the FDSM. The Hamiltonian in this case can be obtained from Eq. (3.23) and can be rewritten as

$$H_{\text{FDSM}}=H^0_{\text{FDSM}}+H_{\text{bro}}+H_{\text{coup}}\ , \tag{5.21}$$

where

$$H_{\text{bro}}=\sum_{ii'\,(r=\text{odd})} B_r^{ii'}P^r(i)\cdot P^r(i') + \sum_{i\,(r=\text{odd})} 2B_r^{ia}P^r(i)\cdot\mathcal{P}^r+\sum_{(r=\text{odd})} B_r^{aa}\mathcal{P}^r\cdot\mathcal{P}^r\ , \tag{5.22}$$

$$H_{\text{coup}}=\sum_{i'} B_1^{1i'}P^1\cdot P^1(i')+2B_1^{1a}P^1\cdot\mathcal{P}^1\ . \tag{5.23}$$

The term H^0_{FDSM} is the Hamiltonian without broken pairs defined in Eq. (4.1). H_{bro} is the interaction between the unpaired nucleons (those which do not form coherent S and D pairs), and H_{coup} is the coupling term between the S and D pair "core" and the unpaired nucleons. If we only retain the lowest order $r=1$ in Eq. (5.22), by using Eq. (4.6) and neglecting the i dependence of all the parameters Eq. (5.22) and (5.23) are simplified:

$$H_{\text{bro}}+H_{\text{coup}}=\gamma I^2+\delta\mathbf{R}\cdot\mathbf{I}\ , \tag{5.24}$$

where I is the total pseudospin of the broken pairs (including the broken pairs in the abnormal level),

$$I=\Sigma I(i)+I(j_0)\ , \tag{5.25a}$$

$$I_\mu(i)=\sqrt{(4/3)i(i+1)}P^1_\mu(i)\ ,$$
$$I_\mu(j_0)=\sqrt{(4/3)j_0(j_0+1)}\mathcal{P}^1_\mu\ , \tag{5.25b}$$

and

$$\mathbf{R}=\sqrt{8/3}P^1 \tag{5.26}$$

is the total pseudoorbital angular momentum which we shall refer to as the core angular momentum. This interpretation follows from comparing Eq. (5.1) to the spectrum of a rigid rotor. (Note: to better conform to the notation prevalent in high spin physics, we have replaced **L** in Eq. (5.1) by **R** in this section.)

With these simplifications, the FDSM Hamiltonian in the SU_3 limit becomes

$$H_{\rm rot}=H^0_{\rm rot}+\alpha\mathbf{R}^2+\gamma I^2+\delta\mathbf{R}\cdot\mathbf{I}\ , \tag{5.27}$$

where

$$H^0_{\rm rot}=E_{\rm g.s.}(N_1)+\Delta_{SU_3}(\lambda,\mu)+\Delta_{Sp_6}(u_1)+\Delta_{\mathcal{SU}_2}(\nu_0)\ , \tag{5.28}$$

and

$$E_{\rm g.s.}(N_1)=E_0(N_1)-\beta C_{SU_3}(2N_1,0)\ , \tag{5.29}$$

$$\Delta_{SU_3}(\lambda,\mu)=\beta[C(2N_1,0)-C(\lambda,\mu)]\ , \tag{5.30}$$

$$\Delta_{Sp_6}(u_1)=G_2[C_{Sp_6}(u_1)-C_{SP_6}(0)]\ , \tag{5.31}$$

$$\Delta_{\mathcal{SU}_2}(\nu_0)=G_0[C_{\mathcal{SU}_2}(\nu_0)-C_{\mathcal{SU}_2}(0)]\ . \tag{5.32}$$

In the above expressions, $E_{\rm g.s.}(N_1)$ is the ground state energy, $\Delta_{SU_3}(\lambda,\mu)$ is the band-head excitation energy for the SU_3 representations (λ,μ), $\Delta_{Sp_6}(u_1)$ is the energy required to break $u_1/2$ S or D pairs of nucleons in the normal parity levels, and $\Delta_{\mathcal{SU}_2}(\nu_0)$ is the energy required to break $\nu_0/2S$ pairs of nucleons in the abnormal-parity level. The Hamiltonian of Eq. (5.27) has the following dynamical symmetry:

$$\underset{u_1}{(Sp_6}\supset\underset{n_1}{U_1}\times\underset{\lambda,\mu}{SU_3})\times(\underset{\nu_0}{\mathcal{SU}_2}\supset\underset{n_0}{\mathcal{U}_1}\times\underset{I_0}{\mathcal{SO}_3})\times\underset{I_1}{SO^i_3}\supset\underset{R}{SO^R_3}\times\underset{I}{SO^I_3}\supset\underset{J}{SO^J_3}\ . \tag{5.33}$$

Using this symmetry, one can obtain *analytical solutions* which contain most of the important physical features of high-spin physics. These include rotational alignment, Coriolis antipairing, multiple band crossing, and the associated backbendings.[23] That this is so should not be surprising, since Eq. (5.27) is similar to the particle-rotor model.[24] However, the FDSM Hamiltonian (5.27) is fully microscopic, and has been derived without the explicit use of macroscopic concepts such as deformation. It is the S and D pairs in the normal-parity levels which play the role of a deformed core. This suggests that the basic concepts of the geometrical models (e.g., deformation) may be derived via the FDSM from the spherical shell model. It should be noted that the quadrupole coupling between the "core" and the decoupled nucleons $[P^2(k=1)\cdot P^2(i)]$ is beyond the dynamical symmetry of Eq. (5.27). Its effect can be incorporated as a symmetry-breaking term. Clearly, the treatment we have outlined here for the broken pairs can be applied equally well in other symmetry limits (e.g., SO_6) of the FDSM.

VI. THE THREE VIBRATIONAL LIMITS OF THE FDSM

There are three vibrational limits within the FDSM. They are the SU_2 limit of the $SP_6\times\mathcal{SU}_2$ symmetry and the $SO_5\times SU_2$ and SO_7 limits of the $SO_8\times\mathcal{SU}_2$ symmetry. We now discuss these cases.

A. The SU_2 limit

The SU_2 limit stands for the $SU_2\times SO_3\times\mathcal{SU}_2$ dynamical symmetry chain shown in Fig. 3. The general Hamiltonian follows for this limit by setting $s_3=0$ (i.e., $B_2=G_2$) in Eq. (4.7). As we have previously mentioned, this physically corresponds to the dominance of pairing. Analytical solutions result for the two following situations (see Table VII and Fig. 3).

(i) For $g_0=0$, then

$$H_{SU_2}=H^0_{\rm vib}+(G_0-G_2)C_{SU_2}+\tfrac{3}{8}(B_1-B_2)\mathbf{L}^2\ , \tag{6.1a}$$

$$H^0_{\rm vib}=H_0+\mathcal{G}_0C_{\mathcal{SU}_2}+G_2C_{Sp_6}\ , \tag{6.1b}$$

where H_0 is defined in Eq. (4.7b).

(ii) For $\epsilon_0=\epsilon_1$ and $\eta_0=\eta_1=0$,

$$H_{SU_2}=H^0_{\rm vib}+v_1C_{SU_2}+g_0C_{SU^T_2}+\tfrac{3}{8}(B_1-B_2)\mathbf{L}^2\ , \tag{6.2a}$$

$$H^0_{\rm vib}=h_0+\epsilon_0n+\eta n(n-1)/2+v_0C_{\mathcal{SU}_2}+G_2C_{Sp_6}\ . \tag{6.2b}$$

The difference between (i) and (ii) is the following: In (i), n_0 and n_1 (the pair number of nucleons in the abnormal-parity level and normal-parity levels, respectively) are both good quantum numbers, while in (ii) neither are good quantum numbers (the sum $n=n_0+n_1$ is a good quantum number in both cases). From Eq. (4.1d), the requirement $g_0=0$ is a short hand notation for $g_0\langle S^\dagger\mathcal{S}+\mathcal{S}^\dagger S\rangle\sim 0$. Thus, situation (i) corresponds to a neglected of pair scattering between normal and abnormal-parity orbitals. This is justified physically when there is a large energy gap between these orbitals. Situation (ii) is the opposite of (i), since in this case the normal- and abnormal-parity levels are assumed degenerate. The eigenvectors of the Hamiltonian in the SU_2

limit are as follows:

$$|u_1(\nu_0 I)N_0 N_1(\nu_1\sigma_1 L);JM\rangle \quad \text{[case (i)]}\ , \tag{6.3a}$$

$$|u_1(\nu_0 I)N\sigma_N(\nu_1\sigma_1 L);\nu JM\rangle \quad \text{[case (ii)]}\ , \tag{6.3b}$$

where N is the total number of nucleon pairs, ν $(=\nu_0+\nu_1)$ is the seniority of SU_2^T, and σ_N and σ_1 stand for additional quantum numbers. In the $(S^\dagger, D^\dagger, \mathcal{S}^\dagger)$ subspace, we have $u_1=\nu_0=0$, $I=0$, $L=J$, and $\nu=\nu_1$. Let

$$\nu_1=2\kappa\ . \tag{6.4}$$

In this case, the labelings (6.3) can be shortened as $|N_0N_1(\kappa\sigma_1);LM\rangle$ for case (i) and $|N\sigma_N(\kappa\sigma_1);LM\rangle$ for case (ii). In both cases, the expression for the energies is given in Table X with the parameters listed in the column labeled $SU_2\times SO_3$ limit. The formula shown at the top of Table X is, in fact, the IBM formula for U_5 limit but now with a microscopic fermion basis, the d-boson number N_d should be replaced by one-half of the seniority κ [see Eq. (6.4)]. The ground state energy in this limit is

$$E_{\text{g.s.}}(N_1)=E_0(N_1)=A_0N_1^2-2B_0N_1+C_0 \tag{6.5}$$

which is the same as that defined in Eq. (5.7) and (5.8). For case (ii), $E_{\text{g.s.}}$ is just a function of N since $\epsilon_0=\epsilon_1$ and $\eta_0=\eta_1=0$ (and therefore $A_0=B_0=0$). For (i), similar to the SU_3 case, the N_1 value of the ground state N_{1g} is determined by minimizing the ground state energy and has the same form as Eq. (5.14), i.e.,

$$N_{1g}=a_{\text{vib}}+b_{\text{vib}}N \tag{6.6a}$$

with

$$a_{\text{vib}}=\frac{2\Delta\epsilon+\eta_1-\eta_0}{4(\eta_1+\eta_0)}\ ,\quad b_{\text{vib}}=\frac{\eta_0}{(\eta_1+\eta_0)}\ . \tag{6.6b}$$

Some $B(E2)$ formulas for this limit are given in Table XI.

The allowed angular momenta for a given SU_2 seniority can be decided in the following way. Here we take case (a) as an illustrative example and consider only the situation where $N_1\le\Omega_1/3$. For this case, there is a one-to-one correspondence between the IBM and the FDSM basis[17] which can be understood as follows:

$$\begin{aligned}&(\text{FDSM})\quad |\mathrm{Sp}_6\supset \mathrm{SU}_2\times\mathrm{SO}_3\rangle = M_1\,|\mathrm{Sp}_6\supset\mathrm{SU}_3\supset\mathrm{SO}_3\rangle\\ &\qquad\qquad\qquad\qquad\uparrow\downarrow\ \text{(one to one)}\\ &(\text{IBM})\quad |\mathrm{U}_6\supset\mathrm{U}_5\supset\mathrm{SO}_5\supset\mathrm{SO}_3\rangle = M_2\,|\mathrm{U}_6\supset\mathrm{SU}_3\supset\mathrm{SO}_3\rangle\ ,\end{aligned} \tag{6.7}$$

where M_1 and M_2 are unitary matrices. Therefore, the listing of the possible angular momenta for the $Sp_6\supset SU_2\times SO_3$ limit can be taken over from that for the IBM U_5 limit. More precisely, suppose that

$$f_{N_D(\tau n_\Delta)LM}(s^\dagger,d_\mu^\dagger)\,|\,0\rangle$$

is an irreducible basis of the IBM $U_5\supset SO_5\supset SO_3$, where τ is the SO_5 quantum number, and n_Δ counts the triplets of d bosons coupled to angular momentum zero. Then

$$f_{N_D(\tau n_\Delta)LM}(S^\dagger,D_\mu^\dagger)\,|\,0\rangle\ ,$$

where N_D is the number of D pairs, will be a basis for the Sp_6 FDSM which has the same angular momentum content as the basis $|\mathrm{Sp}_6\supset\mathrm{SU}_2\times\mathrm{SO}_3\rangle$. However, it is not an $Sp_6\supset SU_2\times SO_3$ irreducible basis, since it does not have a definite SU_2 seniority. This is due to the failure of $Sf(D_\mu)\,|\,0\rangle$ to vanish, where $f(D_\mu)$ is a function of D_μ. We can introduce "dressed" D pairs denoted as D' by requiring $Sf(D'^\dagger_\mu)\,|\,0\rangle=0$. It can be shown that

$$\begin{aligned}D'^\dagger_\mu \equiv D^\dagger_\mu &+2S^\dagger(\Omega_1-n_1)^{-1}P_\mu-(S^\dagger)^2(\Omega_1-n_1-1)^{-1}\\ &\times(\Omega_1-n_1-2)^{-1}(-)^\mu D_{-\mu}\ ,\end{aligned} \tag{6.8}$$

which satisfies the following commutation relationship with S:

$$\begin{aligned}[S,D'^\dagger_\mu]=&-(S^\dagger)^2(\Omega_1-n_1-3)^{-1}\\ &\times(\Omega_1-n_1-4)^{-1}(-)^\mu D_{-\mu}S\ .\end{aligned} \tag{6.9}$$

Now the basis

$$f_{N_D(\tau n_\Delta)LM}(S^\dagger,D'^\dagger_\mu)\,|\,0\rangle$$

has definite SU_2 seniority $\nu=2\kappa$, but not definite D pair number. Instead, it is a mixture of states with different N_D and

$$\kappa=(N_D)_{\max}\ . \tag{6.10}$$

By analogy to the IBM U_5 limit,[9] the angular momentum range in the SU_2 limit for given $N_1\le\Omega_1/3$ is

TABLE X. Energy formulas for the vibrational limits. The following energy formula,

$$E=E_{\text{g.s.}}+\epsilon N_d+\alpha N_d(N_d-1)/2+\beta(N_d-\tau)(N_d+\tau+3)+\gamma(L(L+1)-6N_d)\ ,$$

is for the IBM U_5 limit. For the SO_7 limit, N_d should be replaced by the quantum number $\bar\kappa=N_1-w/2$, where w is the number of nucleons which do not form D pairs. For the $SU_2\times SO_3$ and $SO_5\times SU_2$ limits, N_d should be replaced by κ $(=\nu/2)$, where ν is the SU_2 seniority.

Parameter	$SU_2\times SO_3$ limit	$SO_5\times SU_2$ limit	SO_7 limit
ϵ	$\Omega_1(G_2-G_0)-\Omega_0 g_0$ $+9(B_1-B_2)/4$	$\Omega_1(G_2-G_0)-\Omega_0 g_0$ $+6B_1/5+14B_3/5-4B_2$	$[\Omega_1(G_2-G_0)-4B_2$ $+6B_1/5+14B_3/5]-2(G_2-G_0)(N_1-1)$
α	$2(G_0-G_2)$	$2(G_0-G_2+B_3-B_2)$	$2(B_3-B_2)$
β	0	B_2-B_3	G_2-B_3
γ	$\frac{3}{8}(B_1-B_2)$	$\frac{1}{5}(B_1-B_3)$	$\frac{1}{5}(B_1-B_3)$

TABLE XI. Some vibrational $B(E2)$ formulas. $B_L \equiv B[E2, L_1 \rightarrow (L_1-2)]$; $B_{2'} \equiv B(E2, 2_2 \rightarrow 2_1)$; $B_{0'} \equiv B(E2, 0_2 \rightarrow 2_1)$. The quadrupole transition operator is qP^2 for the FDSM and $q(d^{\dagger}s+s^{\dagger}d)$ for the IBM U_5 limit.

$B(E2)$	U_5	$SO_5 \times SU_2$	$SU_2 \times SO_3$	SO_7
B_2/q^2	N	$N_1 \frac{\Omega_1-N_1}{\Omega_1-1}$	$N_1 \frac{\Omega_1-N_1}{\Omega_1-1}$	$N_1 \frac{\Omega_1+6-N_1}{\Omega_1+7-2N_1}$
B_4/q^2	$2(N-1)$	$2(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-3}$	$2(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-2}$	$2(N_1-1)\frac{\Omega_1+7-N_1}{\Omega_1+9-2N_1}$
$B_{2'}/q^2$	$2(N-1)$	$2(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-3}$	$(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-2}\frac{2\Omega_1+3}{\Omega_1-2}$	$2(N_1-1)\frac{\Omega_1+7-N_1}{\Omega_1+9-2N_1}$
$B_{0'}/q^2$	$2(N-1)$	$2(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-3}\frac{\Omega_1+4}{\Omega_1-1}$	$(N_1-1)\frac{\Omega_1-1-N_1}{\Omega_1-2}\frac{2\Omega_1+3}{\Omega_1-1}$	$2(N_1-1)\frac{\Omega_1+2-2N_1}{\Omega_1+7-2N_1}\frac{\Omega_1+7-N_1}{\Omega_1+9-2N_1}$

$$\kappa = N_1, N_1-1, \ldots, 0 , \tag{6.11a}$$

$$\tau = \kappa, \kappa-2, \ldots, \ 1 \text{ or } 0 , \tag{6.11b}$$

$$\lambda = \tau - 3n_{\Delta} , \quad n_{\Delta} = 0, 1, 2, \ldots , \tag{6.11c}$$

$$L = \lambda, \lambda+1, \ldots, 2\lambda-2, 2\lambda . \tag{6.11d}$$

The quantum numbers $(\tau, n_{\Delta}, \lambda)$ are now the additional quantum numbers σ_1 of Eq. (6.3). It should be emphasized that for the SU_2 vibrator, τ is merely an additional quantum number rather than the SO_5 seniority (because Sp_6 has no SO_5 subgroup). This reduction rule may also be applied to case (ii) by replacing N_1 by N in (6.11a) since

$$| N\sigma_N(\kappa\sigma_1); LM \rangle = \sum_{N_1} C_{N_1}^{\sigma_N} | N_0 N_1(\kappa\sigma_1); LM \rangle . \tag{6.12}$$

When $N_1 > \Omega_1/3$, by arguments similar to the SU_3 case discussed in Sec. V, some high-seniority states (i.e., $\kappa > \Omega_1/3$) are forbidden. Nevertheless, the low-lying states have low seniorities which satisfy the restriction $\kappa \leq \Omega_1/3$, and for those states the reduction rule of Eq. (6.11) is valid.

B. The $SO_5 \times SU_2$ limit

The $SO_5 \times SU_2$ limit stands for the $SO_5 \times SU_2 \times \mathcal{SU}_2$ dynamical symmetry chain in Fig. 3(b). By setting $B_2 = G_2$ in (4.8) we also obtain a vibrational limit (see Table VII). As for the SU_2 vibration, two cases may be distinguished (see Fig. 3(b)]:

(i) For $g_0 = 0$, then

$$H_{SO_5} = H^0_{vib} + (G_0 - G_2) C_{SU_2} + (B_3 - B_2) C_{SO_5} + \tfrac{1}{5}(B_1 - B_3)\mathbf{L}^2 , \tag{6.13a}$$

$$H^0_{vib} = H^0 + \mathcal{G}_0 C_{\mathcal{SU}_2} + G_2 C_{SO_8} . \tag{6.13b}$$

(ii) For $\epsilon_0 = \epsilon_1$ and $\eta_0 = \eta_1 = 0$,

$$H_{SO_5} = H^0_{vib} + v_1 C_{SU_2} + g_0 C_{SU_2^T} + (B_3 - B_2) C_{SO_5} + \tfrac{1}{5}(B_1 - B_3)\mathbf{L}^2 , \tag{6.14a}$$

$$H^0_{vib} = h_0 + \epsilon_0 n + \eta n(n-1)/2 + v_0 C_{\mathcal{SU}_2} + G_2 C_{SO_8} . \tag{6.14b}$$

Thus, this Hamiltonian is similar to Eqs. (6.1) and (6.2), but with an additional term proportional to C_{SO_5}. The energy formula for this limit is given in Table X with the parameters given in the third column and the d boson number N_d replaced by κ. The reduction rule is as that in the SU_2 limit [Eq. (6.1)] except now there is no restriction $\kappa \leq \Omega_1/3$. The κ value can go up to $\Omega_1/2$. The ground-state energy and the value of N_1 can be obtained as before by Eqs. (6.5) and (6.6). Some $B(E2)$ formulas are given in Table XI for this limit.

C. The SO_7 limit

The SO_7 limit stands for the $SO_7 \times \mathcal{SU}_2$ dynamical symmetry shown in Fig. 3(b). The general Hamiltonian is obtained by setting $g_0 = 0$ and $G_0 = B_2$. This condition is different from the previous two vibrational limits where the pairing interaction is always dominant and g_0 need not be zero. Here, the strengths of the monopole pairing and quadrupole-quadrupole interactions must be identical and the pairing interaction between the normal and abnormal parity levels must be weak.

The Hamiltonian in this limit is given by Eq. (4.16). For the zero heritage case, the irrep of SO_7 is classified by the quantum numbers $[(\Omega_1 - w)/2, 0, 0]$ (see Table VI), where w is the number of nucleons which do not form D pairs. For the ground state, $w = 2N_1$, which means no D pairs, i.e., $D_{\mu} | SO_7, \text{ g.s.} \rangle = 0$ [note $D_{\mu}(S^{\dagger})^{N_i} | 0 \rangle \neq 0$]. Defining $\bar{\kappa} = N_1 - w/2$, the eigenvectors can be expressed as

$$| N_0 N_1(\bar{\kappa}\sigma_1); Lm \rangle ,$$

where σ_1 stands for $(\tau, n_{\Delta}, \lambda)$ as before. The energy formula is

$$E_{SO_7} = E_{\text{g.s.}} + (G_2 - B_2)\bar{\kappa}(\Omega_1 - 2N_1 + \bar{\kappa} + 5) + (B_3 - G_2)\tau(\tau+3) + \tfrac{1}{5}(B_1 - B_3)\mathbf{L}^2 . \tag{6.15}$$

The reduction rule is the same as that in the $SO_5 \times SU_2$ case [Eq. (6.11)], except κ now is replaced by $\bar{\kappa}$. Equation (6.15) can also be written in the same form as the

energy formula of Table X with the parameters given in the fourth column, N_d replaced by $\bar{\kappa}$ and

$$E_{\text{g.s.}} = E_0(N_1) - 5(G_2 - B_2)N_1 \ . \tag{6.16}$$

The N_1 value can also be obtained from Eq. (6.6) except the constant a_{vib} is different:

$$a_{\text{vib}} = \frac{2\Delta\epsilon + \eta_1 - \eta_0 - 5(G_2 - B_2)}{4(\eta_1 + \eta_0)} \ . \tag{6.17}$$

The $B(E2)$ values for some low-lying transitions in the SO_7 limit are shown in Table XI.

D. Distinguishing features of the three fermion vibrational limits and the IBM U_5 limit

As we have seen, all three fermion vibrational limits can be described by an energy formula (which is formally equivalent to that of the IBM U_5 limit). However, the fermion and boson dynamical symmetries differ fundamentally due to the Pauli principle. Firstly, some of the allowed boson states are forbidden in the FDSM. Secondly, the electromagnetic transition rates for the three fermion vibrational limits differ from the IBM U_5 limit by Pauli correction factors (see Table XI). Furthermore, the Pauli factors for all three limits are different and approach unity when the normal-parity shell degeneracy $\Omega_1 \to \infty$. It will be interesting to see whether these limits can be experimentally distinguished. Obviously, finding evidence for any of the three fermion vibrational limits will be independent evidence for the dynamical symmetries discussed in this paper.

Among these three vibrational limits the SO_7 limit is the most interesting since there is no counterpart in the IBM. The conditions for the realization of the SO_7 symmetry require that monopole pairing between normal- and abnormal-parity levels be suppressed ($g_0 \langle \mathcal{S}^\dagger S + S^\dagger \mathcal{S} \rangle \approx 0$). The physical realization of this condition is most likely for medium-mass nuclei where the spin-orbit interaction has not pushed the abnormal-parity orbital deeply into the lower shell. The signatures for SO_7 are that (1) it occurs in SO_8 shells, (2) it has a vibrational spectrum which is compressed linearly in N_1 (note: N_1 is proportional to N), and (3) there are some crucial $B(E\lambda)$ ratios which distinguish it from the SO_5 and SO_6 limits. A recent survey of systematic energies and transition probabilities in Ru and Pd isotopes suggests the occurrence of SO_7 symmetry. Details of this study have been published separately.[25]

VII. EFFECTIVE INTERACTIONS FOR THE FDSM

It is well known that some *truncation* procedure is imperative in microscopic nuclear structure physics since solving the full shell-model Hamiltonian (infinite Hilbert space with bare residual interactions) is impossible. To find a truncation scheme which renders the nuclear structure problem tractable and still possesses the primary aspects of the physics is a central problem of nuclear structure physics. Such efforts will necessarily lead to "effective interactions" (hereafter designated as EI) in nuclei.

As is well known (summarized by Kirson[26]), the EI is intimately connected with the truncation of the Hilbert space. In this section we shall discuss the meaning of the FDSM from the EI point of view. Also, we shall compare the FDS-EI with the more traditional ones, such as the pairing plus quadrupole $(P+Q)$-EI (Ref. 27) and the shell model (SM)-EI.[28–30]

In principle, given a truncated basis, the effective interaction can be calculated from the bare nucleon-nucleon interaction using many-body theory.[29] In practice, it is difficult to obtain effective interactions which can accurately reproduce data, especially in heavy nuclei. Therefore, one has to either resort to a simple model based on physical considerations for the effective interaction; or simply treat the two-body matrix elements of the effective interaction as parameters and determine them empirically. The latter scheme, pioneered immediately after the inception of the shell model theory by Talmi,[30] is now a standard procedure to obtain the shell model effective interaction (SM-EI). On the other hand, both the FDSM and the $P+Q$ interactions are models of effective interactions which may be regarded as different simplifications of the "real" effective interactions.

First we shell discuss the FDS-EI and $(P+Q)$-EI. Before proceeding, it is instructive to briefly review the main assumptions of the $P+Q$ model. The Hamiltonian of the $P+Q$ model can be obtained from the general Hamiltonian of Eq. (3.1) with the following simplifications. (1) Assume that the higher-order terms in V_p and V_Q can be neglected; (2) assume that V_Q is a separable residual interaction which has only a *spatial* (no spin) dependence; (3) assume that the matrix elements in Eq. (3.2) can be parametrized (see the following discussions). For example, by retaining only the lowest order of V_p (monopole pairing) and assuming that the two body matrix element of V_p is proportional to the level degeneracy Ω_j,

$$\langle j^2 0 \mid V_P \mid j'^2 0 \rangle = 2\sqrt{\Omega_j \Omega_{j'}} G_0 \ , \tag{7.1}$$

Eq. (3.2a) becomes

$$V_p = G_0 S^{T\dagger} \cdot S^T \ , \tag{7.2}$$

where

$$S^{T\dagger} = \sum_j \sqrt{\Omega_j/2} [a_j^\dagger a_j^\dagger]_0^0 \ . \tag{7.3}$$

On the other hand, in the spirit of assumptions (2) and (3), Eq. (3.2b) can be approximated by

$$V_Q = \sum_{r=\text{even}} \chi_r Q^r \cdot Q^r \ , \tag{7.4}$$

where χ_r is a parameter representing the strength of the r multipole of V_Q and Q^r is a multipole operator defined as

$$Q_\mu^r = \sum_{j_1 j_2} \frac{(j_1 \| Y_r \| j_2)}{\sqrt{2r+1}} [a_{j_1}^\dagger \tilde{a}_{j_2}]_\mu^r \ . \tag{7.5}$$

In Eq. (7.4), parity conservation demands that r must be even. Therefore, the lowest order nontrivial term of V_Q

is the quadrupole term ($r=2$), since the monopole term is a constant, and therefore the $P+Q$ Hamiltonian is

$$H_{PQ}=\sum_j e_j a_j^\dagger a_j + G_0 S^{T\dagger} S^T + \chi_2 Q^2 \cdot Q^2 \ . \tag{7.6}$$

Although the $P+Q$ model has been applied successfully to many problems, it suffers the usual difficulties of standard shell model theory when a large number of interacting valence nucleons is present in the system.

The FDSM can also be derived from the general Hamiltonian of Eq. (3.1), as we have demonstrated. There are some similarities between the two models: (1) each is based on multipole expansion of the nuclear force and the EI's are expressed in terms of the pairing plus multipole interactions; (2) they have the *same* monopole pairing interaction; (3) each truncates the multipole expansion and parametrizes the matrix elements.

The main difference between these two models is the manner of truncation and parametrization of the matrix elements. In the $P+Q$ model, the assumption of spin independence, separability, and a special form of the radial part of the interaction are introduced in order to obtain a simple $Q\cdot Q$ interaction. The reduced matrix element of Q can then be expressed as

$$(j_1\|Q^{(2)}\|j_2)=Q_0\sqrt{5}\hat{j}_1(l_1020\,|\,l_20)U(j_12\tfrac{1}{2}l_2;j_2l_1) \ , \tag{7.7}$$

where Q_0 is a parameter, and all higher multipole interactions are neglected.

In the FDSM, none of the above assumptions for the $(P+Q)$-EI are made. In the pairing interaction, monopole and quadrupole terms are retained, and the truncation of the multipole expansion and the parametrization are achieved by symmetry consideration, namely the basic assumptions (2) and (3) as discussed in Sec. III. Under these assumptions, the FDS-EI also has a quadrupole-quadrupole interaction which is denoted as $P^2\cdot P^2$. In addition, there are other multipole interactions $P^r(\alpha)\cdot P^r(\alpha')$ with $r=1$ and 3 [see the discussion preceding Eq. (3.21)]. It can be shown that the reduced matrix elements of $P^r(\alpha)$ are

$$(j_1\|P^r(i)\|j_2)=\hat{r}\,\hat{j}_1 U(j_1rki;j_2i) \ , \tag{7.8a}$$

$$(j_1\|P^r(k)\|j_2)=\hat{r}\,\hat{j}_1 U(j_1rik;j_2k) \ . \tag{7.8b}$$

In the k-active coupling scheme, for example, the reduced matrix element of the quadrupole operator P^2 is

$$(j_1\|P^2(k)\|j_2)=\sqrt{5}\hat{j}_1 U(j_12i1;j_21) \ . \tag{7.8c}$$

Comparing Eqs. (7.7) and (7.8c), we see that the two expressions are quite different. The Clebsch-Gordan coefficient (which is a result of the assumption that the $P+Q$ interaction has only spatial dependence) of (7.7) is not present in (7.8c), and the Racah coefficients are not the same. These differences are the primary reasons that the operators of (7.8c) do not couple the truncated k-i, S-D shell model space to the rest of the space. As a result, the FDSM is the first model which can provide *analytic solutions* for the shell model Schrödinger equation in a system with a large number of valence nucleons outside the closed shell. On the other hand, we see that (7.7) strongly couples the S-D space to the rest of the shell model space. Thus, a straightforward application of the $(P+Q)$-EI in a shell model calculation (i.e., diagonalization of the EI within a spherical shell model basis) for a system of many valence nucleons is beyond reach at this stage. In fact, most applications of such an interaction are carried out via theoretical schemes such as the Hartree-Fock-Bogoliubov (HFB) method.[31]

Since the FDS-EI and the $(P+Q)$-EI are different, a natural question is which EI is more realistic? To answer this question, one should first note that the truncated space as well as the methods of application for these two EI's are very different. Because the EI is directly a result of the truncated space, different truncations can result in different wavefunctions and effective interactions. Consequently, as emphasized by Kirson,[26] it is not always meaningful to carry out a direct comparison of the wave functions and the EI's when these EI's exist in different truncated Hilbert spaces. Seen in this light, it is not surprising that the $(P+Q)$-EI and the FDS-EI are different. The only relevant test is whether the effective interaction within the truncated space reproduces the appropriate physical observations. As we have seen in the previous sections, the FDS-EI within the k-i truncated basis does describe the well known low energy collective modes of even-even nuclei, suggesting that such an EI is a good caricature of the realistic one, even though in form it differs from the more conventional EI's.

We now turn to the question of the relations between the FDS-EI and the shell-model effective interaction (SM-EI).[32] There are several reasons why this comparison is of interest: (1) unlike the $(P+Q)$-EI, the truncated space for the SM-EI is well defined (i.e., one major shell); (2) the SM-EI is empirically obtained and closer to the realistic effective interaction than the $(P+Q)$-EI (certainly true in the s-d shell and at least for the beginning of the shell for heavy nuclei); (3) the FDSM is also a shell model in a (further) k-i truncated basis in one major shell. Thus the FDSM and the shell model with an effective SM-EI share a common foundation.

In comparing the SM-EI and FDS-EI we should recall two points.

(i) If the space left out by the FDSM is not negligible then, as emphasized by Kirson,[26] the SM-EI and FDS-EI could be quite different and still describe the same physics.

(ii) The shell model EI is obtained empirically only near closed shells (except for the s-d shell where a smooth change of the EI through the shell is demanded by the data). Contrary to the shell model EI, the FDS-EI is expected to be valid for describing low-lying collective states of nuclei, where there are many valence nucleons. It is not obvious that the EI which is appropriate for the beginning of the shell should be the same as that in the mid-shell, although this is the most economical assumption. Of course, the FDS-EI presented here may be oversimplified, since its details remain to be studied. It is hoped that, by comparing the SM-EI and the

FDS-EI, a better EI for heavy nuclei in midshell may emerge. This work is now in progress.

VIII. SUMMARY AND DISCUSSION

In this paper we have proposed a microscopic nuclear structure model called the fermion dynamical symmetry model (FDSM). By assuming that coherent S and D pairs are the most important building blocks in low-energy collective states, we find that a variety of dynamical symmetries exist due to the nuclear shell structure in nuclei.

Several important ingredients of the FDSM have been emphasized.

(1) The FDSM is deeply rooted in the shell structure of nuclei. It is, in fact, a prescription for solution of the shell model through a radical symmetry-dictated truncation.

(2) The FDSM is a model of nucleon-nucleon effective interactions. As discussed in Secs. III and VII, we believe that the Hamiltonian for the FDSM is potentially more useful than that for the pairing plus quadrupole model since it allows a more general residual interaction, and can have benchmark dynamical symmetries with analytical solutions, even in regions of strong deformation.

(3) The crucial interplay between the normal and the abnormal parity levels allows the $Sp_6 \supset SU_3$ symmetry, which Ginocchio was compelled to discard in his model, to be resurrected. Thus the model has an axially-symmetric rotational limit.

(4) All the nuclear dynamical symmetries exhibited phenomenologically by the IBM are recovered by the FDSM from a more fundamental fermionic level without the necessity of boson mapping. The energy formulas for each limiting case in the FDSM with no broken pairs ($u_1 = \nu_0 = 0$) are similar to the IBM ones. However, due to the Pauli effects and the shell structure, there are some differences: (i) The boson number N is replaced by N_1 (i.e., the number of nucleon pairs in the normal-parity levels); (ii) the $\lambda + \mu > 2\Omega_1/3$ representations of SU_3 and the $\nu \geq 2\Omega_1/3$ representations of SU_2 are not allowed due to Pauli effects; (iii) the matrix elements of some physical quantities like electromagnetic-transition rates, and transfer operators, may differ from the IBM expressions for the same quantities by a Pauli factor.

(5) A *new* vibrational symmetry limit (SO_7) has been predicted, and observed. This symmetry is a transition symmetry, lying between the SO_5 and SO_6 limits of SO_8.

(6) The FDSM suggests a systematic connection between dynamical symmetries and shell structure. Although the n-p interaction has yet to be included, Table IV already reflects global nuclear systematics. For example, rotational nuclei often occur in a region where at least one type of valence nucleons (neutrons or protons) occupies the Sp_6 shells. This is in agreement with the fact that only the Sp_6 symmetry has an SU_3 limit (see Table IV). Thus the widespread occurrence of axially-symmetric rotation in the actinides is understood as a consequence of fermion shell symmetry since both valence neutrons and protons have Sp_6 symmetry. On the other hand, in the regions where neither valence neutrons nor protons possess Sp_6 symmetry (e.g., in shell 6 and shell 4, see Table IV), nuclei rarely behave as good rotors (the few exceptions may be regarded as due to the symmetry being broken).

It is widely believed that the n-p interaction is responsible for nuclear deformation.[33] This is not necessarily in contradiction to the idea expressed here that the occurrence of rotational motion is strongly related to the symmetry of the valence shells. Actually, both must play a role. The symmetry of the valence shells determines the *possible* collective modes, while the effective interaction determines which ones are actually realized. For example, an Sp_6 shell has a propensity to collective rotational modes because of the SU_3 subgroup, but unless the quadrupole-quadrupole term dominates the pairing in the effective interaction the SU_2 (vibrational) subchain of Sp_6 will be realized instead. For an n-p system, even through both neutrons and protons possess SU_3 symmetry (SU_3^{π} and SU_3^{ν}), the whole system need not have SU_3 symmetry (denoted as $SU_3^{\pi+\nu}$), unless there is a strong quadrupole-quadrupole n-p interaction. In this sense, one can say that it is the n-p interaction which determines the occurrence of deformed nuclei. However, if both neutrons and protons are in SO_8 shells, even strong n-p interaction will not lead to $SU_3^{\pi+\nu}$ modes. Instead, we may expect $SO_6^{\pi+\nu}$ which corresponds to a γ-soft rotor. For example, this is in agreement with the experimental observation of SO_6 nuclei in the $A = 130$ region.[34] Axial-symmetric rotors can occur in such regions only if the effective interaction severely breaks the SO_8 symmetry. The fact that very few nuclei ($^{128-132}$Nd and $^{126-130}$Ce are examples) are found to behave rotationally without having valence nucleons occupying Sp_6 shells, indicates that in most cases the shell symmetry (Sp_6 or SO_8) is not seriously broken, and that it plays a crucial role in determining where different collective modes can occur.

(7) The model can accommodate states with *broken pairs,* and thus encompasses important aspects of nuclear structure which are not accessible within boson models. A particle-rotor model Hamiltonian which can describe high-spin physics is obtained microscopically from the FDSM.

(8) Although we have emphasized even-even systems in this paper, odd-mass and odd-odd nuclei are naturally described within the FDSM by the odd-heritage (or spinor) representations. Therefore, the model provides a unified description of high and low spin states of even-even, odd-even, and odd-odd nuclei.

Several approximations employed in the FDSM require further elaboration.

(i) *The n-p interaction.* The symmetry limits presented here are the symmetries of the shell structure for valence neutrons and protons. The implication is that one may choose the shell model truncation scheme in accordance with that dictated by the symmetries of Sp_6 or SO_8. In practical applications, the n-p interaction should be *explicitly* introduced. It is worth emphasizing, however, that in some cases the basic features of the symmetries remain the same in the presence of n-p interaction. For

example, SU_3 and SO_6 symmetry limits can still be obtained when neutrons and protons possess the same symmetry and interact via, say, a quadrupole-quadrupole interaction $P_2^{\pi}\cdot P_2^{\nu}$ (the group chains will be $SU_3^{\pi}\times SU_3^{\nu}\supset SU_3^{\pi+\nu}$ and $SO_6^{\pi}\times SO_6^{\nu}\supset SO_6^{\pi+\nu}$). More generally, the symmetries will be broken by the n-p interaction but this can be handled by numerical diagonalization within the direct product space of the neutron-proton k-i bases. It is particularly important to understand the situations, such as for the rare-earth nuclei, where neutrons and protons would possess different symmetries in the absence of n-p interactions. The details of this study will be published in the next paper of this series.

(ii) *D pairs in the abnormal-parity level.* In this model we assume an S condensate for the abnormal-parity level. The component of the abnormal-parity level incoherent D pairs (in the conventional sense) is neglected because it breaks the symmetry. However, this may not be a bad approximation. Due to parity conservation, there is only one component associated with the abnormal parity level in the wave function of a coherent D pair (i.e., $|(j_0)^2,2\rangle$, where j_0 is the angular momentum of the abnormal parity level), but there are many components from the normal-parity orbitals. Because of the collectivity, the amplitude for each component should not be large.[35] An estimate can be made from the D-pair wave function obtained by Bonsignori *et al.*[36] where the probability of this configuration is found to be roughly one-tenth. Therefore, ignoring the component of the intruder orbital in coherent D pairs may not affect the results very much. It should be stressed, however, that this does not mean that the abnormal level plays *no* role for the low lying states. In fact, it is precisely the existence of the abnormal level which allows nuclei near midshell able to have large deformation and rotational-like behavior.[21] The effect of the abnormal level is to absorb the excess nucleons so that the highest SU_3 representation can exist near midshell without violating the Pauli principle. Moreover, it should be noted that the FDSM neglects the contribution of abnormal parity level to the *coherent* D pairs. Noncollective pairs with all possible J's ($J\geq 2$) in the abnormal-parity level are included if the configurations with $v_0\neq 0$ are taken into account. Likewise, the higher angular momentum pairs in the normal-parity levels are considered if configurations with $u_1\neq 0$ are taken into account. These configurations play important roles in high-spin physics and will be discussed in more detail in a separate paper of this series.

(iii) *Symmetry-breaking terms.* Realistic nuclear Hamiltonians do not have perfect symmetries. The symmetry limits are the idealized mathematical representations of the physical collective modes. There are symmetry-breaking terms such as the single-particle energy splitting term, and the quadrupole-quadrupole interaction between the S-D pairs ("core") and the decoupled particles $[P^2(k)\cdot P^2(i)]$ which may be important. This is especially true for the odd-mass system and high-spin states. Nevertheless, the symmetry limits can be used as starting points, and provide a natural and systematic way to truncate the shell model space. Their main advantage is simplicity. One can use the analytical formulas to test for correct physical trends without having to carry out complicated calculations. These limits provide broad classification schemes which are useful in understanding the global systematics of nuclear structure, and the relationship of the phenomenology to the underlying shell structure. It is this aspect which has been emphasized in the present paper. If the simplest limiting cases (corresponding to the pure k-i configuration) are already capable of describing the basic feature of collective motion, we may conclude that the corresponding symmetry basis, which is a severely truncated shell model, is a good basis for describing collective motion. We can then take the most reliable effective interaction, which includes all the necessary symmetry breaking terms, and carry out the numerical diagonalization within the limited basis. Since the k-i basis is severely truncated and is just a generalization of an L-S coupling scheme, existing shell model codes appear to be capable of making these calculations with minimal alteration. By this procedure, it should be possible to carry out microscopic shell model calculations for strongly collective modes in heavy nuclei. This aspect is currently being pursued.[37]

Although the initial testing and applications of the FDSM hold considerable promise, there are several open questions which require further study.

(i) What is the effective interaction which is appropriate for the truncated k/i basis? As we have argued in Sec. VII, this effective interaction may be different from the interaction appropriate for the entire shell. Establishing such an effective interaction is crucial for applying the FDSM to realistic nuclei.

(ii) Although essentially neglected in this paper, a new symmetry G_3 is predicted by the FDSM for shells 3 and 5 (Table IV). This symmetry should be investigated. Since in this mass region neutrons and protons are in the same shell, the FDSM must be extended to incorporate the isospin before it can be applied quantitatively to these cases.

(iii) Since the shell structure and Pauli effects make the fermion dynamical symmetries different from the boson dynamical symmetries [point (4) above], it would be desirable to see if this could be tested experimentally, particularly the N_1 dependence and the Pauli factors in the transition probabilities of different vibrational limits (see Table XI).

(iv) The FDSM as presented here is based on symmetries for a single valence shell. How stable are these symmetries and their associated collective modes against subshell closures? For example, the rare-earth neutrons are expected to exhibit SP_6 symmetry based on the structure of the full shell. However, if the subshell closure at $N=114$ is taken seriously, the three orbitals being filled for $N>114$ ($p_{3/2}$,$f_{5/2}$,$p_{1/2}$) can form an SO_8 algebra with an SO_6 subgroup. This, in conjunction with the expected SO_8 symmetry for rare-earth protons, would provide a convenient explanation of the γ-soft (SO_6) behavior of this region.

(v) Can this symmetry scheme, or a related one, be extended to incorporate multiple major shells within a single "supershell," thereby providing a dynamical symme-

try description of giant resonances, superdeformations, and low-lying negative-parity collective states?

In summary, although several features of the FDSM require further study, the preliminary indications are that it is a powerful method for describing the microscopic structure of collective excitations in heavy nuclei. Perhaps the most attractive feature of the model is that since it is fully microscopic, it is fully testable. If the method survives the detailed scrutiny to which it is currently being subjected, we may expect it to provide important insights into the nature of collective motion in nuclear physics.

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