Model-independent analysis of heavy-ion elastic scattering

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(Received 29 September 1986)

I propose a fast algorithm for performing unconstrained phase-shift analyses of (low energy) heavy-ion elastic-scattering data. The method is applied, as an example, to $^{16}O+^{28}Si$ elastic scattering at $E_{c.m.}=21.1$ MeV. A study of general properties of the error matrix is performed, and criteria for the feasibility of a model-independent analysis are discussed. Some recommendations for future experiments and model-dependent analyses are made.

INTRODUCTION

Until recently, analyses of heavy-ion reaction data were performed almost exclusively in model-dependent frameworks like the optical model, coupled-channel approaches, or constrained phase-shift analysis. Although one can extract the partial-wave amplitudes from the cross-section information unambiguously for elastic processes involving an asymptotically known Coulomb potential (at least in principle, if the cross section is known accurately enough), this step was never made in an entirely model-independent fashion. The main reason is that the quality of the available cross-section data did not allow for a meaningful fit with as large a set of parameters as would be required in such a model-independent analysis. Recently, though, various attempts have been made to obtain phase shifts from the cross section, mainly with the purpose to identify resonances from the energy dependence of the amplitudes. For instance, a study of the phase shifts of $^{12}C+^{12}C$ has been performed recently in both the elastic and inelastic channels.[1,2] Since measurements were available over a wide angular range and at small energy intervals, possible phase-shift ambiguities were believed to be removed by the requirement of continuity with energy. (Furthermore, in symmetric spin 0 systems such as $^{12}C+^{12}C$ only even partial waves contribute, and the number of parameters is about half of what it would be for comparable nonsymmetric cases.) With the continuously increasing accuracy of experimental data, an unconstrained determination of the scattering amplitude becomes sufficiently reliable in some situations (low energy or strong absorption) to be of much physical interest, particularly if the results are used to supplement those of other approaches.

The purpose of this paper is to discuss feasibility and reliability of unconstrained analysis of heavy-ion elastic-scattering data. First, I will describe a fast algorithm, essentially, a Newton-Raphson method, to determine the partial-wave amplitudes. Application of the method to the $^{16}O+^{28}Si$ elastic cross section will be presented. Then I will study the error matrix and discuss to what extent various parts of the elastic-scattering angular distribution determine the partial-wave amplitudes. Ambiguities in the phase shifts as well as the scattering amplitude are briefly discussed. Finally, I will make some suggestions for future experiments and for model descriptions of heavy-ion reactions.

A FAST ALGORITHM FOR UNCONSTRAINED PHASE-SHIFT ANALYSIS

First, I briefly review the heavy-ion scattering problem.[3] The elastic-scattering cross section is determined by the scattering amplitude which is a function of the center-of-mass scattering angle θ,

$$f(\theta)=f_N(\theta)+f_C(\theta)\ . \tag{1}$$

Here, the Coulomb amplitude, $f_C(\theta)$, is given by

$$f_C(\theta)=\frac{-\gamma}{2k}\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}\frac{e^{-2i\gamma\ln[\sin(\theta/2)]}}{\sin^2(\theta/2)}\ , \tag{2}$$

with wave number k and Sommerfeld parameter $\gamma=Z_1Z_2\mu e^2/\hbar^2k$ for heavy ions with reduced mass μ and charges Z_1 and Z_2. The amplitude $f_N(\theta)$ is defined to be the difference between the total and the Coulomb amplitude and hence incorporates all the nuclear force effects. It can be partial-wave expanded in terms of the amplitudes s_l,

$$\begin{aligned}f_N(\theta)&=f_N(\theta,s_l)\\&=\frac{1}{2ik}\sum_{l=0}^{l_{\max}}(2l+1)(s_l-1)s_l^C P_l(\cos\theta)\ ,\end{aligned} \tag{3}$$

where s_l^C is the Coulomb partial-wave amplitude

$$s_l^C=\frac{\Gamma(l+1+i\gamma)}{\Gamma(l+1-i\gamma)}\ . \tag{4}$$

Since all nuclear effects are finite in range, the sum in (3) runs over a finite number of partial waves only. By the definition (3) of the amplitudes s_l, the Coulomb phase is removed and s_l approaches 1 asymptotically. The complex variables s_l $(0\le l\le l_{\max})$ are the ones we want to obtain from a phase-shift analysis; hence, there are $2l_{\max}+2$ parameters.

The algorithm I am proposing is a Newton-Raphson iteration for the amplitudes s_l to reproduce the cross section at the m measured angles θ_i $(i=1,\ldots,m)$

$$\sigma(\theta_i)=|f_N(\theta_i,s_l)|^2+|f_C(\theta_i)|^2 +2\mathrm{Re}[f_N^*(\theta_i,s_l)f_C(\theta_i)] . \quad (5)$$

It is rapidly converging essentially because the Coulomb-nuclear interference term, which is linear in the amplitude s_l, dominates over $|f_N(\theta)|^2$ for a large fraction of the angular distribution. A sequence of values $s_l^{(n)}$ that converges to s_l is generated using the assumption that the cross section is locally linear in s_l. That is, we obtain $s_l^{(n+1)}$ from $s_l^{(n)}$ by solving for $\Delta s_l \equiv s_l^{(n+1)}-s_l^{(n)}$ in the linear equations

$$2\,\mathrm{Re}\left[\sum_l A_{\theta_i l}(s^{(n)})\Delta s_l\right]=\sigma(\theta_i)-\sigma^{(n)}(\theta_i) , \quad (6)$$

with $\sigma^{(n)}$ the approximation to the cross section obtained from $s_l^{(n)}$ and

$$A_{\theta_i l}(s^{(n)})=[f_C^*(\theta_i)+f_N^*(\theta_i,s^{(n)})] \times\frac{1}{2ik}(2l+1)s_l^C P_l(\cos\theta_i) . \quad (7)$$

To simplify notation I identify the vector Δs and the matrix A with

$$\Delta s=\begin{bmatrix}\Delta s_l\\ \Delta s_l^*\end{bmatrix},\quad A=\begin{bmatrix}A_{\theta_i l} & A^*_{\theta_i l}\end{bmatrix} , \quad (8)$$

respectively. Then Eq. (6) becomes

$$A(s^{(n)})\Delta s=\Delta\sigma^{(n)} . \quad (9)$$

In favorable situations the matrix A is nonsquare (more data than parameters) and we have to solve the linear equation in a least squares manner. This is a standard problem which involves simple matrix manipulation.[4] I will discuss the solution briefly only for later reference.

To find the appropriately weighted least-squares solution one has to introduce the (diagonal) error matrix W for the measured cross section [$W_{ii}=1/e(\theta_i)^2$, $e(\theta_i)$ is the error in $\sigma(\theta_i)$], and we have to minimize

$$S=(\Delta\sigma-A\,\Delta s)^T W(\Delta\sigma-A\,\Delta s) . \quad (10)$$

The solution for Δs is found after some trivial algebra,

$$\Delta s=(A^\dagger WA)^{-1}A^\dagger W\,\Delta\sigma . \quad (11)$$

If we have obtained $s^{(\infty)}$ we can determine the error matrix,[5] W_s, for the amplitudes from $A=A(s^{(\infty)})$,

$$W_s=A^\dagger WA . \quad (12)$$

If the linear approximation is sufficiently accurate, diagonalization of W_s yields eigenvectors w_i ($i=1,\ldots,2l_{\max}+2$) which represent the major axes of the error ellipsoid in phase-shift space. The corresponding eigenvalues, λ_i, give the inverse square of the length of each of these axes.

A useful quantity in the type of analysis we are pursuing is a norm for vectors in phase-shift space. We can write

$$||\Delta s||^2=\sum g_{ll'}\Delta s_l^*\Delta s_{l'} . \quad (13)$$

where $g_{ll'}$ is a metric, yet to be defined. Although $g_{ll'}=\delta_{ll'}$ would be an obvious choice, I will argue that

$$g_{ll'}=(2l+1)\delta_{ll'} \quad (14)$$

is a more physical one (in addition to being more convenient in the context of later discussion). If $||\Delta s||$ as it appears in (13) is expressed in terms of the angular amplitude, $\Delta f_N(\theta)$, using Eq. (3), one obtains

$$||\Delta s||^2=2k^2\int d\cos\theta\,|\Delta f_N(\theta)|^2=\frac{k^2}{\pi}\int d\Omega\,|\Delta f_N(\theta)|^2 . \quad (15)$$

Hence the metric (14) for the partial wave amplitudes implies an angular independent measure for the norm of the angular amplitude.

APPLICATION TO ELASTIC SCATTERING DATA

A completely model-independent analysis of heavy-ion elastic scattering obviously requires the availability of high quality data: a full angular distribution, a fine angular grid, and small statistical and systematic errors. In addition the scattering energy cannot be too high or the number of partial waves that are affected by the nuclear potential (and hence the number of parameters) becomes an unreasonably large fraction of the number of data points. These considerations led me to choose the $^{16}O+^{28}Si$ data at $E_{\mathrm{c.m.}}=21.1$ MeV given in Ref. 6. Another reason for selecting this set of data is that it has received much attention in the past few years and many different model analyses have been performed.[6,7] It would be of interest to compare the results for the amplitudes obtained here with those of previous studies.

The 21.1 MeV $^{16}O+^{28}Si$ angular distribution is actually composed of two independent measurements, one at the more forward and one at the more backward angles (for details see Ref. 6). The normalization for the former measured region was obtained from the condition that $\sigma/\sigma_{\mathrm{Ruth}}$ approaches 1 as $\theta\to 0$; the normalization at the latter was obtained by matching the two data sets in the overlap region. This procedure can result in systematic errors of the order of 10%. Clearly, it is extremely difficult to deal with such errors if their nature is not exactly known. For the present paper I will ignore the possibility of occurrence of systematic errors in the data under study. Let me only remark that the normalization of a set (or part of a set) of measurements can easily be included as a free parameter in the type of phase-shift analysis I am proposing. Since the cross section is linear in the extra parameter this does not lead to additional complications.

The $^{16}O+^{28}Si$ data that I will consider consist of cross-section measurements at 72 angles ranging from about 30° to 180°. The partial wave that will experience a nuclear potential strength which is less than 2% of the Coulomb is found from semiclassical arguments to lie between $16\hbar$ and $17\hbar$. I do not pretend to be able to determine the nuclear amplitudes with a precision of more than 2% and I choose $l_{\max}=16\hbar$. Hence there are 34 parameters. This is about half the number of available cross section measurements.

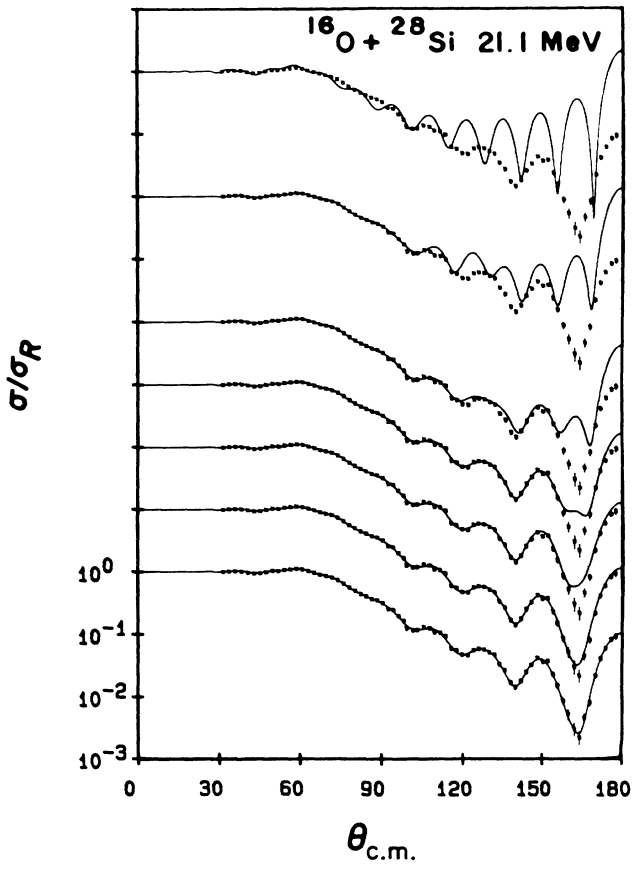

FIG. 1. A comparison of the measured angular distribution with the ones computed in the fitting procedure. From top to bottom: the initial guess and the results of six successive iterations (solid lines). Data are from Ref. 6.

TABLE I. Partial wave amplitudes for $^{16}O+^{28}Si$ elastic scattering at $E_{c.m.}=21.1$ MeV resulting from the unconstrained analysis.

l	$\lvert s_l \rvert$	arg(s_l) (deg)
0	0.061	59.4
1	0.092	−92.2
2	0.082	−81.8
3	0.069	−15.6
4	0.104	72.0
5	0.114	−150.1
6	0.026	28.9
7	0.027	112.7
8	0.046	26.4
9	0.058	100.1
10	0.116	68.7
11	0.149	11.8
12	0.353	−4.7
13	0.682	−10.0
14	0.881	−8.9
15	0.974	−6.3
16	0.992	−2.6

To obtain fits to the angular distribution I have implemented the method described in the preceding section on a personal computer (IBM PC-AT, 512k RAM) using the linear equation solver f04jgf of the NAG-PC library[8] (the program is coded in FORTRAN). Figure 1 shows the angular distributions obtained from the initial guess and the six successive iterations that were required to converge satisfactorily to the data. I started with partial wave amplitudes given by a sharp cutoff model with cutoff angular momentum $l_0=13$ ($s_l=0$ for $l<13$, $s_l=1$ for $l\geq 13$). Each iteration took, on the average, approximately 60 sec of central-processing-unit (CPU) time. Details on the resulting phase shifts are given in Table I.

Several remarks concerning the fitting procedure should be made at this point. Some initial experimentation with the program had shown rather large oscillations at very forward angles ($0°<\theta<30°$) where no measurements are available. In order to avoid these, I forced the fit to be close to 1 in this region by adding some "measurements" with magnitude 1 and error of the order of a few percent. I will return to this oscillatory behavior later when I discuss the error matrix. In Table II I present the χ^2 per data point for each of the iterations. It can be seen that it decreases each time by an order of magnitude. Also $||\Delta s||$ decreases rapidly going from one iteration to the next. This, together with the observation that the data at smaller angles are fitted in earlier iterations, seems to indicate that the cross section becomes more and more sensitive to changes in the partial wave amplitudes as θ increases. I will investigate this important point in more detail in a later section. Finally I want to mention the remarkable fact that the phase of the amplitude as given in Table I for large l is slightly negative. Although I have not performed a very extensive study, this seems to be consistently the outcome of the analysis. One would expect the phase to be positive asymptotically since the nuclear potential is attractive. This effect might be a compensation for the omission of small contributions from partial waves with $l>16$. On the other hand, the effect might be a physical one and caused by the absorption that has already set in at $l\cong 16$.

ERROR ANALYSIS

First I will discuss the reliability of the fit presented in the preceding section. Then I will turn to a more general study of errors in the analysis of elastic scattering data.

TABLE II. χ^2 per data point for the cross sections obtained in the iteration procedure.

	χ^2/m
Initial guess	13 500
Iteration 1	1680
2	110
3	5.93
4	3.22
5	1.02
6	0.60

TABLE III. Lowest eigenvalues of the error matrix W_s.

4.4×10^{-5}	117
5.0×10^{-4}	168
0.2	298
1.5	391
44.9	835
54.2	907

I have diagonalized the error matrix W_s [cf. Eq. (12)] at the values $s_l^{(6)}\cong s_l^{(\infty)}$ obtained in the last iteration of the fit to the $^{16}O+^{28}Si$ cross section. The lowest eigenvalues, λ_i, are listed in Table III. It turns out that exceedingly small values for λ_i occur and hence enormously large errors $e_i=1/(\lambda_i)^{1/2}$ will be present ($e_1\cong150$!). To investigate the origin of these large errors I have computed the cross section using

$$s_l(i)=s_l^{(\infty)}+\Delta s_l(i),\quad \Delta s(i)=w_i/\sqrt{\lambda_i}\ , \tag{16}$$

where w_i is the eigenvector corresponding to λ_i (note that the amplitudes $s_l(i)$ do not necessarily satisfy unitarity). This set of phase shifts is on the edge of the 60% confidence region and should provide a description of the data with χ^2 not much larger than 1 (at least if the linear approximation is valid). The results for $i=1,\ldots,4$ are presented in Fig. 2 where the computed cross section is compared with the data for each case. In Table IV I give $\Delta S_l(i)$ for the same values of i. From Fig. 2 one can observe enormous oscillations at the forward angles. These are clearly unphysical but one cannot exclude their existence from just the data. If one replaces the cross section by its linear approximation around $s_l^{(\infty)}$ [cf. Eq. (6)] the data are reproduced almost perfectly everywhere; hence, the discrepancies one can observe in Fig. 2 at backward angles are due to (actually, remarkably small) nonlinearities. To move back into the region of 60% confidence one only needs to make a small correction on $\Delta s(i)$. Although the condition of unitarity would severely reduce the 60% confidence region, it seems that only some knowledge of the behavior of the cross section for

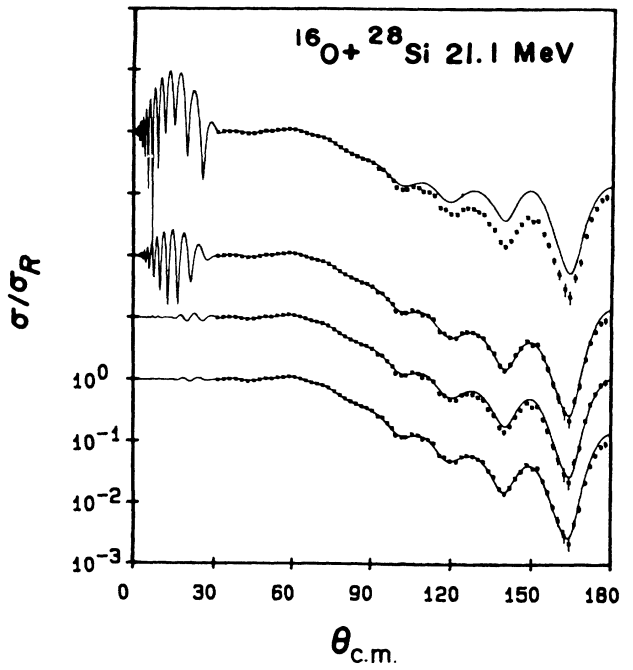

FIG. 2. Angular distributions obtained from the amplitudes (16) corresponding to the lowest four eigenvalues of the error matrix (solid lines). The data are displayed for comparison.

TABLE IV. Error vectors $\Delta s_l(i)$ corresponding to the smallest eigenvalues of W_s and hence the largest errors.

	$i=1$		$i=2$		$i=3$		$i=4$	
l	Re(Δs_l)	Im(Δs_l)	Re(Δs_l)	Im(Δs_l)	Re(Δs_l)	Im(Δs_l)	Re(Δs_l)	Im(Δs_l)
0	54.133	38.414	11.225	−15.818	0.369	0.401	0.163	−0.172
1	−45.615	−45.401	−13.279	13.425	−0.312	−0.366	−0.159	0.096
2	27.273	54.336	15.942	−7.995	0.068	0.296	0.138	−0.046
3	0.991	−55.455	−16.390	−0.278	0.016	−0.146	−0.051	−0.014
4	−29.893	39.154	11.605	8.864	0.079	−0.060	−0.010	−0.025
5	41.844	−5.989	−1.770	−12.510	−0.309	0.003	−0.000	0.119
6	−25.048	−24.471	−7.399	7.529	0.276	0.379	0.161	−0.095
7	−6.834	27.242	8.280	2.099	0.252	−0.584	−0.241	−0.100
8	21.282	−3.467	−1.027	−6.545	−0.684	0.002	−0.016	0.281
9	−7.051	−14.301	−4.450	2.174	0.216	0.688	0.289	−0.099
10	−8.912	6.684	2.101	2.821	0.580	−0.289	−0.126	−0.261
11	5.102	5.393	1.751	−1.589	−0.349	−0.452	−0.202	0.146
12	3.575	−3.191	−0.999	−1.096	−0.442	0.276	0.131	0.168
13	−1.455	−2.054	−0.682	0.535	0.105	0.278	0.144	−0.091
14	−1.137	0.787	0.203	0.397	0.178	−0.131	−0.034	−0.089
15	0.134	0.758	0.191	−0.045	−0.016	−0.190	−0.058	0.031
16	0.172	0.138	0.020	−0.058	−0.042	−0.054	−0.008	0.034

$0 \leq \theta \leq 30°$ can lead to desirable confidence limits on the partial-wave amplitudes.

A particularly salient feature of the error matrix obtained for the $^{16}O+^{28}Si$ data set, which may be generally observed, is the enormous range in eigenvalues ($\sim 10^{-5}$ -10^{6}), even if measurements would be extended to 0° with the same trends in statistical errors ($\sim 10-10^{6}$). This implies large errors in certain directions in phase-shift space and exceedingly small errors in others. Hence, correlations are important and errors cannot be represented appropriately by diagonal elements of the error matrix. However, nonlinear effects seem to be small in the example analyzed here (probably due to the accurate measurements at backward angles) and the few smallest eigenvalues of the error matrix and their corresponding eigenvectors are reliable measures for the uncertainties.

Despite the fact that errors in the phase shifts are considerable, it turns out that the total elastic scattering amplitude, $f(\theta)$, is well determined at angles where data are available. I computed the error $\Delta f_N(\theta)$ in the amplitude from improved error vectors derived from those given in Table IV and it appeared that the magnitude of $\Delta f_N(\theta)$ is small compared to the total amplitude for $\theta > 30°$. The error in the phase of $f(\theta)$ is smaller than 10° everywhere in that angular range. Note, though, that the error analysis will not provide information on ambiguities stemming from secondary minima in χ^2. At $\theta \approx 160°$, where a minimum in the cross section occurs, I detected such an ambiguity. At the position of that minimum the phase of the total amplitude was able to make a rapid change of 2π. A more thorough investigation of ambiguities will be subject of a future publication.

I am now going to address the following question. What (diagonal) error matrix, W, for the cross section data will yield a hyperspherical or nearly hyperspherical [under the norm given by Eqs. (13) and (14)] 60% confidence region for the amplitudes? The answer to this problem is relevant in two situations. First of all, it describes the perfect experiment: one would like to perform a set of measurements and obtain an error matrix of that form, since only then would one most effectively exploit experimental information to pin down the scattering amplitude. Second, in certain model descriptions (usually with a limited set of parameters) the best fit to the data will yield a χ^2 much larger than 1 and the calculated cross section is inconsistent with the data in a statistical sense. Despite that, one hopes that the model will describe some of the physics although it does not reproduce all of nature's details. If "physics" stands here for "scattering amplitude" one obviously does not want to minimize χ^2, or equivalently [cf. Eq. (10)]

$$S = \delta\sigma^T W \delta\sigma \tag{17}$$

and obtain a fit closest to the measurements (where "closest" is determined by how accurately an experimentalist is willing to measure at certain angles), rather one would like to find an amplitude that is not too far removed from the physical one. Hence, instead, one would like to minimize the distance $||\delta s||$ between the model and physical amplitudes, or equivalently

$$S' = ||\delta s||^2 = \delta s^\dagger G \, \delta s \ , \tag{18}$$

where G is related to the metric (14),

$$G = \frac{1}{2} \begin{bmatrix} (2l+1)\delta_{ll'} & \Theta \\ \Theta & (2l+1)\delta_{ll'} \end{bmatrix} . \tag{19}$$

That these two prescriptions are not the same one may see from the linear relation between $\delta\sigma$ and δs which is correct if we are not too far away from the real amplitude (δs small), $\delta\sigma = A \, \delta s$. Indeed (17) and (18) are only equivalent if $G = A^\dagger W A$; hence, only if the error matrix for the amplitudes equals G.

The diagonal elements of the error matrix W can be written as $W_{ii} = w(\theta_i) = 1/e(\theta_i)^2$. I will consider the situation in which cross-section measurements are performed at equally spaced angles θ_i which are close enough together to justify the use of the continuum limit for θ. The quantity S in Eq. (17) can now be written as

$$S = \int d\theta \, w(\theta) \delta\sigma(\theta)^2 \ . \tag{20}$$

To make the discussion more transparent I will express S' in (18) in terms of infinitesimal changes in the angular amplitude,

$$S' = \frac{k^2}{\pi} \int d\Omega \ | \delta f_N(\theta) |^2 \ . \tag{21}$$

I will allow for small but otherwise general changes in the amplitude $f_N(\theta)$. This corresponds to changes in the phase shifts for all l and hence $l_{\max} \to \infty$. Since δf is related to $\delta\sigma$ in a simple fashion (for small δf),

$$\delta\sigma(\theta) = 2 \, \mathrm{Re}\{[f_C^*(\theta) + f_N^*(\theta)] \delta f_n(\theta)\} \ , \tag{22}$$

the error matrix, W_f, for f_N does not show correlations for amplitudes at different values of θ. Because (22) relates the real variables $\delta\sigma(\theta)$ to twice as many complex variables $\delta f_N(\theta)$, it is obvious that half of the eigenvalues of W_f equal 0. This is an expression of the fact that we cannot determine the amplitudes from the cross section unambiguously if we make no assumptions on the properties of the amplitude [like finite $l_{\max}$, cf. Eq. (3)]. From (20)—(22) it can be seen immediately that all nonzero eigenvalues of W_f are degenerate if

$$w(\theta) = \frac{\sin\theta}{\sigma(\theta)} \ . \tag{23}$$

This value of $w(\theta)$ corresponds to the following angle dependence for the error in cross-section measurements,

$$\frac{\dot{e}(\theta)}{\sigma(\theta)} = \frac{1}{\sigma(\theta)} \frac{1}{\sqrt{w(\theta)}} = \frac{1}{\sqrt{\sigma(\theta)\sin\theta}} \ . \tag{24}$$

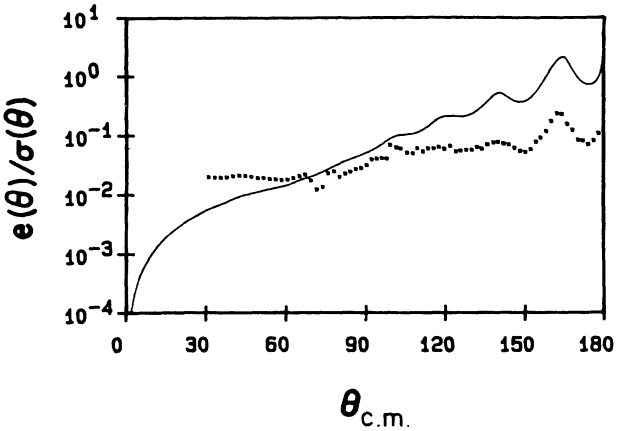

FIG. 3. The angular dependence of the ideal relative error in the cross section (24) (arbitrarily normalized; solid line). The relative error in the data is also plotted (dots).

If one wants to acquire an equal amount of information on the scattering amplitude at all angles, (24) specifies the required relative accuracy of cross-section measurements. The factor $\sin\theta$ in (24) stems from the cylindrical symmetry of the scattering problem which allows one to perform measurements at a fixed azimuthal angle ϕ [it would be absent if I had considered $e(\Omega)$]. The term $1/\sqrt{\sigma}$ will automatically appear as a statistical error in experimental situations since σ is proportional to the count rate. Therefore, one can neatly rephrase the simple result (24):

The flux of information on the scattering amplitude released in a scattering process is independent of the scattering angle.

In nuclear (heavy ion) reactions, the subject of the present study, one knows that a finite number of partial waves are contributing to the nuclear amplitude and the properties of the error matrix as discussed above will be modified. Although all its zero eigenvalues will become finite, they may still be very small. How small will depend on the particulars of the reaction (the exact form of the scattering amplitude) and it is difficult to make general quantitative statements. It is expected, and numerical studies seem to indicate, however, that experimental errors of the form (24) will lead, if not to the smallest, to a very small spread in eigenvalues of the amplitude's error matrix. In Fig. 3 I have plotted the relative error (24) against θ. The values for the $^{16}O+^{28}Si$ data are also given in that plot. It appears (at least from the viewpoint of a model-independent approach) that in the experiment too much emphasis has been put on the backward angles and relatively too little on the forward ones.

Even in case l_{max} is finite, Eq. (24) provides an upper limit (which becomes better the larger l_{max}) for the relative changes in the cross section that are caused by arbitrary excursions of fixed length $||\delta s||$ away from the scattering amplitude. Hence (24) confirms what I observed in the numerical studies: at the backward angles the cross section is the most sensitive to small changes in the scattering amplitude. Turning this around, (24) implies that a model that accurately reproduces data at the backward but not the forward or intermediate angles is likely to yield a worse amplitude then a model that accurately describes all the data except those at the very backward angles.

CONCLUSIONS AND RECOMMENDATIONS FOR EXPERIMENTS

I have tried to show in this paper that a fit to elastic scattering angular distributions, using the many partial wave amplitudes as free parameters, can be obtained with little effort and can provide an excellent description of the data. An analysis of statistical errors can also easily be performed and is not quickly spoiled by nonlinearities. Although I have not studied systematic errors here, I have pointed out that one can deal with some of those straightforwardly within the presented framework. Despite the fact that errors in the partial wave amplitudes are quite large, it turns out that the total amplitude is well determined at angles where measurements are available. A careful study of ambiguities has to be performed and will be the subject of a future publication. Preliminary investigations seem to suggest though that ambiguities will only substantially affect the amplitude at backward angles.

Experimental attention should be more focused on small scattering angles to increase the reliability of an unbiased determination of the partial-wave amplitudes. It is very probably difficult to adhere to the "ideal" accuracy for cross-section measurements as expressed in Eq. (24). However, any experimental information at forward angles would already be of much interest.

In case a model is not capable of providing a consistent description of elastic scattering data ($\chi^2 \gg 1$), it is very likely that a better agreement with the (unknown) scattering amplitude is obtained if a least squares fit is performed *after* the measured error $e(\theta_i)$ is replaced by $\text{const} \times [\sigma(\theta_i)/\sin\theta_i]^{1/2}$.

The methods outlined in this paper can be extremely powerful if used in combination with scattering models. For instance, amplitudes of partial waves that are only affected by the tail of the nuclear potential may be reliably obtained from an optical potential. Models may provide information on the behavior of the nuclear part of the amplitude as $\theta \to 0$ and, hence, ease the problem of the lack of data in that region. If a model description of an angular distribution is obtained, one can study corrections to the model's S matrix using the algorithm presented here. Studies in that spirit to identify resonances have already been performed.[9] They can be extended to include more partial waves and an error analysis can provide information on the soundness of reached conclusions.

ACKNOWLEDGMENTS

I wish to thank Hay Yiung Cheung, Steve Sterbenz, Peter Braun-Munzinger, and Alex Lande for stimulating discussions. This work was supported in part by U.S. Department of Energy Contract DE-AC02-76ER03074.

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