Quantal equivalent local potentials from resonating group method n+α nonlocal interactions

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A quantal method to construct equivalent local potentials for the resonating group method nonlocal kernels is presented. The method requires two linearly independent solutions of the resonating group method integrodifferential equation. This Wronskian equivalent local potential is generally smooth, well behaved at all energies and partial waves, and is free of singularities. Therefore it can be used for interpreting purposes where the semiclassical method of Horiuchi either fails (low energies, small size of nuclei, comparable ranges of nonlocality and nucleus, etc.) or is multivalued. The necessity and importance of renormalization of the wave functions are discussed. The kernel need not be renormalized. Results are given for an n+α system and the l, E, r, and parity dependence of the system is investigated.

I. INTRODUCTION

Equivalent local potentials (ELP's) have been widely used to gain an intuitive physical understanding of complicated nonlocal interactions, in particular, those obtained in the microscopic resonating group model (RGM) approach. In most investigations the trivial equivalent local potential (TELP) or semiclassical Wentzel-Kramers-Brillouin (WKB) approximations have been successfully used. Examples of successes of these ELP's in extracting important information and features of internuclei interactions in investigations which employ the RGM have been the following.

(i) The elucidation of the parity dependence of the internucleus potential by Tang and co-workers by means of ELP's obtained in the Born approximation[1] or by using the TELP.[2]

(ii) The discovery of the orthogonality of the RGM wave functions to the (almost) Pauli forbidden states and the subsequent development of the orthogonality condition model (OCM) by Saito[3] and the development of ELP's based on the OCM.[4,5] These have played an important role in cluster model studies of light nuclei.

(iii) Horiuchi's more recent derivation of a semiclassical ELP (Ref. 6) for RGM intercluster interaction which enabled him and his collaborators to show how the basic properties of the RGM nonlocal potentials manifest themselves in terms of a more easily visualized and intuitively understandable local potential.[7] In particular, questions concerning the parity dependence of the ELP (confirming the results of Refs. 1 and 2), the size and shape of different exchange contributions in the RGM and their energy dependence, and the controversy concerning deep and shallow potentials[8] have been elucidated in this way.

Despite these successes, however, both the TELP and the WKB approximations have their drawbacks. The former, although exact, is usually ill behaved and therefore, in general, cannot be used for interpreting the nonlocal interaction. The TELP has singularities at the node of the wave function (which is real in the RGM case) and in addition rapidly varies as a function of all variables r, l, and E. Consequently, it does not lend itself to an intuitive and clear interpretation of the intercluster interaction. This is even the case for nonlocalities which are considerably simpler than those occurring in the RGM studies, e.g., the Frahn-Lemmer-type[9] of nonlocality which has been widely used in nuclear reactions.

On the other hand, it has been found that the ELP in the WKB approximation reproduces the RGM phase shifts rather accurately.[10] Usually, changes of the potential strength within 5% suffice for a nearly exact reproduction even for very light systems like x+^{4}He (where x is any Os-shell nucleus).[11] The one exception is the lightest system, i.e., the n+^{4}He scattering in which case a modification of about 30% in the strength of the potential was required for the s wave.[11,12] In other cases at low energies the WKB approximation even breaks down and results in multivalued potentials.[13] In such cases an exact quantal determination of the ELP is required.

To circumvent the aforementioned difficulties, an exact or quantal ELP can be constructed using the Wronskian of any two independent solutions of the nonlocal interaction.[14,9] This quantal Wronskian ELP is generally smooth, well behaved, and free of singularities. Further, it has a close relationship with the ELP of Horiuchi derived in the partial wave WKB approximation,[6] as has been demonstrated in Ref. 9 for Frahn-Lemmer-type nonlocality. In that case Horiuchi's ELP calculated in the three-dimensional WKB approximation reduces to the ELP of Perey and Buck.[15] If the WKB approximation is applied to the nonlocal interactions after the partial wave expansion, the resulting ELP's have a stronger l dependence and at higher energies closely approximate the Wronskian ELP. The potentials derived in the partial wave and three-dimensional WKB approximations closely resemble each other in the outer regions, but differ inside the classical turning point. However, their phase shifts do not differ much. In general the partial wave ELP's are more accurate.

The Wronskian ELP has been recently applied[16] to n+α scattering at 10 and 20 MeV using the semimicroscopic nonlocal n+α interactions of Lassaut and Vinh

Mau for comparison with the approximate ELP's of Horiuchi in the partial wave WKB approximation, the Perey-Buck, and the Peirls-Vinh Mau[17] approximations. Such comparison was possible since the $n+\alpha$ interaction of Lassaut and Vinh Mau can be written as a sum of Frahn-Lemmer-type terms. The ELP of Horiuchi (three-dimensional WKB approximation) although inadequate even in this case, does not fail to the same extent as for the RGM interaction which has a much more complicated nonlocal structure. For an $n+\alpha$ interaction of the RGM type (or even more complicated ones), the quantal method therefore appears to be the only suitable alternative at low energies.

In this paper we consider a one-channel RGM kernel for $n+\alpha$ scattering employed by Thompson and Tang[1] to determine the ELP's phase equivalent to the RGM potentials in the Born approximation. In earlier work Thompson *et al.*[2] employed the TELP to interpret their results. To get rid of the singularities a hard core with a radius given by the outermost node of the wave function is employed, i.e., the region where the clusters overlap, was thus avoided. Since, however, this is precisely the region of greatest physical interest, we prefer to employ the Wronskian ELP in this paper at energies where Horiuchi's WKB potential fail, for purposes of physical interpretation. It also has the advantage of allowing the calculation of the damping factor. Such an approach was also advocated by Ali, Ahmad, and Ferdous[18] in their recent and very comprehensive survey of the literature concerning the $n+\alpha$ interaction. For the physical interpretation it is particularly useful that the Wronskian ELP's tend to the partial wave WKB equivalent local potentials of Horiuchi at higher energies. This allows one to present a unified physical picture over the whole energy range.

In Sec. II the formalism of the RGM $n+\alpha$ interaction is briefly summarized, the renormalization of the wave function is discussed, and the method of constructing the Wronskian ELP is presented. The results of the calculations are presented in Sec. III and their physical interpretation is discussed. Our conclusions are summarized in Sec. IV.

II. FORMALISM

The work of Tang and collaborators[1,2] indicates that the single channel RGM calculation of $n+\alpha$ scattering results only in a moderately good fit to the data and has been superseded by coupled channel RGM calculations. Noncentral nucleon-nucleon forces have also been used.[19] The wave function of the $n+\alpha$ system is assumed to be

$$\Psi_{n\alpha}=A[\phi_\alpha\chi(\mathbf{R}_\alpha-\mathbf{r}_5)\zeta(\sigma,\tau)] , \tag{2.1}$$

where A is the antisymmetrization operator and ϕ_α describes the spatial behavior of the α cluster

$$\phi_\alpha=\exp\left[-\tfrac{1}{2}a\sum_{i=1}^{4}(\mathbf{r}_i-\mathbf{R}_\alpha)^2\right] . \tag{2.2}$$

The width parameter a is determined by the fit to the α-particle root-mean-square (rms) radius while $\mathbf{R}_\alpha$ is the position vector of the c.m. of the α cluster. The function $\chi(r)$ describes the relative motion of the $n+\alpha$ system and is determined from the variational principle

$$\delta\int\Psi^*_{n\alpha}(H-E_T)\Psi_{n\alpha}d\tau=0 , \tag{2.3}$$

where H is the Hamiltonian of the system and E_T the total energy. The nucleon-nucleon interaction is given by

$$V_{ij}=-V_0e^{-\lambda r_{ij}^2}(w+mP^r_{ij}+bP^\sigma_{ij}-hP^r_{ij}) \tag{2.4}$$

with

$$V_0=72.98\text{ MeV},\quad \lambda=0.46\text{ fm}^{-2}$$

and

$$w+m+b+h=1 ,$$

$$w+m-b-h=0.63 .$$

For simplicity, a pure Serber force is taken as in Ref. 1, i.e., $w=m$, $b=h$, and the Coulomb interaction has been omitted.

From Eq. (2.3) one obtains the integral equation

$$\int[H(\mathbf{R},\mathbf{R}')-EN(\mathbf{R},\mathbf{R}')]\chi(\mathbf{R}')d\mathbf{R}'=0 \tag{2.5}$$

or equivalently

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2+V_D(\mathbf{R})-E\right]\chi(\mathbf{R})+\int K(\mathbf{R},\mathbf{R}')\chi(\mathbf{R}')d\mathbf{R}'=0 , \tag{2.6}$$

where E is the relative energy of the two clusters in the c.m. system, i.e., $E=E_T-E_\alpha$ and $N(\mathbf{R},\mathbf{R}')$ represents the norm kernel while $V_D(R)$ represents the direct potential. In operator notation we have

$$(H-EN)\chi=0 . \tag{2.7}$$

Explicit expressions for $K(\mathbf{R},\mathbf{R}')$ in the case of $n+\alpha$ scattering are given in Ref. 2. From this kernel, Thompson *et al.*[1] constructed ELP's which in the Born approximation yield the same scattering amplitude as the RGM calculations.

The partial wave counterpart of Eq. (2.6) is

$$\left\{\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]+E-V_D(r)\right\}x_l(r)=\int_0^\infty K_l(r,r')x_l(r')dr' . \tag{2.8}$$

A wave function equivalent ELP, the so-called TELP, can then be constructed as follows:

$$V_l^{\text{TELP}}(E,r)=V_D(r)+\frac{1}{x_l(r)}\int_0^\infty K_l(r,r')x_l(r')dr' , \tag{2.9}$$

to assist in the interpretation of the nonlocal RGM interaction, and was employed by Thompson *et al.*[2] for this purpose. However, the TELP as mentioned, apart from the singularities arising from the zeros of the (real) function $x_l(r)$, in general, shows a rapid and irregular dependence on E, l, and r. Thompson *et al.*[2] used, instead, a phase equivalent potential defined by

$$V_l^*(E,r)=\begin{cases} V_l^{\mathrm{TELP}}(E,r), & r>r_0 \\ +\infty, & r<r_0 \end{cases} \tag{2.10}$$

where r_0 is the position of the outermost singularity of the TELP.

It is clear that with this choice the most interesting region where the clusters overlap can be excluded (inside the repulsive core). The Born ELP's proved to be more useful in this respect. A further difficulty with the TELP of Eq. (2.9) arises from the fact that the wave functions of Eqs. (2.6) and (2.9) do not have the correct probabilistic interpretation, i.e., strictly speaking, the RGM TELP is not off shell equivalent. In the single channel case one can obtain

$$\langle\Psi\,|\,\Psi\rangle=\langle\chi\,|\,N\,|\,\chi\rangle\ , \tag{2.11}$$

with $N=1-N^{\mathrm{ex}}$ being the overlap kernel and N^{ex} the norm kernel. It is seen that the function

$$\chi_N=\sqrt{1-N^{\mathrm{ex}}}\chi \tag{2.12}$$

has the proper probability density interpretation. A elegant discussion on this matter was given by Schmid in a review talk at the Karlsruhe conference.[20] Before we describe how this problem is tackled we briefly describe how the E- and l-dependent Wronskian ELP is constructed. By employing the Wronskian of two independent solutions of the integrodifferential equation (2.8), which are taken to be the regular $u_l(r)$ and irregular $v_l(r)$ solutions, we construct[9,14]

$$V_l^{\mathrm{ELP}}(r)=-\frac{1}{2}\frac{W_l''(u,v)}{W_l(u,v)}+\frac{3}{4}\left[\frac{W_l'(u,v)}{W_l(u,v)}\right]^2+\frac{1}{W_l(u,v)}\int_0^\infty K_l(r,r')[u_l'(r)v_l(r')-u_l(r')v_l'(r)]dr'\ . \tag{2.13}$$

While for the non-RGM kernels the construction of the ELP is straightforward and without any serious numerical difficulties, this is not the case for the RGM cases. The local equivalent obtained with the unnormalized [according to Eq. (2.13)] is wildly behaved and has a Wronskian W which can be even negative. Thus is constructing the Wronskian ELP—as well as the trivial—the properly normalized solutions must be used.

For this we write (we omit the l)

$$\begin{aligned} u_N(r)&=(1-N^{\mathrm{ex}})^{1/2}u(r)\ , \\ v_N(r)&=(1-N^{\mathrm{ex}})^{1/2}v(r)\ . \end{aligned} \tag{2.14}$$

Following Saito we define[21]

$$N^{\mathrm{ex}}\phi_\alpha=\eta_\alpha\phi_\alpha\ . \tag{2.15}$$

Eigenvalues with $\eta_\alpha=1$ are the Pauli forbidden states (FS's) while those with $\eta_\alpha<1$ are the partly Pauli forbidden states (PFS's). We may now rewrite Eq. (2.14) as follows:

$$\begin{aligned} u_N(r)&=u+(\sqrt{N}-1)u \\ &=u+\sum_\alpha[(1-\eta_\alpha)^{1/2}-1]\,|\,\phi_\alpha\rangle\langle\phi_\alpha\,|\,u\rangle\ , \end{aligned} \tag{2.16}$$

and similarly for v_N. This procedure is quite practical for cases where u_N does not differ much from u. We found, however, that the irregular solution $v(r)$ has a completely different behavior than $v_N(r)$; the use of (2.16) in this case gave rise to numerical problems. Instead, the following renormalization procedure has been adopted. We firstly define the function $\tilde{u}$ by

$$\tilde{u}=Nu\ , \tag{2.17}$$

which is free from δ-function (square root) singularities. Then we get

$$\begin{aligned} \tilde{u}&=\sqrt{N}\,u_N \\ &=u_N+\sum_\alpha[(1-\eta_\alpha)^{1/2}-1]\,|\,\phi_\alpha\rangle\langle\phi_\alpha\,|\,u_N\rangle\ . \end{aligned} \tag{2.18}$$

Noting that for the Pauli forbidden states, $\langle\phi_\alpha\,|\,u_N\rangle=0$, we get

$$\tilde{u}=u_N+\sum_{\substack{\alpha \\ \eta_\alpha\neq1}}[(1-\eta_\alpha)^{1/2}-1]\,|\,\phi_\alpha\rangle\langle\phi_\alpha\,|\,\sqrt{N}\,|\,u\rangle\ . \tag{2.19}$$

Thus finally

$$u_N=\tilde{u}-\sum_{\substack{\alpha \\ \eta_\alpha\neq1}}A_\alpha\,|\,\phi_\alpha\rangle\ , \tag{2.20}$$

with

$$A_\alpha=[(1-\eta_\alpha)^{1/2}-1]\sqrt{1-\eta_\alpha}\langle\phi_\alpha\,|\,u\rangle\ . \tag{2.21}$$

The advantage of Eq. (2.20) over (2.16) is obvious. The Pauli forbidden states which contribute to much of the difference between u_N and u are included exactly by means of Eq. (2.17), while using Eq. (2.20) one makes the refinements stemming from the PFS. Further, while in (2.16) we renormalize by modifying u (which might differ considerably from u_N, as is the case for the irregular solution), in Eq. (2.20) we start from $\tilde{u}$ which always is a good approximation to u_N.

One further point should be made here. Equation (2.13) is applicable when not only the wave functions $u_l(r)$ and $v_l(r)$ are renormalized but also the kernel $K_l(r,r')$. To overcome this difficulty we employ instead the relation

$$V^{\mathrm{ELP}}(r)=k^2-\frac{l(l+1)}{r^2}-\frac{1}{2}\frac{W''(u,v)}{W(u,v)}+\frac{3}{4}\left[\frac{W'(u,v)}{W(u,v)}\right]^2-\frac{W(u',v')}{W(u,v)}\ . \tag{2.22}$$

The derivation of this relation is given in the Appendix.

The nonlocal wave function $u_l^{\rm NL}$ can then be recovered by using the solution

$$u_l^{\rm NL}(r)=f_l(r)u_l^L(r)\ , \tag{2.23}$$

where $u_l^L(r)$ is the regular wave function corresponding to the effective local interaction $V_l^{\rm ELP}(r)$ and $f_l(r)$ the damping (or Perey) factor,

$$f_l(r)=\sqrt{W(u_N,v_N)}\underset{r\to\infty}{\to}1\ . \tag{2.24}$$

In other words, the $V^{\rm ELP}$ can provide us not only with on-shell information but also, by means of Eq. (2.23), the wave function of the system. However, a wave function equivalent velocity dependent potential (EVDP) satisfying the velocity-dependent wave equation can be readily obtained. Recall first the equivalent form of Eq. (2.6).

$$\left[\nabla\frac{m}{M(r)}\nabla+k^2\right]\chi(\mathbf{r})=V^{\rm EVDP}(\mathbf{r})\chi(\mathbf{r}) \tag{2.25}$$

with the local part (in its partial wave form) given by[9]

$$V_l^{\rm EVDP}(r)=\frac{m}{M_l(r)}\left\{V_l^{\rm ELP}(r)+\frac{1}{2}\frac{M_l''(r)}{M_l(r)}-\frac{3}{4}\left[\frac{M_l'(r)}{M_l(r)}\right]^2-k^2\left[1-\frac{M_l(r)}{m}\right]+\frac{1}{r}\frac{M_l'(r)}{M_l(r)}\right\}. \tag{2.26}$$

For the RGM kernels we derive [we work as in the case of Eq. (2.22)]

$$V_l^{\rm EVDP}(r)=k^2-\frac{1}{W(u,v)}\times\left[\frac{l(l+1)}{r^2}+\frac{W(u',v')}{W(u,v)}-\frac{1}{r}\frac{W'(u,v)}{W(u,v)}\right], \tag{2.27}$$

which is of the form used in nuclear optical model calculations.

In conclusion, we may construct on- and off-shell equivalent interactions using the (renormalized) regular and irregular solutions and their derivatives only. In order to get these solutions, Eq. (8) is transformed first into an integral equation of Voltera type for u_l and v_l, or equivalently for the Jost solutions,[22]

$$f_l^{\pm}(k,r)=f_l^{\pm(0)}(k,r)-\int_r^\infty dr' g_l(k,r,r')V_D(r')f_l^{\pm}(k,r')-\int_r^\infty dr' g_l(k,r,r')\int_0^\infty dr'' K_l(r',r'')f_l^{\pm}(k,r'') \tag{2.28}$$

from which we get

$$u_l(k,r)=\frac{1}{2ik}(F_l^-f_l^+-F_l^+f_l^-) \tag{2.29}$$

with

$$V_l(k,r)=\frac{1}{2|F_l(k)|^2}(F_l^+f_l^-+F_l^-f_l^+)\ , \tag{2.30}$$

$$F_l(k)=F_l^+e^{i\delta l^{(k)}}\ . \tag{2.31}$$

The Jost solutions $f_l^{\pm(0)}(k,r)$, Green's functions $g_l(k,r,r')$, and the Jost function $F_l^{\pm}(k)$ are given in Ref. 25 and will not be repeated here.

III. RESULTS

The Schmid-Wildermuth parametrization[23] of the two-body interaction, Eq. (2.4), has been used. In order to avoid unessential numerical complications the Coulomb interaction has been omitted while the width parameter a was taken to be 0.514 fm^{-2}, which gives the correct value of 1.48 fm for the α-particle rms radius and a binding energy of -27.41 MeV, which is quite close to the experimental value. The norm kernel eigenvalues are[24] $\eta_\alpha=(-\frac{1}{4})^\alpha$ with $\alpha=0,2,4$ for even partial waves and $\alpha=1,3,5,\ldots$ for odd partial values. In all calculations the first five were used for renormalization, the rest being too insignificant.

A typical example for a trivial ELP obtained using the unnormalized wave function [Eq. (2.9)], as well as the one obtained after renormalization, is shown in Fig. 1 for $E=10$ MeV and for $l=0$. It is seen that there is a broad singularity in the contact region which rules out the use of TELP without employing any arbitrary extrapolation across the singularity. Further, the difference between the normalized and unnormalized TELP is remarkable; while in the interior region the normalized TELP is more at-

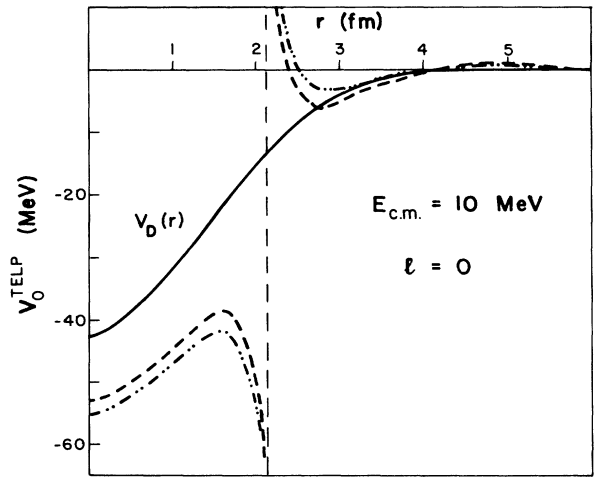

FIG. 1. The trivial equivalent local potential for $l=0$ and $E_{\rm c.m.}=10$ MeV before (— — —) and after (——··——) renormalization.

tractive and only about 6% deeper, in the important region of 3—4 fm the renormalized TELP is comparatively shallower by more than 50%. For more complicated nonlocal kernels, such as those of $\alpha+\alpha$ or ^{3}He+^{4}He, one must expect amplification of such phenomena. For higher partial wave channels, where there are no forbidden states, the renormalized and unnormalized TELP's are quite close to each other except in regions close to singularities.

In Fig. 2 we present the results for Wronskian ELP for $l=0$ and for c.m. energies 10, 20, and 50 MeV. As already mentioned, it was impossible to obtain an ELP before renormalization of the wave functions, the reason being the behavior of the irregular wave function in the interior region. We shall return to this point later. The difference between the V_D and the ELP is quite large emphasizing once more the importance of the antisymmetrization effects. The energy dependence is important in the outer region of the ELP and confirms the results obtained previously by Horiuchi concerning the energy dependence of the interaction and the question of deep and shallow potentials.[8] The characteristic dip of the potentials is around the position of the node of the wave function (singularity of the TELP). One of the main concerns in constructing an ELP is to prohibit the relative wave function from occupying an FS which is equivalent to orthogonalizing the wave function to the FS. This orthogonalization must be reflected in a deep local potential sufficient to support all the FS and in accordance with the Levinson theorem, the latter demanding that the relation $\delta_l(0)-\delta_l(\infty)=(\eta_B+\eta_F)\pi$, where the η_B are the physical bound states and η_F are the FS's. It is noted that similar dips in the interaction have been obtained in Ref. 16 using the nonlocal n+α potential of Lassaut and Vinh Mau[25] calculated in an antisymmetrized folding model for different effective nucleon-nucleon interactions. However, while the node of the wave function is around $r=2$ fm the characteristic dip was in the region of $r\simeq 1$ fm, which means that the dip is not related to the position of the wave function node, but from the dynamics of the nonlocality considered. It is interesting to observe the repulsion which appears in the range 3.5—6 fm at all energies, which, in turn, causes the effective mass $M(r)$ to be slightly greater than m in the same region.

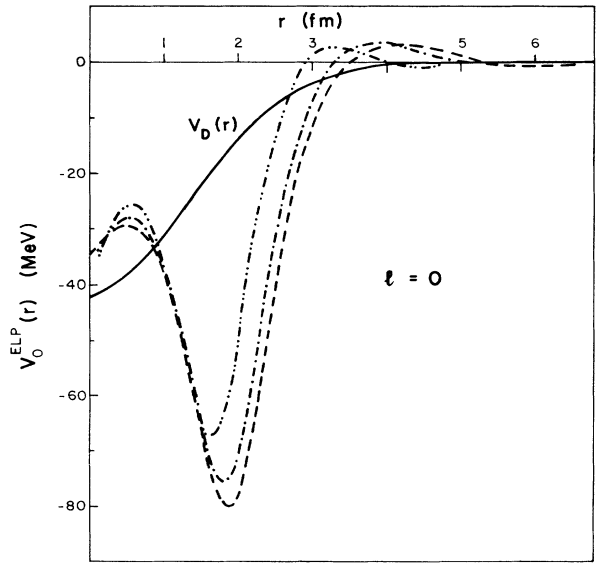

FIG. 2. The Wronskian equivalent local potential for $l=0$ and for energies $E=10$ MeV (— — —), $E=20$ MeV (—·—·—·), and $E=50$ MeV (——··——).

In Fig. 3 the results obtained for the $l=1$ channel are presented for the same energies, i.e., 10, 20, and 50 MeV. The (renormalized) TELP at 20 MeV is also shown. We notice that the energy dependence is rather weak in this channel, but shows the same trends as in the $l=0$ case, i.e., insignificant energy dependence in the interior region, and that higher energies give shallower potentials. Note that the exact ELP and the trivial ELP differ considerably in the interior region ($\lesssim 1.5$ fm). The l dependence and parity dependence are shown in Fig. 4 for $E_{\text{c.m.}}=10$ MeV and for partial waves $l=0$, 1, and 2. The l dependence is quite important, while the parity dependence is limited to the outside region where the $l=0$ and 2 interactions exhibit the same behavior, namely, the height of the repulsion and its spatial extension is the same.

The regular wave function results for $l=0$ and $E=20$ MeV are given in Fig. 5 together with the corresponding Wronskian $W_{l=0}(r)$. The $u_{l,N}(r)$ represents the nonlocal wave function after being renormalized as described in Sec. II. It is seen that the $u_{l,N}$ and the unnormalized wave function $u_l(r)$ do not differ much. Although the $u_l^{\text{ELP}}(r)$ differs from the exact wave function for the system, the $u_{l,N}(r)$ can be recovered from $u_l^{\text{ELP}}(r)$ using the relation (2.23).

It is worth noting that the nodes of all wave functions are preserved. Similar results are obtained at all energies. For $l=1$ the difference between the various wave functions is much less. This is due to the fact that in this channel only PFS's are present. For the $l=0$ case the po-

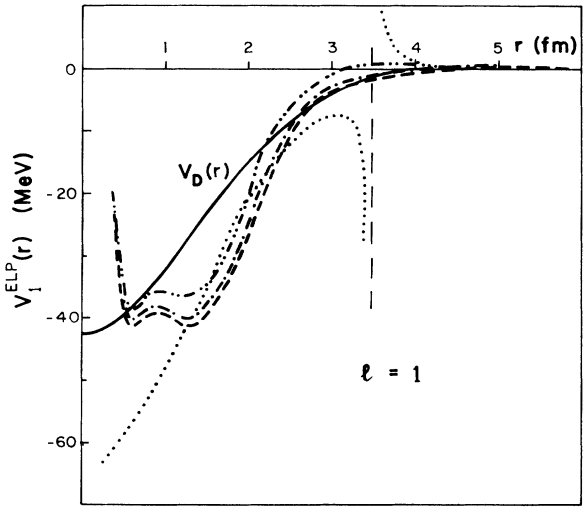

FIG. 3. Same as for FIG. 2 but for $l=1$. The trivial ELP (· · · ·) for 20 MeV is also shown.

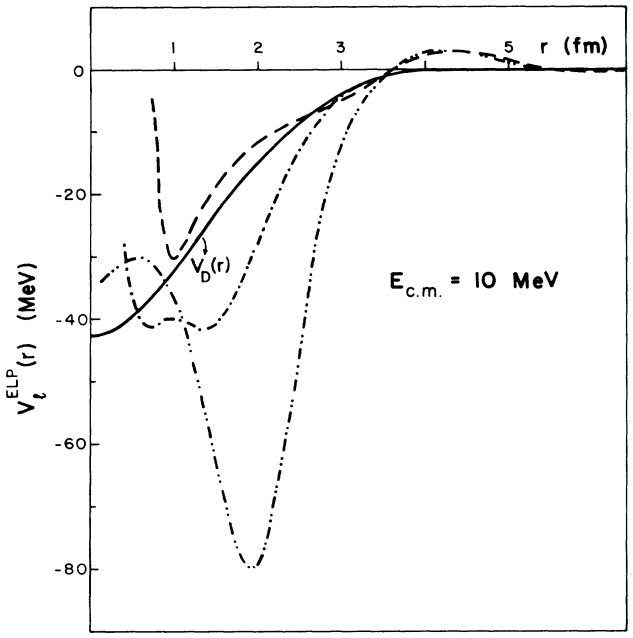

FIG. 4. The Wronskian equivalent local potential for $l=0$ (——··——), $l=1$ (—·—·—·), and $l=2$ (— — —). The l dependence and parity dependence ($r \gtrsim 3.6$) should be noted.

sition of node exhibits very weak energy dependence, the difference between the 1 and 50 MeV cases being ~ 0.2 fm. This is not the case with the $l=1$ channel, where for the 1 MeV case no node appears in the region less than 10 fm, while for higher energies it was found that one or two nodes appear having a strong energy dependence.

In Fig. 6 the behavior of the irregular solution for $l=0$ and $E=20$ MeV is shown. The unnormalized wave function differs completely in the interior region where even

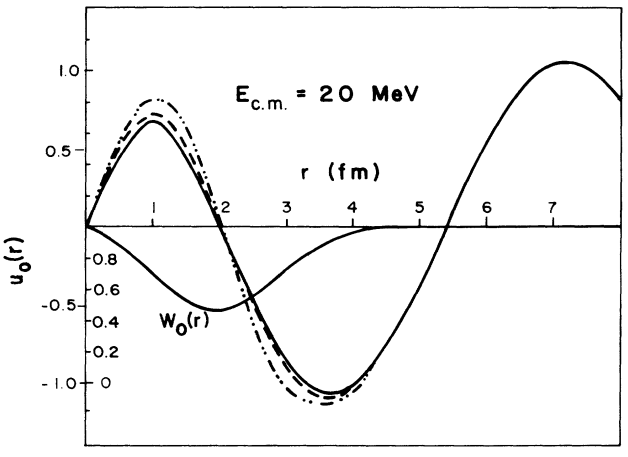

FIG. 5. Regular wave function and Wronskian results for $l=0$ and $E=20$ MeV: the normalized nonlocal $u_{l,N}$ (——), the unnormalized $u_l(r)$ (— — —), and the ELP $U_l^L(r)$ (——··——) wave functions. The Wronskian results are plotted in the lower region for comparison.

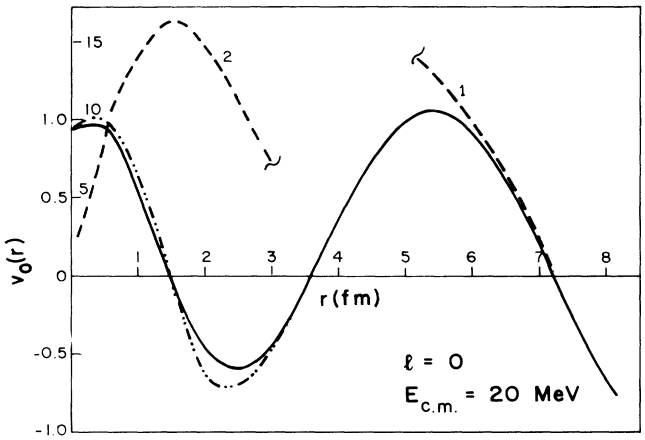

FIG. 6. Same as for Fig. 5 but for the irregular solution. The interior scale in the upper half corresponds to part 2 of the $v_l(r)$ (— — —).

the node is not present and with large values for its magnitude. However, after normalization the irregular solution has normal behavior. As a consequence of such large deviations no Wronskian could be constructed and hence no ELP from Eq. (2.13) could be obtained. Furthermore, the iterative procedure in solving the RGM integrodifferential equation fails due to this behavior of the irregular solution.

Finally we mention that the phase shifts obtained by solving Eq. (2.8) are practically the same as the one given in Ref. 2, while the phase shifts obtained using V^{ELP} differ from the exact values by less than 0.5% the small difference being attributed to purely numerical inaccuracy as well as to the fact that only five terms have been used in the renormalization series, Eq. (2.20).

IV. CONCLUSION

Our main conclusion is that the use of the fully quantal Wronskian is not only possible for the RGM nonlocal interactions but also unavoidable when the WKB approximation of Horiuchi either fails (at low energies, small size of nuclei, comparable ranges of nonlocality and nucleus, etc.) or is multivalued.

The other alternative, namely the construction of the TELP (although on- and off-shell equivalent to the original nonlocal interaction), is only of academic interest because of the singularities occurring in the overlap region and its irregular behavior. The latter point is clearly demonstrated in Fig. 1, where information about the physically interesting region cannot be obtained from the TELP. An arbitrary extrapolation or the introduction of hard cores by Thompson *et al.*[2] to construct an effective interaction is not a solution as both are not based on an exact dynamical calculation and, in addition, the off-shell equivalence is destroyed and, unlike the case of the Wronskian ELP, cannot be restored by employing the damping factor.

The constructed ELP's are all smooth at all energies and partial waves and easily reveal the intercluster charac-

teristics and the dependence on l, E, and r. The energy dependence of the interaction is clearly demonstrated in Figs. 2 and 3, while the l and parity dependences are shown in Fig. 4. The r dependence for $l=0$ and $l=1$ is also interesting. For $l=0$ there is a characteristic dip in the interaction in the region 1.6—2.0 fm. This attraction seems to be necessary to produce the Pauli forbidden state. In $l=1$ case, where there is no forbidden state, a repulsive core appears at very short distances—a phenomenon which was previously observed in the investigation of the Frahn-Lemmer-type nonlocality for the n + Ca system and for $l>0$ (Ref. 9) as well as in the study of the semimicroscopic nonlocal n+α interactions of Lassaut and Vinh Mau.[16]

We must emphasize here the importance of the renormalization procedure; for $l=0$ the unnormalized Wronskian varies wildly and no effective local interaction can be constructed. The cause of such behavior is found to be the irregular wave function which displays completely different characteristics in the interior region as compared to the characteristics of the renormalized one. For the same reason the Voltera-type equation was employed in order to obtain the two independent solutions $u_l(r)$ and $v_l(r)$.

The fact that the constructed ELP is not off-shell equivalent is not a major difficulty since the ELP is mainly intended for interpretation purposes. When, however, the off-shell behavior is needed, a wave function equivalent velocity dependent interaction can be constructed using Eq. (2.27).

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APPENDIX

We may use the relations

$$W=f^2, \quad \frac{W'}{W}=2\frac{f'}{f},$$

and

$$\frac{W''}{W}=2\left[\frac{f'}{f}\right]^2+2\frac{f''}{f}$$

to rewrite the Wronskian terms of Eq. (2.13) as follows:

$$-\frac{1}{2}\frac{W''}{W}+\frac{3}{4}\left[\frac{W'}{W}\right]^2=2\left[\frac{f'}{f}\right]^2-\left[\frac{f''}{f}\right], \tag{A1}$$

and thus

$$I\equiv\frac{1}{W}\int_0^\infty K(r,r')[u'(r)v(r')-u(r')v'(r)]dr'$$
$$=V^{\mathrm{ELP}}(r)-2\left[\frac{f'}{f}\right]^2+\left[\frac{f''}{f}\right]. \tag{A2}$$

On the other hand, using Eq. (2.8) we get

$$I=-\frac{W(u',v')}{W(u,v)}+\left[k^2-\frac{l(l+1)}{r^2}\right]. \tag{A3}$$

It is easy to see that Eqs. (A2) and (A3) are identical. For this we set

$$u(r)=f(r)u_L(r),$$
$$v(r)=f(r)v_L(r), \tag{A4}$$

where u_L and v_L are the regular and irregular solutions for the interaction $V^{\mathrm{ELP}}(r)$. Using Eq. (A4) the Wronskian $W(u',v')$ can easily be evaluated:

$$W(u',v')=ff''(u_Lv_L'-v_Lu_L')+2f'^2(u_L'v_L-v_L'u_L)$$
$$+ff'(u_L''v_L-v_L''u_L)+f^2(u_L''v_L'-v_L''u_L'). \tag{A5}$$

Thus, using the relations

$$W(u_L,v_L)=1, \quad W'(u_L,v_L)=0,$$

and

$$u_L''v_L'-v_L''u_L'=k^2-\frac{l(l+1)}{r^2}-V^{\mathrm{ELP}}, \tag{A6}$$

we get

$$-\frac{W(u',v')}{W(u,v)}+\left[k^2-\frac{l(l+1)}{r^2}\right]=\frac{f''}{f}-2\left[\frac{f'}{f}\right]^2+V^{\mathrm{ELP}}, \tag{A7}$$

or using (A1) we finally get

$$V^{\mathrm{ELP}}(r)=k^2-\frac{l(l+1)}{r^2}-\frac{1}{2}\frac{W''(u,v)}{W(u,v)}$$
$$+\frac{3}{4}\left[\frac{W'(u,v)}{W(u,v)}\right]^2-\frac{W(u',v')}{W(u,v)}. \tag{A8}$$

In this way the nonlocal kernel is eliminated from the calculation in constructing the quantal equivalent local potential.

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