PHYSICAL REVIEW C VOLUME 35, NUMBER 2 FEBRUARY 1987

Deser-Goldberger-Baumann-Thirring formula for π^-p atoms

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(Received 2 September 1986)

A recent measurement of the level shift in the π^-p atom obtains a value in apparent contradiction with the currently accepted π^-p scattering length. The inferred equivalence of level shift and scattering length involved the use of the Deser-Goldberger-Baumann-Thirring formula. We have checked this formula for the case of π^-p atoms and find that it is accurate well beyond the degree of the present discrepancy. Thus the disagreement between the recent experiment and previous analyses persists.

The Deser-Goldberger-Baumann-Thirring (DGBT) formulas

$$\delta E/E_0 = -4\,\mathrm{Re}a(\pi^-)/r_B\ , \tag{1a}$$

$$\Gamma/E_0 = 8\,\mathrm{Im}a(\pi^-)/r_B = 8qr_B\,|\,a(\pi^0\leftarrow\pi^-)/r_B\,|^2 \tag{1b}$$

are used routinely to relate the $1s$ hadronic energy shift (δE) and level width (Γ) of the π^-p atom to $a(\pi^-)$ and $a(\pi^0\leftarrow\pi^-)$, the elastic ($\pi^-\mathrm{p}\rightarrow\pi^-\mathrm{p}$) and charge exchange ($\pi^-\mathrm{p}\rightarrow\pi^0\mathrm{n}$) amplitudes at the π^-p threshold.[1] The quantities r_B ($=1/m\alpha$) and E_B ($=-\alpha/2r_B$) are the Bohr radius and energy of the 1-s level of the π^-p atom; m is the π^-p reduced mass, q is the center-of-mass momentum of the π^0n system at the π^-p threshold, and α is the fine-structure constant.

An extension of the DGBT formula has been given by Trueman,[2] who rederived the DGBT formula via an effective-range expansion about threshold. Trueman included the Coulomb field in the expansion, obtaining small electrostatic corrections omitted in Ref. 1. He also extended the formulas to arbitrary angular momenta. In a further extension Rasche and Woolcock[3] included explicitly the π^0n and γn channels through the K-matrix formalism.

Recent measurements of both the scattering lengths[4] and level shifts[5] of the π^-p atom are not in agreement with Eq. (1a) and its extensions. The discrepancy, noted in Ref. 5, is 3 standard deviations. Recent calculations in perturbative quantum chromodynamics (QCD) find better agreement with the scattering length inferred from the atomic shift measurement than with the more traditional values inferred from many measurements.[6] It is thus of importance to reexamine Eqs. (1) in the context of a coupled channel formalism such as that recently used to analyze low energy data.[7] The corrections due to vacuum polarization and relativity were examined in detail in Ref. 3 and will not be repeated here.

We check the validity of the DGBT formulas by exact (numerical) solution of a potential model which includes both the π^0n and γn reaction channels, the mass splitting between the different charge states, and the finite charge radius of the system.[8] The major corrections of Refs. 2 and 3 are thus included exactly, within the range of validity of the model.

The work is a continuation of the coupled-channel calculation of Siegel and Gibbs,[7] which provides a good representation of the low-energy πN scattering and photoproduction data as well as the Panofsky ratio. Instead of the separable potentials of Ref. 7 we use local potentials which are nearly equivalent at low energies. We have used both exponential and Yukawa potentials. Our conclusions are the same for both; for brevity we include numerical results only for the exponential. The potential ranges and strengths are given in Table I. These values are sufficiently accurate for a test of the validity of Eqs. (1).

We begin by calculating the left-hand side of Eq. (1a). The rest mass of the π^-p system is 3.3108 MeV greater than that of the π^0n; E_0, the Bohr energy of the $1s$ atomic state, is only 3.2 keV. Hence, the $1s$ atomic state of the π^-p system lies well above the π^0-n threshold. Because of the coupling to the π^0n and γn channel the atomic state is unstable; this leads to a very narrow resonance in the π^0-n scattering cross section. The position and width of the resonance peak of this (experimentally inaccessible) process give the level shift and width of the π^-p atom.

The coupled Schrödinger equations are solved numerically. The asymptotic form of the solution in the π^-p channel is a Whittaker function of the second kind (i.e., a decaying below-threshold Coulomb function); in the γn and π^0n channels the asymptotic forms are Hankel functions.

The results of the calculation are shown in Fig. 1. The

TABLE I. Strengths (S) and ranges (α) used in the exponential potential: $S\exp(-\alpha r)$.

$S(I=\frac{1}{2})=-135$ MeV	$\alpha(I=\frac{1}{2})=400$ MeV/c
$S(I=\frac{3}{2})=350$ MeV	$\alpha(I=\frac{3}{2})=600$ MeV/c
$S(\pi^-\mathrm{p},\gamma\mathrm{n})=11.5$ MeV	$\alpha(\pi^-\mathrm{p},\gamma\mathrm{n})=200$ MeV/c

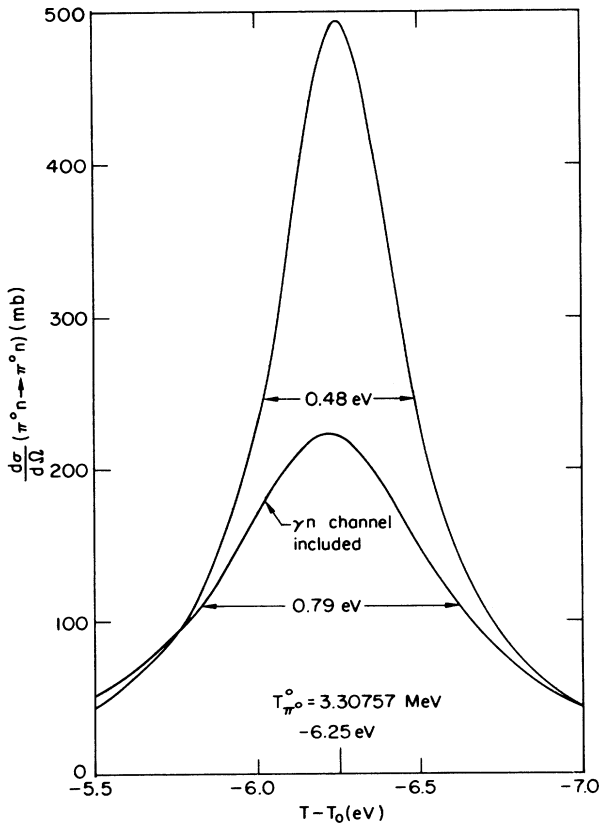

FIG. 1. π^0n elastic scattering for incident π^0 energy near the ground state energy of the π^-p atom. The effect of the γn channel is to lower the resonance peak causing Γ to increase. The position of the peak is nearly unchanged.

cross section is isotropic, as it is totally dominated by s-wave scattering. The abscissa is the π^0n center-of-mass kinetic energy measured relative to T_0 (=3.307 557 MeV), the kinetic energy of the π^-p (Bohr) ground state. The very large cross section is due to the low incident energy of the π^0.

To examine the influence of the γn channel we have computed π^0n elastic scattering with and without including it. The γn channel is seen to have little effect on the position of the peak (and hence the level shift) which remains $\delta E \simeq -6.25$ eV. The open γn channel removes flux from the elastic channel, lowering the peak value of the resonance. Consequently, the width, Γ, of the resonance increases from 0.48 to 0.79 eV when the γn channel is included.

The scattering length, needed for the right-hand side of Eq. (1a), was evaluated from the numerical solution of the coupled Schrödinger equations near threshold. There are several (not exactly equivalent) definitions of the scattering length depending on the manner of inclusion or neglect of the Coulomb force. The γn channel is included only in (e).

(a) Both the point-Coulomb interaction and the mass splitting between the channels are included. The scattering length is evaluated by fitting the near-threshold amplitude to an effective-range formula modified to include the Coulomb interaction as given in Ref. 2.

(b) Same as (a) but with the Coulomb interaction omitted. A coupled-channel calculation must still be performed. In this case the scattering length is simply the value of the π^--p amplitude at threshold; as in (a) it is easily evaluated by fitting the near-threshold amplitudes to an effective-range formula. The scattering length corresponding to the charge-exchange reaction, which is exothermic, is defined by Eq. (13) of Ref. 1.

(c) Same as (a) but with finite charge radius. The electrostatic potential in the π^-p channel corresponds to a uniformly charged sphere of radius 0.7 fm.

(d) Coulomb interaction and multiplet mass splitting omitted. In this case the equations are diagonal in the isospin basis and so are particularly easy to solve.

(e) The full three-channel calculation with analysis as in (a).

The results of our calculations for the five cases are presented in Table II. On the basis of these calculations we have confirmed that the DGBT formula is satisfactory for relating π^-p scattering lengths and $1s$ atomic parameters. Corrections due to multiplet mass splitting, Coulomb corrections to the scattering length, and finite

TABLE II. Scattering lengths, level shifts, and level widths. The rows correspond to the models used for calculating the scattering lengths as is described in the text. (pC—point Coulomb; ms—mass splitting included; fC—finite size Coulomb; nC—no Coulomb; nm—no mass splitting; γ—nγ channel included) The DGBT widths are calculated from $a(\pi^0 \leftarrow \pi^-)$ in (b) and (d) and from Im$a(\pi^-)=0.0029$ μ^{-1} for (a) and (c) and 0.0048 μ^{-1} in (e).

Model	$a(\pi^-)$ (μ^{-1})	$a(\pi^0 \leftarrow \pi^-)$ (μ^{-1})	$1s$ shifts DGBT (eV)	$1s$ shifts π^0n scattering (eV)	$1s$ widths DGBT (eV)	$1s$ widths π^0n scattering (eV)
(a)pC, ms	0.0759		−6.24	−6.25	0.48	0.48
(b)nC, ms	0.0768	0.118	−6.31		0.46	
(c)fC, ms	0.0756		−6.21	−6.22	0.48	0.48
(d)nC, ms	0.0750	0.119	−6.17		0.50	
(e)pC, ms, γ	0.0755		−6.21	−6.22	0.79	0.79
Pure isospin amplitudes (d):			$a_1=0.1590$ μ^{-1}			
			$a_3=-0.0931$ μ^{-1}			

size effects are at, or below, the 0.008 eV level.

If the discrepancy between the measured values of δE and Γ and the DGBT formula persists as the experimental tolerances are narrowed, this would seem to imply a variation in the π^-p amplitude between threshold and 3.1 keV below threshold which would be most unusual—and difficult to understand.

This work was supported by Associated Western Universities and the U.S. Department of Energy.

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