Algebraic analysis of physical and spurious states in Dyson boson mapping

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Dyson boson mapping of a system of $2n$ fermions distributed among k single-particle states uses a space generated by two-index ideal bosons. We define a unitary group U(k) to classify this ideal-boson space. The physical subspace is shown to correspond to the antisymmetric irreducible representation $[1^{2n}]$. This classification enables one to introduce a "Majorana" operator S which is a linear combination of one- and two-body Casimir operators of U(k). A zero eigenvalue of S characterizes the physical subspace while positive eigenvalues identify all other irreducible representations that occur. The Hermitian boson image of the Hamiltonian does not connect physical and spurious states because it is a function of the generators of U(k). All the nonphysical eigenvalues and eigenstates of a physical boson Hamiltonian (Dyson boson image of a physical fermion Hamiltonian) which could be Hermitian or non-Hermitian, can be removed without affecting the physical eigenvalues and eigenvectors, by adding a suitable multiple of S to the Hamiltonian.

The Dyson boson mapping method has been introduced to study low lying collective states.[1,2] Recently, it has been used to study the microscopic support for the interacting boson (IBM) model.[3] In order to define such a boson mapping, an ideal boson space is introduced. The physical boson states, i.e., those that are in one-to-one correspondence with the fermion states, span a subspace of the ideal boson space. Subsequently, working in the entire boson space, instead of the physical subspace, introduces the problem of identifying the nonphysical (spurious) states. The spurious states cannot easily be identified and removed except in certain special cases.[2–4]

Geyer *et al.*[5] have shown that even in the general case, diagonalization of a physical boson operator in the full ideal boson space still produces all the correct physical eigenvalues and their corresponding physical eigenvectors. In this paper we go one step further by introducing an operator S to remove all spurious states. This operator turns out to be effectively the same as the one introduced by Janssen *et al.*[1]

The ideal boson space B is defined using ideal boson creation operators $b^\dagger_{\alpha\beta}$ with corresponding Hermitian conjugate $b_{\alpha\beta}$. They have the following properties

$$b^\dagger_{\alpha\beta}=-b^\dagger_{\beta\alpha}\ ,\qquad [b_{\alpha\beta},b^\dagger_{\alpha'\beta'}]=\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'}\ , \tag{1}$$

where $\alpha,\beta=1,2,\ldots,k$. Then,

$$B=\mathrm{span}\{\,|0),b^\dagger_{\alpha\beta}\,|0),b^\dagger_{\alpha_1\beta_1}b^\dagger_{\alpha_2\beta_2}\,|0),\ldots\}\ . \tag{2}$$

To classify the basis states in B, let us find appropriate groups for which B forms a representation space. Let

$$G_1=\mathrm{span}\{b^\dagger_{\alpha\beta}b_{\alpha'\beta'}\}\ ; \tag{3}$$

then the generators of G_1 form a unitary algebra in $k(k+1)/2$ dimensions. Thus, $G_1=\mathrm{U}[k(k+1)/2]$. Since G_1 does not connect the boson spaces with different boson numbers, each boson space with a given number n of bosons is invariant under G_1. Since it is generated by boson operators, it belongs to the symmetric irreducible representation $[n]$ of G_1. Thus we have decomposed the ideal boson space B into subspaces

$$B_n=\mathrm{span}\left\{\prod_{i=1}^{n}b^\dagger_{\alpha_i\beta_i}\,|0)\right\}\ . \tag{4}$$

In order to distinguish physical and nonphysical boson spaces, it is useful to introduce a subalgebra G_2 of G_1 defined by

$$G_2=\mathrm{span}\left\{\sum_\gamma b^\dagger_{\alpha\gamma}b_{\beta\gamma}\right\}\ . \tag{5}$$

To determine the group structure of G_2, we exploit the isomorphic fermion algebra $\hat{G}_2$ whose boson image under Dyson boson mapping is G_2. As is clear from Eq. (9), the fermion algebra $\hat{G}_2=\mathrm{span}\{c^\dagger_\alpha c_\beta\}$ gives G_2 under the Dyson mapping. Here $c^\dagger_\alpha$ and its Hermitian conjugate c_α are fermion creation and destruction operators. Since $\hat{G}_2$ forms a unitary algebra U(k), G_2 also forms U(k). The tensor property of $b^\dagger_{\alpha\beta}$ with respect to G_2 is also found in a similar way. The fact that the fermion pair operator $c^\dagger_\alpha c^\dagger_\beta$ carries an irreducible tensor property $[1^2]$ with respect to $\hat{G}_2$ implies its boson image $[c^\dagger_\alpha c^\dagger_\beta]_D\equiv B^\dagger_{\alpha\beta}$ also carries the same tensor property with respect to G_2. Also, it is simple to show that $B^\dagger_{\alpha\beta}$ and $b^\dagger_{\alpha\beta}$ have the same commutation relations with respect to the generators of G_2. Thus we have shown that $B^\dagger_{\alpha\beta}$ and $b^\dagger_{\alpha\beta}$ are irreducible tensor operators belonging to $[1^2]$ of G_2.

Having determined the tensor properties of ideal boson creation operators to be $[1^2]$, ideal boson space with n bosons is easily decomposed into the irreducible representations of G_2 by studying the reduction of the n-boson tensor operator

$$T^{(n)}_{(\bar{\alpha},\bar{\beta})}=b^\dagger_{\alpha_1\beta_1}\cdots b^\dagger_{\alpha_n\beta_n}\ . \tag{6}$$

As remarked previously, it is totally symmetric under G_1. Under G_2, it is a symmetrized product of antisymmetric pair tensors, i.e.,

$$T^{(n)}_{(\bar{\alpha},\bar{\beta})} \propto \sum_{P\in S_n} p_{i_1\cdots i_n}\left[\begin{bmatrix}\alpha_{i_1}\\ \beta_{i_1}\end{bmatrix}\times\begin{bmatrix}\alpha_{i_2}\\ \beta_{i_2}\end{bmatrix}\times\cdots\times\begin{bmatrix}\alpha_{i_n}\\ \beta_{i_n}\end{bmatrix}\right], \quad (7)$$

where ([ ]×[ ]× ⋯ ×[ ]) denotes the tensor product of antisymmetric pair tensors. From the well-known results on representation of the general linear group using irreducible tensors,[6] basis states belonging to a given irreducible representation of G_2 have a definite permutation symmetry $[f]=[f_1,\ldots,f_k]$ which can be used to label the G_2 irreducible representations. Thus the G_2 irreducible representations are labeled by the permutation symmetries of the ideal boson indices $(\alpha_1\beta_1\cdots\alpha_n\beta_n)$, which we sometimes denote as the "G_2 symmetry." Now, the decomposition of n-boson space into the irreducible representations of G_2 is reduced to finding the allowed G_2 symmetries in the nth rank tensor given by Eq. (7).

Perhaps, the reduction of symmetrized product of antisymmetric pair tensors is less familiar than the reduction of the symmetrized product of symmetric pair tensors used both in the SU(6)⊃SU(3) reduction of the IBM boson space[7] and in the Sp(3,R)⊃SU(3) classification of the symplectic shell model space.[8] The above reductions are closely related by conjugate symmetry.[9,10] The conjugate symmetry $[\tilde{f}]$ of $[f]$ is found by interchanging rows and columns of the Young tableau corresponding to $[f]$. Applying the above result with the restriction that the number of rows cannot exceed k, n-boson space B_n is decomposed into the irreducible subspace $B_{n[f]}$. The allowed G_2 symmetries $[f]$ are given as

$$\left\{[f]=[f_1,\ldots,f_k]\,|\,f_1=f_2\geq f_3=f_4\geq\cdots\geq f_{k-1}=f_k\geq 0 \text{ and } \sum_i f_i=2n\right\}. \quad (8)$$

An explicit example for the $k=8$ case is shown in Table I where the dimension of each space is also shown. It is easily verified that the dimensions of the decomposed spaces add up correctly.

The Dyson mapping of bifermion operators is given as

$$\begin{aligned} c^\dagger_\alpha c^\dagger_\beta &\to B^\dagger_{\alpha\beta}=b^\dagger_{\alpha\beta}-\sum_{\gamma,\delta} b^\dagger_{\alpha\gamma}b^\dagger_{\beta\delta}b_{\gamma\delta}\,,\\ c_\alpha c_\beta &\to b_{\beta\alpha}\,,\\ c^\dagger_\alpha c_\beta &\to \sum_\gamma b^\dagger_{\alpha\gamma}b_{\beta\gamma}\,. \end{aligned} \quad (9)$$

This maps the $2n$-fermion states $c^\dagger_{\alpha_1}c^\dagger_{\beta_1}\cdots c^\dagger_{\alpha_n}c^\dagger_{\beta_n}|0\rangle$ onto $B^\dagger_{\alpha_1\beta_1}\cdots B^\dagger_{\alpha_n\beta_n}|0)$. The space spanned by these Dyson boson images of n-pair fermion states is defined to be the physical boson space. Similarly, the Dyson boson image of any fermion operator constructed using the bifermion operators is defined to be a physical boson operator. Since $2n$-fermion states carry $[1^{2n}]$ symmetry with respect to $\hat{G}_2$, their boson images also belong to the $[1^{2n}]$ irreducible representation of G_2. Alternatively, one can show explicitly that physical boson states are proportional to the antisymmetrized ideal boson states,[1]

$$B^\dagger_{\alpha_1\beta_1}\cdots B^\dagger_{\alpha_n\beta_n}|0)\propto\sum_p(-1)^p p\, b^\dagger_{\alpha_1\beta_1}\cdots b^\dagger_{\alpha_n\beta_n}|0)\,, \quad (10)$$

where p is a permutation operator on the $2n$ objects

TABLE I. Decomposition of ideal boson space using subgroup chain $G_1=\mathrm{U}(28)\supset G_2=\mathrm{U}(8)$ for the $j=\frac{7}{2}$ (i.e., $k=8$) case. The numbers in parentheses are the dimensions of the space and the last column gives the eigenvalues of S. The physical boson spaces correspond to completely antisymmetric spaces.

n	U(28) $[n]$	dim$[n]$	U(8) $[F]$	dim$[F]$	$\langle S\rangle_{[F]}$
1	[1]	(28)	$[1^2]$	(28)	0
2	[2]	(406)	$[2^2]$	(336)	12
			$[1^4]$	(70)	0
3	[3]	(4060)	$[3^2]$	(2520)	36
			$[2^2,1^2]$	(1512)	20
			$[1^6]$	(28)	0
4	[4]	(31 465)	$[4^2]$	(13 860)	72
			$[3^2,1^2]$	(15 120)	52
			$[2^4]$	(1764)	40
			$[2^2,1^4]$	(720)	28
			$[1^8]$	(1)	0

$(\alpha_1\beta_1 \cdots \alpha_n\beta_n)$. Thus, all the physical boson states have a definite G_2 symmetry and belong to the completely antisymmetric irreducible representation of G_2. For example, in the $k=8$ case, the boson spaces with more than four bosons are all nonphysical, and in general the physical boson spaces constitute a relatively small fraction of the ideal boson space, as one can observe from Table I.

Now, we inquire how one can determine the G_2 symmetry of a given n-boson state if it is known to have a definite G_2 symmetry. This can be done by defining a G_2-scalar operator. Consider the following fermion operator:

$$\hat{S}=\hat{N}^2_H-\hat{N}^2_{NH} , \tag{11}$$

where

$$\hat{N}^2_H=\sum_{\alpha,\beta} c^\dagger_\alpha c_\alpha c^\dagger_\beta c_\beta,\quad \hat{N}^2_H=\sum_{\alpha} c^\dagger_\alpha c_\alpha-\sum_{\alpha,\beta} c^\dagger_\alpha c^\dagger_\beta c_\alpha c_\beta .$$

In fact, the above $\hat{S}=0$ because $\hat{N}^2_{NH}$ is simply the $\hat{N}^2_H$ written in normal ordered form. Its Dyson image, found from Eq. (9), is

$$S=\sum_{\alpha,\beta,\gamma,\delta} b^\dagger_{\alpha\beta}b_{\alpha\beta}b^\dagger_{\gamma\delta}b_{\gamma\delta}-\sum_{\alpha,\beta} B^\dagger_{\alpha\beta}b_{\alpha\beta}-\sum_{\alpha,\beta} b^\dagger_{\alpha\beta}b_{\alpha\beta}$$
$$\equiv N^2-N-K , \tag{12}$$

where

$$N=\sum_{\alpha,\beta} b^\dagger_{\alpha\beta}b_{\alpha\beta},\quad K=\sum_{\alpha,\beta} B^\dagger_{\alpha\beta}b_{\alpha\beta} .$$

The above K is the same as $\hat{S}$ of Janssen *et al.*[1] Thus it is interesting that the boson image of Eq. (11) gives the same operator used in Eq. (3.26) of Ref. 1. Using the results of Ref. 1, K and the permutation operator can be expressed as

$$K=N-2\sum_{i=1}^{n}\sum_{k\neq i}^{n}(p_{\alpha_i\beta_k}+p_{\alpha_k\alpha_i}) , \tag{13}$$

$$\sum_{i<j}^{2n}p_{ij}=-\frac{N}{2}+\sum_{i=1}^{n}\sum_{k\neq i}^{n}(p_{\alpha_i\alpha_k}+p_{\alpha_i\beta_k}) , \tag{14}$$

so that

$$K=-2\sum_{i<j}^{2n}p_{ij} , \tag{15}$$

$$S=2\left\{\frac{N(N-1)}{2}+\sum_{i<j}^{2n}p_{ij}\right\} . \tag{16}$$

The eigenvalue of S for the n-boson states with G_2 symmetry $[f]$ is

$$\langle S\rangle_{[f]}=2\left\{\left\langle\sum p_{ij}\right\rangle_{[f]}+n(2n-1)\right\} . \tag{17}$$

The eigenvalue $\langle\sum p_{ij}\rangle_{[f]}$ is calculated using the fact that $\sum p_{ij}$ is invariant with respect to the permutation group, so that from Shur's lemma its irreducible matrix representation $[\sum p_{ij}]_{[f]}$ must be a multiple of the identity matrix, i.e.,

$$\left[\sum p_{ij}\right]_{[f]}=\left\langle\sum p_{ij}\right\rangle_{[f]} I_{d_{[f]}\times d_{[f]}} , \tag{18}$$

where $d_{[f]}$ is the dimension of irreducible representation $[f]$ of S_{2n}. Taking the trace of Eq. (18) and using the result that traces of all $[p_{ij}]_{[f]}$ have the same value $\chi_{[f]}$,

$$\left\langle\sum p_{ij}\right\rangle_{[f]}=n(2n-1)\chi_{[f]}/d_{[f]} , \tag{19}$$

where $\chi_{[f]}$ is commonly called a character corresponding to the single two-cycle class of the permutation group S_{2n}.

The physical interpretation of S is clear from Eq. (16). It is a Majorana type operator whose eigenvalues depend only on G_2 symmetries; in particular, it has eigenvalue zero for physical boson states.

Equations (17) and (19) are not very useful in actual calculation of eigenvalues of S because the characters, in general, are not easy to compute. Equations (12) and (17) imply S is composed of one- and two-body boson operators and it is scalar with respect to G_2. Thus, we expect S could also be written as a linear combination of one- and two-body Casimir operators of G_2. It is given as

$$S=N^2-N(k+1)+\mathscr{C}^2_{u(k)} . \tag{20}$$

The eigenvalues for $\mathscr{C}^2_{u(k)}$ are[7]

$$\langle\mathscr{C}^2_{u(k)}\rangle_{[f]}=\sum_{i=1}^{k}f_i(f_i+k+1-2i) , \tag{21}$$

from which eigenvalues of S can easily be calculated. The last column in Table I shows the eigenvalues of S for up to four bosons. Hence, given an n-boson state that is known to have good G_2 symmetry, we can determine its G_2 symmetry by calculating its eigenvalue of S and comparing it to the known eigenvalues.

Let us inquire under what conditions the nonphysical states can be removed by adding a suitable multiple of S to the Hamiltonian. Suppose we are given a number-conserving fermion Hamiltonian $\hat{H}=\hat{H}_0+\hat{V}$, where $\hat{H}_0$ and $\hat{V}$ are one- and two-body operators, respectively. For $\hat{V}$, there exist two different Dyson boson images.[3] The first is found by mapping the normal ordered form of $\hat{V}$, denoted by

$$\hat{V}_{NH}=\hat{V}(c^\dagger c^\dagger,cc) . \tag{22}$$

Alternatively, we can rearrange the fermion creation and destruction operators, so that $\hat{V}$ contains only $c^\dagger c$ type bifermion operators, to get a form denoted by

$$\hat{V}_H=\hat{V}(c^\dagger c) . \tag{23}$$

Let us denote the Dyson boson images of $\hat{H}$ by

$$h_{NH}=h_0+[\hat{V}_{NH}]_D , \tag{24}$$

$$h_H=h_0+[\hat{V}_H]_D , \tag{25}$$

where $h_0=[\hat{H}_0]_D$ and $[\ ]_D$ denotes Dyson boson image. It is to be noted that h_H is Hermitian whereas h_{NH} is not, due to the nonunitary property of the Dyson boson mapping.

Since $\hat{V}_H$ is a function of only $c^\dagger c$ type operators, $[\hat{V}_H]_D$ and hence h_H is a function of generators of G_2.

Therefore, the eigenfunctions of h_H have definite G_2 symmetry, and using the operator S one can determine the G_2 symmetry of each eigenfunction of h_H. Also, if one is interested only in physical boson states, the energies of nonphysical eigenstates can be shifted up by adding λS ($\lambda > 0$) to h_H such that all the low lying eigenstates are physical. In the large λ limit, the lowest nonphysical state is expected to belong to the $[F]=[2^2,1^{2n-4}]$ irreducible representation, which has unperturbed energy

$$\lambda\langle S\rangle_{[2^2,1^{2n-4}]}=\lambda(8n-4) . \tag{26}$$

One can estimate a suitable value to choose for λ by comparing the unperturbed energy of Eq. (26) to the excitation energies of h_N in physical subspace.

However, for h_{NH} the situation is more complicated because h_{NH} is a function of $B^\dagger_{\alpha\beta}$ and $b_{\alpha\beta}$, which are not generators of G_2, so that h_{NH} will, in general, mix different irreducible representations of G_2. Nevertheless, it is still possible to remove all nonphysical states due to a special property of the physical boson space with respect to physical boson operators—it is an invariant subspace. This property is easily seen from the fact that the Dyson boson images of the bifermion operators given in Eq. (9) close under commutation and form an algebra [SO($2k$)]. The physical boson states span an irreducible representation of the same algebra, since physical boson states are precisely those generated from the vacuum state by applying raising operators. Thus acting on a physical ket with a physical boson operator h_{NH}, which is a function of generators of the algebra, will give another physical ket. The invariance property of physical boson space is most conveniently exploited using an effective operator method for the analysis of h_{NH}.[12]

The effective operator, when restricted to the "model space" M with λS as unperturbed component of the operator $\lambda S + h_{NH}$, is given as

$$h^{\rm eff}=P(\lambda S+h_{NH})P+Ph_{NH}Q \times\left[\frac{1}{E-Q(\lambda S+h_{NH})Q}\right]Qh_{NH}P , \tag{27}$$

where P and Q are the projection operators onto the subspace M and its orthogonal complement space, respectively. Corresponding eigenvectors are given as

$$|E\rangle=\left[1+\sum_{k=1}^{\infty}\left[Q\frac{1}{E-\lambda Sh_{NH}}\right]^k\right]|E^0\rangle , \tag{28}$$

where $|E^0\rangle = P|E\rangle$. To determine the effect of nonphysical states on physical states, let M be the physical boson space. Since $Qh_{NH}P=0$ due to the invariance of physical space under h_{NH}, Eqs. (27) and (28) become

$$h^{\rm eff}=Ph_{NH}P , \quad |E\rangle=|E^0\rangle , \tag{29}$$

where we have used the result $PSP=0$. Therefore, enlarging the physical space to the entire boson space has no effect on physical eigenvalues and eigenvectors. Now consider the effects of physical boson space on nonphysical space. Let the model space M now be the entire nonphysical space. Then the second term in Eq. (27) vanishes because $Ph_{NH}Q=0$ for this choice of model space, so that the effective Hamiltonian for nonphysical space is

$$h^{\rm eff}_{\rm NP}=P(\lambda S+h_{NH})P . \tag{30}$$

In the $\lambda\to 0$ limit, $h^{\rm eff}_{\rm NP}\to Ph_{NH}P$, which means nonphysical eigenvalues of h_{NH} are not affected by the presence of physical states. But the eigenvectors contain physical components, since in general $Q[1/(E-\lambda S)]^k h_{NH}|E^0\rangle \neq 0$. In the large λ limit, the nonphysical energy for $\lambda S + h_{NH}$ is found by solving Eq. (30) with the result

$$E_{\rm NP}\approx\lambda\langle S\rangle_{[f]}+\epsilon , \tag{31}$$

where ϵ is the eigenvalue of h_{NH} within a restricted nonphysical irreducible representation space $[f]$. Hence we have shown that even in the h_{NH} case it is possible to remove all nonphysical eigenvalues of h_{NH} without affecting the physical eigenvalues and corresponding eigenvectors, simply by adding λS to h_{NH}.

We have carried out a systematic analysis of algebraic properties of the ideal boson space which leads naturally to introduction of the operator S. Although Janssen *et al.*[1] introduced S, they did not exploit it systematically except for briefly stating that S can be used to minimize the nonphysical component in the energy eigenstates. Equations (16) and (20) display the operator S in two alternative forms. The former emphasized the physical point of view that S is a Majorana-like operator, and the latter reminds us of its algebraic origin—it is a Casimir operator of G_2. Then we show that, for any number-conserving Hamiltonian, S can be used to identify and remove all the nonphysical (spurious) states without affecting the physical eigenvalues and eigenvectors. Geyer *et al.*[5] also discussed a method of identifying physical and spurious eigenstates by exploiting the invariance of the physical space under every physical boson operator. However, their method is not practical, in general, since unambiguous determinations require calculating matrix elements for many different physical boson operators between all eigenstates of a given physical boson Hamiltonian. However, with our S operator method, one simply calculates the expectation of S for each eigenstate. Then the physical states are characterized by zero expectation value and all the nonphysical states have positive expectation values. Moreover, one need not even be concerned about the problem of identifying spurious states. By introducing the operator S into the boson Hamiltonian, one can simply remove all spurious states from the low lying energy eigenstates.

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