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Solvable model for one-dimensional nuclear matter: Simultaneous eigenstates of spin, isospin, and energy

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The eigenvalue problem is solved exactly for a one-dimensional system composed of many protons and neutrons interacting via delta-function potentials which conserve the total spin and isospin of protons and neutrons. It is shown that simultaneous eigenstates of spin, isospin, and energy, and energy eigenvalues are determined by the solutions of some transcendental coupled equations for given quantum numbers.

We consider a one-dimensional model described by the Hamiltonian $H = H_0 + H_{\rm int}$, which is given by

$$H_0 = \sum_p (p^2/2m)\{p_\uparrow^*(p)p_\uparrow(p) + p_\downarrow^*(p)p_\downarrow(p) + n_\uparrow^*(p)n_\uparrow(p) + n_\downarrow^*(p)n_\downarrow(p)\}\ , \tag{1a}$$

$$\begin{aligned} H_{\rm int} = \sum_{p,q,r} (g/L)\{ & p_\uparrow^*(p+r)p_\uparrow(p)p_\downarrow^*(q-r)p_\downarrow(q) \\ & + n_\uparrow^*(p+r)n_\uparrow(p)n_\downarrow^*(q-r)n_\downarrow(q) \\ & - p_\uparrow^*(p+r)p_\downarrow(p)n_\downarrow^*(q-r)n_\uparrow(q) \\ & - p_\downarrow^*(p+r)p_\uparrow(p)n_\uparrow^*(q-r)n_\downarrow(q) \\ & - p_\uparrow^*(p+r)p_\uparrow(p)n_\uparrow^*(q-r)n_\uparrow(q) \\ & - p_\downarrow^*(p+r)p_\downarrow(p)n_\downarrow^*(q-r)n_\downarrow(q)\}\ , \end{aligned} \tag{1b}$$

where $p_\uparrow^*(q)$ and $n_\uparrow^*(q)$ [$p_\downarrow^*(q)$ and $n_\downarrow^*(q)$] denote the creation operators of an up (down) spin proton and neutron of mass m with momentum $q = (2\pi\hbar/L)\times$integer (L is the length of our system), respectively. The interaction Hamiltonian with a coupling constant g conserves the total spin and isospin of the system defined by

$$\begin{aligned} \mathcal{S}^+ &= \sum_p \{p_\uparrow^*(p)p_\downarrow(p) + n_\uparrow^*(p)n_\downarrow(p)\}\ , \\ \mathcal{S}^- &= \sum_p \{p_\downarrow^*(p)p_\uparrow(p) + n_\downarrow^*(p)n_\uparrow(p)\}\ , \\ \mathcal{S}_Z &= \sum_p \{p_\uparrow^*(p)p_\uparrow(p) - p_\downarrow^*(p)p_\downarrow(p) + n_\uparrow^*(p)n_\uparrow(p) - n_\downarrow^*(p)n_\downarrow(p)\}/2\ , \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathcal{T}^+ &= \sum_p \{p_\uparrow^*(p)n_\uparrow(p) + p_\downarrow^*(p)n_\downarrow(p)\}\ , \\ \mathcal{T}^- &= \sum_p \{n_\uparrow^*(p)p_\uparrow(p) + n_\downarrow^*(p)p_\downarrow(p)\}\ , \\ \mathcal{T}_Z &= \sum_p \{p_\uparrow^*(p)p_\uparrow(p) - n_\uparrow^*(p)n_\uparrow(p) + p_\downarrow^*(p)p_\downarrow(p) - n_\downarrow^*(p)n_\downarrow(p)\}/2\ . \end{aligned} \tag{3}$$

We must therefore look for the simultaneous eigenstate of spin, isospin, and energy.

On the basis of the simultaneous eigenstate[1] of spin and energy in the one-dimensional system[2] of many fermions interacting via delta-function potentials, we assume the following form for the eigenstate of the present N-body system:

$$\begin{aligned} |\Psi^{S=S_Z=N/2-M}_{T=T_Z=N/2-K}\{q_j\},\{\lambda_\alpha^S\},\{\lambda_a^T\}\rangle = & \sum_{1\le l_1<l_2<\cdots<l_M\le N}\ \sum_{1\le m_1<m_2<\cdots<m_K\le N}\ \sum_{\{p_{j,a}^S\},\{Q_{\alpha,\beta}^S\},\{p_{j,a}^T\},\{Q_{a,b}^T\}} \\ & \times \prod_{\beta=1}^{M} \delta\left[\sum_{j=1}^{N} p_{j,\beta}^S - \sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{M} Q_{\alpha,\beta}^S - \lambda_\beta^S\right] \prod_{b=1}^{K} \delta\left[\sum_{j=1}^{N} p_{j,b}^T - \sum_{\substack{a=1\\ a\neq b}}^{K} Q_{a,b}^T - \lambda_b^T\right] \\ & \times \prod_{1\le\alpha<\beta\le M} (Q_{\alpha,\beta}^S - \omega_{\alpha,\beta}^S)^{-1} \prod_{1\le a<b\le K} (Q_{a,b}^T - \omega_{a,b}^T)^{-1} \\ & \times \sum_{\{\mu\}} \prod_{\lambda=1}^{M} \prod_{\substack{j_{\mu_\lambda}=1\\ j_{\mu_\lambda}\neq l_\lambda}}^{N} f^S(j_{\mu_\lambda},\mu_\lambda) \sum_{\{\nu\}} \prod_{e=1}^{K} \prod_{\substack{j_{\nu_e}=1\\ j_{\nu_e}\neq m_e}}^{N} f^T(j_{\nu_e},\nu_e) O^*(\{p_j\};\{l_\lambda\};\{m_e\})|0)\ , \end{aligned} \tag{4}$$

where $S=S_Z=N/2-M$ means that the eigenvalues of the square of spin and its z component are $(N/2-M)(N/2-M+1)$ and $N/2-M$, respectively, and the corresponding meanings in isospin are indicated by $T=T_Z=N/2-K$. N momenta $\{q_j\}$ are introduced as quantum numbers. Similarly, M momenta $\{\lambda_\alpha^S\}$ and K momenta $\{\lambda_a^T\}$ are introduced as quantum numbers concerning spin and isospin. Greek letters μ and ν indicate the permutations $\mu=\binom{1,2,\ldots,M}{\mu_1,\mu_2,\ldots,\mu_M}$ and $\nu=\binom{1,2,\ldots,K}{\nu_1,\nu_2,\ldots,\nu_K}$. The functions $f^S(j,\alpha)$ and $f^T(j,a)$ are defined by

$$f^S(j,\alpha)=(p_{j,\alpha}^S-k_{j,\alpha}^S)^{-1} \text{ and } f^T(j,a)=(p_{j,a}^T-k_{j,a}^T)^{-1} . \tag{5}$$

The integers $\{l_\alpha\}$ and $\{m_a\}$ denote the positions of creation operators concerning down spin and down isospin particles. We define the down isospin particle as a neutron. Then the operator $O^*(\{p_j\};\{l_\alpha\};\{m_a\})$ is composed of $(N-K)$ protons and K neutrons, where M particles among them have down spin; for example,

$$O^*(p_1,\ldots,p_4;1,3;2,3)=p_\downarrow^*(p_1)n_\uparrow^*(p_2)n_\downarrow^*(p_3)p_\uparrow^*(p_4) .$$

The N momenta $\{p_j\}$ are connected with $p_{j,\alpha}^S$, $p_{j,a}^T$, and q_j as

$$p_j=q_j+\sum_{\alpha=1}^{M}p_{j,\alpha}^S+\sum_{a=1}^{K}p_{j,a}^T \quad (1\leqq j\leqq N) . \tag{6}$$

According to this, let us introduce auxiliary quantities k_j,

$$k_j=q_j+\sum_{\alpha=1}^{M}k_{j,\alpha}^S+\sum_{a=1}^{K}k_{j,a}^T \quad (1\leqq j\leqq N) . \tag{7}$$

Corresponding to the restrictions concerning the momenta $p_{j,\alpha}^S$, $Q_{\alpha,\beta}^S$, and $p_{j,a}^T$, $Q_{a,b}^T$ through Kronecker's deltas, we assume

$$\sum_{j=1}^{N}k_{j,\beta}^S=\lambda_\beta^S+\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{M}\omega_{\alpha,\beta}^S \quad (1\leqq\beta\leqq M) , \tag{8a}$$

$$\sum_{j=1}^{N}k_{j,b}^T=\lambda_b^T+\sum_{\substack{a=1\\ a\neq b}}^{K}\omega_{a,b}^T \quad (1\leqq b\leqq K) \tag{8b}$$

for the quantities $k_{j,\beta}^S$, $\omega_{\alpha,\beta}^S$ and $k_{j,b}^T$, $\omega_{a,b}^T$.

If we require the conditions

$$2\cot(L\omega_{\alpha,\beta}^S/2\hbar)=\cot(Lk_{j,\beta}^S/2\hbar)-\cot(Lk_{j,\alpha}^S/2\hbar) \quad (1\leqq\alpha<\beta\leqq M) , \tag{9a}$$

$$2\cot(L\omega_{a,b}^T/2\hbar)=\cot(Lk_{j,b}^T/2\hbar)-\cot(Lk_{j,a}^T/2\hbar) \quad (1\leqq a<b\leqq K) , \tag{9b}$$

then we can prove that

$$\mathscr{S}^+\,|\,\Psi_{T=T_Z=N/2-K}^{S=S_Z=N/2-M}\{q_j\},\{\lambda_\alpha^S\},\{\lambda_a^T\}\rangle=0 ,$$

and

$$\mathscr{T}^+\,|\,\Psi_{T=T_Z=N/2-K}^{S=S_Z=N/2-M}\{q_j\},\{\lambda_\alpha^S\},\{\lambda_a^T\}\rangle=0 ,$$

in the same way as in the case of the proof[1] of the spin eigenstate in the one-dimensional many-fermion system. By successive operations of $\mathscr{S}^-$ and $\mathscr{T}^-$ on the vector (4), we can make[3] the eigenstate vectors corresponding to the $(2S+1)$ eigenvalues of $\mathscr{S}_Z$ and the $(2T+1)$ ones of $\mathscr{T}_Z$. We can rewrite[3] the eigenstate (4) expressed in terms of the configurations of down spin and down isospin particles by the one expressed in terms of the configurations of up spin and down isospin ones, down spin and up isospin ones, and up spin and up isospin ones by working the conditions (8a),(8b) and (9a),(9b).

If we require that

$$\cot(Lk_{j,\alpha}^S/2\hbar)-\cot(Lk_{l,\alpha}^S/2\hbar)=(k_j-k_l)/(mg/2\hbar) , \tag{10a}$$

$$\cot(Lk_{j,a}^T/2\hbar)-\cot(Lk_{l,a}^T/2\hbar)=-(k_j-k_l)/(mg/2\hbar) , \tag{10b}$$

besides (9a) and (9b), then we can prove that

$$(H_0+H_{\text{int}}-E)\,|\,\Psi_{T=T_Z=N/2-K}^{S=S_Z=N/2-M}\{q_j\},\{\lambda_\alpha^S\},\{\lambda_a^T\}\rangle=0 , \tag{11}$$

where the energy eigenvalue E is given by

$$E=\sum_{j=1}^{N}(k_j^2/2m) . \tag{12}$$

Operation of $2m(H_0-E)$ on the eigenvector (4) yields a factor $\sum_{j=1}^{N}(p_j^2-k_j^2)$ in it. Fixing the permutations μ and ν, we can decompose the factor as

$$\begin{aligned}\sum_{j=1}^{N}(p_j^2-k_j^2)\Longrightarrow&\sum_{\alpha=1}^{M}\sum_{\substack{j=1\\ j\neq l_1\sim l_M}}^{N}(p_{j,\mu_\alpha}^S-k_{j,\mu_\alpha}^S)(p_j-p_{l_\alpha}+k_j-k_{l_\alpha})\\ &+\sum_{1\leqq\alpha<\beta\leqq M}\{(Q_{\mu_\beta,\mu_\alpha}^S-\omega_{\mu_\beta,\mu_\alpha}^S)-(p_{l_\beta,\mu_\alpha}^S-k_{l_\beta,\mu_\alpha}^S)+(p_{l_\alpha,\mu_\beta}^S-k_{l_\alpha,\mu_\beta}^S)\}(p_{l_\alpha}-p_{l_\beta}+k_{l_\alpha}-k_{l_\beta})\\ &+\sum_{a=1}^{K}\sum_{\substack{j=1\\ j\neq m_1\sim m_K}}^{N}(p_{j,\nu_a}^T-k_{j,\nu_a}^T)(p_j-p_{m_a}+k_j-k_{m_a})\\ &+\sum_{1\leqq a<b\leqq K}\{(Q_{\nu_b,\nu_a}^T-\omega_{\nu_b,\nu_a}^T)-(p_{m_b,\nu_a}^T-k_{m_b,\nu_a}^T)+(p_{m_a,\nu_b}^T-k_{m_a,\nu_b}^T)\}(p_{m_a}-p_{m_b}+k_{m_a}-k_{m_b})\end{aligned} \tag{13}$$

by making use of (8a) and (8b) and the restrictions through Kronecker's deltas. Contributions from the second and fourth terms vanish[1] because of the conditions (9a) and (9b). The contributions from the first and third terms are classified into vectors of six types. The typical examples are

$$|A\rangle=[\cdots](p^S_{J,\mu_a}-k^S_{J,\mu_a})(p_J-p_{l_a}+k_J-k_{l_a})\cdots p^*_\downarrow(p_{l_a})\cdots p^*_\uparrow(p_J)\cdots|0), \tag{14a}$$

$$|B\rangle=[\cdots](p^T_{J,\nu_a}-k^T_{J,\nu_a})(p_J-p_{m_a}+k_J-k_{m_a})\cdots n^*_\uparrow(p_{m_a})\cdots p^*_\uparrow(p_J)\cdots|0), \tag{14b}$$

$$|C\rangle=[\cdots](p^S_{m_a,\mu_a}-k^S_{m_a,\mu_a})(p_{m_a}-p_{l_a}+k_{m_a}-k_{l_a})\cdots n^*_\downarrow(p_{l_a})\cdots n^*_\uparrow(p_{m_a})\cdots|0), \tag{14c}$$

$$|D\rangle=[\cdots](p^T_{l_a,\nu_a}-k^T_{l_a,\nu_a})(p_{l_a}-p_{m_a}+k_{l_a}-k_{m_a})\cdots n^*_\downarrow(p_{m_a})\cdots p^*_\downarrow(p_{l_a})\cdots|0), \tag{14d}$$

$$|E\rangle=[\cdots]\{(p^S_{m_a,\mu_a}-k^S_{m_a,\mu_a})-(p^T_{l_a,\nu_a}-k^T_{l_a,\nu_a})\}(p_{m_a}-p_{l_a}+k_{m_a}-k_{l_a})\cdots p^*_\downarrow(p_{l_a})\cdots n^*_\uparrow(p_{m_a})\cdots|0), \tag{14e}$$

$$|F\rangle=[\cdots]\{(p^S_{J,\mu_a}-k^S_{J,\mu_a})+(p^T_{J,\nu_a}-k^T_{J,\nu_a})\}(p_J-p_{m_a}+k_J-k_{m_a})\cdots n^*_\downarrow(p_{l_a=m_a})\cdots p^*_\uparrow(p_J)\cdots|0), \tag{14f}$$

where the symbol $[\cdots]$ represents the remaining factors, except the operator $O^*(\{p_j\};\{l_\lambda\};\{m_c\})$ for fixed integers $\{l_\lambda\}$, $\{m_c\}$, and J. We turn our attention to the vector $|E\rangle$. Let us denote l_a and m_a by I and J, which are subjected to $m_{b-1}<I<m_b$ and $l_{\beta-1}<J<l_\beta$. Extracting the factor $f^S(J,\mu_a)f^T(I,\nu_a)$ from $[\cdots]$, we multiply it by two factors in curly brackets. Then the content of the curly brackets changes to $\{(p^T_{I,\nu_a}-k^T_{I,\nu_a})^{-1}-(p^S_{J,\mu_a}-k^S_{J,\mu_a})^{-1}\}$. The sum of the first term over p^T_{I,ν_a} can be taken to become $(-L/2\hbar)\cot(Lk^T_{I,\nu_a}/2\hbar)$ due to such changes of summation variable as $p^S_{I,\mu_a}\to v-p^T_{I,\nu_a}$, $p^S_{J,\mu_a}\to u^S-v+p^T_{I,\nu_a}$, and $p^T_{J,\nu_a}\to u^T-p^T_{I,\nu_a}$. In the case of the second term, achievement of the sum over p^S_{J,μ_a} yields $(-L/2\hbar)\cot(Lk^S_{J,\mu_a}/2\hbar)$, which is permitted by transformation $p^S_{I,\mu_a}\to u^S-p^S_{J,\mu_a}$, $p^T_{I,\nu_a}\to -u^S+v+p^S_{J,\mu_a}$, and $p^T_{J,\nu_a}\to u^S+u^T-v-p^S_{J,\mu_a}$. Thus, the two factors in the curly brackets have been changed to

$$(-L/2\hbar)\{\cot(Lk^T_{I,\nu_a}/2\hbar)-\cot(Lk^S_{J,\mu_a}/2\hbar)\}\ .$$

There exists a vector $|E'\rangle$ corresponding to $|E\rangle$,

$$|E'\rangle=[\cdots]\{(p^S_{I,\mu'_\beta}-k^S_{I,\mu'_\beta})-(p^T_{J,\nu'_b}-k^T_{J,\nu'_b})\}(p_I-p_J+k_I-k_J)\cdots n^*_\uparrow(p_I)\cdots p^*_\downarrow(p_J)\cdots|0), \tag{15}$$

where integers $\{l'_\lambda\}$ and $\{m'_c\}$ are related to $\{l_\lambda\}$ and $\{m_c\}$ in $|E\rangle$ as $l'_\lambda=l_\lambda$ $(1\leqq\lambda\leqq\alpha-1,\ \beta+1\leqq\lambda\leqq M)$, $l'_\lambda=l_{\lambda+1}$ $(\alpha\leqq\lambda\leqq\beta-1)$, $m'_c=m_c$ $(1\leqq c\leqq b-1,\ a+1\leqq c\leqq K)$, and $m'_c=m_{c-1}$ $(b+1\leqq c\leqq a)$, and we have set $I=m'_b$ and $J=l'_\beta$. The two factors in the curly brackets of (15) can be changed to

$$(-L/2\hbar)\{\cot(Lk^T_{J,\nu'_b}/2\hbar)-\cot(Lk^S_{I,\mu'_\beta}/2\hbar)\}$$

by applying the same arguments as for the vector $|E\rangle$. In order to combine this result for $|E'\rangle$ and the one for $|E\rangle$ mentioned above, we make two changes as follows. First, two permutations μ' and ν' in $|E'\rangle$ are changed as $\mu'_\lambda\to\mu_\lambda$ $(1\leqq\lambda\leqq\alpha-1,\ \beta+1\leqq\lambda\leqq M)$, $\mu'_\lambda\to\mu_{\lambda+1}$ $(\alpha\leqq\lambda\leqq\beta-1)$, $\mu'_\beta\to\mu_\alpha$, $\nu'_c\to\nu_c$ $(1\leqq c\leqq b-1,\ a+1\leqq c\leqq K)$, $\nu'_c\to\nu_{c-1}$ $(b+1\leqq c\leqq a)$, and $\nu'_b\to\nu_a$. Then the changed two factors in the curly brackets become

$$(-L/2\hbar)\{\cot(Lk^T_{J,\nu_a}/2\hbar)-\cot(Lk^S_{I,\mu_a}/2\hbar)\}\ ,$$

which equals

$$(-L/2\hbar)\{\cot(Lk^T_{I,\nu_a}/2\hbar)-\cot(Lk^S_{J,\mu_a}/2\hbar)\}$$

according to the conditions (10a) and (10b). Next, we interchange p_I with p_J by transforming v into

$$-v+u^S+u^T+(q_J-q_I)+\sum_{\lambda\neq\alpha}(p^S_{J,\mu_\lambda}-p^S_{I,\mu_\lambda})+\sum_{c\neq a}(p^T_{J,\nu_c}-p^T_{I,\nu_c})\ .$$

Then, the sum of $|E\rangle$ and $|E'\rangle$ is given by

$$|E\rangle+|E'\rangle=[\cdots]'\sum_{u^S,u^T,v}\delta\left[\sum_{j\neq I,J}p^S_{j,\mu_a}-\sum_{\lambda\neq\alpha}Q^S_{\mu_\lambda,\mu_a}-\lambda^S_{\mu_a}+u^S\right]\delta\left[\sum_{j\neq I,J}p^T_{j,\nu_a}-\sum_{c\neq a}Q^T_{\nu_c,\nu_a}-\lambda^T_{\nu_a}+u^T\right]$$
$$\times(-L/2\hbar)[\cot(Lk^T_{I,\nu_a}/2\hbar)-\cot(Lk^S_{J,\mu_a}/2\hbar)]2(k_I-k_J)\cdots p^*_\downarrow(p_I)\cdots n^*_\uparrow(p_J)\cdots|0), \tag{16}$$

where

$$p_I = q_I + \sum_{\lambda\neq\alpha} p^S_{I,\mu_\lambda} + \sum_{c\neq a} p^T_{I,\nu_c} + v$$

and

$$p_J = q_J + \sum_{\lambda\neq\alpha} p^S_{J,\mu_\lambda} + \sum_{c\neq a} p^T_{J,\nu_c} + u^S + u^T - v ,$$

and all factors not necessary for our discussion are contained in the symbol $[\cdots]'$.

Among the vectors produced by the operation of $2mH_{\rm int}$ on the state vector (4), there are two vectors, denoted $|E'_{\rm int}\rangle$ and $|E''_{\rm int}\rangle$,

$$|E'_{\rm int}\rangle = [\cdots](-2mg/L)\sum_r \cdots p^*_\downarrow(p_I - r)\cdots \times n^*_\uparrow(p_J + r)\cdots |0) , \quad (17a)$$

$$|E''_{\rm int}\rangle = [\cdots](-2mg/L)\sum_r \cdots n^*_\uparrow(p_I - r)\cdots \times p^*_\downarrow(p_J + r)\cdots |0) , \quad (17b)$$

where $l'_\lambda = l_\lambda$ $(1\leqq\lambda\leqq\alpha-1,\ \beta+1\leqq\lambda\leqq M)$, $l'_\lambda = l_{\lambda+1}$ $(\alpha\leqq\lambda\leqq\beta-1)$, $l'_\beta = J$, and $m'_c = m_c$ $(1\leqq c\leqq K)$ in (17a). In (17b), $m'_c = m_c$ $(1\leqq c\leqq b-1,\ a+1\leqq c\leqq K)$, $m'_c = m_{c-1}$ $(b+1\leqq c\leqq a)$, $m'_b = I$, and $l'_\lambda = l_\lambda$ $(1\leqq\lambda\leqq M)$. For the vector $|E'_{\rm int}\rangle$, let us first change the permutation μ as $\mu_\lambda\to\mu_\lambda$ $(1\leqq\lambda\leqq\alpha-1,\ \beta+1\leqq\lambda\leqq M)$, $\mu_\lambda\to\mu_{\lambda+1}$ $(\alpha\leqq\lambda\leqq\beta-1)$, and $\mu_\beta\to\mu_\alpha$. Next, we make such displacements as $p^S_{I,\mu_\alpha}\to p^S_{I,\mu_\alpha}+r$ and $p^S_{J,\mu_\alpha}\to p^S_{J,\mu_\alpha}-r$. Then, we can take the sum $\sum_r(p^S_{I,\mu_\alpha}+r-k^S_{I,\mu_\alpha})^{-1}$ to yield $(-L/2\hbar)\cot(Lk^S_{I,\mu_\alpha}/2\hbar)$. After this, we can utilize the same transformation as in the case of the first term in the curly brackets of (14e). Then we can take the sum over p^T_{I,ν_a}, and we have the factor

$$(-L/2\hbar)^2\cot(Lk^S_{I,\mu_\alpha}/2\hbar)\cot(Lk^T_{I,\nu_a}/2\hbar)$$

on the right-hand side of (17a). Similarly treating the rhs of (17b), we get

$$(-L/2\hbar)^2\cot(Lk^S_{J,\mu_\alpha}/2\hbar)\cot(Lk^T_{J,\nu_a}/2\hbar) .$$

By making use of the same transformation of the variable v as in the combination of $|E\rangle$ and $|E'\rangle$, the above two results for $|E'_{\rm int}\rangle$ and $|E''_{\rm int}\rangle$ are combined to become

$$|E'_{\rm int}\rangle + |E''_{\rm int}\rangle = \{\cdots\}(-2mg/L)(-L/2\hbar)^2[\cot(Lk^S_{I,\mu_\alpha}/2\hbar)\cot(Lk^T_{I,\nu_a}/2\hbar) - \cot(Lk^S_{J,\mu_\alpha}/2\hbar)\cot(Lk^T_{J,\nu_a}/2\hbar)]\cdots p^*_\downarrow(p_I)\cdots n^*_\uparrow(p_J)\cdots |0) , \quad (18)$$

where the symbol $\{\cdots\}$ indicates the symbol $[\cdots]'$, the summation symbol, and the Kronecker deltas in (16). Noting that two terms in the square brackets of (18) can be changed to

$$[\cot(Lk^T_{I,\nu_a}/2\hbar) - \cot(Lk^S_{J,\mu_\alpha}/2\hbar)] \times [\cot(Lk^T_{I,\nu_a}/2\hbar) - \cot(Lk^T_{J,\nu_a}/2\hbar)]$$

due to the conditions (10a) and (10b), and then to

$$[\cot(Lk^T_{I,\nu_a}/2\hbar) - \cot(Lk^S_{J,\mu_\alpha}/2\hbar)](k_J - k_I)/(mg/2\hbar) ,$$

we can see that the right hand sides of both (16) and (18) cancel each other.

The same arguments can be applied to the vectors of the other types, and then we have the same conclusion obtained above. In this way Eq. (11) for the state vector (4) has proved to be valid under the conditions (9a),(9b) and (10a),(10b).

Let us summarize our results obtained in our previous discussions for the present system. The state vector of the assumed form (4) with the assumptions (8a) and (8b) becomes the eigenstate vector of spin and isospin when the conditions (9a) and (9b) hold. It also becomes the energy eigenstate when the conditions (10a) and (10b) are fulfilled besides (9a) and (9b). Then it proves to be the simultaneous eigenstate of spin, isospin, and energy under the conditions (9a),(9b) and (10a),(10b). By successive operation of $\mathscr{S}^-$ and $\mathscr{T}^-$ on the eigenvector (4), we can obtain the simultaneous eigenstate vector of given eigenvalues S_Z $(-S\leqq S_Z\leqq S)$ and T_Z $(-T\leqq T_Z\leqq T)$ in the same way[3] as in the one-dimensional many-fermion system.

The conditions (10a) and (10b) mean that we can introduce M auxiliary quantities Λ^S_α and K auxiliary quantities Λ^T_a defined by

$$\Lambda^S_\alpha = k_j/(mg/2\hbar) - \cot(Lk^S_{j,\alpha}/2\hbar)$$

and

$$\Lambda^T_a = -k_j/(mg/2\hbar) - \cot(Lk^T_{j,a}/2\hbar) .$$

Then, the conditions (9a) and (9b) become $\cot(L\omega^S_{\alpha,\beta}/2\hbar) = (\Lambda^S_\alpha - \Lambda^S_\beta)/2$ and $\cot(L\omega^T_{a,b}/2\hbar) = (\Lambda^T_a - \Lambda^T_b)/2$. From these definitions, (7) and (8a),(8b), we can obtain coupled equations for k_j, Λ^S_α, and Λ^T_S. Further analysis for these coupled equations is under consideration. Details will be published elsewhere.

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[1]T. Kebukawa, Prog. Theor. Phys. **73**, 1098 (1985).

[2]J. B. McGuire [J. Math. Phys. **6**, 432 (1965); **7**, 123 (1966)] first solved the eigenvalue problem in the configuration space for the case of one down spin fermion. M. Flicker and E. H. Lieb [Phys. Rev. **161**, 179 (1967)] solved the eigenvalue problem for the case of two spin down fermions. M. Gaudin [Phys. Lett. **24A**, 55 (1967)] and C. N. Yang [Phys. Rev. Lett. **19**, 1312 (1967)] solved the eigenvalue problem for the general case by using the Bethe ansatz. The present author, recently (Ref. 1) solved the same problem in the field theoretical framework. Consequently, the simultaneous eigenstate of spin and energy was represented systematically in a compact form.

[3]T. Kebukawa, Prog. Theor. Phys. **75**, 506 (1986).