PHYSICAL REVIEW C VOLUME 35, NUMBER 2 FEBRUARY 1987

Role of three-body unitarity in π-N scattering

I. R. Afnan and B. C. Pearce
School of Physical Sciences, The Flinders University of South Australia, Bedford Park, South Australia 5042, Australia
(Received 16 May 1986)

We consider the amplitude for π-N scattering within the framework of a Lagrangian of the form suggested by the cloudy bag model with volume coupling. By first exposing two-body, and then three-body unitarity, we derive a set of integral equations that couple the πN to the $\pi\pi$N channel. These equations satisfy two- and three-body unitarity, and can be used to describe π-N scattering both below and above the threshold for pion production. Below this threshold, the equations have the form of the Lippmann-Schwinger equation with the new feature that in the potential, the vertices in the pole diagram are undressed, while those in the crossed diagram are dressed. This feature allows for the proper description of the P_{11} amplitude.

I. INTRODUCTION

The recent interest in the π-N system has been motivated by several outstanding problems. (i) The need for π-N amplitudes as input to pion-nucleus calculation. Here most of the work is based on parametrization of the data in a form that is convenient for use in other calculations. These parametrizations are often in terms of separable amplitudes[1] which fit the data very well but are not unique in terms of their off-shell behavior. For example, in the P_{11} channel, the division of the amplitude into a pole and nonpole has been a great source of uncertainty in pion-deuteron calculations.[2] (ii) The π-N system, as the simplest nuclear system, has become the testing ground for our ideas of the underlying quark structure of hadrons and mesons. In particular, the cloudy bag model[3] has been extensively used for P-wave, and more recently, for S-wave[4] scattering. These models, which treat the nucleon, delta, and other isobars on equal footing, have the advantage of consistency with QCD, the presently accepted theory of strong interactions. In this respect they may be considered more fundamental, and could be used to remove some of the uncertainties in the phenomenological models mentioned above. (iii) There have been some recent questions regarding the nature of some π-N resonances above the threshold for pion production. In particular, there are several models for the Roper or N*(1440) resonance,[5] as a three-quark state,[6] or the result of the π-Δ threshold.[7] To examine these resonances we need to invoke three-body unitarity, as they are highly inelastic.[1]

In an attempt to examine some of the above questions, we will present the framework of a theory that starts with the Lagrangian in terms of quark degrees of freedom and ends with a set of equations for π-N scattering that satisfy two- and three-body unitarity. Although our final results are similar to those obtained by Fuda using projection operators,[8] our analysis is simpler, covariant, and can be extended to include Lagrangians that are nonlinear in the pion field.[3,9] The method used in deriving the equations for π-N scattering are identical to those used for the NN-πNN equations,[10–13] and is based on the classification of the diagrams (that contribute to a given process) according to their irreducibility.[14]

In Sec. II we commence with the Lagrangian for the cloudy bag model[3] and show how we project onto a basis of asymptotic states involving three quark configurations (the baryon B = N, Δ, N*, ...). In this new basis we can write a many body Hamiltonian involving hadrons and mesons, where the interaction is the basic πBB vertex, and includes a π-B interaction derived from the contact term in the original Lagrangian. For the present derivation we neglect terms in the Lagrangian that can change the number of pions by two. However, we will show where these terms would arise in our derivation, and how they will effect our final results. Although the derivation does not explicitly depend on the form of the Lagrangian, we will use the cloudy bag Lagrangian[3] as a guide in our classification of diagrams.

After stating briefly the basic lemma used in our classification scheme, we proceed in Sec. III to a derivation of the two-body equation by exposing the corresponding unitarity cut. We find that the amplitude is divided in a natural way, into a term that has the dressed baryon pole and a background term. The pole term has a dressed B$\to\pi$B′ form factor, and the residue of this amplitude at the baryon pole gives the dressed coupling constant. At this stage the background amplitude results from an interaction that is the sum of all diagrams that are two-particle irreducible. To examine the structure of this two-particle irreducible amplitude, we proceed in Sec. IV to expose the three-particle unitarity cut. Here we find that the effective π-B interaction, which gives the background term, includes the crossed π-N diagram, but with the πB$\to$B vertex dressed. This last observation is crucial[15] for a proper description of the π-N phase shifts in the P_{11} channel and the πNN coupling constant. Also included in this interaction are the π-B scattering through the contact term, and all multiple scattering in the $\pi\pi$B Hilbert space. This latter contribution is included through the three-body $\pi\pi$B amplitude. To include the contribution from three-body unitarity and avoid multidi-

 ©1987 The American Physical Society

mensional integration, we show in Sec. V that the background amplitude, and the full π-B amplitude, satisfy a set of Faddeev-like equations that couple the πB to the $\pi\pi$B Hilbert space. Finally, in Sec. VI we present our concluding remarks, and comment on the terms in the Lagrangian that change the pion number by two.

II. THE LAGRANGIAN

Although our final equations are independent of the detail form of the Hamiltonian, in actual calculations one needs to specify the exact form of the Hamiltonian being used. Furthermore, to make the classification scheme conceptually simpler, the reader should have an explicit Hamiltonian in mind. For this reason we will consider the cloudy bag model[3] Lagrangian as our starting point. In particular, we take the case of volume coupling,[16] as that includes a contact term which allows for S-wave scattering, and plays an important role in the P_{11} channel.[15] This Lagrangian is given, after expansion to second order in f_π^{-2}, by

$$\mathscr{L}=\left[\frac{i}{2}\bar{q}(x)\overleftrightarrow{\partial}q(x)-B\right]\Theta_V-\tfrac{1}{2}\bar{q}(x)q(x)\Delta_S+\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}m_\pi^2\boldsymbol{\phi}^2$$
$$+\frac{1}{2f_\pi}\bar{q}(x)\gamma^\mu\gamma_5\boldsymbol{\tau}\cdot(\partial_\mu\boldsymbol{\phi})q(x)\Theta_V-\frac{1}{4f_\pi^2}\bar{q}(x)\gamma^\mu\boldsymbol{\tau}\cdot(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})q(x)\Theta_V-\frac{1}{6f_\pi^2}[(\partial_\mu\boldsymbol{\phi})^2\boldsymbol{\phi}^2-(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2] \tag{2.1}$$
$$\equiv\mathscr{L}_0+\mathscr{L}_{\pi qq}+\mathscr{L}_{\pi\pi qq}+\mathscr{L}_{\pi\pi\pi\pi}\;, \tag{2.2}$$

where $q(x)$ is the quark field, f_π is the pion decay constant, and B is the bag energy density. Here, Δ_S is a surface delta function, and Θ_V is one, inside, and zero, outside, the bag. In the above, $\mathscr{L}_0$ is the Lagrangian for the MIT bag and the free Lagrangian for the pions. The interaction Lagrangian is the sum of a term that couples the pion to the quark ($\mathscr{L}_{\pi qq}$), the contact term which gives the interaction of the pion with the quark ($\mathscr{L}_{\pi\pi qq}$), and the Lagrangian for π-π interaction ($\mathscr{L}_{\pi\pi\pi\pi}$). We expect at this stage the expansion in f_π^{-2} to be convergent, and the neglected terms to a small contribution. However, it is possible that in some reactions [e.g., (π,2π) (Ref. 9)] we may need to include higher order terms. With the exception of the last term ($\mathscr{L}_{\pi\pi\pi\pi}$), this Lagrangian is identical to that derived from the nonlinear σ model.

Having truncated our Lagrangian, we can quantize it by replacing the quark and pion fields by the corresponding field operators with the appropriate commutation relation. These field operators can be written in terms of creation and annihilation operators as

$$\boldsymbol{\phi}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega_k}}(e^{-ik\cdot x}\mathbf{a}_k+e^{ik\cdot x}\mathbf{a}_k^\dagger)\;, \tag{2.3}$$

$$q(x)=\sum_{n\nu}\psi_{n\nu}b_{n\nu}\;, \tag{2.4}$$

where $b_n^\dagger$ and b_n are the creation and annihilation operators for quarks, while $a_k^\dagger$ and a_k are the corresponding operators for pions. In the above, $\psi_{n\nu}$ is the wave function of the quark in the MIT bag. We note at this stage that in quantizing the quark field we have excluded antiquarks. This is a simplification that could be avoided.

To project onto the baryon spectrum we need to define our asymptotic baryon states in terms of their quark structure. For the present investigation we assume all baryons are three-quark configurations, i.e., our unperturbed bases are

$$|\,\mathrm{B}\rangle=|\,(\mathrm{qqq})\mathrm{B}\rangle\;,$$
$$|\,\pi\mathrm{B}\rangle=|\,\pi,(\mathrm{qqq})\mathrm{B}\rangle\;, \tag{2.5}$$
$$|\,\pi\pi\mathrm{B}\rangle=|\,\pi,\pi,(\mathrm{qqq})\mathrm{B}\rangle\;,$$

where B=N, Δ, N*, ... In defining the above space we have not included states with quark-antiquark ($q\bar{q}$) configurations. This can, in principle, be included, but for the present we will neglect them to maintain consistency with our quantization of the quark fields. By projecting the cloudy bag Lagrangian onto the above basis, we derive the Hamiltonian in the space of baryons and mesons to be

$$H=\sum_\alpha\epsilon_\alpha B_\alpha^\dagger B_\alpha+\int dk\,\omega(k)\mathbf{a}_k^\dagger\cdot\mathbf{a}_k+H_I\;, \tag{2.6}$$

where ϵ_α is the energy of the baryon, while $\omega(k)=(k^2+m_\pi^2)^{1/2}$ is the energy of the pion. At this stage the baryon is static. This is the result of the fact that we have used the bag model for our starting Lagrangian. In the advent of a more realistic model, such as a chiral soliton model, we hope to be able to correct for the center of mass and get a kinetic energy term for the baryon. For the present we assume this has been achieved and we take ϵ_α to include the kinetic energy of the baryon. The interaction Hamiltonian (H_I) resulting from the Lagrangian in Eq. (1.1) is

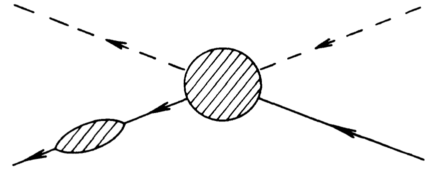

FIG. 1. A diagram that contributes to final state propagator dressing.

$$
\begin{aligned}
H_I = & \sum_{\alpha\beta\mu} \int d^3k (\langle \alpha | H_{\pi qq} | \beta,\mu k \rangle B_\alpha^\dagger B_\beta a_{\mu k} + \langle \alpha,\mu k | H_{\pi qq} | \beta \rangle B_\alpha^\dagger B_\beta a_{\mu k}^\dagger) \\
& + \sum_{\substack{\alpha\beta \\ \mu\nu}} \int d^3k\, d^3k' \langle \alpha,\mu k | H_{\pi\pi qq} | \beta,\nu k' \rangle B_\alpha^\dagger B_\beta a_{\mu k}^\dagger a_{\nu k'} \\
& + \sum_{\substack{\mu\nu \\ \sigma\lambda}} \int d^3k\, d^3k' d^3k'' d^3k''' \langle \mu k,\nu k' | H_{\pi\pi\pi\pi} | \sigma k'',\lambda k''' \rangle a_{\mu k}^\dagger a_{\nu k'}^\dagger a_{\lambda k'''} a_{\sigma k''} \ .
\end{aligned}
\tag{2.7}
$$

Here the matrix elements of the different terms in the interaction Hamiltonian are given in Ref. 15, for the case of the cloudy bag model. We note that in writing the contribution of the contact term to Eq. (1.5) we have only included terms which do not change the number of pions. In this way we avoid the coupling of states that differ by two pions. This constraint will render integral equations for π-N scattering that are practical from a computational point of view. We will see that the neglected terms could be included in perturbation theory. The above Hamiltonian will be used in the following sections to help in the classification of the diagrams.

III. TWO-BODY UNITARITY

Having established the form of the Hamiltonian we will be considering, we turn to the derivation of our equations. The method used in the present analysis is an extension of the method used recently[10] to derive the NN-πNN equations within the framework of a chiral bag model. This involves considering the Green's function, in momentum space, for the process with n initial momenta $p_1, \ldots, p_n$, and l final momenta $q_1, \ldots, q_l$. For the case of scalar particles, this is given by[17]

$$
\begin{aligned}
& G^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n)(2\pi)^4 \delta^4(p_1 + \cdots + p_n - q_1 - \cdots - q_l) \\
& \quad = \int d^4x_1 \cdots d^4x_n d^4y_1 \cdots d^4y_l \exp\left[i \sum_{k=1}^{l} q_k y_k - i \sum_{k=1}^{n} p_k x_k \right] \langle 0 | T[\phi(y_1) \cdots \phi(y_l) \phi(x_1) \cdots \phi(x_n)] | 0 \rangle \ ,
\end{aligned}
\tag{3.1}
$$

where $\phi(x)$ is the field at the space-time point x, and T stands for time ordered products. This Green's function, in perturbation theory, can be taken as the sum of all topologically distinct diagrams that contribute to this process. The corresponding amplitudes are obtained by employing Lehmann-Symanzik-Zimmermann (LSZ) reduction (see Ref. 17), i.e., taking the amplitude $\langle a | T | b \rangle$ as the residue of the corresponding Green's function at the physical masses of all the initial and final particles, i.e.,

$$
\langle q_1, \ldots, q_l | T | p_1, \ldots, p_n \rangle = (-iZ^{-1/2})^{l+n} \prod_{k=1}^{l} (q_k^2 - m^2) G^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n) \prod_{k=1}^{n} (p_k^2 - m^2) \ .
\tag{3.2}
$$

Here, Z is the wave function renormalization, and results from the fact that that $\phi(x)$ is the interacting field.

To classify the diagrams that contribute in perturbation theory to a given amplitude, we need to introduce some basic definitions.[10,11,14]

(i) A k cut is an arc that separates initial from final states in a given diagram, and cuts k particle lines with at least one internal line. An internal k cut is one that cuts internal lines only.

(ii) An amplitude is said to be r-particle irreducible if all diagrams that contribute to the amplitude do not admit any k cuts with $k \leq r$.

(iii) The last cut lemma[14] states that for a given amplitude that is $(r-1)$-particle irreducible there is a unique way of getting an internal r cut closest to either the initial or final states for all diagrams that contribute to the amplitude.

These definitions were first used by Taylor for the three-body problem,[14] and more recently to derive the NN-πNN equations.[10,11]

To illustrate the use of the above definitions and at the same time examine the basic structure of the π-N amplitude, we consider the problem of renormalization of the baryon propagators. The amplitude for π-B ($\mathrm{B} = \mathrm{N}, \Delta, \mathrm{N}^*, \ldots$) scattering $\langle f(\pi \mathrm{B}) | T | i(\pi \mathrm{B}') \rangle$, and the connected diagrams that contribute to this amplitude, can be divided into two classes.

(i) Those for which we have a self-energy contribution on an external baryon leg (see Fig. 1). These diagrams are of the general form

$$
\langle f(\pi \mathrm{B}) | T | i(\pi \mathrm{B}') \rangle = \Sigma_{\mathrm{B}}^{(1)} d_{\mathrm{B}}^{(0)} \langle f(\pi \mathrm{B}) | T | i(\pi \mathrm{B}') \rangle \ ,
\tag{3.3}
$$

where $d_{0\mathrm{B}}$ and $\Sigma_{\mathrm{B}}^{(1)}$ are the undressed baryon propagator [e.g., $d_{0\mathrm{B}} = (\not{p} - m_{0\mathrm{B}})^{-1}$ for baryons with bare mass $m_{0\mathrm{B}}$] and the self-energy of baryon B, which is one-particle irreducible as indicated by the superscript (1). Here, m_{0B} is the mass of the baryon as determined by the MIT bag with possible gluon corrections.

(ii) The rest of the diagrams not belonging to (i). These we denote by $\langle f(\pi\hat{B}) \mid T \mid i(\pi B')\rangle$. The caret on the B indicates that the diagrams that contribute to this amplitude exclude those that have a bubble on the final baryon line.

We now can write the π-B amplitude as

$$\langle f(\pi B) \mid T \mid i(\pi B')\rangle = \Sigma_B^{(1)} d_{0B} \langle f(\pi B) \mid T \mid i(\pi B')\rangle + \langle f(\pi\hat{B}) \mid T \mid i(\pi B')\rangle \ . \quad (3.4)$$

This result can be rewritten as

$$d_{0B}\langle f(\pi B) \mid T \mid i(\pi B')\rangle = d_B \langle f(\pi\hat{B}) \mid T \mid i(\pi B')\rangle \ , \quad (3.5)$$

where d_B is the dressed baryon propagator, and is given by

$$d_B = (d_{0B}^{-1} - \Sigma_B^{(1)})^{-1} = Z_B d_B^R \quad (3.6)$$

with Z_B the wave function renormalization constant and d_B^R the corresponding renormalized propagator. This propagator has a pole at the physical baryon mass with unit residue. In a similar manner we can dress the initial baryon B′ to get

$$Z_B^{1/2} d_B^R \langle f(\pi\hat{B}) \mid T \mid i(\pi B')\rangle d_{B'}^R Z_{B'}^{1/2} = d_{0B}\langle f(\pi B) \mid T \mid i(\pi B')\rangle d_{0B'} \ . \quad (3.7)$$

The π-B amplitude $\langle f(\pi\hat{B}) \mid T \mid i(\pi\hat{B}')\rangle$ as defined above includes a factor of $Z^{1/2}$ for each external baryon line. This is consistent with our definition of the amplitude in terms of the Green's function in Eq. (3.2). The above procedure will give us the π-B amplitude for physical baryons. One could carry the same procedure for the initial and final pion; however, the Hamiltonian we are considering gives no dressing to the pion. This is due to the fact that we have neglected the antiquark contribution in our quantization of the cloudy bag Lagrangian.

Before we can proceed to examine two-body unitarity, we need to expose the one-body intermediate states, which are possible for this system. The connected diagrams that contribute to the π-B amplitude $\langle f(\pi\hat{B}) \mid T \mid i(\pi\hat{B}')\rangle$ can be divided into two classes.

(i) Those with intermediate states of at least two particles (e.g., πB or $\pi\pi$B states). These diagrams are by definition one particle irreducible. We denote this class of diagrams, whose sum is the one particle irreducible amplitude, by $\langle f(\pi\hat{B}) \mid T^{(1)} \mid i(\pi\hat{B}')\rangle$.

(ii) The diagrams not belonging to (i). These are one particle reducible and can be written in terms of the $\pi B \to B'$ amplitude $\langle B' \mid f \mid i(\pi\hat{B})\rangle$ using the last-cut lemma as

$$\sum_{B''} \langle f(\pi\hat{B}) \mid f^{(1)\dagger} \mid \hat{B}''\rangle d_{0B''} \langle B'' \mid f^{(0)} \mid i(\pi\hat{B}')\rangle \ . \quad (3.8)$$

In writing Eq. (3.8) we have used the last cut lemma to expose the final one particle intermediate state. Alternatively, we could expose the first one particle intermediate state and write the diagrams that belong to (ii) as

$$\sum_{B''} \langle f(\pi\hat{B}) \mid f^{(0)\dagger} \mid B''\rangle d_{0B''} \langle \hat{B}'' \mid f^{(1)} \mid i(\pi\hat{B}')\rangle \ . \quad (3.9)$$

We now can write the π-B amplitude as the sum of nonpole (one-particle irreducible) and pole (one-particle reducible) amplitudes as

$$\langle f(\pi\hat{B}) \mid T \mid i(\pi\hat{B}')\rangle = \langle f(\pi\hat{B}) \mid T^{(1)} \mid i(\pi\hat{B}')\rangle + \sum_{B''} \langle f(\pi\hat{B}) \mid f^{(1)\dagger} \mid \hat{B}''\rangle d_{0B''} \langle B'' \mid f^{(0)} \mid i(\pi B')\rangle \quad (3.10)$$

$$= \langle f(\pi\hat{B}) \mid T^{(1)} \mid i(\pi\hat{B}')\rangle + \sum_{B''} \langle f(\pi\hat{B}) \mid f^{(0)\dagger} \mid B''\rangle d_{0B''} \langle \hat{B}'' \mid f^{(1)} \mid i(\pi\hat{B}')\rangle \ . \quad (3.11)$$

To rewrite these equations in terms of physical (i.e., dressed) baryons we need to examine the $\pi B \to B'$ amplitude $\langle B' \mid f^{(0)} \mid i(\pi\hat{B})\rangle$. The diagrams that contribute to this amplitude can be divided into two classes: (i) those that are one particle irreducible, which we denote by $\langle \hat{B}' \mid f^{(1)} \mid i(\pi\hat{B})\rangle$, and (ii) the diagrams that are one particle reducible. These can all be written, using the last-cut lemma, in the form

$$\Sigma_{B'}^{(1)} d_{0B'} \langle B' \mid f^{(0)} \mid i(\pi\hat{B})\rangle \ . \quad (3.12)$$

We now can write the amplitude for $\pi B \to B'$ as

$$\langle B' \mid f^{(0)} \mid i(\pi\hat{B})\rangle = \langle \hat{B}' \mid f^{(1)} \mid i(\pi\hat{B})\rangle + \Sigma_{B'}^{(1)} d_{0B'} \langle B' \mid f^{(0)} \mid i(\pi\hat{B})\rangle \ , \quad (3.13)$$

or

$$d_{0B'} \langle B' \mid f^{(0)} \mid i(\pi\hat{B})\rangle = d_{B'} \langle B' \mid {}^{(1)} \mid i(\pi\hat{B})\rangle \ . \quad (3.14)$$

This result allows us to rewrite Eq. (3.10) in terms of the dressed baryon propagators only as

$$\langle f(\pi\hat{B}) \mid T \mid i(\pi\hat{B}')\rangle = \langle f(\pi\hat{B}) \mid T^{(1)} \mid i(\pi\hat{B}')\rangle + \sum_{B''} \langle f(\pi\hat{B}) \mid f^{(1)\dagger} \mid \hat{B}''\rangle \times d_{B''} \langle \hat{B}'' \mid f^{(1)} \mid i(\pi\hat{B}')\rangle \ . \quad (3.15)$$

In this way we have written the π-N amplitude in terms of a pole part and a nonpole part by exposing the one particle intermediate states using the last-cut lemma. Here we observe that the pole part has contribution from all baryons that have a bare three-quark structure. Furthermore, all pionic corrections to these baryons are included. Thus if the final baryon has a mass greater than the π-N threshold, that baryon is given as a resonance and its width for decay by pion emission can be calculated given $\Sigma_B^{(1)}$.

To expose the two-particle unitarity structure of the equations, we need to classify the diagrams that contribute to the amplitudes for $\pi B' \to \pi B$, $\pi B' \to B$, and the self-energy term [i.e., $\langle f(\pi\hat{B}) | T^{(1)} | i(\pi\hat{B}') \rangle$, $\langle \hat{B} | f^{(1)} | i(\pi\hat{B}') \rangle$, and $\Sigma_B^{(1)}$] according to their irreducibility using the last-cut lemma. However, before we proceed with this task, we need to simplify our notation in two ways. (i) We have shown above how we can always replace the bare baryon propagator d_{0B} by the corresponding dressed one d_B, and in the process we replaced B by $\hat{B}$. This procedure is identical to the removal of all explicit bubbles in baryon lines, and at the same time considers all baryon propagators dressed. From this point on we will assume that all baryon propagators have been dressed, and at the same time suppress the caret on the baryon label. (ii) If we include in our space n distinct baryons (e.g., $n=2$ if $B=N,\Delta$), then our amplitude for a given process is an $n\times n$ matrix in this space. On the other hand, if the truncation in the number of distinct baryons is different in the B and πB Hilbert space, then our amplitude for $\pi B \to B$ will be $m\times n$, where m and n are the number of distinct baryons in the B and πB spaces, respectively. If we further assume that the propagators for the different baryons form a diagonal matrix of the form

$$[d]_{BB'} = \delta_{BB'} d_B \ , \tag{3.16}$$

then Eq. (3.15) can be written in operator form as

$$T_{BB}^{(0)} = T_{BB}^{(1)} + f^{(1)\dagger} d f^{(1)} \ , \tag{3.17}$$

where the superscript (0) denotes the fact that the total π-B amplitude $T_{BB}^{(0)}$ is one-particle reducible. The introduction of subscripts for the amplitudes is superficial at this stage, but will become necessary when we introduce three-body unitarity.

The contribution of two-body unitarity to the π-N amplitude comes from three sources, $T_{BB}^{(1)}$, $f^{(1)}$, and $\Sigma^{(1)}$, and we need to examine each separately, by employing the last-cut lemma. For the self-energy we are basically considering the one-particle irreducible amplitude for $B\to B$ off shell. The diagrams that contribute to this amplitude can be divided into two classes: (i) Those that are two-particle irreducible, which we denote by $\Sigma^{(2)}$, and (ii) the two-particle reducible diagrams. These can be written with the help of the last-cut lemma as

$$f^{(2)} d d_\pi f^{(1)\dagger} \ .$$

This allows us to write the self-energy term as

$$\Sigma^{(1)} = \Sigma^{(2)} + f^{(2)} d d_\pi f^{(1)\dagger} = \Sigma^{(2)} + f^{(1)} d d_\pi f^{(2)\dagger} \ , \tag{3.18}$$

where d_π is the pion propagator given in momentum space as $d_\pi = (p^2 - m_\pi^2)^{-1}$. In Eq. (3.18) the self-energy is written in terms of the one- and two-particle irreducible amplitudes for $B\to\pi B$ (see Fig. 2). We note here that the π-B intermediate states have physical baryons. Also, the matrix Σ is not necessarily diagonal. Thus if we have two or more baryons that have the same quantum numbers, then Σ will have elements that connect those baryons. An example at hand is that of the nucleon and Roper, if both are taken as three-quark configurations. In this case, Σ is a 2×2 matrix and the dressed baryon propagator [given in Eq. (3.16)] is no longer diagonal (see Ref. 15).

To completely expose the two-body unitarity cut in $\Sigma^{(1)}$, we need to turn our attention to $f^{(1)}$. Here again the amplitude can be divided into (i) a part that is two-particle irreducible, which we denote by $f^{(2)}$, and (ii) the rest of the diagrams not belonging to (i). These can be written, using the last-cut lemma, as

$$f^{(2)} d d_\pi T_{BB}^{(1)} \ .$$

Thus the one-particle irreducible $\pi B \to B$ amplitude $f^{(1)}$ can be written as

$$f^{(1)} = f^{(2)} + f^{(2)} d d_\pi T_{BB}^{(1)} \ . \tag{3.19}$$

With this result, which we present diagrammatically in Fig. 3, we can rewrite the baryon self-energy as

$$\Sigma^{(1)} = \Sigma^{(2)} + f^{(2)} d d_\pi f^{(2)\dagger} + f^{(2)} d d_\pi T_{BB}^{(1)} d d_\pi f^{(2)\dagger} \ . \tag{3.20}$$

In this way we reduced the problem of calculating $\Sigma^{(1)}$ and $f^{(1)}$ to that of determining the one-particle irreducible $\pi B \to \pi B$ amplitude $T_{BB}^{(1)}$, and the form factor for $\pi B \to B$, $f^{(2)}$. The structure of this form factor is not relevant to two-body unitarity since it is two-particle irreducible, and one could take $f^{(2)}$ from the basic Hamiltonian, i.e., $f^{(2)}$ can be written in terms of the quark wave function in the MIT bag model.

Finally, to get the two-body unitarity structure of $T_{BB}^{(1)}$, we divide the diagrams that contribute to this amplitude into two classes: (i) those that are two-particle irreducible, and which we denote by $T_{BB}^{(2)}$, and (ii) the diagrams that do not contribute to (i), which can be written, using the last-cut lemma, as

$$T_{BB}^{(2)} d d_\pi T_{BB}^{(1)} = T_{BB}^{(1)} d d_\pi T_{BB}^{(2)} \ .$$

In this way we find that $T_{BB}^{(1)}$ is a solution of the two-body equations with $T_{BB}^{(2)}$ as the potential, i.e.,

$$T_{BB}^{(1)} = T_{BB}^{(2)} + T_{BB}^{(2)} d d_\pi T_{BB}^{(1)} = T_{BB}^{(2)} + T_{BB}^{(1)} d d_\pi T_{BB}^{(2)} \ . \tag{3.21}$$

In this way we have completed the exposure of the two-body unitarity cut, and the solution of Eqs. (3.17) and (3.19)–(3.21) will guarantee that our final π-N amplitude

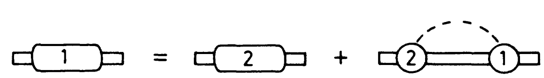

FIG. 2. Diagrammatic illustration of Eq. (3.18) for the one-particle irreducible self-energy.

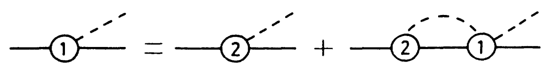

FIG. 3. Diagrammatic illustration of Eq. (3.19) for the one particle irreducible $\pi B \rightarrow B$ form factor.

$T_{BB}^{(0)}$ will satisfy two-body unitarity. This result for the case when B=N was first derived by Mizutani and Koltun[18] using the projection operator technique. It also can be shown[19,20] that the final result for $T_{BB}^{(0)}$ is identical to the solution of the two-body equation as a two potential problem, with the potential given by

$$V = T_{BB}^{(2)} + f^{(2)\dagger} d_0 f^{(2)} , \tag{3.22}$$

where d_0 is the bare baryon propagator, and $T_{BB}^{(2)}$ is any potential.

The main motivation for the present detailed analysis of the two-body unitarity sector was to illustrate the application of this approach to the simple problem of two body scattering below the threshold for pion production, and at the same time generalize the result to include any number of baryons as three-quark states. At this stage we have not analyzed the structure of $T_{BB}^{(2)}$, and can only state that, in lowest order, it will include the contact term in the Hamiltonian as well as the crossed diagram. By examining the three-body unitarity structure of this amplitude we will be able to state more precisely the structure of $T_{BB}^{(2)}$. This has been shown to be essential for the description of pion-nucleon scattering in the P_{11} channel.[15]

IV. THREE-BODY UNITARITY

The motivation for exploring the three-particle sector of the π-N amplitude is twofold: Firstly, we need to get the exact form for $T_{BB}^{(2)}$ and, in particular, the strength of the coupling in the crossed diagram and the contact term. Secondly, several π-N resonances, most notably the Roper, occur above the threshold for pion production and are highly inelastic. In order to make accurate deductions concerning the quark content of such resonances, it is essential to use a theory that includes three-body unitarity.

To expose the three-particle unitarity cut, it is clear from Eqs. (3.18), (3.19), and (3.21) that we need to examine all amplitudes that are two-particle irreducible (i.e., $\Sigma^{(2)}$, $f^{(2)}$, and $T_{BB}^{(2)}$). For the self-energy term $\Sigma^{(2)}$ we can divide the diagrams that contribute to this amplitude into two classes: (i) those that are three-particle irreducible, which we denote by $\Sigma^{(3)}$, and (ii) the rest of the diagrams not included in (i). These are three-particle reducible and can be written, with the help of the last-cut lemma, as

$$\Gamma^{(2)} G \Gamma^{(3)\dagger} = \Gamma^{(3)} G \Gamma^{(2)\dagger} , \tag{4.1}$$

where $\Gamma^{(i)}$ is the amplitude for $\pi\pi B \rightarrow B$ that is i-particle irreducible, and G is the three-body free propagator $d_\pi d_\pi d$. This allows us to write the self-energy term $\Sigma^{(2)}$ as

$$\begin{aligned} \Sigma^{(2)} &= \Sigma^{(3)} + \Gamma^{(3)} G \Gamma^{(2)\dagger} \\ &= \Sigma^{(3)} + \Gamma^{(2)} G \Gamma^{(3)\dagger} . \end{aligned} \tag{4.2}$$

If we take the constraint imposed on the Hamiltonian, namely that it not include any terms that couple Hilbert spaces that differ by two or more pions, then $\Gamma^{(3)}$ is zero, and so is $\Sigma^{(3)}$. Under this condition the self-energy term $\Sigma^{(2)}$ is zero. However, in expanding the Lagrangian, to second order in f_π^{-1}, we include the contact terms, which gives coupling between the B and the $\pi\pi B$ Hilbert spaces. If we decide to include this coupling in lowest order, its contribution to the self-energy is given by

$$\Sigma^{(2)} = \Gamma^{(3)} G \Gamma^{(3)\dagger} . \tag{4.3}$$

The contribution from $\Sigma^{(3)}$ is zero unless we include the coupling between the B and $\pi\pi\pi B$ Hilbert spaces, and that requires retaining terms of a higher order than f_π^{-2} in the Lagrangian.

In a similar manner we can write the two particle irreducible $\pi B \rightarrow B$ amplitude as

$$\begin{aligned} f^{(2)} &= f^{(3)} + (\Gamma^{(3)} G F^{(2)\dagger})_c \\ &= f^{(3)} + (\Gamma^{(2)} G F^{(3)\dagger})_c , \end{aligned} \tag{4.4}$$

where $F^{(i)}$ is the amplitude for $\pi B \rightarrow \pi\pi B$ that is i-particle irreducible. The subscript c indicates that we should include only connected diagrams. Here again, if we constrain our Hamiltonian to have no coupling between the B and $\pi\pi B$ spaces, then $\Gamma^{(i)}=0$ for $i \geq 2$ and $f^{(2)} = f^{(3)} = f^{(4)} = \cdots$. In this case $f^{(2)}$ can only include diagrams with no intermediate states. In other words, $f^{(2)}$ is the basic interaction that couples the B to the πB Hilbert space as included in the Hamiltonian, and it is given in terms of the quark wave function. In the event that we need to include that contribution from the contact term that couples the B to the $\pi\pi B$ space, then Eq. (4.4) can be used as the basis for a perturbation expansion. We will present later an expression for $F^{(2)\dagger}$ which will be the basis of a multiple scattering expansion for $f^{(2)}$ that includes the coupling of the B to the $\pi\pi B$ channels. From the above analysis we see that, for the Hamiltonian in Eq. (2.7), both $\Sigma^{(2)}$ and $f^{(2)}$ have no contribution to three-body unitarity.

With the main contribution to three-body unitarity coming from $T_{BB}^{(2)}$, we consider the diagrams that contribute to this amplitude. These can be divided into those that are three-particle irreducible, which we denote by $T_{BB}^{(3)}$. These include, among other diagrams, the contribution from the contact term (see Fig. 4). The rest of the diagrams that are three-particle reducible can be written as

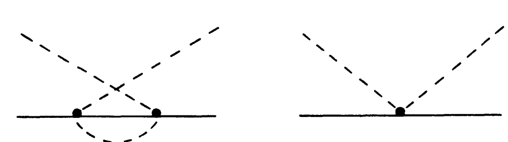

FIG. 4. Some of the diagrams that contribute to $T_B^{(3)}$.

$$(F^{(3)}GF^{(2)\dagger})_c=(F^{(2)}GF^{(3)\dagger})_c \ . \tag{4.5}$$

The $\pi\pi$B$\rightarrow\pi$B amplitude $F^{(3)}$ has a connected and disconnected part. The disconnected part is given by

$$F_d^{(n)}=\sum_{ij}\overline{\delta}_{ij}d_\pi^{-1}(i)f^{(n-1)}(j) \ , \tag{4.6}$$

with $i,j=1,2$ labeling the pions, $\overline{\delta}_{ij}=1-\delta_{ij}$, and $f^{(n)}(j)$ as the n-particle irreducible amplitude for πB$\rightarrow$B with the jth pion being absorbed. On the other hand, the connected part of $F^{(3)}$, because of its three-particle irreducibility, requires the coupling of the πB to the $\pi\pi\pi$B Hilbert spaces. If we neglect this coupling, as it is not included in our Hamiltonian, then the connected part of $F^{(3)}$ is zero. We now can write the two-particle irreducible π-B amplitude as

$$\begin{aligned}T_{\rm BB}^{(2)}&=T_{\rm BB}^{(3)}+(F_d^{(3)}GF^{(2)\dagger})_c\\&=T_{\rm BB}^{(3)}+(F^{(2)}GF_d^{(3)\dagger})_c \ .\end{aligned} \tag{4.7}$$

The second term on the right hand side (rhs) of Eq. (4.7) includes terms of the form

$$(F_d^{(3)}GF_d^{(2)\dagger})_c \ ,$$

which are given diagrammatically in Fig. 5, and have the form of the crossed diagram with the unsymmetric feature that one of the vertices is one-particle irreducible while the other is two-particle irreducible. To resolve this problem of lack of symmetry in the vertices, we need to consider the second term, i.e.,

$$(F_d^{(3)}GF_c^{(2)\dagger}) \ ,$$

and, in particular, we need to examine the structure of $F_c^{(2)}$, the amplitude for $\pi\pi$B$\rightarrow\pi$B. The diagrams that contribute to this amplitude can again be divided according to their irreducibility, and if we apply the last-cut lemma to the three-particle reducible class of diagrams, we get

$$\begin{aligned}F_c^{(2)}&=F_c^{(3)}+(F^{(3)}GM^{(2)})_c\\&=F_c^{(3)}+(F^{(2)}GM^{(3)})_c \ ,\end{aligned} \tag{4.8}$$

where $M^{(i)}$ is the i-particle irreducible amplitude for $\pi\pi$B$\rightarrow\pi\pi$B. The second term on the rhs of Eq. (4.8) will give us the mechanism for coupling the πN to the ρN channel. Here, ρ stands for the π-π interaction. Making use of Eqs. (3.19) and (4.6), we can write the disconnected $\pi\pi$B$\rightarrow\pi$B amplitude $F_d^{(2)}$ as

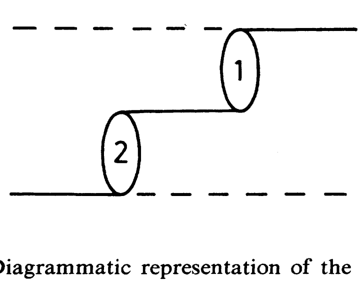

FIG. 5. Diagrammatic representation of the connected part of $F_d^{(3)}GF_d^{(2)\dagger}$.

$$\begin{aligned}F_d^{(2)}=\sum_{ij}\overline{\delta}_{ij}d_\pi^{-1}(i)f^{(1)}(j)&=F_d^{(3)}+(F_d^{(3)}GM_d^{(2)})_d\\&=F_d^{(3)}+(F_d^{(2)}GM_d^{(3)})_d \ ,\end{aligned} \tag{4.9}$$

where

$$M_d^{(n)}=\sum_{ij}\overline{\delta}_{ij}d_\pi^{-1}(i)T_{\rm BB}^{(n-1)}(j)+d^{-1}t^{(n-1)} \ . \tag{4.10}$$

Here, $t^{(n)}$ is the n-particle irreducible π-π amplitude, and $T_{\rm BB}^{(n)}(j)$ is the n-particle irreducible π-B amplitude with the jth pion interacting with the baryon. Combining the results of Eqs. (4.8) and (4.9), we can write,

$$\begin{aligned}F^{(2)}&=F^{(3)}+F^{(3)}GM^{(2)}\\&=F^{(3)}+F^{(2)}GM^{(3)} \ .\end{aligned} \tag{4.11}$$

Using this result in Eq. (4.7), we get

$$T_{\rm BB}^{(2)}=T_{\rm BB}^{(3)}+(F^{(3)}GF^{(3)\dagger})_c+(F^{(3)}GM^{(2)}GF^{(3)\dagger})_c \ . \tag{4.12}$$

This π-N potential includes (i) the contact term $T_{\rm BB}^{(3)}$, which at this stage has the strength given by the interaction Hamiltonian, (ii) the crossed diagram $(F^{(3)}GF^{(3)\dagger})_c$, which has the bare πB$\rightarrow$B vertices, and (iii) the last term on the rhs, which includes all the multiple scattering in the $\pi\pi$B Hilbert space in $M^{(2)}$. In particular, it includes scattering of the pion off the baryon and the interaction between the two pions. Both of these effects are important for the description of π-N scattering above the pion production threshold. However, in the P_{11} channel, the behavior of the amplitude in the region of the Roper resonance can influence the phase shifts at lower energies. To understand the role of this last term in more detail, we need to examine the structure of the $\pi\pi$B$\rightarrow\pi\pi$B amplitude $M^{(2)}$. This amplitude has contributions from connected diagrams M_c and disconnected diagrams M_d. Although the disconnected part does not contribute to a physical amplitude, when included in Eq. (4.12), it can lead to a connected diagram that gives a physical contribution to the π-B potential. We now can write

$$M^{(2)}=M_d^{(2)}+M_c^{(2)} \ , \tag{4.13}$$

with $M_d^{(2)}$ given by Eq. (4.10), i.e.,

$$M_d^{(2)}=\sum_{ij}\overline{\delta}_{ij}d_\pi^{-1}(i)T_{\rm BB}^{(1)}(j)+d^{-1}t^{(1)} \ , \tag{4.14}$$

where $t^{(1)}$ is the π-π amplitude, which we assume to satisfy a two-body equation of the form

$$t^{(1)}=t^{(2)}+t^{(2)}d_\pi d_\pi t^{(1)} \ , \tag{4.15}$$

with the potential $t^{(2)}$ given by the last term in the interaction Hamiltonian in Eq. (2.7). We note at this stage that $M_d^{(2)}$ includes the π-B amplitude $T_{\rm BB}^{(1)}$, which we need to determine. Thus, it appears that we have a self-consistency problem. However, we observe that the energy at which $T_{\rm BB}^{(1)}$ is required in Eq. (4.14) is at least m_π less than the energy in $M_d^{(2)}$ and $T_{\rm BB}^{(2)}$ in Eq. (4.12). In this respect the problem of self-consistency is overcome, and to calculate the π-N amplitude in the Roper region we need the amplitude below the threshold for pion production, which we can parametrize as input to the calculation. Making use of Eq. (4.15) and the fact that $T^{(1)}_{\rm BB}$ satisfies a two-body equation, we can write

$$M_d^{(2)}=\sum_{\alpha=1}^{3} M_d^{(2)}(\alpha)=M_d^{(3)}+(M_d^{(3)}GM_d^{(2)})_d \ . \tag{4.16}$$

We now turn to the connected, two-particle irreducible $3\rightarrow 3$ amplitude for $\pi\pi{\rm B}\rightarrow\pi\pi{\rm B}$, $M_c^{(2)}$. The diagrams that contribute to this amplitude can be divided into two classes: (i) Those that are three particle irreducible, which we denote by $M_c^{(3)}$ (these diagrams will play the role of a three-body force in this three-body problem), and (ii) diagrams that are not included in (i). These latter diagrams can be classified using the last-cut lemma and written as

$$(M^{(3)}GM^{(2)})_c=(M^{(2)}GM^{(3)})_c \ . \tag{4.17}$$

Combining the results of Eqs. (4.16) and (4.17), we can write an integral equation for the $3\rightarrow 3$ amplitude which is of the form of the Lippmann-Schwinger equation, i.e.,

$$\begin{aligned} M^{(2)}&=M^{(3)}+M^{(3)}GM^{(2)} \\ &=M^{(3)}+M^{(2)}GM^{(3)} \ , \end{aligned} \tag{4.18}$$

where

$$M^{(3)}=M_d^{(3)}+M_c^{(3)} \ . \tag{4.19}$$

If at this stage we consider Eq. (4.18) as the three-particle Lippmann-Schwinger equation with the potential given by Eq. (4.19), then we can implement Faddeev methods to write our integral equation for the $3\rightarrow 3$ amplitude. In particular, if we take the three-body force $M_c^{(3)}$ to be zero, then the amplitude $M^{(2)}$ can be written as

$$\begin{aligned} M^{(2)}&=\sum_{\alpha\beta} M^{(2)}_{\alpha\beta} \\ &=\sum_{\alpha\beta}[M_d^{(2)}(\alpha)\delta_{\alpha\beta}+M_d^{(2)}(\alpha)GU_{\alpha\beta}GM_d^{(2)}(\beta)] \ , \end{aligned} \tag{4.20}$$

where $U_{\alpha\beta}$ are the Alt-Grassberger-Sandhas[21] (AGS) amplitudes and $M_d^{(2)}(\alpha)$ given is by Eq. (4.14), i.e.,

$$M_d^{(2)}(\alpha)=\begin{cases} \sum_i \bar{\delta}_{ij} d_\pi^{-1}(i)T^{(1)}_{\rm BB}(j) \ \ \text{for } \alpha=j=1,2 \ , & (4.21) \\ d^{-1}t^{(1)}(3) \ \ \text{for } \alpha=3 \ . & (4.22)\end{cases}$$

Here, 1 and 2 are the pions, while 3 is the baryon. From this point on we use the notation that $i,j=1,2$, and $T^{(1)}_{\rm BB}(j)$ is the amplitude for the interaction of the jth pion with the baryon. On the other hand, $\alpha,\beta,\ldots=1,2,3$. With this result we can write the π-B potential as

$$\begin{aligned} T^{(2)}_{\rm BB}&=T^{(3)}_{\rm BB}+(F^{(3)}GF^{(3)\dagger})_c \\ &+\sum_{\alpha\beta}[F^{(3)}GM_d^{(2)}(\alpha)\delta_{\alpha\beta}GF^{(3)\dagger}]_c \\ &+\sum_{\alpha\beta}[F^{(3)}GM_d^{(2)}(\alpha)GU_{\alpha\beta}GM_d^{(2)}(\beta)GF^{(3)\dagger}]_c \ . \end{aligned} \tag{4.23}$$

The AGS (Ref. 21) amplitude $U_{\alpha\beta}$ satisfies the coupled integral equations,

$$\begin{aligned} U_{\alpha\beta}&=\bar{\delta}_{\alpha\beta}G^{-1}+\sum_\gamma \bar{\delta}_{\alpha\gamma}M_d^{(2)}(\gamma)GU_{\gamma\beta} \\ &=\bar{\delta}_{\alpha\beta}G^{-1}+\sum_\gamma U_{\alpha\gamma}GM_d^{(2)}(\gamma)\bar{\delta}_{\gamma\beta} \ . \end{aligned} \tag{4.24}$$

It is clear from Eq. (4.23) that we can combine the second and third terms on the rhs to replace one of the $F^{(3)}$ by $F^{(2)}$. In fact, we can use the AGS equations to iterate Eq. (4.23) and then regroup terms in order to replace all the bare $\pi{\rm B}\rightarrow{\rm B}$ vertices $f^{(2)}$ by the corresponding dressed ones $f^{(1)}$ using Eq. (3.19). After some algebra, we get, for the π-B potential,

$$\begin{aligned} T^{(2)}_{\rm BB}&=T^{(3)}_{\rm BB}+\sum_{ij} F_d^{(2)}(i)G\bar{\delta}_{ij}F_d^{(2)\dagger}(j) \\ &+\sum_{ij}\sum_\alpha F_d^{(2)}(i)G\bar{\delta}_{i\alpha}M_d^{(2)}(\alpha)GU_{\alpha j}GF_d^{(2)\dagger}(j) \ . \end{aligned} \tag{4.25}$$

In this way we have written the π-B potential in terms of the dressed $\pi{\rm B}\rightarrow{\rm B}$ form factor, which by definition should give the experimental coupling constants. The first term on the rhs of Eq. (4.25) is the contact term, which at this stage still has its strength determined by the interaction Hamiltonian. The second term has the crossed diagram, but with the dressed $\pi{\rm B}\rightarrow{\rm B}$ form factors. If we neglect the last term and substitute the rest in Eq. (3.22), we find that the π-N potential consists of a direct pole term with bare form factors, a crossed diagram with the dressed form factor, and a contact term. The two-body Lippmann-Schwinger equation then dresses the form factor for the direct pole term, but gives no further dressing to the crossed diagram. This result is different from the standard procedure adopted in the past, where the coupling constant is taken to be the same in the direct pole term and crossed diagram. We have recently[15] shown that the implementation of the above scheme gives very good results in the P_{11} channel.

The last term on the rhs of Eq. (4.25) gives the contribution to the potential from the multiple scattering effect. In particular, it includes both the contribution due to π-π scattering, and the effect of any π-B resonances that are not included in terms of their quark structure. Furthermore, to satisfy three-body unitarity we need to include this last term.

V. THE πB-$\pi\pi$B EQUATIONS

In the preceding section we wrote the π-B potential $T^{(2)}_{\rm BB}$ in terms of the dressed $\pi{\rm B}\rightarrow{\rm B}$ vertex

$$F_d^{(2)}(i)=\sum_j \bar{\delta}_{ij}d_\pi^{-1}(j)f^{(1)}(i) \ ,$$

and the AGS amplitude $U_{\alpha\beta}$. In this form we can only get the potential by either solving the AGS equations, and then performing the multidimensional integral necessary to calculate the last term on the rhs of Eq. (4.25), or, alternatively, we can use the lowest order multiple scattering expansion for $T^{(2)}_{\rm BB}$. Both of these approaches involve multidimensional integrals, and in the latter case there is no guarantee that three-body unitarity is satisfied. One solution to the above problem is to recast the equations into a set of coupled integral equations that couple the two-body πB and three-body $\pi\pi$B channels.

To derive such a set of equations we need to examine the amplitude that connects these channels (i.e., $\pi\pi B \rightarrow \pi B$, $F^{(0)}$. The diagrams that contribute to this amplitude can be divided into two classes: (i) those that are one-particle irreducible, which we represent by $F^{(1)}$, and (ii) diagrams that do not belong to the first class. The latter we can write, using the last-cut lemma, as

$$f^{(1)\dagger} d \Gamma^{(1)} . \tag{5.1}$$

Since d, the baryon propagator, is dressed, we need to take both f and Γ to be one-particle irreducible. We now can write the connected $\pi\pi B \rightarrow \pi B$ amplitude as

$$F_c^{(0)} = F_c^{(1)} + f^{(1)\dagger} d \Gamma^{(1)} . \tag{5.2}$$

To proceed further, we need to examine $F^{(1)}$ and $\Gamma^{(1)}$. The diagrams that contribute to the $\pi\pi B \rightarrow B$ amplitude $\Gamma^{(1)}$ can be divided into two classes: (i) the diagrams that are two-particle irreducible $\Gamma^{(2)}$, and (ii) the two-particle reducible diagrams which can be written using the last-cut lemma as $f^{(1)} d_\pi d F^{(2)}$. Thus,

$$\Gamma^{(1)} = \Gamma^{(2)} + f^{(1)} d_\pi d F^{(2)} . \tag{5.3}$$

In a similar manner we can classify $\Gamma^{(2)}$ to get

$$\begin{aligned} \Gamma^{(2)} &= \Gamma^{(3)} + \Gamma^{(3)} G M^{(2)} \\ &= \Gamma^{(3)} + \Gamma^{(2)} G M^{(3)} . \end{aligned} \tag{5.4}$$

As already stated at the beginning of Sec. IV, $\Gamma^{(3)}$ and $\Gamma^{(2)}$ are both zero for the Hamiltonian under consideration. In the event that we need to include terms in our Hamiltonian that change the number of pions by two, then $\Gamma^{(3)}$ is taken directly from that Hamiltonian, and in that case $\Gamma^{(2)}$ is given by Eq. (5.4), and is related to the $3 \rightarrow 3$ amplitude $M^{(2)}$. This last fact makes the derivation of a coupled set of equations for the πB-$\pi\pi B$ system more complex. This is the result of the direct coupling between the πB and $\pi\pi\pi B$ channels. To avoid this extra complexity at this stage, we will restrict our analysis to the Hamiltonian in Eq. (2.7), and take $\Gamma^{(2)} = 0$. We now can write the amplitude for $\pi\pi B \rightarrow \pi B$ as

$$F_c^{(0)} = F_c^{(1)} + f^{(1)\dagger} d f^{(1)} d_\pi d F^{(2)} . \tag{5.5}$$

We turn next to the structure of the one-particle irreducible amplitude for $\pi\pi B \rightarrow \pi B$, $F^{(1)}$. Here, again, going through our classification scheme, we get

$$F_c^{(1)} = F_c^{(2)} + T_{BB}^{(1)} d_\pi d F^{(2)} , \tag{5.6}$$

and, therefore,

$$\begin{aligned} F_c^{(0)} &= F_c^{(2)} + (T_{BB}^{(1)} + f^{(1)\dagger} d f^{(1)}) d_\pi d F^{(2)} \\ &= F_c^{(2)} + T_{BB}^{(0)} d_\pi d F^{(2)} \\ &= F_c^{(2)} + T_{BB}^{(0)} d_\pi d F_c^{(2)} + T_{BB}^{(0)} d_\pi d F_d^{(2)} , \end{aligned} \tag{5.7}$$

where we have used Eq. (3.17) in writing the second line of Eq. (5.7). In Fig. 6 we have a diagrammatic representation of the last line of Eq. (5.7). To get the amplitude for $\pi B^* \rightarrow \pi B$ (B^* is a π-B resonance) or $\rho B \rightarrow \pi B$ (ρ is a π-π resonance), we need to take the right hand residue of Eq. (5.7) at the B^* or ρ pole. Before we take this residue, we first observe that the last term on the rhs of Eq. (5.7) is

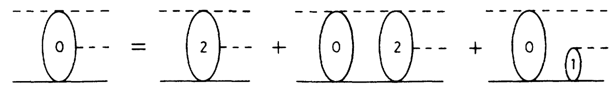

FIG. 6. Diagrammatic illustration of Eq. (5.7).

given in terms of known amplitudes (i.e., the π-B amplitude $T_{BB}^{(0)}$ and the $\pi B \rightarrow B$ vertex $F_d^{(2)} = d_\pi^{-1} f^{(1)}$). Second, to close the system of equations that couple the πB and $\pi\pi B$ space, we need not include this term. Thus we need only take the residue of

$$F_c^{(2)} + T_{BB}^{(0)} d_\pi d F_c^{(2)} = (1 + T_{BB}^{(0)} d_\pi d) F_c^{(2)} . \tag{5.8}$$

To take this residue we need to write $F^{(2)}$ in terms of the amplitudes for π-B with a spectator pion and π-π with a spectator baryon. From Eqs. (4.6), (4.11), and (4.20) we have

$$\begin{aligned} F^{(2)} &= F^{(3)}(1 + G M^{(2)}) \\ &= \sum_{ij} \bar{\delta}_{ij} d_\pi^{-1}(i) f^{(2)}(j) \\ &\quad \times \left[1 + \sum_\alpha G M_\alpha^{(2)} + \sum_{\alpha\beta} G M_d^{(2)}(\alpha) G U_{\alpha\beta} G M_d^{(2)}(\beta) \right] , \end{aligned} \tag{5.9}$$

where we have assumed $F_c^{(3)} = 0$, which is consistent with our Hamiltonian. Since the π-B potential given in Eq. (4.25) is expressed in terms of the dressed $\pi B \rightarrow B$ vertex $F_d^{(2)}(i)$, we should also rewrite Eq. (5.9) in terms of $F_d^{(2)}(i)$, in order to establish a set of coupled equations. This is achieved by employing the AGS equations (4.24) for $U_{\alpha\beta}$ to iterate Eq. (5.9) and then regroup terms using Eqs. (4.9) and (4.24). After some algebra we get

$$F^{(2)} = F_d^{(2)} + \sum_{i\beta} F_d^{(2)}(i) G U_{i\beta} G M_d^{(2)}(\beta) . \tag{5.10}$$

We observe here that the first term on the rhs is disconnected, while the second term is connected and can be used to write Eq. (5.8) as

$$\begin{aligned} (1 + T_{BB}^{(0)} d_\pi d) F_c^{(2)} &= (1 + T_{BB}^{(0)} d_\pi d) \\ &\quad \times \sum_{i\beta} F_d^{(2)}(i) G U_{i\beta} G M_d^{(2)}(\beta) . \end{aligned} \tag{5.11}$$

We can now take the rhs residue of Eq. (5.11) to get the amplitude for $\pi B^* \rightarrow \pi B$ and $\rho B \rightarrow \pi B$, since the amplitude $M_d^{(2)}(\beta)$ has the B^* pole for $\beta = 1,2$ and the ρ pole for $\beta = 3$. Because the π-B and π-π subamplitudes in $M_d^{(2)}$ satisfy the two-body equations, we can use Fredholm theory to write, for $\beta = 1,2$,

$$\begin{aligned} M_d^{(2)}(i) &= \sum_j \bar{\delta}_{ij} d_\pi^{-1}(j) T_{BB}^{(1)}(i) \\ &= \sum_j \bar{\delta}_{ij} d_\pi^{-1}(j) \sum_n \frac{|\phi_n(i)\rangle\langle\phi_n(i)|}{1 - \lambda_n(i)} , \end{aligned} \tag{5.12}$$

where $\lambda_n(i)$ and $\phi_n(i)$ are the eigenvalues and eigenvectors of the kernel of the two-body equations for $T_{BB}^{(1)}(i)$. Making use of the fact that one of the eigenvalues λ_n is one at the pole of the π-B subamplitude, we can get the amplitude for $\pi B^* \to \pi B$ by taking the rhs residue of Eq. (5.11) at the B* pole to get

$$\langle \psi_0 | T^{(0)}_{BB^*} G | \phi_n \rangle d_\pi^{-1} | \chi_\pi \rangle = \langle \psi_0 | T^{(0)}_{BB^*} (d_\pi d | \phi_n \rangle)(|\chi_\pi\rangle) , \quad (5.13)$$

where $|\chi_\pi\rangle$ and $d_\pi d |\phi_n\rangle$ are the wave function of the pion and the B* resonance in the πB^* channel, respectively, while $|\psi_0\rangle$ is the asymptotic wave function for the πB channel. In Eq. (5.13), T_{BB^*} is given by

$$T^{(0)}_{BB^*} = (1 + T^{(0)}_{BB} d_\pi d) \sum_i F_d^{(2)}(i) G U_{ij} , \quad (5.14)$$

where j, in U_{ij}, is the label of the pion interacting first with the baryon in the πB^* channel and is needed to the symmetrization of the equations.

This same procedure can be followed to get the $\rho B \to \pi B$ amplitude. In this case $M_d^{(2)}(3)$ is given, using Fredholm theory for the π-π two-body amplitude, by

$$M_d^{(2)}(3) = d^{-1} t^{(1)}(3) = d^{-1} \sum_n \frac{|h_n(3)\rangle\langle h_n(3)|}{1 - \lambda_n(3)} . \quad (5.15)$$

Here again $\lambda_n(3)$ and h_n are the eigenvalues and eigenvectors of the kernel of the two-body π-π equation. We now can write the amplitude for $\rho B \to \pi B$, by taking the rhs residue of Eq. (5.11) at the ρ pole, as

$$\langle \psi_0 | T^{(0)}_{B\rho} (d_\pi d_\pi | h_n \rangle) | \chi_B \rangle , \quad (5.16)$$

with

$$T^{(0)}_{B\rho} = (1 + T^{(0)}_{BB} d_\pi d) \sum_i F_d^{(2)}(i) G U_{i3} . \quad (5.17)$$

In Eq. (5.16), $|\chi_B\rangle$ and $d_\pi d_\pi | h_n\rangle$ are the wave functions for the B and ρ resonance in the ρB channel, respectively. We can combine the results of Eqs. (5.14) and (5.17) into one equation of the form

$$T^{(0)}_{B\lambda} = (1 + T^{(0)}_{BB} d_\pi d) \sum_i F_d^{(2)}(i) G U_{i\lambda} , \quad (5.18)$$

where for $\lambda = 1,2$ we have the $\pi B^* \to \pi B$ amplitudes, while $\lambda = 3$ gives the $\rho B \to \pi B$ amplitude. The amplitudes for the reversed reactions are given by

$$T^{(0)}_{\lambda B} = \sum_i U_{\lambda i} G F_d^{(2)\dagger}(i)(1 + d_\pi d T^{(0)}_{BB})$$

$$= \sum_i U_{\lambda i} G F_d^{(2)\dagger}(i)(1 + d_\pi d T^{(1)}_{BB}) \quad (5.19)$$

$$+ \sum_i U_{\lambda i} G F_d^{(2)\dagger}(i) d_\pi d f^{(1)\dagger} d f^{(1)} . \quad (5.20)$$

At this stage we have a choice of either deriving coupled equations for the one-particle reducible amplitudes $T^{(0)}_{BB}$ and $T^{(0)}_{\lambda B}$ or for the one-particle irreducible amplitudes $T^{(1)}_{BB}$ and $T^{(1)}_{\lambda B}$. Each of the above options has its advantage in that it reveals different aspects of the equations. We will commence with the latter approach of working with the one-particle irreducible amplitudes. These are given by

$$T^{(1)}_{\lambda B} = \sum_i U_{\lambda i} G F_d^{(2)\dagger}(i)(1 + d_\pi d T^{(1)}_{BB}) , \quad (5.21)$$

while $T^{(1)}_{BB}$ satisfies the equation

$$T^{(1)}_{BB} = T^{(2)}_{BB}(1 + d_\pi d T^{(1)}_{BB}) , \quad (5.22)$$

with $T^{(2)}_{BB}$ given by Eq. (4.25). If we substitute Eq. (4.25) into Eq. (5.22) and make use of Eq. (5.21), we get

$$T^{(1)}_{BB} = \left[T^{(3)}_{BB} + \sum_{ij} F_d^{(2)}(i) \bar{\delta}_{ij} G F_d^{(2)\dagger}(j) \right] (1 + d_\pi d T^{(1)}_{BB}) + \sum_{i\lambda} F_d^{(2)}(i) G \bar{\delta}_{i\lambda} M_d^{(2)}(\lambda) G T^{(1)}_{\lambda B} . \quad (5.23)$$

In this way we have an equation for $T^{(1)}_{BB}$ in terms of $T^{(1)}_{BB}$ and $T^{(1)}_{\lambda B}$. To close the integral equations we need to get an equation for $T^{(1)}_{\lambda B}$. This we get by employing the AGS equations (4.24) to iterate Eq. (5.21) once. We then use Eq. (5.21) to write

$$T^{(1)}_{\lambda B} = \sum_i \bar{\delta}_{\lambda i} F_d^{(2)\dagger}(i)(1 + d_\pi d T^{(1)}_{BB}) + \sum_\alpha \bar{\delta}_{\lambda\alpha} M_d^{(2)}(\alpha) G T^{(1)}_{\alpha B} . \quad (5.24)$$

In Eqs. (5.23) and (5.24) we have a set of coupled equations for the one-particle irreducible amplitude in the πB-$\pi\pi B$ system. These equations could be solved for $T^{(1)}_{BB}$, given the π-B and π-π subamplitudes, as well as the $\pi B \to B$ vertex. At this stage it appears that the input to these equations is exactly what we are striving to determine. In this sense we have a nonlinear problem which needs to be solved self-consistently. However, to calculate $T^{(1)}_{BB}(E)$ using Eqs. (5.23) and (5.24) at energy E, we will need the π-B amplitude $T^{(1)}_{BB}(E - \omega)$ at energy $E - \omega$, where ω is the energy of the spectator pion. In this sense we do not have a self-consistency problem, but an analytic continuation problem in the energy. A simple way of resolving this problem is to parametrize, or calculate using the Hamiltonian in Eq. (2.7), $T^{(1)}_{BB}(E - \omega)$, by ignoring the coupling to the $\pi\pi B$ Hilbert space, and then using that information in conjunction with Eqs. (5.23) and (5.24) to extend the calculation to energies above the pion production threshold. This philosophy has been followed in the NN-πNN system with reasonable success.[2]

To calculate the full π-B amplitude, we need to evaluate $T^{(0)}_{BB}$, which can be evaluated using $T^{(1)}_{BB}$ and Eqs. (3.17) and (3.19). However, we now need $T^{(1)}_{BB}$ off shell to the extent that the evaluation of $f^{(1)}$ using Eq. (3.19) involves an integral over $T^{(1)}_{BB}$ as determined by Eqs. (5.23) and (5.24). To avoid this extra integration we need to write our coupled equations for $T^{(0)}_{BB}$ and $T^{(0)}_{\lambda B}$. Making use of the fact that $T^{(0)}_{BB}$ is a solution of the equation[19,20]

$$T^{(0)}_{BB} = (f^{(2)\dagger} d_0 f^{(2)} + T^{(2)}_{BB})(1 + d_\pi d T^{(0)}_{BB}) , \quad (5.25)$$

with $T^{(2)}_{BB}$ given by Eq. (4.25), we now can write

$$T^{(0)}_{BB} = V_{BB}(1 + d_\pi d T^{(0)}_{BB}) + \sum_{ij\alpha} F_d^{(2)}(i) G \bar{\delta}_{i\alpha} M_d^{(2)}(\alpha) G U_{\alpha j} G F_d^{(2)\dagger}(j) \times (1 + d_\pi d T^{(0)}_{BB}) , \quad (5.26)$$

where

$$V_{\rm BB}=f^{(2)\dagger}d_0f^{(2)}+T^{(3)}_{\rm BB}+\sum_{ij}F_d^{(2)}(i)G\overline{\delta}_{ij}F_d^{(2)\dagger}(j)\ . \tag{5.27}$$

With the help of Eq. (5.19), Eq. (5.26) can be written as

$$T^{(0)}_{\rm BB}=V_{\rm BB}(1+d_\pi dT^{(0)}_{\rm BB})+\sum_{i\lambda}F_d^{(2)}(i)G\overline{\delta}_{i\lambda}M_d^{(2)}(\lambda)GT^{(0)}_{\lambda\rm B}\ . \tag{5.28}$$

To get the corresponding equation for $T^{(0)}_{\lambda\rm B}$, we iterate Eq. (5.19) with the help of the AGS equations. This gives

$$T^{(0)}_{\lambda\rm B}=\sum_i\overline{\delta}_{\lambda i}F_d^{(2)\dagger}(i)(1+d_\pi dT^{(0)}_{\rm BB})+\sum_\alpha\overline{\delta}_{\lambda\alpha}M_d^{(2)}(\alpha)GT^{(0)}_{\alpha\rm B}\ . \tag{5.29}$$

In Eqs. (5.28) and (5.29) we have a set of coupled equations for $T^{(0)}_{\rm BB}$ and $T^{(0)}_{\lambda\rm B}$. The advantage of these equations over Eqs. (5.23) and (5.24) is that these equations are for the full π-B amplitude $T^{(0)}_{\rm BB}$, and their solution on shell does not require the determination of an off-shell amplitude which is then integrated over. More important is the fact that these equations, and particularly the kernel of these equations, can be used to investigate π-n resonances above the threshold for pion production by examining the eigenvalues of the kernel.[23] The π-B potential $V_{\rm BB}$ includes a pole term which involves the bare $\pi{\rm B}\to{\rm B}$ vertex $f^{(2)}$, while the crossed term has the corresponding dressed form factor. Thus in the absence of coupling to the ρB and $\pi{\rm B}^*$ channels [i.e., neglecting the second term on the rhs of Eq. (5.26)], the π-B amplitude is a solution of a two-body equation with the potential having different strengths for the $\pi{\rm B}\to{\rm B}$ vertex in the pole and crossed diagram. This feature of the π-N interaction has not been used until recently.[15]

If we iterate Eqs. (5.28) and (5.29) once, to get the lowest order multiple scattering contribution to $V_{\rm BB}$, we get an effective potential of the form

$$V'_{\rm BB}=V_{\rm BB}+\sum_{ij\lambda}F_d^{(2)}(i)G\overline{\delta}_{i\lambda}M_d^{(2)}(\lambda)\overline{\delta}_{\lambda j}GF_d^{(2)\dagger}(j)\ , \tag{5.30}$$

where the second term on the rhs of Eq. (5.30) is illustrated diagrammatically in Fig. 7. In Fig. 7(a) we have a contribution that corresponds to dressing for the contact term. The inclusion or otherwise of this correction to the contact term should determine its strength. Thus if we chose not to include these corrections, the contact term will have a strength with f_π determined from current algebra to be $\sim$93 MeV. This seems to work very well for the P_{11} channel,[15] where the contact term has a major contribution to the attraction. On the other hand, if the multiple scattering is included, the strength of the bare contact term might have to be a parameter, adjusted to fit a piece of experimental data.

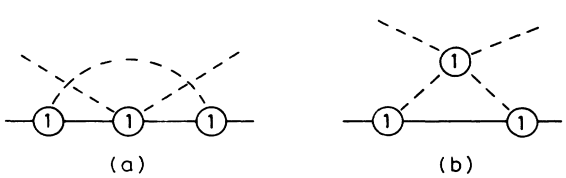

FIG. 7. The lowest multiple scattering contribution to the π-B interaction.

VI. CONCLUSIONS

In the above analysis we have shown that for Lagrangians of the form suggested by the cloudy bag model,[3] we can derive a set of equations that couple the πB to the $\pi\pi$B channel and satisfy two- and three-body unitarity. The essential ingredients in the analysis are the following. (i) The validity of expanding the chiral Lagrangian to second order in the pion field. In this way we can quantize the theory and write the corresponding Hamiltonian in the space of baryons and mesons. The validity of this expansion depends on the convergence of the series in the pion field. For a bag radius $R>0.8$ fm we expect this expansion to be reasonable, while for $R<0.8$ fm convergence problems could become serious.[15] Here we should remember that a large radius for the bag can lead to the fact that quarks and gluons inside the bag cannot be treated perturbatively.[22] In our model the implication of this is the need to include a larger space of baryon configurations and the diagonalization of the gluonic interaction in this basis. We hope that for $0.8<R<1.2$ fm the expansion in the pion field is convergent, and the quarks and gluons inside the bag can be treated perturbatively. We note that the procedure we have presented means that the resulting partial summation of diagrams arising from the Lagrangian includes all terms that contribute to two- and three-body unitarity. (ii) Although the derivation does not rely on the detailed form of the underlying Lagrangian, we assume that perturbation theory is valid to the extent that we can classify diagrams according to their irreducibility.

The resultant equations which are of the form of Faddeev equations can be used to test quark models, such as the cloudy bag model,[3] within the framework of pion-nucleon scattering. In fact, below the threshold for pion production, if we neglect the coupling to the $\pi\pi$B channel, the equations take the form of a two-body equation with the potential given by Eq. (5.27). This potential consists of three terms. The first is the pole term in which the πBB vertex is not dressed, and we have a bare baryon propagator. The second term is the contact term which gives S-wave scattering as well as attraction in the P_{11} channel. Because we are not including the coupling to the $\pi\pi$B channel (i.e., no dressing of the contact term), the strength is given in terms of $f_\pi\simeq93$ MeV. Finally, the last term is the crossed diagram, but now we have to use dressed form factors for the πBB vertex. This potential has been used with great success in the P_{11} channel.[15]

Above the pion production threshold, the equations which include three-body unitarity can be used to investigate π-N resonances which are highly inelastic. In particular, we can examine the eigenvalues of the kernel of Eqs. (5.28) and (5.29) in the complex energy plane[23] to determine if π-N resonances are genuine poles or the result of the π-Δ threshold.[7] This work is presently in progress.

The study of pion-nucleus scattering has been plagued with uncertainty in the off-shell behavior of the π-N amplitude. The most recent example is the role of the P_{11} interaction in pion-deuteron scattering[2] The above theory will be able to give us some quantitative constraint on the off-shell behavior as dictated by quark models of hadrons. Thus the success of the cloudy bag model for the P_{11} channel[15] will give us some constraint as to how this amplitude should be divided into a pole and nonpole part. This in turn will remove some of the uncertainty in the pion-deuteron problem.

Throughout our derivation we considered but neglected terms in the Lagrangian that change the number of pions by two or more. The main reason for not including them at this stage is to guarantee that the final equations be practical from a computational point of view. We have consistently discussed these terms, and indicated how they could be included perturbatively. The inclusion of such terms completely will involve the coupling to channels with more than two pions, i.e., for or more body unitarity. This will make the final equations much more complex, which is not warranted at this stage.

The authors would like to thank the Australian Research Grant Scheme for their financial support.

[1] R. J. McLeod and I. R. Afnan, Phys. Rev. C **32**, 222 (1985); **32**, 1786(E) (1985).

[2] I. R. Afnan and R. J. McLeod, Phys. Rev. C **31**, 1821 (1985).

[3] S. Theberge, G. A. Miller, and A. W. Thomas, Phys. Rev. D **22**, 2838 (1980); **23**, 2106(E) (1981); A. W. Thomas, S. Theberge, and G. A. Miller, *ibid.* **24**, 216 (1981); A. W. Thomas, Adv. Nucl. Phys. **13**, 1 (1984).

[4] E. T. Veit, B. K. Jennings, and A. W. Thomas, Phys. Rev. D **33**, 1859 (1986).

[5] A. Mokhtari *et al.*, Phys. Rev. D **33**, 296 (1986).

[6] E. Umland, I. Duck, and W. von Witsch, Phys. Rev. D **27**, 2678 (1983).

[7] B. Blankleider and G. E. Walker, Phys. Lett. **152B**, 291 (1985).

[8] M. G. Fuda, Phys. Rev. C **30**, 666 (1984); **31**, 1365 (1985); **32**, 2024 (1985).

[9] G. Kälbermann and J. M. Eisenberg, Phys. Rev. D **28**, 66 (1983); **28**, 71 (1983); **28**, 1318 (1983); **29**, 517 (1984).

[10] I. R. Afnan and B. Blankleider, Phys. Rev. C **32**, 2006 (1985).

[11] Y. Avishai and T. Mizutani, Phys. Rev. C **27**, 312 (1983).

[12] A. W. Thomas and A. S. Rinat, Phys. Rev. C **20**, 216 (1979); A. W. Thomas, Ph.D. thesis, Flinders University of South Australia, 1973 (unpublished).

[13] I. R. Afnan and B. Blankleider, Phys. Rev. C **22**, 1638 (1980).

[14] J. G. Taylor, Nuovo Cimento Suppl. **1**, 934 (1963); Phys. Rev. **150**, 1321 (1966).

[15] B. C. Pearce and I. R. Afnan, Phys. Rev. C **34**, 991 (1986).

[16] A. W. Thomas, J. Phys. **7**, L283 (1981); A. Szymacha and S. Tatur, Z. Phys. C **7**, 311 (1981).

[17] See C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[18] T. Mizutani and D. Koltun, Ann. Phys. (N.Y.) **109**, 1 (1977).

[19] I. R. Afnan and A. T. Stelbovics, Phys. Rev. C **23**, 1384 (1981).

[20] T. Mizutani, C. Fayard, G. H. Lamot, and S. Nahabetian, Phys. Rev. C **24**, 2633 (1981).

[21] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).

[22] G. E. Brown and M. Rho, State University of New York at Stony Brook Report No. SPLT/85/193, 1985.

[23] B. C. Pearce and I. R. Afnan, Phys. Rev. C **30**, 2022 (1984).