$2j$th rank tensor polarization in reactions involving a spin j particle

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It is shown that in a reaction in which a spin j particle is produced, the tensor parameters t_q^{2j} are sufficient to determine the reaction amplitudes in spin space. Choosing the initial particles in a pure (spin) state $|i\rangle$ and keeping the spin state $|\alpha\rangle$ of all other coproduced particles fixed, the subset of reaction amplitudes $A_m^j(\alpha,i)$ characterized by (α,i) is completely characterized by $2j$ axes and a complex scalar which may be determined (except for an overall phase and the discrete ambiguities) by measuring the tensor parameters t_q^k with $k=2j$ and the differential cross section. Additional measurements to fix the relative phases between subsets with different (α,i) are also suggested.

I. INTRODUCTION

The density matrix ρ for a spin j system has the well known expansion

$$\rho_{m'm}=(\mathrm{Tr}\rho/2j+1)\left\{1+\sum_{k=1}^{2j}(-1)^q[k]C(jkj;mqm')t_{-q}^k\right\}, \tag{1}$$

where the t_q^k are the spherical tensor parameters which transform under rotations according to

$$(t_q^k)^{\mathrm{II}}=\sum_{q'}D_{q'q}^k(\phi,\theta,\psi)(t_{q'}^k)^{\mathrm{I}}, \tag{2}$$

where I and II designate the coordinate systems before and after the rotation effected through the Euler angles (ϕ,θ,ψ). Further, the parameters satisfy the hermiticity constraint

$$t_q^{k*}=(-1)^q t_{-q}^k . \tag{3}$$

The particular case of spin 1 systems has been extensively studied[1] and several properties such as bounds have been discussed[2] both for $j=1$ and $\frac{3}{2}$ in the context of spin-parity determination of resonances. Incidentally, it may be noted that the normalization chosen in (1) is in accordance with the Madison convention,[3] whereas different normalizations have been used in literature with consequent scaling of bounds. It has been shown (see Dalitz[1]) that when a spin j particle is produced in a reaction which is parity conserving, all the parameters t_q^k (characterizing the particle) vanish for q odd if the z axis is chosen normal to the reaction plane, provided no polarized particle is used initially and no other polarization measurement is made in correlation with that of the particle produced. The role played by polarization in the study of parity violating weak interactions[4] is a saga by itself. Spin correlation measurements were first introduced in the context of N-N scattering,[5] which was followed by the analysis by Schumacher and Bethe.[6] Later, correlation measurements have been discussed by Simonius[7] and Goldstein and Moravcsik,[8] in some generality, with a view to obtain information on reaction amplitudes. In fact, polarization experiments have been employed gainfully (i) to resolve phase shift ambiguities, (ii) to determine pion photoproduction multipole amplitudes, and (iii) to check various dynamical models or theories.[9] It has been reported,[10] e.g., that the Regge pole models which enjoyed a fair amount of success with regard to cross-section data failed crucially to account for the polarization observables. The recent success[11] of Dirac phenomenology in explaining the analyzing power and the spin-rotation parameter in nucleon-nucleus scattering emphasizes, once again, the pivotal role played by polarization measurements. This role is not confined to any energy region like low or intermediate, but is well recognized to be important at high energies also. To quote Craigie *et al.*,[10] "The polarization effects in hadron interactions are usually large, often of the order of 0.1—1.0, not disappearing at large energies, often surprising and often not understood." Thus, while it would be ideal to measure all the polarization observables associated with a given reaction, it is practically (and monetarily) a forbidding task—at least at the present level of spin technology. In view of this and its general theoretical interest, Simonius[7] has obtained necessity conditions for a set of polarization observables to be complete in the spin space. Recently, Goldstein and Moravcsik[12] have obtained sufficiency conditions (up to within discrete ambiguities) as well in the special context of spin-1 particles; they have shown that in a reaction with spin-1 particles the reaction amplitudes may be determined without resorting to measurements of vector polarization (of the spin-1 particle). This, as they point out,[12] is important since "the techniques for vector polarization are different from those for tensor polarization, and in some respects the former lags behind the latter." We quote them further: "besides exploring what is feasible with currently available techniques, the theorem may also give stimulus to the future evolution of tensor polarization techniques in view of their potential to suffice by themselves. Furthermore, it is most likely that similar theorems can also be found for particles with spins higher than 1, an effort that may be stimulated by the present theorem."

The purpose of this paper is to demonstrate precisely the existence of such a general theorem for reactions involving particles with *arbitrary* spin j. With this end in view we, first of all, establish in the next section, that a

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spherical tensor of arbitrary rank k, $k=\frac{1}{2},1,\frac{3}{2},\ldots,$ may be parametrized in terms of $2k$ unit vectors and a complex scalar. If k is an integer and further if t_q^k satisfies the hermiticity constraint (3), the number of independent vectors reduces to k and the scalar can be chosen to be positive real.[13] Consequently, the density matrix ρ given by (1) is specified completely and equivalently by a set of $j(2j+1)$ axes and $2j$ real scalars.[13] The extension of this geometrical representation to spin systems which do not possess a sharp j value is given in the Appendix. In Sec. III, we first consider the simple reaction $a+b\rightarrow d+c$, where a, b, and c are spinless and d has spin j (which is necessarily an integer because of the conservation of angular momentum) and we show that the reaction amplitudes that describe the process are determined, except for discrete ambiguities, if the tensor parameters t_q^{2j} are measured along with the differential cross section. For the particular case of $j=1$, this result has been derived in Ref. 12 by a different argument. On the other hand, the method employed here is not only valid for any (integer) spin j, but may also be generalized to discuss the more complex case $a+b\rightarrow d+c_1+c_2+\cdots$, where d has an arbitrary spin $j=\frac{1}{2},1,\frac{3}{2},\ldots,$ the initial particles have spins j_a and j_b, and the companions $c_1,c_2,\ldots,$ of d have spins $j_1,j_2,\ldots.$ We show that in order to determine similarly the reaction amplitudes, it is sufficient to measure, apart from the differential cross section, the highest rank tensor parameters, t_q^{2j}, of d (in correlation with each α where, α denotes, collectively, a fixed spin configuration of companions $c_1,c_2,\ldots,$ when the particles a and b are prepared in each pure state $|i\rangle$). Since each set of the amplitudes characterized by (α,i) has an undetermined overall phase, a method of fixing these relative phases is also given by prescribing additional measurements in the spin space. This result provides a generalization of the theorem which was obtained by Goldstein and Moravcsik for spin-1 particles. The above considerations are illustrated in Sec. IV for spin $\frac{1}{2}$ as well as the spin 1 case in detail, and we establish correspondence with the observables of optimal formalism employed in Ref. 12 for $j=1$. We also consider the other interesting cases of massless spin j particles and the anomalous case when all the particles in the final state are spinless.

II. GEOMETRY OF SPHERICAL TENSORS

Consider a spherical tensor σ_q^k of rank k, $k=\frac{1}{2},1,\ldots.$ A ready example of tensors of half-odd integer rank[14,15] is afforded by a pure state $|\psi\rangle$ of spin j, which is, in general, expressible as

$$|\psi\rangle=\sum_m c_m^j\,|jm\rangle\ , \tag{4}$$

where the c_m^j are arbitrary complex numbers (satisfying, at best, a normalization condition). The quantities

$$\tau_m^j=c_m^{j*} \tag{5}$$

do constitute a spherical tensor of rank j since, they transform according to (2) under rotations. Following Schwinger (p. 261, Ref. 15), we define the Hermitian conjugate tensor of a given tensor σ_q^k by

$$\sigma_q^{\dagger k}=(-1)^q\sigma_{-q}^{k*}\ . \tag{6}$$

In general, $\sigma_q^{\dagger k}$ need not be equal to σ_q^k as may be seen in the example given in (5). However, if k is an integer it is possible to choose σ_q^k such that $\sigma_q^k=\sigma_q^{\dagger k}$. An important example of such Hermitian tensors is already found in the tensor parameters t_q^k that characterize a spin j assembly. The definition (6) forbids such a choice if k is a half-odd integer, since the condition $\sigma_q^{\dagger k}=\sigma_q^k$ is covariant under rotations, only for k an integer. An alternative way of seeing this is to observe, from (6), that $\sigma_q^{\dagger\dagger k}=(-1)^{2k}\sigma_q^k$, which forces non-hermiticity on all half-odd rank spherical tensors, except the trivial null tensors ($\sigma_q^k=0$ for all q). We note that by visualizing a density matrix ρ for a spin j assembly as "obtained" through

$$\rho_{m'm}=\frac{1}{N}\sum_{i=1}^{N}c_{m'}^j(i)c_m^{j*}(i)\ , \tag{7}$$

where the index i runs over all the individual spin j particles constituting the assembly, the parameters t_q^k in (1) may readily be seen to be given by

$$t_q^k=\frac{(-1)^j[j]}{N}\sum_{i=1}^{N}[\tau^j(i)\otimes\tau^{\dagger j}(i)]_q^k\ , \tag{8}$$

where we have used the standard shorthand notation

$$(A^{k_1}\otimes B^{k_2})_q^k=\sum_{q_1}C(k_1k_2k;q_1q_2q)A_{q_1}^{k_1}B_{q_2}^{k_2}\ , \tag{9}$$

where $A_{q_1}^{k_1}$ and $B_{q_2}^{k_2}$ are any two spherical tensors. Irrespective of the value of j and of the quantities τ_m^j $(=c_m^{j*})$, it is clear, from (9), which represents a hermitized product, that $t_q^k=t_q^{\dagger k}$, as it should be. If ρ can be brought to the diagonal form ρ^0 through a rotation, the spin assembly is said to be oriented.[16] Further, if the vector polarization[17] $\mathbf{p}=\mathrm{Tr}(\mathbf{J}\rho)/j\,\mathrm{Tr}\rho$ is nonvanishing, the eigenstates of ρ are simply the $|jm\rangle$ states with respect to the z axis, which may be chosen parallel or antiparallel to $\mathbf{p}$. In this coordinate system, which may be referred to as the Lakin frame, $t_{\pm1}^1=0$ since

$$\begin{aligned} t_{\pm1}^1&=\mp[3/2j(2j+1)]^{1/2}(p_x\pm ip_y)\ ,\\ t_0^1&=[3/j(2j+1)]^{1/2}p_z\ . \end{aligned} \tag{10}$$

It may be pointed out that for tensors $\sigma_q^1\neq\sigma_q^{\dagger1}$, $\sigma_1^1=0$ does not imply $\sigma_{-1}^1=0$ and *vice versa*. In such cases, it is still possible to define a Cartesian vector $\mathbf{p}$ through (10), but such a vector would have the form $\mathbf{a}+i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real vectors. Another example of such a non-Hermitian σ_q^1 is provided by the spin-dependent amplitude in pion photoproduction on nucleon.

Considering pure states of the form (4), Biedenharn[18] has argued that it is always possible to find a coordinate system (by rotation) such that any chosen c_m^j can be made zero. This suggests that for any arbitrary σ_q^k one might seek a coordinate system where $\sigma_k^k=0$. Considering, in particular, $k=\frac{1}{2}$, we may express $\sigma_q^{1/2}$ in the form

$$\begin{aligned} \sigma_{1/2}^{1/2}(\hat{\mathbf{q}})&=-\sin(\theta/2)\exp(i\phi/2)\ ,\\ \sigma_{-1/2}^{1/2}(\hat{\mathbf{q}})&=\cos(\theta/2)\exp(-i\phi/2)\ , \end{aligned} \tag{11}$$

whence

$$\sigma_{1/2}^{\dagger 1/2}(\hat{\mathbf{q}})=i\cos(\theta/2)\exp(i\phi/2)\ ,$$
$$\sigma_{-1/2}^{\dagger 1/2}(\hat{\mathbf{q}})=i\sin(\theta/2)\exp(-i\phi/2)\ , \quad (12)$$

where $\hat{\mathbf{q}}$ denotes the unit vector with polar angles (θ,ϕ). Clearly, $\sigma_{1/2}^{1/2}$ $(\sigma_{1/2}^{\dagger 1/2})$ is zero when the z axis is chosen parallel (antiparallel) to $\hat{\mathbf{q}}$.

Given, in general, a set σ_q^k, in a coordinate system labeled I, we seek a frame II where

$$(\sigma_k^k)^{\mathrm{II}}=\sum_q D_{qk}^k(\phi,\theta,\psi)(\sigma_q^k)^{\mathrm{I}}=0\ . \quad (13)$$

We show that there are, in general, $2k$ coordinate systems in which $\sigma_k^k=0$. Making use of the well-known Wigner expression[19] for the $D_{qk}^k(\phi,\theta,\psi)$, we recast (13) into the polynomial form

$$f(z)=\sum_{r=0}^{2k}\alpha_r z^r=0\ , \quad (14)$$

where the complex variable z and the coefficients α_r are given by

$$z=\cot(\theta/2)\exp(-i\phi),\ \ \alpha_r=\binom{2k}{r}^{1/2}\sigma_{r-k}^k\ . \quad (15)$$

The definition of z given in (14) establishes a bijective mapping between (θ,ϕ) (with the domain of definition being $0\le\theta\le\pi$, $0\le\phi<2\pi$) and the entire complex plane. Thanks to the fundamental theorem of algebra, $f(z)$ has $2k$ roots z_i, $i=1,\ldots,2k$, and every root z_i corresponds uniquely to a unit vector $\hat{\mathbf{q}}_i(\theta_i,\phi_i)$, so that when the z axis is chosen parallel to $\hat{\mathbf{q}}_i$, the parameter $\sigma_k^k=0$. The remark that the polynomial $f(z)$ is completely characterized by its roots z_i (except for an overall multiplying factor) and hence by the directions $\hat{\mathbf{q}}_i$ equips us with an elegant geometrical representation for σ_q^k. Indeed, given the unit vectors $\hat{\mathbf{q}}_i$, $i=1,\ldots,2k$, we construct a spherical tensor of rank k using the recursive form

$$S_q^k=[S^{k-1/2}(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k-1})\otimes\sigma^{1/2}(\hat{\mathbf{q}}_{2k})]_q^k\ , \quad (16)$$

and observe that the analogous polynomial equation obtained for S_q^k has the same set of roots z_i and, hence, the same set of axes $\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k}$. Consequently, σ_q^k can differ from S_q^k by, at most, an overall complex multiplicative factor, say a_k. We may, therefore, represent an arbitrary spherical tensor, σ_q^k, in the form

$$\sigma_q^k=a_k S_q^k(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k})\ . \quad (17)$$

We note that $S_q^k(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k})$ as defined in (16) is completely symmetric with respect to all the vectors $\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k}$. We observe that if σ_q^k is given the representation (17), we have

$$\sigma_q^{\dagger k}=a_k^* S_q^{\dagger k}(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k})\ , \quad (18)$$

where $S_q^{\dagger k}$ is given recursively through

$$S_q^{\dagger k}(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k})=[S^{\dagger k-1/2}(\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k-1})\otimes\sigma^{\dagger 1/2}(\hat{\mathbf{q}}_{2k})]_q^k \quad (19)$$

using the parametrization given in (12) for $\sigma_m^{\dagger 1/2}(\hat{\mathbf{q}})$. It may be noted that while $\sigma_k^k=0$ when the z axis is chosen parallel to $\hat{\mathbf{q}}_i$, $i=1,\ldots,2k$, $\sigma_k^{\dagger k}$ vanishes when the z axis is chosen antiparallel to $\hat{\mathbf{q}}_i$, $i=1,\ldots,2k$.

Considering the particular case of spherical tensors defined in (5) for a pure state $|\psi\rangle$, we observe that an overall phase factor in the representation (17) is irrelevant and that $|a_j|^2$ is fixed by the normalization requirement. The pure state $|\psi\rangle$ may, therefore, be represented by $2j$ points on a unit sphere; thereby, we recover the famous observation of Majorana.[20] In this context, we parenthetically note that while deriving conditions on the tensor parameters t_q^k that represent a pure spin j state, Biedenharn[18] has also suggested a representation (for pure states) in terms of what he designates as Poincaré vectors. The representation (17) allows for a representation of not only pure states, but also any (arbitrary) spherical tensor of rank k in terms of $2k$ unit vectors $\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2k}$ and a complex scalar a_k; these may be visualized geometrically as a set of $2k$ points on a sphere of radius $|a_k|$ so long as the phase is not required to be specified. As we have already remarked, σ_q^k is necessarily non-Hermitian if k is not an integer. If k is an integer and, moreover, $\sigma_q^k=\sigma_q^{\dagger k}$, the tensor needs only k axes $\hat{\mathbf{Q}}_1,\ldots,\hat{\mathbf{Q}}_k$ and a positive real scalar for its specification.[13] The connection between (17), and this result is shown in the appendix, where an extension of the geometrical representation of a density matrix for spin systems with a nonsharp j value is also outlined.

III. 2jTH RANK TENSOR POLARIZATION IN REACTIONS WITH A SPIN j PARTICLE

The basic representation (17) deduced in the preceding section will now be used to detail a program for the determination of the reaction amplitudes in a process involving a spin-j particle by measurements of only the tensor parameters t_q^{2j} and the differential cross section. In the discussion to follow, it will be understood that the measurements are performed for a fixed kinematical configuration of particles in some given channel. The arguments rely entirely on the transformation properties of the reaction amplitudes under rotations. The reaction may possibly be further characterized by parity conservation, time reversal invariance, etc., which may reduce the number of independent reaction amplitudes. We will not, however, assume any such conservation law in the proof.

Consider first a reaction $a+b\to d+c$ where all the particles are spinless, except d, which has a spin j (integer). The reaction is completely described by $2j+1$ complex amplitudes A_m^j and we write the transition operator in the form

$$T=\sum_m(-1)^m A_{-m}^j\,|jm;00\rangle\langle 00|\ , \quad (20)$$

which ensures that A_m^j transform as spherical tensors of rank j. Denoting the operator $|jm;00\rangle\langle 00|$ by T_m^j, a spherical tensor operator which "produces" the spin j particle, it is clear that

$$T=(T^j\cdot A^j)\ , \quad (21)$$

in which the rotational invariance of the T operator is manifest. The density matrix, $\rho=TT^\dagger$, for the spin j particle is explicitly given by

$$\rho_{mm'}=(-1)^{m-m'}A^{j}_{-m}A^{j*}_{-m'} , \tag{22}$$

so that

$$\mathrm{Tr}\rho=\sum_{m=-j}^{+j}|A^{j}_{m}|^{2} \tag{23}$$

and

$$\mathrm{Tr}(\rho)t^{k}_{q}=(-1)^{j+k}[j](A^{j}\otimes A^{\dagger j})^{k}_{q} , \tag{24}$$

where the t^{k}_{q} denote the tensor parameters characterizing the spin j particle. Considering $k=2j$, and making use of the representations (17) and (18) for A^{j}_{m} and $A^{\dagger j}_{m}$, respectively, we note that the t^{2j}_{q} is characterized by $2j$ axes $\hat{\mathbf{Q}}_1,\ldots,\hat{\mathbf{Q}}_{2j}$, which are the same as the unit vectors $\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2j}$, as shown in the Appendix, i.e.,

$$t^{2j}_{q}=P_{2j}[\cdots(\hat{\mathbf{Q}}_1\otimes\hat{\mathbf{Q}}_2)^2\otimes\cdots)^{2j-1}\otimes\hat{\mathbf{Q}}_{2j}]^{2j}_{q} , \tag{25}$$

where

$$P_{2j}=\frac{|a_j|^2[j]}{\mathrm{Tr}(\rho)(\sqrt{2})^{j}} . \tag{26}$$

The positive real P_{2j} and the differential cross section (proportional to $\mathrm{Tr}\rho$) together readily determine $|a_j|^2$. The axes $\hat{\mathbf{Q}}_1,\ldots,\hat{\mathbf{Q}}_{2j}$ may be obtained as solutions of polynomial equation (14) (of degree $4j$), which has $4j$ roots z_i corresponding to $\pm\hat{\mathbf{Q}}_i$, $i=1,\ldots,2j$. Thus the amplitudes A^{j}_{m} are determined, except for the phase of a_j, which is not an observable, and the ambiguity in the sign of $\hat{\mathbf{q}}_i=\pm\hat{\mathbf{Q}}_i$, $i=1,\ldots,2j$. These may be identified with the discrete ambiguities[21] associated with the determination of the reaction amplitudes. We note that all the t^{k}_{q} for $k=1,\ldots,2j-1$ are then easily determined through (24) once $\hat{\mathbf{q}}_1,\ldots,\hat{\mathbf{q}}_{2j}$ and $|a_j|$ are determined from measurements of t^{2j}_{q} and $\mathrm{Tr}(\rho)$. The measurement of $t^{k\neq 2j}_{q}$ is thus rendered redundant as far as the continuous ambiguities are concerned. One may, however, need to use some of these parameters to resolve the discrete ambiguities or to check for the consistency of the experimental measurements or to minimize errors and uncertainties associated with them.

Having proved the result in the simple case, we now consider the complex reaction $a+b\to d+c_1+c_2+\cdots$, where d has an arbitrary spin $j=\frac{1}{2},1,\frac{3}{2},\ldots$, the initial particles a,b have spins j_a,j_b and the companions $c_1,c_2,\ldots$ of d have spins $j_1,j_2,\ldots$. The transition operator in the general case may be written, as in the form (18), as

$$T=\sum_{m}\sum_{m_am_b}\sum_{m_1m_2\cdots}(-1)^{m}A^{j}_{-m}(m_1,m_2,\ldots;m_am_b)\times|jm;j_1m_1,j_2m_2\cdots\rangle\langle j_am_aj_bm_b| , \tag{27}$$

where, *however*, the states $|j_im_i\rangle$ are defined through

$$(\mathbf{J}_i\cdot\hat{\mathbf{P}}_i)|j_im_i\rangle=m_i|j_im_i\rangle,\quad i=a,b;1,2,\ldots \tag{28}$$

with $\hat{\mathbf{P}}_i$ denoting the respective quantization axes for the particles. It should be emphasized that if $\hat{\mathbf{P}}_i$ are all kept fixed and *only* the quantization axis $\hat{\mathbf{P}}$ associated with d is rotated, it follows from (27) that A^{j}_{m} do transform as spherical tensors of rank j for each set of indices $\alpha=\{m_1,m_2,\ldots\}$ and $i=\{m_a,m_b\}$. A well known example in which the amplitudes are specified by using different quantization axes is afforded by the helicity formalism, where the independent quantization axes are chosen parallel to the respective momenta. Our choice of $\hat{\mathbf{P}}_i$ is not necessarily tied up with momenta and, in particular, when $\hat{\mathbf{P}}$ (associated with d) is rotated, no change in the kinematical configuration is envisaged. With these remarks, let $\langle jm;\alpha|T|i\rangle$ denote the amplitudes $A^{j}_{m}(\alpha,i)$, where, as indicated above, $|i\rangle$ denotes the pure spin state in which the initial particles are prepared and α denotes collectively a particular (fixed) spin state of the companions $c_1,c_2,\ldots$. Considering the density matrix for the spin j particle that is diagonal in (α,i), it follows that

$$\rho^{j}_{mm'}(\alpha,i)=\langle jm;\alpha|T|i\rangle\langle jm';\alpha|T|i\rangle^{*} , \tag{29}$$

and the associated (Hermitian) tensor parameters t^{k}_{q} are given by

$$\mathrm{Tr}\rho(\alpha,i)t^{k}_{q}(\alpha,i)=(-1)^{j+k}[j][A^{j}(\alpha,i)\otimes A^{\dagger j}(\alpha,i)]^{k}_{q} , \tag{30}$$

which, except for the symbols (α,i), is precisely of the same form as given by (24). The analysis employed earlier in the simple case is now seen to hold here as well for each set characterized by (α,i). Thus, measurements of $\mathrm{Tr}\rho(\alpha,i)$ and $t^{2j}_{q}(\alpha,i)$ determine the reaction amplitudes $A^{j}_{m}(\alpha,i)$, except for the discrete ambiguities and the phase of $a_j(\alpha,i)$. The number of measurements, viz., of $\mathrm{Tr}\rho(\alpha,i)$ and $t^{2j}_{q}(\alpha,i)$ being $2(2j+1)$, it is obvious that the total number of such measurements as we cover all (α,i) adds up to

$$2(2j+1)(2j_a+1(2j_b+1)\prod_{r}(2j_r+1)=2N ,$$

where N denotes the number of complex amplitudes in (27) that describe the reaction. However since the relative phases of $a_j(\alpha,i)$, which are $n=N/(2j+1)-1$ in number are left undetermined, we have to resort to measurements of tensor parameters which are in sectors off diagonal in (α,i) to fix the relative phases. It is interesting that the $2N$ measurements do not constitute a complete set[22] in the spin space, although the total number of complex amplitudes is just N. This circumstance arises due to the sesquilinear nature of the equations ($\rho^{f}=T\rho^{i}T^{\dagger}$). In fact, the relationship among the bilinear products of amplitudes (bicoms) and the observables is central to the optimal formalism developed by Moravcsik and co-workers.[8]

To resolve these additional ambiguities in the general spin j case considered here, we may arrange the $n'=N/(2j+1)$ sets of $A^{j}_{m}(\alpha,i)$ [each set with different m but same (α,i)] sequentially, such that two immediate neighbors $a_j(\alpha,i)$ and $a_j(\alpha',i')$ differ in only one index. If we determine the $n'-1$ relative phases between the immediate neighbors, the additional ambiguities, which are at most $n'-1$, would be then resolved. For, then, the phase between any two $\{A^{j}_{m}(\alpha,i)\}$ and $\{A^{j}_{m}(\alpha',i)\}$ could be determined by moving along the sequence connecting (α,i) to (α',i'). The phrase "at most" has been used because the question of determining a relative phase may not arise if a set $\{A_m^j(\alpha,i)\}$ vanishes for some (α,i), or is determined in terms of other amplitudes due to the operation of one or more symmetry principles. To determine these $n'-1$ relative phases, one may, conveniently, measure some nonvanishing t_q^{2j} associated with the density matrix $\rho(\alpha,i;\alpha'i')$, where $(\alpha,i),(\alpha',i')$ are two immediate neighbors. It is advantageous, whenever possible, to choose q to be of the highest value since this would avoid clumsy summations.

In summary, therefore, we have demonstrated that the measurements of tensor parameters $t_q^{2j}(\alpha,i)$ of a spin j particle and the differential cross section $\mathrm{Tr}\rho(\alpha,i)$ in a reaction $a+b\to d(j)+c_1+c_2+\cdots$, for all (α,i), determine the reaction amplitudes, except for an overall phase associated with each (α,i) and the discrete ambiguities. In addition, the $n'-1$ relative phases of $a_j(\alpha,i)$ may be resolved by making additional measurements, say, of $t_{2j}^{2j}(\alpha,i;\alpha',i')$ for immediate neighbors (α,i) and (α',i'). Only the overall phase associated with the transition operator (27) remains unknown. The Goldstein-Moravcsik theorem ensues as a special case when we consider $j=1$. The present analysis not only generalizes the Goldstein-Moravcsik theorem, but also provides an alternative proof of the same. Finally, we note that the proof also satisfies the necessity conditions laid down by Simonius.[7,23]

IV. SPECIAL CASES

The indices (α,i) will be hereafter dropped for convenience.

A. $j=\frac{1}{2}$

In this simplest case, the illustration of the theorem proved in the preceding section is straightforward. The relevant and the only tensor parameters are of rank 1, and the polynomial equation (14) reads

$$t_1^1 z^2+\sqrt{2}zt_0^1+t_{-1}^1=0 . \tag{31}$$

Since the t_q^1 is Hermitian, the two solutions of (31) are easily seen to correspond to $\hat{\mathbf{p}}$ and $-\hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ is defined through (10). We note that

$$\mathrm{Tr}(\rho)t_q^1=(-1)^{1/2}\sqrt{2}[A^{1/2}(\hat{\mathbf{p}})\otimes A^{\dagger 1/2}(\hat{\mathbf{p}})]_q^1 , \tag{32}$$

where $A_m^{1/2}(\hat{\mathbf{p}})=a_{1/2}\sigma_m^{1/2}(\hat{\mathbf{p}})$ and $A_m^{\dagger 1/2}(p)=a^*_{1/2}\sigma_m^{\dagger 1/2}(\hat{\mathbf{p}})$. The scalar $|a_{1/2}|$ is determined from (26), i.e., $P_1=2^{1/4}|a_{1/2}|^2/\mathrm{Tr}\rho$.

B. $j=1$

This is the particular case considered earlier by Goldstein and Moravcsik[12] using a different approach. The correspondence with their notation for the observables is established by noting that $\Lambda=\sqrt{2}t_0^2$, $R^+-R^-=-\sqrt{2/3}\mathrm{Re}(t_1^2)$; $I^+-I^-=-\sqrt{2/3}\mathrm{Im}(t_1^2)$, $R^0=(1/\sqrt{3})\mathrm{Re}(t_2^2)$, and $I^0=(1/\sqrt{3})\mathrm{Im}(t_2^2)$. The tensor parameters are defined following the Madison convention.[3]

Their reaction amplitudes A,B,C are related to the tensor amplitudes A_m^1 through

$$A=-A_{-1}^1,\ \ B=-A_0^1,\ \ C=-A_1^1 . \tag{33}$$

Considering the t_q^k with $k=2$, the polynomial equation (14) is written as

$$t_2^2z^4+2t_1^2z^3+\sqrt{6}t_0^2z^2+2t_{-1}^2z+t_{-2}^2=0 , \tag{34}$$

which has four solutions corresponding to $\pm\hat{\mathbf{Q}}_1(\theta_1,\phi_1)$, $\pm\hat{\mathbf{Q}}_2(\theta_2,\phi_2)$. The solution of the fourth degree equation is particularly simple in the frame defined[24] by principal axes (PAAF) associated with the Cartesian components, $p_{\alpha\beta}$, of the second rank tensor t_q^2. In this frame, $t_{\pm1}^2=\mathrm{Im}(t_{\pm2}^2)=0$. Consequently, (34) becomes a quadratic equation in z^2. Depending on the respective intervals $[-\infty,-\sqrt{2/3}]$, $[-\sqrt{2/3},\sqrt{2/3}]$, and $[\sqrt{2/3},\infty]$ to which the ratio $r=(t_0^2/t_2^2)_{\mathrm{PAAF}}$ belongs, the two axes $\hat{\mathbf{Q}}_1,\hat{\mathbf{Q}}_2$ lie in the z_Ax_A, x_Ay_A, or y_Az_A planes, with one of the principal axes acting as respective bisector. The PAAF is unique up to the cubic subgroup of rotations[25] and we may, without loss of generality, label the plane containing $\hat{\mathbf{Q}}_1,\hat{\mathbf{Q}}_2$ as z_Ax_A. We look for the representation of t_q^2 with $P_2>0$ [see (26) relating $|a_1|^2$ to $\mathrm{Tr}(\rho)$]. Hence, if $t_2^2<0$, we have

$$\theta_1=\theta_2=\theta;\ \ \phi_1=0,\ \ \phi_2=\pi , \tag{35}$$

$$P_2=t_2^2(\sqrt{3/2}r-1) , \tag{36}$$

and

$$\sin^2\theta=-2t_2^2/P_2 , \tag{37}$$

while, for $t_2^2>0$,

$$\theta_1=\theta,\ \ \theta_2=\pi-\theta,\ \ \phi_1=\phi_2=0 , \tag{35'}$$

$$P_2=t_2^2(1-\sqrt{3/2}r) , \tag{36'}$$

and

$$\sin^2\theta=2t_2^2/P_2 . \tag{37'}$$

In constructing the reaction amplitudes,

$$A_m^1=a_1[\sigma^{1/2}(\hat{\mathbf{q}}_1)\otimes\sigma^{1/2}(\hat{\mathbf{q}}_2)]_m^1 , \tag{38}$$

we see that Eqs. (33)—(37) determine $\hat{\mathbf{Q}}_1,\hat{\mathbf{Q}}_2$ up to a common sign factor which, indeed, is the source of discrete ambiguities in the determination of A_m^1. Since $P_2>0$, we are left with only two possible choices for A_m^1 (recall that flipping one of the axes, say, $\hat{\mathbf{Q}}_1$, changes the sign of P_2). If $\hat{\mathbf{q}}_1=\hat{\mathbf{Q}}_1$, $\hat{\mathbf{q}}_2=\hat{\mathbf{Q}}_2$ is a choice that yields positive value for P_2; obviously, the only other choice is given by $\hat{\mathbf{q}}_1=-\hat{\mathbf{Q}}_1$, $\hat{\mathbf{q}}_2=-\hat{\mathbf{Q}}_2$. Denoting the amplitudes obtained from these choices, respectively, $A_m^1(+)$ and $A_m^1(-)$, it may readily be seen that $A_m^1(+)=A_m^{\dagger1}(-)$. In the notation of Goldstein and Moravcsik,[12] this ambiguity corresponds to $A\leftrightarrow -C^*$, $B\leftrightarrow -B^*$. Both $A_m^1(+)$ and $A_m^1(-)$ yield the same value for t_q^2 and $\mathrm{Tr}(\rho)$. However, as may be easily seen, their predictions do differ when it comes to vector polarization. More explicitly, writing

$$\mathrm{Tr}(\rho)t_q^2=-\sqrt{3}(A^1\otimes A^{\dagger1})_q^2 , \tag{39}$$

it is clear that, if $A_m^1 \to +A_m^{\dagger 1}$,

$$\mathrm{Tr}(\rho)t_q^2 \to -\sqrt{3}(A^{\dagger 1}\otimes A^{\dagger\dagger 1})_q^2 = -\sqrt{3}(A^{\dagger 1}\otimes A^1)_q^2 = -\sqrt{3}(A^1\otimes A^{\dagger 1})_q^2 = \mathrm{Tr}\rho t_q^2 , \quad (40)$$

which displays the ambiguity. However, considering

$$\mathrm{Tr}(\rho)t_q^1 = \sqrt{3}(A^1\otimes A^{\dagger 1})_q^1 , \quad (41)$$

it is clear that the transformation $A_m^1 \to A_m^{\dagger 1}$ changes the sign of t_q^1. Thus a measurement of t_0^1 (even approximately since only the sign is crucial) would eliminate the discrete ambiguity.

The above analysis also shows that a discrete ambiguity does not arise only in the special situation $A_m^1 = aY_{1m}(\hat{\mathbf{q}})$; in this case, $t_q^1 = 0$, and the particle d is then produced in the pure state $|10\rangle$ defined with respect to $\hat{\mathbf{q}}$ as the quantization axis.

The manifestation of the discrete ambiguities in the parameters $t_q^{k\neq 2j}$ when we consider a reaction with spin j particles may also be studied. To that end, we notice that, again, $P_{2j} > 0$ [see Eq. (26)], when, given the amplitudes $A_m^j(\hat{\mathbf{q}}_1, \ldots, \hat{\mathbf{q}}_{2j})$, other amplitudes which yield the same t_q^{2j} are obtained by flipping an even number of vectors in A_m^j. If two of them, say $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2$, are flipped, we conveniently rewrite (17) as

$$A_m^j = [A^1(\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2)\otimes A^{j-1}]_m^j , \quad (42)$$

and in this form, Eq. (24) for t_q^k assumes the form

$$\mathrm{Tr}(\rho)t_q^k = [j](-1)^{j+k}\sum_{\eta_1\eta_1'} C(jk;\eta_1\eta_1')\mathscr{C}_q^k(j;\eta_1\eta_1') , \quad (43)$$

where

$$C(jk;\eta_1\eta_1') = \begin{bmatrix} 1 & j-1 & j \\ 1 & j-1 & j \\ \eta_1 & \eta_1' & k \end{bmatrix} ,$$

$$\mathscr{C}_q^k(j;\eta_1\eta_1') = [(A^1\otimes A^{\dagger 1})^{\eta_1}\otimes(A^{j-1}\otimes A^{\dagger j-1})^{\eta_1'}]_q^k . \quad (44)$$

The change $\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2 \to -\hat{\mathbf{q}}_1, -\hat{\mathbf{q}}_2$ transforms $A_m^1(\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2)$ to $A_m^{\dagger 1}(\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2)$, and the expression (43) is transformed into

$$\mathrm{Tr}(\rho)t_q^k = [j](-1)^{j+k}\sum_{\eta_1\eta_1'}(-1)^{\eta_1}C(jk;\eta_1,\eta_1')\mathscr{C}_q^k . \quad (43')$$

This expression may easily be generalized when we flip l pairs of vectors, say $\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2;\hat{\mathbf{q}}_3,\hat{\mathbf{q}}_4;\ldots;\hat{\mathbf{q}}_{2l-1},\hat{\mathbf{q}}_{2l}$. In this case, we write (17) as

$$A_m^j = [A^l(\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2;\ldots;\hat{\mathbf{q}}_{2l-1},\hat{\mathbf{q}}_{2l})\otimes A^{j-l}]_m^j , \quad (45)$$

where A_m^l has the recursive form

$$A_m^l = [A^1(\hat{\mathbf{q}}_1,\hat{\mathbf{q}}_2)\otimes A^{l-1}]_m^l . \quad (46)$$

It is obvious that Eq. (30) may be written by repeated recombination as

$$\mathrm{Tr}(\rho)t_q^k = \sum\sum_{\eta_1\eta_1';\ldots;\eta_l\eta_l'}\sum C(jk;\eta_1\eta_1';\ldots;\eta_l\eta_l') \times \mathscr{C}_q^k(j;\eta_1\eta_1';\ldots;\eta_l\eta_l') . \quad (47)$$

Under the inversion $\hat{\mathbf{q}}_i \to (-)\hat{\mathbf{q}}_i$, $i=1,\ldots,2l$, observing again that $A_m^1(\hat{\mathbf{q}}_i,\hat{\mathbf{q}}_{i+1}) \to A_m^{\dagger 1}(\hat{\mathbf{q}}_i,\hat{\mathbf{q}}_{i+1})$, we see that (47) goes to

$$\mathrm{Tr}(\rho)t_q^k = \sum\sum_{\eta_1\eta_1';\ldots;\eta\eta'}\sum (-1)^{\eta_1+\cdots+\eta_l}C(jk;\eta_1\eta_1';\ldots;\eta\eta') \times \mathscr{C}_q^k(j;\eta_1\eta_1';\ldots;\eta_l\eta_l') . \quad (47')$$

It is also clear that, for $k=2j$, η_i attain their maximum value $\eta_i = 2$, thereby causing no change in t_q^{2j}.

C. Massless spin j particles

We now proceed to discuss the interesting case when the particle, d, is massless. It is known[26] that a massless spin j particle can possess only two helicities, $\pm j$. Choosing the helicity axis to quantize the spin j, we obtain from (30) that

$$\mathrm{Tr}(\rho) = |A_j^j|^2 + |A_{-j}^j|^2 , \quad (48)$$

$$\mathrm{Tr}(\rho)t_{2j}^{2j} = (-1)^{2j}[j]A_j^jA_{-j}^{j*} , \quad (49)$$

and

$$t_0^{2j} = (-1)^{2j}[j]C(jj\,2j;j,-j,0) . \quad (50)$$

While all other components are zero. We may determine $|A_j^j|$, $|A_{-j}^j|$ and their relative phase from (48) and (49). A special case arises when the parity operator connects d to its antiparticle. In this situation only A_j^j or A_{-j}^j survives and it is sufficient to measure $\mathrm{Tr}(\rho)$ to determine the reaction amplitude.

D. An anomalous case

Finally, we consider the anomalous situation in which all the particles in the final state are spinless. Clearly, this calls for a new analysis; no polarization parameter exists to be measured in the final state and the information on the reaction amplitudes has to be derived from the cross-section and asymmetry measurements. Denote the reaction, compactly, by $a+b \to c(\mathbf{k}_1,\mathbf{k}_2,\ldots)$, where a and b possess spins j_a and j_b, and c collectively denotes the particles in the final state, all of which are spinless and are monitored, say, by their momenta $\mathbf{k}_1,\mathbf{k}_2,\ldots$. The T operator obtains the simple form

$$T = \sum_{m_am_b} A(m_a,m_b)|00\rangle\langle j_am_a;j_bm_b| , \quad (51)$$

from which it follows that, for each preparation of the initial system as a pure state $|m_a;m_b\rangle$, the cross section

$$\mathrm{Tr}(\rho) = |A(m_a;m_b)|^2 \quad (52)$$

determines $|A(m_a;m_b)|$. To determine the $N-1$ relative phases of N amplitudes $A(m_a;m_b)$, one may choose the method outlined in preceding section.

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APPENDIX

The Hermitian tensor parameters t_q^k (k necessarily an integer) have been shown in Ref. 13 to be representable by the form

$$t_q^k = P_k[\cdots(\hat{\mathbf{Q}}_1 \otimes \hat{\mathbf{Q}}_2)^2 \otimes \hat{\mathbf{Q}}_3)^3 \otimes \cdots)^{k-1} \otimes \hat{\mathbf{Q}}_k]_q^k \;, \quad \text{(A1)}$$

where P_k is real, and

$$(\hat{\mathbf{Q}})_q = \sqrt{4\pi/3}\, Y_{1q}(\hat{\mathbf{Q}}) \;. \quad \text{(A2)}$$

To see the transition from the general form (17) [or (18)] to (A1), it is convenient to consider for σ_q^k the equivalent, mixed representation

$$\sigma_q^k = a_k M_q^k(\hat{\mathbf{q}}_1, \ldots, \hat{\mathbf{q}}_{2k}) \;, \quad \text{(A3)}$$

where M_q^k intertwines $\sigma^{1/2}$ and $\sigma^{\dagger 1/2}$ recursively, as given by its definition,

$$M_q^k = [M^{k-1/2}(\hat{\mathbf{q}}_1, \ldots, \hat{\mathbf{q}}_{2k-1}) \otimes \xi^{1/2}(\hat{\mathbf{q}}_{2k})]_q^k \;, \quad \text{(A4)}$$

with the notation

$$\xi^{1/2}(\hat{\mathbf{q}}_r) = \sigma^{1/2}(\hat{\mathbf{q}}_r), \quad r \text{ odd} \quad \text{(A5)}$$

and

$$\xi^{1/2}(\hat{\mathbf{q}}_r) = -i\sigma^{\dagger 1/2}(\hat{\mathbf{q}}_r), \quad r \text{ even} \;. \quad \text{(A6)}$$

When the (Hermitian) tensor t_q^k is represented through the new form (A4), it is straightforward to see that (A1) follows by simply demanding that the vectors in (A4) corresponding to the roots z_i of $f(z)$ in (14) be pairwise degenerate and that a_k must be real. For, writing (A4) as

$$t_q^k = a_k M_q^k(\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_1; \ldots; \hat{\mathbf{Q}}_k, \hat{\mathbf{Q}}_k) \;, \quad \text{(A7)}$$

it follows from the relation

$$-i[\sigma^{1/2}(\hat{\mathbf{Q}}) \otimes \sigma^{\dagger 1/2}(\hat{\mathbf{Q}})]_q^1 = (1/\sqrt{2})(\hat{\mathbf{Q}})_q \quad \text{(A8)}$$

that t_q^k is indeed given by (A1). However, the representation (A1) is unique up to the choice $\pm\hat{\mathbf{Q}}$. It changes sign upon $\hat{\mathbf{Q}}_i \leftrightarrow -\hat{\mathbf{Q}}_i$. Consequently, P_k may be always chosen, without any loss of generality, to be positive by flipping one of the axes $\hat{\mathbf{Q}}_i$, if necessary.

The representation (A1) has been employed[13] to obtain a geometric representation for polarized spin j systems in terms of $\sum_{k=1}^{2j} k = j(2j+1)$ axes and $2j$ positive scalars. We may now invoke the general representation (17) to extend this geometric representation to spin systems which do not possess a sharp j value, i.e., when

$$[\rho, \mathbf{J}^2] \neq 0 \;. \quad \text{(A9)}$$

Such a situation could arise when the particles are intrinsically not in an eigenstate of $\mathbf{J}^2$, or when the system is obtained by coupling two or more spins. In any case, if the assembly is characterized by several j values $j_1, j_2, \ldots$, writing

$$\rho = \sum_{jmj'm'} \rho_{jm;j'm'} |jm\rangle\langle j'm'| \,, \quad j,j' = j_1, j_2, \ldots \,, \quad \text{(A10)}$$

we set

$$T_m^j = |jm\rangle, \quad T_m^{\dagger j} = (-1)^m \langle j - m| \quad \text{(A11)}$$

and define the spherical tensor operators

$$\mathcal{T}_q^k(j,j') = (-1)^{j'}[j](T^j \otimes T^{\dagger j'})_q^k \;, \quad \text{(A12)}$$

which have the property

$$\langle j_r m_r | \mathcal{T}_q^k(j,j') | j_s m_s \rangle = [k] C(j_s k j_r; m_s q m_r) \delta_{j_r,j} \delta_{j_s,j'} \;, \quad \text{(A13)}$$

and are thus consistent with the Madison convention[3] adopted for spin j assemblies. In fact, $\mathcal{T}_q^k(j,j) \equiv T_q^k(\mathbf{J})$. In the operator basis provided by $\mathcal{T}_q^k(j,j')$, we resolve ρ in the standard way as

$$\rho = \sum_{j,j'} \sum_k [t^k(j,j') \cdot \mathcal{T}^k(j,j')] \quad \text{(A14)}$$

where the spherical tensor parameters $t_q^k(j,j')$ are given by

$$t_q^k(j,j') = \frac{(-1)^{j-j'}[k]}{[j][j']} \sum_m C(jkj'; mqm') \rho_{jm;j'm'} \;. \quad \text{(A15)}$$

It may be seen from (A11) that although k is an integer, $t_q^k \neq t_q^{\dagger k}$, except in the sectors diagonal in j. In general, they satisfy

$$t_q^{k*}(j,j') = (-1)^q t_{-q}^k(j',j) \;, \quad \text{(A16)}$$

or, more concisely, $t_q^{\dagger k}(j,j') = t_q^k(j',j)$. Consequently, in determining the geometry of such a system, (A1) may be employed for the sectors $j = j'$, and for sectors off diagonal in j, the general representation (17) has to be used. However, in view of (A16), it is sufficient to enumerate and find the axes for $j \le j'$. Finally, we note that if, in particular, the spin assembly happens to be a coupled system of two spins j_1 and j_2, the range of j,j' is fixed by $|j_1 - j_2| \le j,j' \le j_1 + j_2$.

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