Lippmann-Schwinger equation in a soluble three-body model: Surface integrals at infinity

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At real energies E, the derivation of the Lippmann-Schwinger integral equation from the Schrödinger equation involves various surface integrals at infinity in configuration space. Plausible assumptions about the values of these surface integrals made originally by Gerjuoy imply that the many-particle ($n>2$) Lippmann-Schwinger equation generally has nonunique solutions. This paper evaluates these surface integrals in the same one-dimensional three-body model (of McGuire) employed recently to demonstrate the nonuniqueness explicitly. The computed values of the surface integrals agree precisely with Gerjuoy's hypotheses. These results further confirm the conclusion that the many-particle Lippmann-Schwinger equation has nonunique solutions in actual three-dimensional collisions, and support the belief that the aforesaid derivation of the real energy Lippmann-Schwinger equation is mathematically sound.

I. INTRODUCTION

Recently,[1] we have established the equivalence of several alternative interpretations of the Lippmann-Schwinger (LS) equation.[2] In so doing, we have deduced a number of relations between (a) the values of certain volume integrals involving Green's functions at complex energies $E+i\epsilon$ in the limit $\epsilon\to 0$, and (b) the values of various surface integrals at infinity in configuration space, involving the corresponding Green's functions at purely real energies E. The values of the aforesaid volume integral limits as $\epsilon\to 0$, known in the literature as Lippmann's identities, were derived originally by Lippmann[3] using operator algebra techniques. The values of the aforesaid surface integrals at real energies were inferred originally by Gerjuoy[4] from plausible assumptions about the behavior at infinity of real energy Green's functions and relevant scattering solutions of the Schrödinger equation; granting that these surface integrals have the values Gerjuoy inferred, it is quite trivial to show[4]—via conventional mathematical operations, without operator algebra manipulations—that at real energies the many-particle ($n>2$) LS equation has nonunique solutions, as originally deduced by Foldy and Tobocman,[5] also via operator algebra techniques.

Also recently, we have examined[6] the LS equation in a one-dimensional model first studied by McGuire,[7] involving three equal-mass particles 1,2,3 which move on the same straight line and interact via pairwise attractive δ-function potentials of equal strength. In this model, the scattering solutions to the Schrödinger equation can be written exactly, in closed form. Thereby in paper II we were able to demonstrate explicitly, without any approximations, that in the McGuire model the scattering solutions to the Schrödinger equation do satisfy the LS equation. We additionally were able to show, however, that the McGuire model real energy LS equation corresponding to an incident wave $\psi_i(E)$ in the i channel was satisfied not only by scattering solutions of the Schrödinger equation with incident part $\psi_i(E)$, but also by scattering solutions of the Schrödinger equation with incident part $\psi_f(E)$ corresponding to an incident wave in the $f\neq i$ channel. Here, and elsewhere in this paper unless otherwise stated, we employ the notation used in papers I and II. In particular, $\psi_i(E)$ is given (in the center of mass system) by Eqs. (2.8) of paper II, and represents beams of particles 3 incident on beams of bound particle pairs 1,2. The final f channel may have particles 1 incident on bound pairs 2,3 or may have particles 2 incident on bound pairs 3,1; for $f=1$ incident on bound 2,3 the definition of the f channel we will employ henceforth, $\psi_f(E)$ is given by Eq. (2.12) of paper II. The results of paper II also make it manifest (in the McGuire model, of course) that the i channel real energy "inhomogeneous" LS equation

$$\Psi=\psi_i(E)-G_i^{(+)}(E)V_i\Psi \tag{1.1}$$

has nonunique solutions because the corresponding i channel "homogeneous" LS equation

$$\Psi=-G_i^{(+)}(E)V_i\Psi \tag{1.2}$$

is satisfied by solutions of the inhomogeneous LS equation for $f\neq i$ channels. More specifically, it is shown in paper II that Eq. (1.2) is satisfied by the particular solution $\Psi=\Psi_f$ of

$$\Psi=\psi_f(E)-G_f^{(+)}(E)V_f\Psi\ , \tag{1.3}$$

which is "everywhere outgoing" except for its incident part $\psi_f(E)$; with the incident f channel specified as above, Ψ_f is given by Eq. (2.13) of paper II.

As we have discussed in paper I, the real energy LS equation (1.1) is not the only possible interpretation of the implied limit $\epsilon\to 0$ in the usual operator form[2] of the LS equation,

$$\Psi=\psi_i-\frac{1}{H_i-E-i\epsilon}V_i\Psi \ . \tag{1.4}$$

It is consistently possible to define the i channel "physical" scattering solution $\Psi(E)\equiv\Psi_i(E)$ of the Schrödinger equation,

$$(H-E)\Psi(E)=0 \ , \tag{1.5}$$

corresponding to incident wave $\psi_i(E)$ as

$$\Psi_i(E)=\lim_{\epsilon\to 0}\Psi(E+i\epsilon) \ , \tag{1.6}$$

where $\Psi(E+i\epsilon)$ solves Eq. (1.4) written in the form

$$\Psi(E+i\epsilon)=\psi_i(E)-G_i(E+i\epsilon)V_i\Psi(E+i\epsilon) \ . \tag{1.7}$$

It is easily seen[1] that Eq. (1.7) has a unique solution $\Psi(E+i\epsilon)$. Moreover, under very reasonable assumptions (fully detailed in paper I) concerning the uniform convergence of the integrals which form the last terms on the right-hand sides of Eqs. (1.1) and (1.7), the limit $\Psi_i(E)$ defined by Eq. (1.6) exists, satisfies Eq. (1.1), and is "everywhere outgoing" except for an incident part $\psi_i(E)$. In this sense, solutions to the LS equation (1.4) can be said to be unique, even in multiparticle systems. However, this result in no way contradicts the previously stated result that in multiparticle systems the real energy LS equation (1.1) has solutions other than the solution $\Psi=\Psi_i$ specified by Eq. (1.6).

The immediately preceding assertion seemingly has been rejected by Mukherjee,[8] who insists that solutions to the multiparticle real energy LS equation (1.1) are unique; in so insisting, Mukherjee has criticized[8] as erroneous the results of both Gerjuoy[4] and Lippmann.[2] Mukherjee has especially criticized Lippmann's identities[3] and Gerjuoy's inferred values of the aforementioned surface integrals at infinity in configuration space. In large part, our study reported in paper II was undertaken to refute Mukherjee's criticisms and concomitant predictions of uniqueness. As we have explained, the results of paper II are completely consistent with the nonuniqueness predictions of Ref. 4 and 5, and are correspondingly inconsistent with Mukherjee's claims. Simultaneously, the results of paper II certainly strongly support—though they obviously do not prove—the correctness of Gerjuoy's postulated values for his surface integrals at infinity, and therefore the validity of his assumptions about the behavior at infinity of Green's functions and scattering solutions. Nevertheless, Benoist-Gueutal,[9] though fully aware of the results of paper II, has criticized the mathematics employed in Ref. 4 as lacking in rigor, indeed as crucially based on multiple integrals which are not well defined; accordingly, Benoist-Gueutal[9] questions the aforementioned surface integral values, as well as the implications thereof (e.g., nonuniqueness of solutions) for the theory of the multiparticle real energy LS equation (1.1).

Although we do not believe Benoist-Gueutal's criticisms are consequential, for reasons which have been given in a response[10] to her paper,[9] those criticisms now have led us to examine Gerjuoy's surface integrals in the McGuire three-particle model;[7] in this model these surface integrals—like the volume integrals studied in paper II—can be evaluated in closed analytic form, without approximation and without the need to make assumptions about the asymptotic behavior of Green's functions and solutions of the real energy LS equation (1.1). The new results of this present paper completely confirm (i) the postulated values of the aforesaid surface integrals, and (ii) the assumptions about the behavior at infinity of Green's functions and scattering solutions from which the aforesaid surface integral values originally were inferred.[4] Concomitantly, the McGuire model results of the present paper once again strongly support, although they cannot prove, (a) the validity of the conclusions drawn in paper I and Ref. 4 [e.g., about the equivalence of alternative interpretations of Eq. (1.4) and the nonuniqueness of solutions to its real energy interpretation (1.1)], and (b) the validity of the mathematics employed in paper I and in Ref. 4, including the validity of the assumptions (concerning, e.g., the behavior of Green's functions and scattering solutions) on which those papers depend. Our present exact McGuire model calculations of surface integrals, performed without assumptions about asymptotic behavior at infinity but restricted to a system of three one-dimensional particles, also supplement and reinforce the surface integral evaluations for a system of three three-dimensional particles carried out by one of the present authors and Glöckle,[11] who, however, had to rely on much the same assumptions about asymptotic behavior at infinity as were employed by Gerjuoy.

The organization of this paper is as follows. In Sec. II, for the reader's convenience, we briefly recapitulate the main McGuire model results we require from paper II; for full details about the model and the derivations of these results, the reader should consult paper II. In Sec. III we derive the asymptotic form of the "incident" i channel Green's function $G_i^{(+)}(E)$ appearing in the real energy LS equation (1.1), via the method of steepest descents applied to the exact integral representation of $G_i^{(+)}(E)$. Using this asymptotic form, together with the exact formulas from paper II for various solutions of Eq. (1.1), the surface integrals discussed in Ref. 4 and paper I are calculated exactly in Sec. IV, and are seen to have their predicted values.

II. McGUIRE MODEL: PREVIOUS RESULTS

In the McGuire model three equal mass (mass m) particles 1,2,3 move along the x axis; their coordinates are x_1,x_2,x_3 and they interact via equal strength attractive δ function interactions. Convenient relative coordinates are

$$X_\mu=\frac{1}{\sqrt{2}}(x_\nu-x_\sigma) \ , \qquad Y_\mu=(\tfrac{2}{3})^{1/2}[x_\mu-\tfrac{1}{2}(x_\nu+x_\sigma)] \ , \tag{2.1}$$

where μ, ν, and σ are any cyclic permutation of (1,2,3). We concentrate on laboratory system collisions wherein the incident i channel contains beams of particles 3 impinging on beams of bound particle pairs 1,2. For such collisions the incident wave is expressed most simply in terms of X_3,Y_3. Therefore, in order to conveniently visualize the collisions in the two-dimensional center of mass system, we introduce $(x,y)\equiv(X_3,Y_3)$; also $(u,v)\equiv(X_1,Y_1)$, $(r,s)\equiv(X_2,Y_2)$. The relations between the coordinate pairs (x,y), (u,v), and (r,s) are given by Eqs. (2.5) of paper II. In terms of these coordinates the Hamiltonian in the center of mass system (with which we henceforth shall be solely concerned) takes the form

$$H=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]-g\delta(x)-g\delta(u)-g\delta(r)\ , \qquad (2.2)$$

where $g>0$. The three lines $x=0$, $u=0$, $r=0$ on which various δ functions in H are nonvanishing divide the whole x,y plane into six 60° sectors, labeled I—VI in Fig. 1 (reproduced here from the correspondingly numbered figure in our former paper II). The incident wave in the i (3 incident on bound 1,2) channel is

$$\psi_i=e^{iky}w(x)\ , \qquad (2.3a)$$

where we take $k>0$ and

$$w(x)=\sqrt{\alpha}e^{-\alpha|x|}\ . \qquad (2.3b)$$

The corresponding bound state energy of the pair 1,2 is

$$E_b=-\hbar^2\alpha^2/2m\ , \qquad (2.4a)$$

where $\alpha=mg/\hbar^2>0$; the total energy of the three particles in the center of mass system is

$$E=\hbar^2(k^2-\alpha^2)/2m\ . \qquad (2.4b)$$

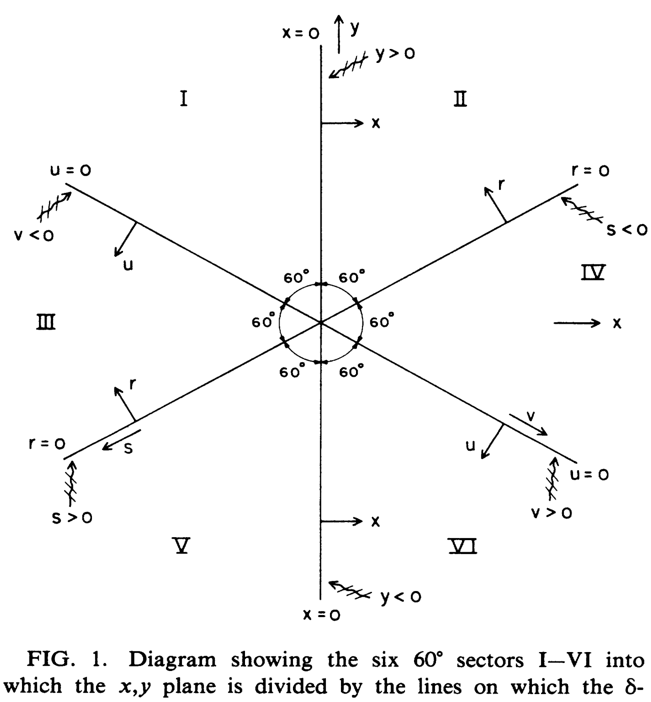


FIG. 1. Diagram showing the six 60° sectors I—VI into which the x,y plane is divided by the lines on which the δ-function interactions need not vanish; namely, the lines on which the original locations x_1,x_2,x_3 of the three particles are not all different. The arrows show the directions of positive x,r,u and y,s,v, at these lines: $x=0$, $r=0$, $u=0$, respectively. The signs of y,s,v at the opposing ends of these respective lines are also indicated.

The incident wave ψ_i of Eq. (2.3a) satisfies

$$(H_i-E)\psi_i=0\ , \qquad (2.5a)$$

where E is given by Eq. (2.4b) and the "initial" i-channel Hamiltonian H_i is

$$H_i=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]-g\delta(x)\ . \qquad (2.5b)$$

For the same E the "scattering solution" $\Psi\equiv\Psi_i$ to the Schrödinger equation

$$(H-E)\Psi=0\ , \qquad (2.6)$$

whose incident part is ψ_i and whose scattered part

$$\Phi_i=\Psi_i-\psi_i \qquad (2.7)$$

is everywhere outgoing, can be written as

$$\begin{aligned}
&\Psi_{i\mathrm{I}}=Ae^{iky}e^{\alpha x}\ ,\\
&\Psi_{i\mathrm{II}}=Ae^{iky}e^{-\alpha x}\ ,\\
&\Psi_{i\mathrm{III}}=Be^{iky}e^{\alpha x}+Ce^{iks}e^{-\alpha r}\ ,\\
&\Psi_{i\mathrm{IV}}=Be^{iky}e^{-\alpha x}+Ce^{ikv}e^{\alpha u}\ ,\\
&\Psi_{i\mathrm{V}}=\sqrt{\alpha}e^{iky}e^{\alpha x}+Ce^{iks}e^{\alpha r}+De^{ikv}e^{-\alpha u}\ ,\\
&\Psi_{i\mathrm{VI}}=\sqrt{\alpha}e^{iky}e^{-\alpha x}+De^{iks}e^{\alpha r}+Ce^{ikv}e^{-\alpha u}\ ,
\end{aligned} \qquad (2.8)$$

where the subscripts I—VI denote the form of Ψ_i in the corresponding sectors of Fig. 1. In Eq. (2.7), A,B,C,D are constants, given by Eqs. (2.11) of paper II; the exact values of A,B,C,D are not required in this paper, as will be seen.

In the f channel (wherein particles 2 and 3 are bound to each other, but neither is bound to particle 1; recall Sec. I), for collisions wherein beams of particle 1 impinge on bound particle pairs 2,3, the incident wave analogous to ψ_i of Eq. (2.3a) is

$$\psi_f=e^{ikv}w(u)=\sqrt{\alpha}e^{ikv}e^{-\alpha|u|}\ . \qquad (2.9)$$

The scattering solution Ψ_f solving Eq. (2.6) for this incident wave is given by

$$\begin{aligned}
&\Psi_{f\mathrm{I}}=\sqrt{\alpha}e^{ikv}e^{\alpha u}+Ce^{iky}e^{\alpha x}+De^{iks}e^{-\alpha r}\ ,\\
&\Psi_{f\mathrm{II}}=Be^{ikv}e^{\alpha u}+Ce^{iky}e^{-\alpha x}\ ,\\
&\Psi_{f\mathrm{III}}=\sqrt{\alpha}e^{ikv}e^{-\alpha u}+De^{iky}e^{\alpha x}+Ce^{iks}e^{-\alpha r}\ ,\\
&\Psi_{f\mathrm{IV}}=Ae^{ikv}e^{\alpha u}\ ,\\
&\Psi_{f\mathrm{V}}=Be^{ikv}e^{-\alpha u}+Ce^{iks}e^{\alpha r}\ ,\\
&\Psi_{f\mathrm{VI}}=Ae^{ikv}e^{-\alpha u}\ ,
\end{aligned} \qquad (2.10)$$

with A, B, C, and D the same constants as in Eqs. (2.8).

In terms of H_i from Eq. (2.5b), the outgoing Green's function in the i channel is defined by

$$G_i^{(+)}(E)=\lim_{\epsilon\to 0}(H_i-E-i\epsilon)^{-1},\ \ \epsilon>0\ . \qquad (2.11)$$

The explicit coordinate space representation of $G_i^{(+)}$ is given by

$$G_i^{(+)}(x,y;x',y';E)=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_\Gamma dk_y \frac{e^{ik_y(y-y')}}{p}\left[e^{ip|x-x'|}-\frac{\alpha}{\alpha+ip}e^{ip|x+x'|}\right], \quad x,x'<0 \text{ or } >0 \tag{2.12a}$$

$$=-\frac{1}{2\pi}\frac{m}{\hbar^2}\int_\Gamma dk_y e^{ik_y(y-y')}\frac{e^{ip|x-x'|}}{\alpha+ip}, \quad x<0<x' \text{ or } x'<0<x \tag{2.12b}$$

where

$$p=(k^2-\alpha^2-k_y^2)^{1/2} . \tag{2.13a}$$

The contour of integration Γ is as shown in Fig. 2 (reproduced here from Fig. 3 of paper II). We assume, as in paper II and as is sufficient for our purposes, that $k^2>\alpha^2$, i.e., recalling Eq. (2.4b), that the total energy in the center of mass system is positive. Correspondingly, the phase of p from Eq. (2.13a) is specified everywhere in the k_y plane by the directions in which the cuts are drawn and by the understanding that near $k_y=0$

$$\arg[(k^2-\alpha^2)^{1/2}-k_y]\cong\arg[k_y+(k^2-\alpha^2)^{1/2}]\cong 0 . \tag{2.13b}$$

In what follows we frequently find it convenient to write

$$K=(k^2-\alpha^2)^{1/2}, \quad K>0 . \tag{2.13c}$$

We are not able to perform the integrals over k_y in Eqs. (2.12). However, from (2.12) we are able to deduce the asymptotic behavior of $G_i^{(+)}$ on the circle at infinity in $\mathbf{r}'=(x',y')$ space for fixed x,y; this asymptotic behavior is all we require in order to evaluate the surface integrals at infinity discussed in Sec. I.

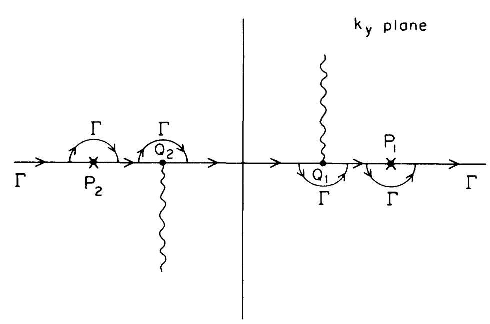


FIG. 2. The contour of integration Γ for the integrals of Eq. (2.12). The termini Q_1 and Q_2 of the branch cuts are at $k_y=\pm(k^2-\alpha^2)^{1/2}$. The poles P_1 and P_2 lie at $k_y=\pm k$. The figure is drawn on the assumption that $k^2>\alpha^2$; by definition, $k>0$.

III. ASYMPTOTIC FORM OF GREEN'S FUNCTION

A. Method of steepest descents

We proceed to derive the asymptotic behavior of $G_i^{(+)}$. Consider first the case $x<0<x'$, where, from Eq. (2.12b),

$$G_i^{(+)}(E)\equiv G_i^{(+)}(x,y;x',y';E)$$
$$=-\frac{1}{2\pi}\frac{m}{\hbar^2}\int_\Gamma dk_y e^{ik_y(y-y')}\frac{e^{-ip(x-x')}}{\alpha+ip} . \tag{3.1}$$

Equation (3.1) holds for arbitrary y,y'; for the present, however, suppose $y>y'$. We introduce the new quantities $a=x'-x$, $b=y-y'$, $a>0$, and $b>0$, and recalling Eq. (2.13c) define the new integration variable $u=k_y/K$ [which, of course, is wholly unrelated to u in Eq. (2.2)]. Then Eq. (3.1) becomes

$$G_i^{(+)}(E)=-\frac{K}{2\pi}\frac{m}{\hbar^2}\int_{-\infty}^{\infty}du\frac{e^{iKf(u)}}{\alpha+iK(1-u^2)^{1/2}} , \tag{3.2a}$$

where

$$f(u)=bu+a(1-u^2)^{1/2} . \tag{3.2b}$$

The integration contour, shown in Fig. 3, now runs along the real u axis, except near the branch points now at $u=\pm1$ and near the poles now at $\pm k/K$. The phases of $1-u$ and $1+u$ are determined by Eq. (2.13) and the directions of the branch cuts. In what follows, it frequently will be more convenient to work with $u-1$ rather than $1-u$; we will specify that $u-1=e^{-i\pi}(1-u)$. Thus, on the real u axis, when $u>1$,

$$\arg(1-u)=\pi, \quad \arg(1+u)=0 ,$$
$$\arg(u-1)=0, \quad \arg(1-u^2)^{1/2}=\pi/2 ; \tag{3.3a}$$

when $-1<u<1$,

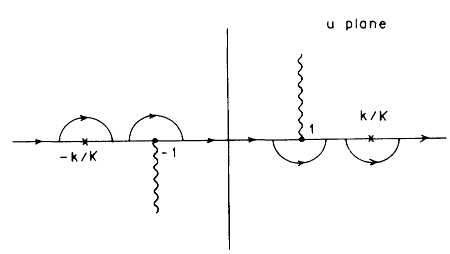


FIG. 3. The contour of integration for the integral of Eqs. (3.2).

$$\arg(1-u)=\arg(1+u)=0\ ,$$
$$\arg(u-1)=-\pi,\quad \arg(1-u^2)^{1/2}=0\ ; \tag{3.3b}$$

when $u<-1$,

$$\arg(1-u)=0,\quad \arg(1+u)=\pi\ ,$$
$$\arg(u-1)=-\pi,\quad \arg(1-u^2)^{1/2}=\pi/2\ . \tag{3.3c}$$

As $\mathbf{r}'=(x',y')$ approaches infinity within the fourth quadrant of the x',y' plane for fixed x,y the quantities a,b each become infinite in magnitude and retain positive sign. Therefore the integrand in Eq. (3.2a) is in a suitable form for evaluating the asymptotic behavior of $G_i^{(+)}(E)$ by the method of steepest descents.[12] The saddle points of $f(u)$ are at $f'(u)=0$, i.e., at the roots of

$$b-\frac{au}{(1-u^2)^{1/2}}=0\ ; \tag{3.4a}$$

Eq. (3.4a) is solved by

$$u=\pm\frac{b}{(a^2+b^2)^{1/2}} \tag{3.4b}$$

From Eq. (3.3b), however, we see that whenever $-1<u<1$, $(1-u^2)^{1/2}$ is real and positive. Consequently, the only saddle point of $f(u)$ in the u plane of Fig. 3 is $u_s=b(a^2+b^2)^{-1/2}$; the root $u=-b(a^2+b^2)^{-1/2}$ of (3.4a) lies on Riemann sheets beneath the u plane of Fig. 3.

In the method of steepest descents, one determines the contours through u_s along which $\exp[iKf(u)]$ in Eq. (3.2a) has constant phase;[12] along any such contour $|\exp[iKf(u)]|$ necessarily varies monotonically in each direction away from u_s, since there is only one saddle point. The desired "steepest descents" contour is the one along which $|\exp[iKf(u)]|$ decreases in each direction away from the saddle point; for $f(u)$ of form (3.2b), the decrease obviously will be exponential. For the method of steepest descents to be applicable to Eq. (3.2a), it is necessary that the contour of Fig. 3 be deformable to the steepest descents contour without crossing branch cuts and without traversing regions at infinity in the u plane where the integrand diverges. As we shall see, the method of steepest descents indeed is applicable to Eq. (3.2a), but the necessary contour deformation can cross a pole in the u plane, whose residue then must be taken into account.

The phase of $\exp[iKf(u)]$ remains constant along any contour through $u=u_s$ for which

$$\mathrm{Re}f(u)=\mathrm{Re}f(u_s)=(a^2+b^2)^{1/2}\equiv f(u_s)\ . \tag{3.5}$$

In the neighborhood of $u=u_s$,

$$f(u)=f(u_s)+\tfrac{1}{2}(u-u_s)^2f''(u_s)+\cdots\ , \tag{3.6a}$$

$$f''(u_s)=-\frac{a}{(1-u_s^2)^{3/2}}=-\frac{(a^2+b^2)^{3/2}}{a^2}\ . \tag{3.6b}$$

In the vicinity of $u=u_s$, therefore, to keep $\mathrm{Re}f(u)$ consistent with Eq. (3.5), the second term in Eq. (3.6a) must be pure imaginary; to ensure that $|\exp[iKf(u)]|$ decreases in each direction away from $u=u_s$, the sign of this now purely imaginary second term must be positive. Hence, since $f''(u_s)$ is real and negative according to Eq. (3.6b), $(u-u_s)^2$ must be negative imaginary near $u=u_s$. In other words, we have determined that the desired steepest descents contour makes an angle of 45° with the real axis at $u=u_s$, in a direction such that on this contour near $u=u_s$

$$\arg(u-u_s)=-\pi/4,\quad \mathrm{Re}u>u_s\ ,$$
$$\arg(u-u_s)=3\pi/4,\quad \mathrm{Re}u<u_s\ . \tag{3.6c}$$

At infinity in the u plane, as we have defined the relative phases of $u-1$ and $1-u$,

$$f(u)=bu+ia(u^2-1)^{1/2}\ . \tag{3.7}$$

Let $u=Re^{i\theta}$. Then, by analytic continuation of the phases for $u-1$ and $u+1$ in Eqs. (3.3), taking into account the directions of the cuts in Fig. 3, we see that, at large R, for the first quadrant, where $0<\theta<\pi/2$,

$$\arg(u-1)=\arg(u+1)=\theta\ ,$$
$$(u^2-1)^{1/2}=\mathrm{R}e^{i\theta}=u\ ;$$

for the second quadrant where $\pi/2<\theta<\pi$,

$$\arg(u-1)=-2\pi+\theta,\quad \arg(u+1)=\theta\ ,$$
$$(u^2-1)^{1/2}=-Re^{i\theta}=-u\ ;$$

for the third quadrant where $\pi<\theta<3\pi/2$,

$$\arg(u-1)=-2\pi+\theta,\quad \arg(u+1)=\theta\ ,$$
$$(u^2-1)^{1/2}=-Re^{i\theta}=-u\ ;$$

for the fourth quadrant where $3\pi/2<\theta<2\pi$,

$$\arg(u-1)=\arg(u+1)=-2\pi+\theta\ ,$$
$$(u^2-1)^{1/2}=Re^{i\theta}=u\ .$$

Thus at infinity in the u plane we have the following, from Eq. (3.7): For the first and fourth quadrants,

$$f(u)=(b+ia)u=i(a-ib)u=i\rho Re^{-i\gamma}e^{i\theta}\ ; \tag{3.8a}$$

for the second and third quadrants,

$$f(u)=(b-ia)u=-i(a+ib)u=-i\rho Re^{i\gamma}e^{i\theta}\ , \tag{3.8b}$$

where we have written (remembering $a>0$, $b>0$)

$$a\pm ib=\rho e^{\pm i\gamma},\quad \rho=(a^2+b^2)^{1/2}\ ,$$
$$\gamma=\tan^{-1}(b/a),\quad 0<\gamma<\pi/2\ . \tag{3.8c}$$

Equation (3.8a) implies that if the steepest descents contour approaches infinity in either the first or fourth quadrant, then the phase angle θ on the contour must asymptotically approach values obeying either

$$\theta-\gamma=0\quad\text{or}\quad \theta-\gamma=2\pi\ . \tag{3.9a}$$

Only when the relations (3.9a) hold as $R\to\infty$ can $\mathrm{Re}f(u)$ have the finite value prescribed by Eq. (3.5), while simultaneously keeping $\mathrm{Im}f(u)>0$, as is necessary for $|\exp[iKf(u)]|$ to approach zero as $u\to\infty$ along the contour. Neither of the relations (3.9a) can be satisfied in the fourth quadrant, but $\theta-\gamma=0$ can be satisfied in the first quadrant. Similarly, Eq. (3.8b) requires that the phase angle θ on the contour must asymptotically approach

$$\theta+\gamma=\pi \quad \text{or} \quad \theta+\gamma=-\pi\ ; \tag{3.9b}$$

Eq. (3.9b) cannot be satisfied in the third quadrant, but $\theta+\gamma=\pi$ can be satisfied in the second quadrant. Therefore the steepest descents contour $u\equiv Re^{i\theta}$ starting at $u=u_s$ as prescribed by Eq. (3.6c) reaches infinity $(R\to\infty)$ along the line $\theta=\gamma$ in the first quadrant, and along the line $\theta=\pi-\gamma$ in the second quadrant. In fact, because $\text{Re}f(u_s)$ from Eq. (3.5) is >0, one sees that at infinity the steepest descents contour approaches $\theta=\gamma$ from below in the first quadrant (i.e., $\theta<\gamma$) and approaches $\theta=\pi-\gamma$ from above in the second quadrant (i.e., $\theta<\pi-\gamma$).

Using Eq. (3.3c), it is obvious that for real $u<-1$, $\text{Re}f(u)$ from Eq. (3.2b) cannot equal the positive $f(u_s)$ from Eq. (3.5); in other words, the steepest descents contour cannot cross the real u axis to the left of the cut at $u=-1$. Using Eq. (3.3a), it is equally obvious that, for real $u>1$, Eq. (3.5) is satisfied only at

$$u=u_C=\left[1+\frac{a^2}{b^2}\right]^{1/2} . \tag{3.10}$$

It also is easy to see that on the real axis in the interval $-1<u<1$ the here purely real $f(u)$ from Eq. (3.2b) satisfies Eq. (3.5) at $u=u_s$ only. Thus in going from $u=u_s$ to its first quadrant $\theta=\gamma$ asymptote at infinity, the steepest descents contour first moves into the fourth quadrant [recall Eq. (3.6c)] and then crosses the real u axis into the first quadrant at $u=u_C$ to the right of the cut at $u=1$. In going from $u=u_s$ to its second quadrant $\theta=\pi-\gamma$ asymptote at infinity, the steepest descents contour never crosses into the third quadrant.

From the preceding two paragraphs, the steepest descents contour C through $u=u_s$—a contour which is completely specified by Eqs. (3.5) and (3.6c)—must have the form sketched in Fig. 4. Equation (3.8a) shows that $|\exp[iKf(u)]|$ vanishes exponentially at infinity for all θ in the first quadrant; Eq. (3.8b) shows that $|\exp[iKf(u)]|$ vanishes exponentially at infinity for all θ in the second quadrant. Consequently, the contour of Fig. 3 can be deformed to the steepest descents contour of Fig. 4 without crossing branch cuts and without traversing regions at infinity where the integrand of Eq. (3.2a) diverges. The sole remaining question is whether this deformation crosses the pole at $u=k/K$, in which event the residue at that pole will have to be included in the steepest descents estimate of the integral (3.2a); because the steepest descents contour never crosses into the third quadrant, the contour deformation from Fig. 3 to Fig. 4 cannot cross the pole at $u=-k/K$.

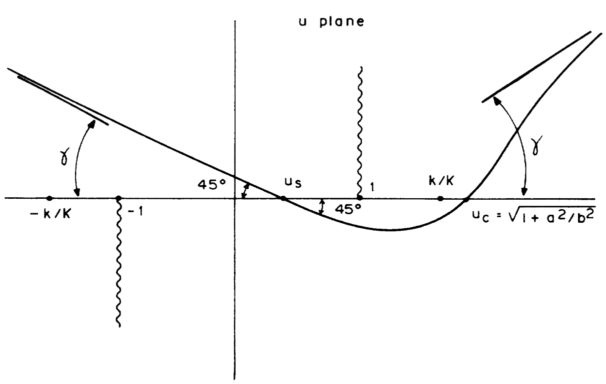


FIG. 4. Sketch of steepest descents contour C for the case that the deformation from Fig. 3 to C does not cross the pole at $u=k/K$.

Since the contour of Fig. 3 runs below the pole at $u=k/K$, the deformation to C will not cross the $u=k/K$ pole if the deformed contour C also runs below $u=k/K$. Consequently, the deformation crosses the pole at $u=k/K$ when (but only when)

$$u_C=(1+a^2/b^2)^{1/2}<k/K=(1-\alpha^2/k^2)^{-1/2} \tag{3.11a}$$

i.e., [recalling Eq. (3.8c) and the definitions of a,b], when (but only when)

$$\tan\gamma=\frac{b}{a}=\frac{y-y'}{x'-x}>\left[\frac{k^2}{\alpha^2}-1\right]^{1/2} . \tag{3.11b}$$

As will be seen, we will require the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ as $\mathbf{r}'\equiv(x',y')$ approaches infinity for fixed x,y. Since $a>0,b>0$ by definition, Eq. (3.11b) is pertinent only to the behavior of $G_i^{(+)}$ as $\mathbf{r}'\to\infty$ in the fourth quadrant of the x',y' plane. In this quadrant, however, Eq. (3.11b) evidently is always satisfied for $\mathbf{r}'\to\infty$ along directions parallel or nearly parallel to the y' axis, no matter how large k/α is. For such directions $\gamma\to\pi/2$ and the contour C of Fig. 4 begins and ends on asymptotes which are almost parallel to the imaginary u axis. Figure 4 happens to be drawn for a smaller γ (i.e., for $\mathbf{r}'\to\infty$ in the fourth x',y' quadrant along a direction more inclined toward the x' axis) which does not satisfy the inequality (3.11b).

When the deformation from Fig. 3 to C of Fig. 4 crosses the pole at $u=k/K$, it does so in a direction such that the residue contribution must be added to the steepest descents contribution in order to obtain the correct estimate of $G_i^{(+)}$ from Eq. (3.2a). Let us first calculate the residue contribution (when applicable). We note that at $u=u_p=k/K$,

$$(1-u_p^2)^{1/2}=i(k^2/K^2-1)^{1/2}=\frac{i\alpha}{K} , \tag{3.12a}$$

using Eq. (2.13c) and the specification (3.3a) for the phase of $(1-u^2)^{1/2}$ when $u>1$. Also, still for $u>1$,

$$\frac{1}{\alpha+iK(1-u^2)^{1/2}}=\frac{\alpha-iK(1-u^2)^{1/2}}{\alpha^2+K^2(1-u^2)}$$
$$=\frac{-K(u^2-1)^{1/2}-\alpha}{K^2}\times\frac{1}{(u+k/K)(u-k/K)} .$$

Hence the residue contribution to (3.2a) is

$$\mathscr{R}=2\pi i\left[-\frac{mK}{2\pi\hbar^2}\right]e^{ikb}e^{-\alpha a}\frac{-2\alpha}{K^2}\frac{1}{2k/K}$$

$$=\frac{im\alpha}{\hbar^2 k}e^{-\alpha a}e^{ikb}\ . \qquad (3.12b)$$

The steepest descents estimate of the integral (3.2a) is obtained on the assumption that for large a,b in (3.2b) the integral over the entire steepest descents contour can be replaced by its predominant contribution (namely the contribution from the neighborhood of $u=u_s$), with $f(u)$ there replaced by its quadratic approximation (3.6a). From Eq. (3.6c), the steepest descents integration contour near $u=u_s$ is along $u-u_s=\omega e^{-i\pi/4}$, where the real variable ω now can be thought to run from $-\infty$ to ∞. At $u=u_s<1$, where the specification (3.3b) pertains,

$$\alpha+iK(1-u^2)^{1/2}=\alpha+\frac{iKa}{(a^2+b^2)^{1/2}}$$

Consequently, the steepest descents estimate of (3.2a) is

$$J=-\frac{K}{2\pi}\frac{m}{\hbar^2}\frac{1}{\alpha+iKa/(a^2+b^2)^{1/2}}e^{iK(a^2+b^2)^{1/2}}\int_{-\infty}^{\infty}d\omega\, e^{-i\pi/4}e^{-K\omega^2[(a^2+b^2)^{3/2}/2a^2]}$$

$$=-\frac{m}{\hbar^2}\left[\frac{K}{2\pi}\right]^{1/2}e^{-i\pi/4}\frac{1}{[iK+\alpha(a^2+b^2)^{1/2}/a]}\frac{e^{iK(a^2+b^2)^{1/2}}}{(a^2+b^2)^{1/4}}\ . \qquad (3.13)$$

The Green's function $G_i^{(+)}$ of Eq. (2.11) satisfies

$$(H_i-E)G_i^{(+)}(x,y;x',y';E)=1=\delta(x-x')\delta(y-y')\ . \qquad (3.14a)$$

Thus $G_i^{(+)}(x,y;x',y';E)$ at points x,y can be interpreted as the solution to Eq. (2.5a) generated by a unit point source at x',y'. However, it is known that[4,13]

$$G_i^{(+)}(x,y;x',y';E)=G_i^{(+)}(x',y';x,y;E)\ . \qquad (3.14b)$$

Thus $G_i^{(+)}(x,y;x',y';E)$ can also be interpreted as the solution to the primed version of Eq. (2.5a) generated by a unit point source at x,y. On this basis it is intuitively obvious that the asymptotic results (3.12b) and (3.13)—for x',y' in the fourth quadrant when x,y is in the second or third quadrant [remember that Eq. (3.1) holds for $x<0<x'$]—in essence must yield the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ when x',y' approaches infinity in the first quadrant. In other words, although the above derivations of (3.12b) and (3.13) apparently have made very strong use of the inequality $b>0$, in essence Eqs. (3.12b) and (3.13) should hold at infinity in the first x',y' quadrant, where $b=y-y'<0$ for any given fixed y. This assertion can be verified directly from Eq. (3.2a), carrying through the appropriately modified (for $b<0$ instead of $b>0$) version of the steepest descents calculation we have described. The verification can be performed more easily, however, as follows.

Suppose $y-y'<0$ in Eq. (3.1). Then we now define $y'-y=b>0$, in which event Eq. (3.1) will again yield Eq. (3.2a), except that now Eq. (3.2b) is replaced by

$$f(u)=-bu+a(1-u^2)^{1/2}\ . \qquad (3.15)$$

Next, let $-u$ be the new variable of integration in Eq. (3.2a), i.e., in Eq. (3.2a) integrate over $-u$ rather than u. Then from Eqs. (3.3), together with the $u\to-u$ reflection symmetry of the cuts and integration contour in Fig. 3, it is obvious that this $u\to-u$ variable of integration transformation once again yields Eq. (3.2a), with $f(u)$ now once again given by Eq. (3.2b), not Eq. (3.15); moreover, b in Eq. (3.2b) still is >0, although now $b=y'-y$, not $y-y'$. Therefore Eqs. (3.12b) and (3.13) once again yield the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$, but now in the first x',y' quadrant consistent with $b=y'-y>0$. More simply put, we now have shown that, when $x<0<x'$, Eqs. (3.12b) and (3.13) yield the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ in the first quadrant ($y'-y>0$) and in the fourth quadrant ($y-y'>0$), provided we redefine b in those equations as $b=|y-y'|$.

B. Interpretation of steepest descents results

Equation (3.13) have a simple and useful interpretation. Were it not for the $g\delta(x)$ interaction in Eq. (2.5b), $G_i^{(+)}(x,y;x',y';E)$ satisfying Eq. (3.14a) would be identical with the two-dimensional free space Green's function $G_0^{(+)}(x,y;x',y';E)$ satisfying

$$(H_0-E)G_0^{(+)}(x,y;x',y';E)=\delta(x-x')\delta(y-y')\ , \qquad (3.16a)$$

$$H_0=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]. \qquad (3.16b)$$

$G_0^{(+)}$ is easily found.[14] Specifically,

$$G_0^{(+)}(x,y;x',y';E)=\frac{2m}{\hbar^2}\frac{i}{4}H_0^{(1)}(K\rho)\ , \qquad (3.17a)$$

where

$$\rho=[(x-x')^2+(y-y')^2]^{1/2}=(a^2+b^2)^{1/2} \qquad (3.17b)$$

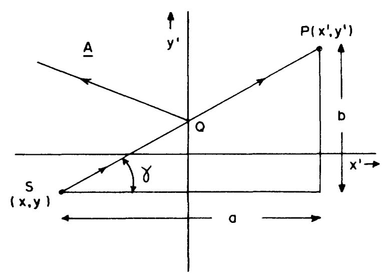
in (3.13), and where $E=\hbar^2K^2/2m$, consistent with Eq. (2.4b) and our previous definition of $K=(k^2-\alpha^2)^{1/2}$. From the known[15] asymptotic behavior of the Hankel function $H_0^{(1)}$, $G_0^{(+)}$ at large ρ is given by

$$G_0^{(+)}(x,y;x',y';E)\equiv G_0^{(+)}(\rho;E)=\frac{im}{\hbar^2\sqrt{2\pi K}}e^{-i\pi/4}\frac{e^{iK\rho}}{\rho^{1/2}}\ . \qquad (3.18)$$

Our result (3.13) for $G_i^{(+)}$ is not identical with $G_0^{(+)}$ of Eq. (3.18) because waves leaving the source at x,y ($x<0$) cannot reach x',y' ($x'>0$) without encountering the δfunction interaction at $x'=0$ in the x',y' plane, an interaction which is not encountered during free space propagation. Consider that portion of the wave leaving the source S at (x,y) and traveling along the ray SP shown in Fig. 5(a), drawn from S to the receiving point P at (x',y'). In Fig. 5(a) we have chosen to place the source in the third quadrant and the receiver in the first quadrant, so that we have the circumstances $a=x'-x>0$ and $b=y'-y>0$ discussed following Eq. (3.15); these values of a,b also are shown in Fig. 5(a). The wave vector $\mathbf{K}$ along the direction SP has the following components:

$$(K_x,K_y)=(K\cos\gamma,\ K\sin\gamma)$$
$$=\left[\frac{Ka}{(a^2+b^2)^{1/2}},\frac{Kb}{(a^2+b^2)^{1/2}}\right]. \tag{3.19}$$

With H_i, as with H_0, these K_x and K_y components of $\mathbf{K}$ propagate independently and freely at points not on the line $x'=0$. At Q in Fig. 5(a), where SP intersects the y'



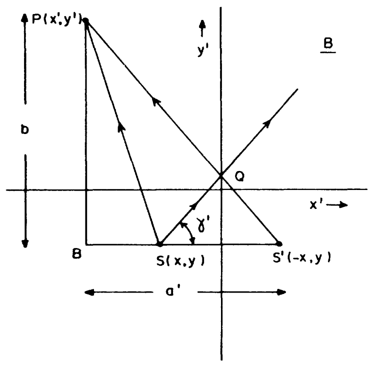


FIG. 5. Diagrams showing the propagation of waves leaving the source at (x,y) and reaching (x',y'). In (a) $x<0<x'$, and only the transmitted ray SQP reaches (x',y'). In (b), $x'<x<0$, and P is reached by a direct ray SP and a reflected ray SQP. In each case, we have put the source in the third quadrant, and have taken $y'>y$. In (a) the quantities $b=y'-y$ and $a=x'-x$ are shown. In (b) the quantity $b=y'-y$ is shown, but a's relevant value depends on whether we are considering the direct ray SP or the reflected ray seemingly reaching P from the reflected source S'; we have shown the relevant value of a for the reflected ray only, namely $a\equiv a'=|x'+x|$. The reflection point Q lies on the y' axis, where and only where the interaction $g\delta(x')$ is nonvanishing.

axis, the K_y propagation is unaffected because the δ-function interaction in H_i is independent of y, but the K_x propagation along the positive x' direction is both reflected and transmitted. The reflection and transmission coefficients are obtained from Eq. (3.7) of paper II, which gives the solution of

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}-g\delta(x')-\frac{\hbar^2K_x^2}{2m}\right]\phi=0\ , \tag{3.20a}$$

for a plane wave of wave number K_x and unit amplitude approaching $x'=0$ from $x'=-\infty$. This solution is

$$\phi(x',K_x)=e^{iK_xx'}-\frac{\alpha}{\alpha+iK_x}e^{-iK_xx'},\quad x'<0$$
$$\phi(x',K_x)=\frac{iK_x}{\alpha+iK_x}e^{iK_xx'},\quad x'>0\ . \tag{3.20b}$$

Thus the transmission and reflection coefficients are, respectively,

$$T_r=\frac{iK_x}{\alpha+iK_x}\ ,$$
$$R_r=-\frac{\alpha}{\alpha+iK_x}\ . \tag{3.21}$$

Moreover, because the values of K_x,K_y in the transmitted wave are identical with K_x,K_y in the incident wave, the transmitted wave is not refracted, i.e., SQP is a straight line, as shown in Fig. 5(a).

We conclude that the amplitude at x',y' for the circumstances of Fig. 5(a) should be the right-hand side of Eq. (3.18) multiplied by T_r from Eq. (3.21). Recalling the definitions (3.17b) and (3.19), it is immediately verified that the just-stated simple prescription yields precisely the result (3.13) obtained from the steepest descents evaluation of (3.2a). Of course, our simple prescription has not accounted for the residue contribution (3.12b), nor could it be expected to, because the exponentially decreasing factor $e^{-\alpha b}=\exp[\alpha(x-x')]$ in (3.12b) corresponds to propagation along the x' direction with an imaginary K_x; for imaginary wave numbers, the foregoing ray-geometric visualization of the wave propagation (to x',y' from a point source at x,y when there is a δ-function interaction along the y' axis) is totally unsuitable.

Nevertheless, the result (3.12b) also has a simple and useful interpretation. The integral representation (2.12) for $G_i^{(+)}$ satisfying Eq. (2.11) was derived from the expansion [Eq. (3.11) of paper II]

$$(H_i-\lambda)^{-1}=\sum_e \chi_e(y)\chi_e^*(y')G_1(x;x';\lambda-E_e)\ , \tag{3.22a}$$

where $\lambda=E+i\epsilon$, $\epsilon>0$; $G_1(x;x';\sigma)$ is the Green's function satisfying

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-g\delta(x)-\sigma\right]G_1(x;x';\lambda)=\delta(x-x')\ ,$$

and $\chi_e(y)$ are the complete orthonormal set of eigenfunctions of the operator $-(\hbar^2/2m)(\partial^2/\partial y^2)$, i.e., $\chi_e(y)$ are the plane waves $(2\pi)^{-1/2}\exp(ik_yy)$ having energy $E_e=\hbar^2k_y^2/2m$. However, an equally valid expansion, though slightly less convenient for the purpose of arriving at Eq. (2.12), is

$$(H_i-\lambda)^{-1}=\sum_t \phi_t(x)\phi_t^*(x')G_y(y;y';\lambda-E_t)\ , \qquad (3.22b)$$

where the $\phi_t(x)$, having eigenvalue E_t, are the complete orthonormal set of eigenfunctions satisfying Eq. (3.20a), and $G_y(y;y';\lambda)$ is the Green's function satisfying

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}-\lambda\right]G_y(y;y';\lambda)=\delta(y-y')\ . \qquad (3.23a)$$

It is trivial to see [compare the derivation of Eq. (3.10) of paper II] that

$$G_y(y;y';\lambda)=\frac{mi}{\hbar^2\kappa}e^{i\kappa|y-y'|}\ , \qquad (3.23b)$$

where $\kappa=(2m/\hbar^2)^{1/2}\sqrt{\lambda}$, $0<\arg\kappa<\pi$. For $E_t>0$, the eigenfunctions $\phi_t(x)$ in Eq. (3.22b) are precisely the functions $\phi(x;K_x)$ given by Eq. (3.20b) and having energy $E_t=\hbar^2K_x^2/2m$. These $\phi_t(x)$ for positive E_t are not complete, however; the bound state eigenfunction $w(x)$ of Eq. (2.3b) must be included in the expansion (3.22b). This bound state contribution to Eq. (3.22b) is

$$\frac{im\alpha}{\hbar^2}e^{-\alpha(|x|+|x'|)}\frac{e^{i\kappa|y-y'|}}{\kappa}\ , \qquad (3.24a)$$

where, recalling Eq. (2.4a), we now have

$$\kappa=\left[\frac{2m}{\hbar^2}\right]^{1/2}\left[\lambda+\frac{\hbar^2\alpha^2}{2m}\right]^{1/2}$$
$$=\left[\frac{2m}{\hbar^2}\right]^{1/2}\left[E+\frac{\hbar^2\alpha^2}{2m}+i\epsilon\right]^{1/2}\ . \qquad (3.24b)$$

Evidently from Eq. (2.4b), in the limit $\epsilon\to 0$, $\kappa\to k$. Thus for the cases we have examined thus far, namely $x<0<x'$ and $y-y'$ either >0 or <0, the bound state contribution (3.24a) is precisely identical to the residue contribution (3.12b), remembering that $a=x'-x$. It is true that our derivation implies that the residue contribution is present only when Eq. (3.11a) is satisfied, whereas the bound state contribution (3.24a) obtained from (3.22b) is valid for all x,y and x',y'. This difference between the results (3.12b) and (3.24a) is illusory, however. For fixed source point x,y in the second or third quadrant, both (3.12b) and (3.24a) become exponentially small as $x'\to\infty$ in the first or fourth quadrants. In other words, as $\mathbf{r}'=(x')$ approaches infinity in the first or fourth quadrants, the bound state contribution (3.24a)—like the residue contribution (3.12b)— is negligibly small unless the approach to infinity is along directions differing infinitesimally from parallelism with the y' axis. Since these directions lie within the domain satisfying (3.11a) [compare (3.11b)], in the asymptotic regime at infinity the effective asymptotic domains of (3.12b) and (3.24a)—i.e., the x',y' domains at infinity where (3.12b) or (3.24a) must be added to (3.13) to get the correct asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$—are identical. Moreover, the previously interpreted (3.13)—together with this now-demonstrated complete identity between (3.24a) [from (3.22b)] and (3.12b) as $\mathbf{r}'$ becomes infinite along directions in the first or fourth quadrants of the x',y' plane—shows that the asymptotic limit of $G_i^{(+)}(x,y;x',y';E)$ does not contain terms proportional to $e^{-\alpha|u'|}$ or $e^{-\alpha|u|}$ even when $\mathbf{r}'\to\infty$ along directions parallel to the fourth quadrant $u'=0, v'>0$ line in the primed (x',y' plane) analogue of Fig. 1. In other words, recalling Eqs. (2.3b) and (2.9), in the first and fourth $\mathbf{r}'=(x',y')$ quadrants where (3.12b) and (3.13) are pertinent, the asymptotic limit of $G_i^{(+)}(x,y;x',y';E)$ from Eq. (3.2a) (valid for $x<0<x'$ and arbitrary y,y') has a negligibly small projection on bound states of the rearrangement f channel. This result illustrates (and thereby confirms) the assumption—originally made in Ref. 4 and importantly employed in paper I—that an incident channel i Green's function "does not propagate" in rearrangement f channels.

C. Asymptotic behavior for $x<0$, $x'<0$

To this point, we have obtained the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ for $x<0<x'$ only. However, our above-described interpretations of Eqs. (3.12b) and (3.13) make it apparent that the asymptotic behavior of $G_i^{(+)}$ in other x,x' domains now can be written down without having to resort to the somewhat complicated, time-consuming steepest descents analysis used to derive Eqs. (3.12b) and (3.13). In particular, consider the case that x and x' are both <0. Suppose for the moment that the source S at (x,y) lies in the third quadrant and the receiving point P lies in the second quadrant, as shown in Fig. 5(b). Now two rays reach P from S: a direct ray SP and a ray SQP which arrives at P via a mirrorlike (angle of reflection equals angle of incidence) reflection at Q. The direct ray contribution in the asymptotic ($x',y'\to\infty$) limit is precisely our previously obtained Eq. (3.18). The reflected ray appears to be reaching P from the image source S' at $(-x,y)$, but its amplitude is reduced by the reflection coefficient at Q of the incident ray SQ. This reflection coefficient still is R_r of Eq. (3.21); however, K_x,K_y in the ray SQ initially incident on the mirror are now given by [refer to Fig. 5(b)]:

$$(K_x,K_y)=(K\cos\gamma',\ K\sin\gamma')$$
$$=\left[K\frac{a'}{(a'^2+b^2)^{1/2}},K\frac{b}{(a'^2+b^2)^{1/2}}\right]\ ,$$
$$a'=|x'+x|\ ,\quad b=y'-y\ , \qquad (3.25)$$

instead of the previous (3.19).

Therefore, for $x<0, x'<0$, the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ as $x',y'\to\infty$ in the second quadrant should be

$$\frac{im}{\hbar^2\sqrt{2\pi K}}e^{-i\pi/4}\left[\frac{e^{iK\rho}}{\sqrt{\rho}}-\frac{\alpha}{\alpha+iK|x'+x|/\rho_r}\frac{e^{iK\rho_r}}{\sqrt{\rho_r}}\right], \qquad (3.26a)$$

$$\rho_r=[(x'+x)^2+(y'-y)^2]^{1/2}\ , \qquad (3.26b)$$

plus—when $x',y'\to\infty$ along directions differing infinitesimally from directions parallel to the y' axis—the bound state contribution (3.24a) with $K=k$; in (3.26a) ρ is defined by Eq. (3.17b).

It is evident from Fig. 5(b) and our previous interpretation of Eqs. (3.12b) and (3.13) that the above result (3.26a) [supplemented near the y' axis by (3.24a)] also holds when $x<0, x'<0$ and $x',y'\to\infty$ in the third quadrant. It is similarly evident that—in addition to the always applicable bound state contribution (3.24a) near the y' axis—the asymptotic behavior as $x',y'\to\infty$ of $G_i^{(+)}(x,y;x',y';E)$ when $x>0, x'>0$ must be identical in every respect with the result (3.26a), while the asymptotic behavior when $x'<0<x$ must be identical with the result (3.13). As it happens, we do not require the results for $x>0$ in the calculations described in later sections of this paper. We do not use the result (3.26a), however. Therefore, to obviate any possible doubt about the validity of our almost trivial ray-geometric prescription which yielded (3.26a), we shall verify (3.26a) by carrying through the steepest descents estimate of $G_i^{(+)}(x,y;x',y';E)$ when $x<0, x'<0$.

Our starting point, replacing our previously employed (3.1), now is Eq. (2.12a). We first observe that the first term on the right-hand side of Eq. (2.12a) [the term without the factor $\alpha(\alpha+i\rho)^{-1}$] can be integrated exactly, without need for a steepest descents estimate. The free space Green's function $G_0^{(+)}$ of Eq. (3.16a) is defined by

$$G_0^{(+)}(x,y;x',y';E)=\lim_{\epsilon\to 0} G_0(x,y;x',y';E+i\epsilon)\ , \tag{3.27}$$

where $\epsilon>0$ and where the Green's function $G_0(E+i\epsilon)$ on the right-hand side of Eq. (3.27) satisfies Eq. (3.16a) with $E+i\epsilon$ replacing E. With this understanding, and recognizing that the eigenfunctions $e^{ik_x x}e^{ik_y y}$ of H_0 in Eq. (3.16b) form a complete set, we infer from Eq. (3.16a) that

$$G_0(x,y;x',y';E+i\epsilon)=\frac{1}{(2\pi)^2}\frac{2m}{\hbar^2}\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_x\frac{e^{ik_y(y-y')}e^{ik_x(x-x')}}{k_x^2+k_y^2-\kappa^2}\ , \tag{3.28}$$

where, much as in Eq. (3.23b), κ now equals $(2m/\hbar^2)^{1/2}(E+i\epsilon)^{1/2}$, $0<\arg\kappa<\pi$.

The integrals in Eq. (3.28) run along the real k_x and real k_y axes. As a function of k_x, the poles of the integrand lie at $k_x=\pm(\kappa^2-k_y^2)^{1/2}$, where the phase of this square root still must be specified as a function of k_y. In the k_y plane, $(\kappa^2-k_y^2)^{1/2}$ has branch points at $\pm\kappa$; since k_y in Eq. (3.28) runs over all real k_y, the branch cuts through $k_y=\pm\kappa$ must not intersect the real k_y axis. Recall that for simplicity we have limited ourselves to $k^2>\alpha^2$, i.e., $E>0$. Then, κ lies in the first quadrant for any $\epsilon>0$, and $-\kappa$ lies in the third quadrant; correspondingly, cuts in the k_y plane running up from κ and down from $-\kappa$ will not cross the real k_y axis. With the cuts so drawn, the phase of $(\kappa^2-k_y^2)^{1/2}$ is specified for all k_y by the prescription, much as in Eq. (2.13b), that near $k_y=0$ and for ϵ infinitesimal (i.e., for $\arg\kappa$ infinitesimally >0),

$$\arg(\kappa-k_y)\cong\arg(k_y+\kappa)\cong 0\ . \tag{3.29a}$$

Indeed, by the reasoning given in Eqs. (3.13b)—(3.16c) of paper II, it now can be seen that

$$0<\arg(\kappa^2-k_y^2)^{1/2}<\pi/2 \tag{3.29b}$$

for all purely real k_y. Therefore it now is evident that for $x-x'>0$ (<0) the integral over k_x in Eq. (3.28) can be performed by closing the contour in the upper (lower) half k_x plane, and then computing the residue at $k_x=(\kappa^2-k_y^2)^{1/2}$ [$k_x=-(\kappa^2-k_y^2)^{1/2}$]. We find that, for either sign of $x-x'$, Eq. (3.28) reduces to

$$G_0(x,y;x',y';E+i\epsilon)=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_{-\infty}^{\infty}dk_y\frac{1}{\bar{p}}e^{ik_y(y-y')}e^{i\bar{p}|x-x'|}\ , \tag{3.30}$$

where $\bar{p}=(\kappa^2-k_y^2)^{1/2}$. As $\epsilon\to 0$, $\kappa\to(k^2-\alpha^2)^{1/2}$ by Eq. (2.4b); the cuts at $\pm\kappa$ coincide with the points Q_1,Q_2 of Fig. 2, and the integral along the real k_y axis must be deformed around Q_1,Q_2 as on the contour Γ of Fig. 2. There are no poles at $k_y=\pm k$ in the integrand of Eq. (3.30), but, of course, there is no reason why the contour for the integral (3.30) cannot follow Γ at $k_y=\pm k$ as well as at $k_y=\pm(k^2-\alpha^2)^{1/2}$. In other words, the integration contour over all real k_y in Eq. (3.30) deforms to Γ of Fig. 2 as $\epsilon\to 0$. Simultaneously, for every k_y on Γ, $\bar{p}$ in Eq. (3.30) approaches p of Eq. (2.13a) in magnitude and phase as $\epsilon\to 0$. We conclude that the first term on the right-hand side of Eq. (2.12a) is precisely $G_0^{(+)}(x,y;x',y';E)$ defined by Eq. (3.27) and given in explicit closed form by Eq. (3.17a).

As for the second term on the righ-hand side of Eq. (2.12a), introducing the same new integration variable $u=k_y/K$ employed in Eq. (3.1) obviously reduces this second term to

$$-\frac{1}{2\pi}\frac{mi\alpha}{\hbar^2}\int_{-\infty}^{\infty}du\frac{e^{iKf(u)}}{(1-u^2)^{1/2}[\alpha+iK(1-u^2)^{1/2}]}\ , \tag{3.31}$$

where $f(u)$ is defined by Eq. (3.2b), now with $a=|x+x'|$ and $b=y-y'$; for definiteness we take $b>0$, i.e., $x',y'\to\infty$ in the third quadrant for fixed x,y (recall now $x<0, x'<0$), but it is apparent from our previous discussion of Eq. (3.15) that the steepest descents result for $y-y'>0$ also immediately yields the asymptotic behavior of (3.31) when $x',y'\to\infty$ in the second quadrant. The contour for Eq. (3.31) is as drawn in Fig. 3, and the poles and branch cuts for Eq. (3.31) also are as previously found for Eq. (3.2a) and shown in Fig. 3. Therefore the steepest descents contour for Eq. (3.31) is reached by the same contour deformation as previously.

Thus, comparing Eqs. (3.2a) and (3.31), it is apparent that—excluding the residue contribution—the steepest descents contribution to (3.31) is

$$J'=\frac{i\alpha}{K}\frac{1}{(1-u_s^2)^{1/2}}J\ , \tag{3.32}$$

where J is given by Eq. (3.13) and $u_s=b(a^2+b^2)^{-1/2}$ as before. Substituting in Eq. (3.32), it is seen that J' of Eq. (3.32) is identical to the second $\rho_r^{-1}\exp(iK\rho_r)$ term in Eq. (3.26a). Similarly, the residue contribution to (3.31), using (3.12a), is

$$\mathscr{R}'=\frac{i\alpha}{K}\frac{1}{(1-k^2/K^2)^{1/2}}\mathscr{R} , \tag{3.33}$$

where $\mathscr{R}$ is precisely the previously obtained residue contribution (3.12b). Of course, as previously this residue contribution is obtained only when the inequality (3.11b) holds. In short, recalling our result—via Eq. (3.30)—for the first term on the right-hand side of Eq. (2.12a), the steepest descents estimate of $G_i^{(+)}(x,y;x',y';E)$ for $x<0, x'<0$ is exactly as deduced in Eq. (3.26a) from our ray-geometric prescription, supplemented (for $x',y'\rightarrow\infty$ along directions nearly parallel to the y' axis) by the bound state contribution (3.24a) with $K=k$.

IV. SURFACE INTEGRALS AT INFINITY

Now that the asymptotic behavior of $G_i^{(+)}(x,y;x',y';E)$ at large x',y' has been deduced, we can evaluate the values of the various surface integrals at infinity (in x',y' space) discussed in paper I and Ref. 4. These surface integrals all involve expressions of the form

$$\mathscr{I}[X,Y]=\int d\mathbf{r}'[Y(\mathbf{r}')T(\mathbf{r}')X(\mathbf{r}')-X(\mathbf{r}')T(\mathbf{r}')Y(\mathbf{r}')] , \tag{4.1}$$

where $T(\mathbf{r}')$ is the kinetic energy operator in the incident channel Hamiltonian $H_i(\mathbf{r}')$ appearing in the LS equation (1.4), or, equivalently, $T(\mathbf{r}')$ is the free space (interaction-free) Hamiltonian $H_0(\mathbf{r}')$. The integral (4.1) is integrated over the entire volume of the configurations space specified by the particle coordinates employed in H (or H_i); the volume integral over all space always can be converted to a surface integral at infinity, however, using Green's theorem (or, equivalently, integration by parts). For reasons discussed in paper I and Ref. 10, the volume integrals over all space in Eq. (4.1) are best regarded as the limits $R\rightarrow\infty$ of integrals over spherical volumes of radius R; therefore the corresponding surface integrals at infinity, after application of Green's theorem, should be similarly regarded as the limits $R\rightarrow\infty$ of integrals over spherical surfaces of radius R.

The surface integral relations we wish to verify are stated in Eqs. (2.14)–(2.17) of paper I. In the McGuire model these relations take the following forms:

$$\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \psi_i(x',y')]=\psi_i(x,y) , \tag{4.2a}$$

$$\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \Psi_i(x',y')]=\psi_i(x,y) , \tag{4.2b}$$

$$\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \psi_f(x',y')]=\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \Psi_f(x',y')]=0 , \tag{4.2c}$$

$$\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \Phi_i(x',y')]=\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\ \Phi_f(x',y')]=0 , \tag{4.2d}$$

where the quantities $G_i^{(+)}$, ψ_i, Ψ_i, ψ_f, and Ψ_f all have been defined above [e.g., by Eq. (2.8)], where $\Phi_i=\Psi_i-\psi_i$ and $\Phi_f=\Psi_f-\psi_f$, and the surface integrals are over the spherical surface at infinity in $\mathbf{r}'=(x',y')$ space, of course; $T(\mathbf{r}')$ now is the primed version of H_0 from Eq. (3.16b), and the "sphere" at infinity now is the circle of radius R ($R\rightarrow\infty$) in the primed version of the x,y plane shown in Fig. 1. The outward drawn normal at any point on this "sphere" lies along the radius vector from the origin to that point. Thus, introducing polar coordinates in the plane ($x'=r'\cos\phi'$, $y'=r'\sin\phi'$), in the McGuire model Eq. (4.1) reduces to

$$\mathscr{I}[X,Y]=\lim_{R\rightarrow\infty}-\frac{\hbar^2}{2m}\int_0^{2\pi}d\phi' r'\left[Y\frac{\partial X}{\partial r'}-X\frac{\partial Y}{\partial r'}\right] , \tag{4.3}$$

where the integral in Eq. (4.3) is along the arc of the circle $r'=R$.

We will apply Eq. (4.3) to the left-hand sides of Eqs. (4.2). For the present and until further notice we take $x<0$ in Eq. (4.3), which means that we can employ the specific asymptotic formulas for $G_i^{(+)}(x,y;x',y';E)$ that were displayed in the preceding section. Because these asymptotic formulas differ for $x'<0$ and $x'>0$, the reader is reminded that in Eq. (4.3) the interval $\pi/2<\phi'<3\pi/2$ encompasses the domain $x'<0$ where Eq. (3.26a) applies, whereas Eq. (3.13) must be used in the intervals $0<\phi'<\pi/2$ and $3\pi/2<\phi'<2\pi$ where $x'>0$. We also remark (as will be obvious from what follows) that in Eqs. (3.13) and (3.26a) we can replace ρ and ρ_r from Eqs. (3.17b) and (3.26b) by their leading terms, which simplifies the applications of Eq. (4.3) to the surface integrals of Eqs. (4.2). Specifically, as $x',y'\rightarrow\infty$ for fixed x,y it is sufficient for our present purposes to employ

$$\rho=r'-(x\cos\phi'+y\sin\phi')+\cdots , \tag{4.4a}$$

$$\rho_r=r'+(x\cos\phi'-y\sin\phi')+\cdots . \tag{4.4b}$$

Now let us evaluate the left-hand side of Eq. (4.2a). Consider first the corresponding contribution to the integral (4.3) from the interval $\pi/2<\phi'<3\pi/2$. Employing Eqs. (4.4) in Eq. (3.26a), at large r' in this interval,

$$G_i^{(+)}\cong\frac{im}{\hbar^2\sqrt{2\pi K}}e^{-i\pi/4}\frac{e^{iKr'}}{\sqrt{r'}}\times\left[e^{-iK(x\cos\phi'+y\sin\phi')}-\frac{\alpha}{\alpha-iK\cos\phi'}e^{iK(x\cos\phi'-y\sin\phi')}\right] , \tag{4.5a}$$

ignoring for the moment the residue contribution to $G_i^{(+)}$. In this same interval, recalling Eqs. (2.3),

$$\psi_i(x',y')=\sqrt{\alpha}e^{\alpha r'\cos\phi'}e^{ikr'\sin\phi'},\quad \pi/2<\phi'<3\pi/2 . \tag{4.5b}$$

Thus (still for $x<0$ and still excluding the residue component of $G_i^{(+)}$), the contribution to the left-hand side of Eq. (4.2a) from the domain $x'<0$ is

$$\lim_{r'=R\rightarrow\infty}-\frac{\hbar^2}{2m}\int_{\pi/2}^{3\pi/2}d\phi' r'\frac{im}{\hbar^2\sqrt{2\pi K}}e^{-i\pi/4}\sqrt{\alpha}e^{\alpha r'\cos\phi'}e^{ikr'\sin\phi'}\times\frac{e^{iKr'}}{\sqrt{r'}}\left[e^{-iK(x\cos\phi'+y\sin\phi')}-\frac{\alpha}{\alpha-iK\cos\phi'}e^{iK(x\cos\phi'-y\sin\phi')}\right][iK-(\alpha\cos\phi'+ik\sin\phi')] , \tag{4.6a}$$

plus terms in the integrand of higher order in $1/r'$. The essential point to note is that the integral in (4.6a) is of the form

$$\mathscr{M}=\sqrt{r'}e^{iKr'}\int_{\pi/2}^{3\pi/2}d\phi' e^{\alpha r'\cos\phi'}e^{ikr'\sin\phi'}F(x,y,\phi')\ , \tag{4.6b}$$

where F is bounded, well behaved, and independent of r'. In the domain $\pi/2<\phi'<3\pi/2$, $\cos\phi'<0$ and $\exp(\alpha r'\cos\phi')$ vanishes as $r'\to\infty$. Thus the only possibly nonvanishing contribution to (4.6b) in the limit $r'\to\infty$ comes from the immediate neighborhood of $\phi'=\pi/2$ and $3\pi/2$, where $\cos\phi'\to 0$.

In fact, Eq. (4.6b) implies

$$|\mathscr{M}|\le\hat{C}\sqrt{r'}\int_{\pi/2}^{3\pi/2}d\phi' e^{\alpha r'\cos\phi'}\ , \tag{4.7a}$$

where $\hat{C}$ is some finite number. With the change of variable $\phi'=\pi/2+\gamma$, we obtain

$$\begin{aligned}\int_{\pi/2}^{3\pi/2}d\phi' e^{\alpha r'\cos\phi'}&=\int_0^{\pi}d\gamma\, e^{-\alpha r'\sin\gamma}\\ &=2\int_0^{\pi/2}d\gamma\, e^{-\alpha r'\sin\gamma}\ ,\end{aligned} \tag{4.7b}$$

taking advantage of the symmetry of $\sin\gamma$ in $0<\gamma<\pi$. However, in the domain $0\le\gamma\le\pi/2$, $(\sin\gamma)/\gamma$ is monotonically decreasing, as is easily proved. In Eq. (4.7b), therefore,

$$\begin{aligned}&\sin\gamma\ge 2\gamma/\pi\ ,\\ &e^{-\alpha r'\sin\gamma}\le e^{-2\alpha r'\gamma/\pi}\ .\end{aligned} \tag{4.7c}$$

Consequently, combining Eqs. (4.7),

$$|\mathscr{M}|\le 2\hat{C}\sqrt{r'}\int_0^{\pi/2}d\gamma\, e^{-2\alpha r'\gamma/\pi}=\frac{\hat{C}\pi}{\alpha\sqrt{r'}}(1-e^{-\alpha r'})\ . \tag{4.8}$$

It follows that the integral in (4.6a) vanishes as $O(1/\sqrt{r'})$ as $r'\to\infty$, i.e., that (still ignoring the residue component of $G_i^{(+)}$) the interval $\pi/2<\phi'<3\pi/2$ makes a vanishing contribution to the left-hand side of Eq. (4.2a). Obviously, this result would not have been altered by retention of higher order terms $O(1/r')$ in Eqs. (4.4). Equally obviously, the contribution to the left-hand side of Eq. (4.2a) from the intervals $0<\phi'<\pi/2$ and $3\pi/2<\phi'<2\pi$ (where $x'>0$) also vanishes when the residue component of $G_i^{(+)}$ is excluded; the asymptotic form (3.13) shares with Eq. (3.26a) the fundamental property that leads to the aforesaid vanishing, namely that as $r'\to\infty$ the asymptotic behavior of $G_i^{(+)}$ is—except for r'-independent factors—proportional to $(r')^{-1/2}\exp(iKr')$, as in Eq. (4.5a). Of course, for $x'>0$, Eqs. (2.3) yield

$$\psi_i(x',y')=\sqrt{\alpha}e^{-\alpha x'}e^{iky'}=\sqrt{\alpha}e^{-\alpha r'\cos\phi'}e^{ikr'\sin\phi'}\ ,\quad 0<\phi'<\pi/2,\ \ 3\pi/2<\phi'<2\pi\ , \tag{4.9}$$

instead of Eq. (4.5b); Eq. (4.9) again guarantees that $\psi_i(x',y')$ is exponentially decreasing at large r' in the integration intervals to which Eq. (4.9) pertains.

In other words, any nonvanishing contribution to the left-hand side of Eq. (4.2a) must come from the residue component of $G_i^{(+)}$, which to this point has been ignored in the evaluation of (4.2a). We have seen that for all values of x,y,x',y' this residue component is given by Eq. (3.24a) with $K=k$. In particular, at large r' this residue component of $G_i^{(+)}$ is (remembering that $x<0$)

$$\frac{im}{\hbar^2}\frac{\alpha}{k}e^{\alpha x}e^{-iky}e^{-i\alpha r'\cos\phi'}e^{ikr'\sin\phi'},\quad 0<\phi'<\pi/2\ , \tag{4.10a}$$

$$\frac{im}{\hbar^2}\frac{\alpha}{k}e^{\alpha x}e^{-iky}e^{\alpha r'\cos\phi'}e^{ikr'\sin\phi'},\quad \pi/2<\phi'<\pi\ , \tag{4.10b}$$

$$\frac{im}{\hbar^2}\frac{\alpha}{k}e^{\alpha x}e^{iky}e^{\alpha r'\cos\phi'}e^{-ikr'\sin\phi'},\quad \pi<\phi'<3\pi/2\ , \tag{4.10c}$$

$$\frac{im}{\hbar^2}\frac{\alpha}{k}e^{\alpha x}e^{iky}e^{-\alpha r'\cos\phi'}e^{-ikr'\sin\phi'},\quad 3\pi/2<\phi'<2\pi\ . \tag{4.10d}$$

Employing now Eqs. (4.5b), (4.9), and (4.10) in Eq. (4.3) as applied to Eq. (4.2a), we find that the residue contribution to the left-hand side of Eq. (4.2a) is

$$-\alpha\sqrt{\alpha}r'e^{\alpha x}e^{iky}\left[\int_{\pi}^{3\pi/2}d\phi' e^{2\alpha r'\cos\phi'}\sin\phi' + \int_{3\pi/2}^{2\pi}d\phi' e^{-2\alpha r'\cos\phi'}\sin\phi'\right]\ , \tag{4.11}$$

in the limit $r'=R'\to\infty$. There is no contribution to Eq. (4.11) from the first and second quadrants $(0<\phi'<\pi)$ because in those quadrants $\psi_i(x',y')$ and the residue component of $G_i^{(+)}$ depend identically on x',y'; in the third and fourth quadrants these x',y' dependences are not identical [compare, e.g., Eqs. (4.10c) and (4.5b)]. Evaluating the integrals in (4.11), we see that the left-hand side of Eq. (4.2a) reduces to

$$\begin{aligned}\mathscr{I}[G_i^{(+)}(x,y;x',y';E),\psi_i(x',y')]&=\lim_{r'=R\to\infty}-\alpha\sqrt{\alpha}r'e^{\alpha x}e^{iky}\left[\frac{e^{2\alpha r'\cos\phi'}}{-2\alpha r'}\Bigg|_{\pi}^{3\pi/2}+\frac{e^{-2\alpha r'\cos\phi'}}{2\alpha r'}\Bigg|_{3\pi/2}^{2\pi}\right]\\ &=\sqrt{\alpha}e^{\alpha x}e^{iky}=\psi_i(x,y),\quad x<0\ .\end{aligned} \tag{4.12}$$

In connection with the result (4.12), we note the following. In the limit $r'\to\infty$, the nonvanishing contributions to the integrals in (4.11) come solely from the immediate neighborhood of $\phi'=3\pi/2$, where $\cos\phi'\sim 0$; therefore, consistent with the discussion following Eqs. (3.24), Eq. (4.11) has not been made erroneous by the fact that the integrals in (4.11) run over the entire angular range π to 2π, rather than over angular ranges falling within the steepest descents criterion (3.11b) for appearance of the residue contribution in the asymptotic limit of $G_i^{(+)}$. Our interpretation of this residue contribution, starting from the expansion (3.22a), suggests that the residue contribution has been obtained exactly, without neglect of higher order terms; $\psi_i(x',y')$ in Eqs. (4.5b) and (4.9) unquestionably is exact. Moreover, Eqs. (4.10), as well as Eqs. (4.5b) and (4.9), have not involved any "leading term" approximations [such as Eqs. (4.4)] to ψ_i or the residue component of $G_i^{(+)}$. Thus it appears that (4.11) is an "exact" evaluation of the contribution the residue component of $G_i^{(+)}$ makes to the left-hand side of Eq. (4.2a). Even if Eqs. (4.10a) represent only the leading terms to the residue component of $G_i^{(+)}$, however [i.e., even if Eq. (3.12b) has neglected higher order terms which are, e.g., $O(1/\sqrt{r'})$ compared to the right-hand side of (3.12b)], it is evident from our final result (4.12) that those higher order terms would make a vanishing contribution to the left-hand side of Eq. (4.2a) in the limit $r'\to\infty$. Equation (4.12) has been derived for $x<0$ only; but the obvious symmetry (also remarked on in paper II) of the McGuire model under reflection about the $x=0$ axis, as manifested in Eqs. (2.3) and (2.12), makes it obvious that our verification of Eq. (4.2a) for $x<0$ will carry through to the case $x>0$. We conclude that the predicted surface integral relation (4.2a) does hold in the McGuire model.

With the foregoing detailed verification of Eq. (4.2a) in hand, it is easy to see that Eqs. (4.2b) and (4.2c) also will hold in the McGuire model; it is sufficient to sketch the analysis. Consider first Eq. (4.2b), where Ψ_i is given by Eqs. (2.8) and is everywhere bounded. Thus, from the derivation leading to Eq. (4.6a) and thence to Eq. (4.8), it is evident that the ray-geometric saddle point portion of $G_i^{(+)}$ [as displayed, e.g., in Eq. (4.5a)] will make a contribution to the left-hand side of Eq. (4.2b) which—after integrating over q'—will be $O(1/\sqrt{r'})$ as in Eq. (4.8), i.e., will vanish as $r'\to\infty$. Once again, therefore, the residue component of $G_i^{(+)}$ makes the only possibly nonvanishing contribution to the left-hand side of Eq. (4.2b). This residue component is exponentially decreasing as $r'\to\infty$ at fixed ϕ', except at angles ϕ' very near $\phi'=\pi/2$ and $\phi'=3\pi/2$ [recall Eqs. (4.10)]. Hence, referring to Fig. 1, in evaluating the left-hand side of Eq. (4.2b) we need be concerned only with $\Psi_{i\mathrm{I}}$ and $\Psi_{i\mathrm{II}}$ (needed near $\phi'=\pi/2$) and $\Psi_{i\mathrm{V}}$ and $\Psi_{i\mathrm{VI}}$ (needed near $\phi'=3\pi/2$). Apart from the constant factor A, $\Psi_{i\mathrm{II}}$ (the appropriate form of Ψ_i near $\phi'=\pi/2$ in the first quadrant) is precisely ψ_i, and similarly for $\Psi_{i\mathrm{I}}$ (the appropriate form of Ψ_i near $\phi'=\pi/2$ in the second quadrant). As before, assume for the present that $x<0$, so that Eqs. (4.10) explicitly display the ϕ' dependence of the residue component of $G_i^{(+)}$ at large r'. Then, just as in Eq. (4.11), there is no residue contribution to the left-hand side of Eq. (4.2b) from the neighborhood of $\phi'=\pi/2$, because in that neighborhood $\Psi_i(x',y')$, like $\psi_i(x',y')$, has the same dependence on x',y' as the residue component of $G_i^{(+)}$. Near $\phi'=3\pi/2$, $\Psi_{i\mathrm{V}}$ and $\Psi_{i\mathrm{VI}}$ are the appropriate forms of Ψ_i in the third and fourth quadrants, respectively. Each of $\Psi_{i\mathrm{V}}$ and $\Psi_{i\mathrm{VI}}$ equals ψ_i plus terms proportional to the constants C and D, as displayed in Eqs. (2.8); from Eqs. (2.5) of paper II, or even more simply from the directions of positive r and u shown in Fig. 1, these C and D terms in $\Psi_{i\mathrm{V}}$ and $\Psi_{i\mathrm{VI}}$ are seen to vanish exponentially as $x',y'\to\infty$ in the neighborhood of $\phi'=3\pi/2$. In other words, at infinity near $\phi'=3\pi/2$, $\Psi_i(x',y')$ becomes identical with $\psi_i(x',y')$; correspondingly, near $\phi'=3\pi/2$ the evaluation of the residue component contribution to the left-hand side of Eq. (4.2b) becomes identical with the calculation carried through in Eqs. (4.11) and (4.12). In this fashion Eq. (4.2b) is verified for $x<0$; that Eq. (4.2b) will also hold for $x>0$ then follows from the same McGuire model symmetry argument employed in the verification of Eq. (4.2a).

As for Eq. (4.2c), by now it is evident that a nonvanishing contribution to $\mathscr{I}[G_i^{(+)},\ \psi_f]$ and $\mathscr{I}[G_i^{(+)},\ \Psi_f]$ can come only from the residue component of $G_i^{(+)}$, and then only from integration of the appropriate versions of Eq. (4.3) in the neighborhoods of $\phi'=\pi/2$ and $\phi'=3\pi/2$. But $\psi_f(x',y')$ of Eq. (2.9) vanishes exponentially at infinity, except near $u'=0$, i.e. (referring to Fig. 1), except near $\phi'=5\pi/6$ and $11\pi/6$. Consequently, $\mathscr{I}[G_i^{(+)},\psi_f]=0$. Turning next to $\mathscr{I}[G_i^{(+)},\Psi_f]$, and again referring to Fig. 1, the relevant forms of Ψ_f near $\phi'=\pi/2$ are $\Psi_{f\mathrm{I}}$ (second quadrant) and $\Psi_{f\mathrm{II}}$ (first quadrant); near $\phi'=3\pi/2$ the relevant forms are $\Psi_{f\mathrm{V}}$ (third quadrant) and $\Psi_{f\mathrm{VI}}$ (fourth quadrant). From Eqs. (2.10), the only terms in $\Psi_{f\mathrm{I}}$ and $\Psi_{f\mathrm{II}}$ which do not vanish exponentially near $\phi'=\pi/2$ are the terms proportional to C; in their respective quadrants these portions of $\Psi_{f\mathrm{I}}$ and $\Psi_{f\mathrm{II}}$ are precisely $C\psi_i$. Hence, just as in the calculation of $\mathscr{I}[G_i^{(+)},\Psi_i]$, the neighborhood of $\phi'=\pi/2$ makes a vanishing contribution to $\mathscr{I}[G_i^{(+)},\Psi_f]$. Near $\phi'=3\pi/2$, both $\Psi_{f\mathrm{V}}$ and $\Psi_{f\mathrm{VI}}$ are exponentially decreasing at infinity, because neither of these forms of Ψ_f contain a component proportional to ψ_i, as did $\Psi_{i\mathrm{V}}$ and $\Psi_{i\mathrm{VI}}$. Consequently, $\mathscr{I}[G_i^{(+)},\ \Psi_f]$ also equals 0, i.e., Eq. (4.2c) has been verified. For $\mathscr{I}[G_i^{(+)},\ \Psi_f]$ (but not for $\mathscr{I}[G_i^{(+)},\ \psi_f]$) this verification pertains only to $x<0$, since in concluding that there is no residue component contribution to $\mathscr{I}[G_i^{(+)},\Psi_f]$ near $\phi'=\pi/2$ we have made explicit use of the fact that Eqs. (4.10a) and (4.10b) are proportional to $\psi_i(x',y')$ in their respective quadrants; as we have explained previously, Eqs. (4.10) hold only for $x<0$. From Eq. (3.24a), however, it is readily seen that the x',y' dependences (though not the x,y dependences) of the residue component of $G_i^{(+)}$ are the same for $x>0$ as for $x<0$. Thus for $x>0$, as for $x<0$, the neighborhood of $\phi'=\pi/2$ makes a vanishing contribution to $\mathscr{I}[G_i^{(+)},\Psi_f]$, i.e., Eq. (4.2c) holds for all x. Recalling that $\Phi_i=\Psi_i-\psi_i$, subtracting Eq. (4.2b) from (4.2a) immediately yields $\mathscr{I}[G_i^{(+)},\Phi_i]=0$, and similarly for the remaining equality in Eq. (4.2d).

The foregoing completes the verification of Eqs. (4.2). In paper I it was shown that the relations (4.2d) can be deduced (in any two-body collision involving three three-dimensional particles interacting via short range potentials, not merely in the McGuire model) from precise formulations of the requirements that Φ_i, Φ_f—the "scattered parts" of Ψ_i,Ψ_f, respectively—must be "everywhere outgoing at infinity"; these formulations are embodied in Eqs. (2.54) and (2.59a) of paper I. In paper I and Ref. 4, however, it also is argued that the relations (4.2d) are to be expected, because one expects that when both X and Y are "everywhere outgoing at infinity" the general three-particle analogue of the McGuire model integrand in Eq. (4.3) will cancel at every point on the surface at infinity in configuration space. This argument is completely borne out by the McGuire model results obtained in this section. As we have chosen ψ_i in Eq. (2.3a), it advances along the positive y direction, i.e., $\psi_i(x',y')$ is incoming at $\phi'=3\pi/2$ in Fig. 1, and is outgoing at $\phi'=\pi/2$. Correspondingly, as we have seen, the entire contribution to the left-hand side of Eq. (4.2a) comes from $\phi'=3\pi/2$. The above result that $\mathscr{I}[G_i^{(+)},\psi_f]=0$ even though $\psi_f(x',y')$ of Eq. (2.9) is incoming at $\phi'=5\pi/6$ illustrates the additional argument, also made in paper I and Ref. 4, that where Y represents propagation in a bound state, the surface integral $\mathscr{I}[G_i^{(+)},Y]$ will vanish unless $G_i^{(+)}$ itself can propagate in that bound state, i.e., unless $G_i^{(+)}$ has a non-negligible projection on the same bound state. The McGuire $G_i^{(+)}$ of Eqs. (2.11) and (2.12) cannot be expected to propagate in the bound state $w(u)$ of Eq. (2.9), because in the i-channel Hamiltonian H_i of Eq. (2.5b) the interaction $g(u)$—which appears in the complete Hamiltonian of Eq. (2.2) and which is needed to maintain $w(u)$—has been dropped.

Indeed, our steepest descents estimate of $G_i^{(+)}$ in Sec. III has explicitly demonstrated that, at large x',y', $G_i^{(+)}(x,y;x',y';E)$ does propagate in ψ_i but does not propagate in ψ_f; only one pole can be crossed in deforming the integration contour of Fig. 3 to the steepest descents contour of Fig. 4, and the residue at this pole is proportional to $\psi_i(x',y')$, not to $\psi_f(x',y')$. The reader will note, moreover, that our derivation of $G_i^{(+)}$ in paper II, leading to Eqs. (2.12), did not use an expansion for $G_i^{(+)}$ wherein the bound state contribution of ψ_i to $G_i^{(+)}$ is immediate and explicit, as, e.g., would have been the case had we chosen to derive Eqs. (2.12) from the expansion (3.22a); in other words, our finding that the steepest descents estimate of $G_i^{(+)}$ contains a residue contribution proportional to ψ_i but not to ψ_f is not explainable as an artifact of having initially used an expansion for $G_i^{(+)}$ obviously favoring projections on ψ_i over projections on ψ_f. These considerations (and the considerations in the preceding paragraph) also make understandable the result that the portion of $G_i^{(+)}$ behaving asymptotically like the free space Green's functions $G_0^{(+)}$ of Eqs. (3.16) and (3.17) makes no contribution to $\mathscr{I}[G_i^{(+)},\psi_i]$ [recall Eqs. (4.5)–(4.8)]; $G_0^{(+)}$, defined in terms of an interaction-free Hamiltonian, cannot propagate in any bound states. It now also is apparent that, for any fixed x,y and x'',y'',

$$\mathscr{I}[G_i^{(+)}(x,y;x',y';E),G_0^{(+)}(x'',y'';x',y';E)]=\lim_{r'=R\to\infty}-\frac{\hbar^2}{2m}\int_0^{2\pi}d\phi' r'\left[G_0^{(+)}\frac{\partial G_i^{(+)}}{\partial r'}-G_i^{(+)}\frac{\partial G_0^{(+)}}{\partial r'}\right]=0\;, \qquad (4.13)$$

because $G_0^{(+)}$ does not propagate in bound states and because at all angles ϕ' on the circle at infinity in x',y' space the ray-geometric saddle point portion of the everywhere outgoing $G_i^{(+)}(x,y;x',y';E)$ has the same r' dependence in lowest order—namely $\sim e^{iKr'}/\sqrt{r'}$—as does the everywhere outgoing $G_0^{(+)}(x'',y'';x',y';E)$ [compare Eqs. (3.18), (3.26), and (4.5a)], so that the integral in (4.13) cancels in lowest (and the only possibly nonvanishing) order.

Equation (4.13) is a McGuire model illustration of the general claim embodied in Eq. (2.60c) of paper I, namely that $\mathscr{I}[X,Y]$ vanishes whenever X and Y are outgoing Green's functions at real energy, for systems of three three-dimensional (not merely one-dimensional) particles interacting via short range forces. It was shown in Ref. 4 that Eq. (4.13) implies

$$G_i^{(+)}(\mathbf{r}';\mathbf{r}'';E)=G_0^{(+)}(\mathbf{r}',\mathbf{r}'';E)-\int d\mathbf{r}\,G_0^{(+)}(\mathbf{r};\mathbf{r}'';E)\times V(\mathbf{r})G_i^{(+)}(\mathbf{r},\mathbf{r}';E)\;; \qquad (4.14)$$

where $V\equiv H_i-H_0$, and where $G_i^{(+)}$ and $G_0^{(+)}$ again pertain to systems of three three-dimensional particles interacting via short range forces; Eq. (4.14) is a particular example of Eq. (2.60b) of paper I. Benoist-Gueutal's criticisms[9] of past derivations of the inhomogeneous and homogeneous LS equations (1.1) and (1.2) rest in part on explicitly expressed doubts about relations (between real energy Green's functions) such as (4.14) or Eq. (2.60b) of paper I. The results of this section mean that, in the McGuire model at least, doubts about Eq. (4.14) are unfounded.

In short, the present section of this paper thoroughly confirms—in the McGuire model at least—the predictions made in Ref. 4 and paper I concerning the values of various relevant surface integrals at infinity $\mathscr{I}[X,Y]$ which appear in straightforward derivations of the real energy LS equation (1.2) from the Schrödinger equation (1.5); correspondingly, this section confirms the qualitative reasoning underlying Gerjuoy's predictions concerning the values of $\mathscr{I}[X,Y]$. The implications of these confirmations, which already have been stated in the Introduction, need not be repeated in this section.

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