Class of Jost-like functions

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A class of functions having the same analytic properties as the Jost function is introduced. In momentum space these functions can be calculated through solutions of some auxiliary integral equations. The present approach is applicable for both local and nonlocal potentials and suggests an unified calculational scheme for bound states and scattering problems.

I. INTRODUCTION

The Jost functions[1,2] $f_{\pm}(k)$ have played a very important role in the development of formal scattering theory. For a local potential, they are conveniently defined by the $r=0$ limit of the irregular solutions $f_k^{(\pm)}(r)$ of the Schrödinger equation in coordinate space,[2]

$$f_{\pm}(k)=f_k^{(\pm)}(0) , \tag{1.1}$$

where k^2 is the energy in units $\hbar=2m=1$ and m is the reduced mass. The on-shell S and t matrices are defined by

$$S(k)\equiv e^{2i\delta}=\frac{f_-(k)}{f_+(k)} , \tag{1.2}$$

$$t(k)\equiv -e^{i\delta}\sin\delta/k = \frac{f_+(k)-f_-(k)}{2ikf_+(k)} , \tag{1.3}$$

where δ is the phase shift.

The principal properties of the Jost functions for a local potential are the following:[2,3]

(i) The functions $f_+(k)$ $[f_-(k)]$ are entire analytic functions of k regular in $\mathrm{Im}k>0$ (<0) and are continuous, with a continuous derivative, in $\mathrm{Im}k\geq 0$ (≤ 0); however, for a certain class of potentials, $f_+(k)$ $[f_-(k)]$ may have a cut along the $\mathrm{Im}k$ axis between $-ia$ and $-i\infty$ (ia and $i\infty$), where a is a positive constant.

(ii) The zeros of f_+ in the upper half k plane lie only on the $\mathrm{Im}k$ axis and correspond to bound states, whereas the zeros of f_- in the upper half k plane lie on the $\mathrm{Im}k$ axis (virtual states) or symmetrically on both sides of the $\mathrm{Im}k$ axis (resonances).

(iii) For $\mathrm{Im}k\geq 0$,

$$\lim_{|k|\to\infty} f_+(k)=1 .$$

(iv) For real energies, $f_{\pm}(k)$ carry the scattering phase

$$f_{\pm}(k)=|f_{\pm}(k)|\exp(\mp i\delta) .$$

The analytic continuation of $f_{\pm}(k)$ in the whole k plane is done through[2]

$$f_+^*(k^*)=f_-(k) . \tag{1.4}$$

For a nonlocal potential the properties of the Jost functions are somewhat altered on the real k^2 axis from those listed above for a local potential, which are discussed in detail in Sec. III. We shall see that for nonlocal potentials the Jost functions may have spurious poles at real k^2. In this paper we suggest a unified class of Jost-like functions $F_{\pm}(k)$ valid both for local and nonlocal potentials, which may have spurious poles at real k^2. These functions are related to usual Jost functions $f_{\pm}(k)$ by

$$F_{\pm}(k)=g(k^2)f_{\pm}(k) , \tag{1.5}$$

where $g(k^2)$ is analytic in k^2, except for possible poles at real k^2 and has the property that $\lim_{|k|\to\infty} g(k^2)=1$. $F_{\pm}(k)$ of Eq. (1.5) satisfies properties (i)—(iv) above, but may have spurious poles at certain real k^2 both for local and nonlocal potentials. This generalization of the Jost function will make the appearance of such poles for nonlocal potentials much less anomalous.

We would like to point out that the functions $F_{\pm}(k)$ naturally arise in a recent discussion of solution of scattering[4,5] and bound state[6] equations in momentum space. In momentum space these functions are easily calculated through the iterative solution of auxiliary integral equations. The present approach provides an easy way of defining and calculating Jost-like functions $F_{\pm}(k)$ for nonlocal potentials where it is neither convenient to introduce nor easy to calculate the usual Jost functions by coordinate space techniques. The present approach is equally applicable to both local and nonlocal potentials.

The plan of the paper is as follows. In Sec. II we give a brief review of our approach, on which the present discussion of the class of Jost-like functions $F_{\pm}(k)$ is based. In Sec. III we give a brief account of Jost functions for nonlocal potentials, and in Sec. IV we present the new class of functions $F_{\pm}(k)$ and discuss their properties. In Sec. V we present numerical results for $F_{\pm}(k)$ for the Reid[1] S_0 potential. To the best of our knowledge, this is the first numerical calculation of the Jost function for such a local potential. Finally, in Sec. VI we present a brief summary.

II. AUXILIARY SCATTERING EQUATIONS

We follow Refs. 5 and 6 to present our auxiliary equations.[7,8] The S wave Lippmann-Schwinger equation (in units $\hbar=2m=1$) can be written as[2]

$$\langle p\,|\,t(E)\,|\,r\rangle=\langle p\,|\,V\,|\,r\rangle+\frac{2}{\pi}\int q^2dq\langle p\,|\,V\,|\,q\rangle(k^2-q^2+i\epsilon)^{-1}\times\langle q\,|\,t(E)\,|\,r\rangle\ , \tag{2.1}$$

where $k^2=E$, and E is the total center of mass energy. In Eq. (2.1) and throughout the rest of the paper the momentum space integration limits are from 0 to ∞. Equation (2.1) has a singular kernel. The solution of Eq. (2.1) is expressed in terms of the solution of the following auxiliary equation:[5]

$$\langle p\,|\,\Gamma(E)\,|\,r\rangle=\langle p\,|\,V\,|\,r\rangle+\frac{2}{\pi}\int q^2dq\langle p\,|\,A(E)\,|\,q\rangle\times(k^2-q^2+i\epsilon)^{-1}\langle q\,|\,\Gamma(E)\,|\,r\rangle\ , \tag{2.2}$$

by[5]

$$\langle p\,|\,t(E)\,|\,r\rangle=\left[\frac{\langle p\,|\,\Gamma(E)\,|\,k\rangle}{\langle k\,|\,\Gamma(E)\,|\,k\rangle}t(k)\frac{\langle r\,|\,\Gamma(E)\,|\,k\rangle}{\langle k\,|\,\Gamma(E)\,|\,k\rangle}\right]+\left[\langle p\,|\,\Gamma(E)\,|\,r\rangle-\frac{\langle p\,|\,\Gamma(E)\,|\,k\rangle\langle k\,|\,\Gamma(E)\,|\,r\rangle}{\langle k\,|\,\Gamma(E)\,|\,k\rangle}\right], \tag{2.3}$$

where

$$t(k)\equiv\langle k\,|\,t(E)\,|\,k\rangle=\frac{\langle k\,|\,\Gamma(E)\,|\,k\rangle}{1-\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\langle q\,|\,\Gamma(E)\,|\,k\rangle}\ . \tag{2.4}$$

Here,

$$\langle p\,|\,A(E)\,|\,q\rangle=[\langle p\,|\,V\,|\,q\rangle-\langle p\,|\,V\,|\,k\rangle\gamma(k,q)]\ , \tag{2.5}$$

where $\gamma(k,q)$ is an arbitrary function with the property

$$\gamma(k,k)=1\ . \tag{2.6}$$

Though the kernel of Eq. (2.2) is not singular, the integration limit in it extends to infinity. But as the integrand involved tends to zero sufficiently rapidly at infinity, this does not cause any special problem and usually an equation like Eq. (2.2) is termed nonsingular. It is well known that the on-shell t matrix $t(k)$ of Eq. (2.4) has a pole at the bound state energy and it is expected that it corresponds to a zero of the denominator of (2.4).

It might be suspected that the denominator of (2.4) may develop spurious zeros for an arbitrary γ which are canceled by a simultaneous zero of the numerator of (2.4).

We show in the following that even for an arbitrary γ the zero of the denominator of Eq. (2.4) always corresponds to a bound state of the system.[7,8] At the zero of the denominator of Eq. (2.4) given by

$$1-\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\langle q\,|\,\Gamma(E)\,|\,k\rangle=0\ , \tag{2.7}$$

the half-on-shell version of Eq. (2.2) can easily be shown to reduce to

$$\langle p\,|\,\Gamma(E)\,|\,k\rangle=\frac{2}{\pi}\int q^2dq\langle p\,|\,V\,|\,q\rangle(k^2-q^2+i\epsilon)^{-1}\times\langle q\,|\,\Gamma(E)\,|\,k\rangle\ . \tag{2.8}$$

Defining

$$\Psi_k(q)\equiv(k^2-q^2+i\epsilon)^{-1}\langle q\,|\,\Gamma(E)\,|\,k\rangle\ , \tag{2.9}$$

it is easy to see that Eq. (2.8) reduces to the homogeneous Schrödinger equation in momentum space:

$$\Psi_k(p)=\frac{2}{\pi}(k^2-p^2+i\epsilon)^{-1}\int q^2dq\langle p\,|\,V\,|\,q\rangle\Psi_k(q)\ , \tag{2.10}$$

which has solutions at bound state energies. For local Hermitian potentials these solutions occur only at negative energies. But for nonlocal potentials Eq. (2.10) may have continuum bound state solutions at positive energies. We shall come to this point later. All solutions of Eq. (2.2) which also satisfy Eq. (2.7) will yield the momentum space bound state wave functions via Eq. (2.9). This idea was used to develop[6] a new iterative method for solving momentum space bound state equations, which has recently been used by Dzhibuti and Tsiklauri[9] to calculate three- and four-body bound states using the hyperspherical harmonics approach in momentum space. Though the on-shell k at negative energies is purely imaginary, in Ref. 6 k in the equivalent of Eq. (2.5) [see k in Eq. (2.3) of Ref. 6] is taken to be an arbitrary positive momentum. At negative energies the proof sketched above holds for any arbitrary positive subtraction point in Eq. (2.5). This gives a very practical calculational scheme at negative energies not only for binding energies and bound state wave functions (as in Refs. 6 and 9), but also for relevant t matrix elements, where in Eq. (2.5) the subtraction point is to be taken to be a positive momentum variable. However, for the purpose of defining the functions $F_\pm$ in the present paper k in Eq. (2.5) will be taken to be on-shell momentum at both positive and negative energies unless

otherwise specified.

The solution for the t matrix written in the form (2.3) has certain advantages as has been pointed out in Refs. 5 and 6. The expression in the first square bracket of (2.3) is exact half-on-shell and is usually a good approximation to the full t matrix; the expression in the last square bracket in this equation is a correction to the first term, is zero half-on-shell, and can possibly be neglected for doing approximate calculation. A decomposition similar to (2.3) was first written by Kowalski and Noyes[10] for a special choice[5] of the function γ.

In this paper we shall show in the next section that for a wide class of functions $\gamma(k,q)$ the denominator of Eq. (2.4) has all the properties of the Jost function $f_+(k)$, and that for $\gamma(k,q)=1$ this denominator is identically equal to the Jost function $f_+(k)$.

III. JOST FUNCTION FOR NONLOCAL POTENTIALS

The extension of the Jost function to the case of a symmetric nonlocal potential is problematic because of the existence of so called spurious and continuum bound states.[3] We show in this section that for symmetric nonlocal potentials the denominator of Eq. (2.4) with $\gamma=1$ is identically equal to the Jost function discussed by Mulligan and collaborators.[3]

First, following Ref. 3, we present the definition of the Jost function in this case for angular momentum $L=0$, though a generalization to other L's is straightforward. The physical wave function $\psi_k^{(+)}$ and its counterpart $\psi_k^{(-)}$ satisfy[3]

$$\psi_k^{(\pm)}(r)=\sin kr+\int_0^\infty\int_0^\infty G^{(\pm)}(r,r';k)V(r',s)\psi_k^{(\pm)}(s)ds\,dr', \tag{3.1}$$

with the Green's function

$$G^{(\pm)}(r,r';k)=-k^{-1}e^{\pm ikr_>}\sin(kr_<)\ .$$

The function $\psi_k^{(+)}$ satisfies the boundary condition

$$\psi_k^{(+)}(r)\underset{r\to\infty}{\to}\sin kr-kt(k)e^{ikr}\ , \tag{3.2}$$

with

$$t(k)=-\frac{e^{i\delta}\sin\delta}{k}=\frac{1}{k^2}\int_0^\infty\int_0^\infty \sin kr\ V(r,s)\psi_k^{(+)}(s)ds\,dr\ . \tag{3.3}$$

The regular solution of the Schrödinger equation ϕ_k is defined by

$$\phi_k(r)=k^{-1}\sin kr+\int_0^r\int_0^\infty G(r,r';k)V(r',s)\phi_k(s)ds\,dr'\ , \tag{3.4}$$

with

$$G(r,r';k)=k^{-1}\sin k(r-r')\ .$$

The regular solution ϕ satisfies the boundary condition

$$\lim_{r\to0}r^{-1}\phi_k(r)\to1\ . \tag{3.5}$$

The Jost solutions $f_k^{(\pm)}(r)$ satisfy

$$f_k^{(\pm)}(r)=e^{\pm ikr}-\int_r^\infty\int_0^\infty G(r,r';k)V(r',s)f_k^{(\pm)}(s)ds\,dr'\ , \tag{3.6}$$

and the boundary condition

$$\lim_{r\to\infty}e^{\pm ikr}f_k^{(\mp)}(r)\to1\ . \tag{3.7}$$

The Jost functions $f_\pm(k)$ are defined by

$$f_\pm(k)=f_k^{(\pm)}(0)\ . \tag{3.8}$$

For the most general case of local and nonlocal (symmetric) potentials, it has been shown that[3]

$$f_\pm(k)=D^{(\pm)}(k)/D(k)\ , \tag{3.9}$$

where $D^{(\pm)}(k)$ and $D(k)$ are the Fredholm determinants associated with the Fredholm equations (3.1) and (3.6) satisfied by $\psi_k^{(\pm)}$ and $f_k^{(\pm)}$. For the special case of the local potential, $D(k)=1$ and $f^{(\pm)}(k)=D^{(\pm)}(k)$. In the case of nonlocal (symmetric) potentials, one has spurious state $[D(k)=0,\ D^{(\pm)}(k)\neq0]$ and continuum states $[D(k)=D^{(\pm)}(k)=0]$ at real positive energies in addition to the physical states at real negative energies, and this makes the definition of phase shift through Eq. (3.3) a delicate task. However, despite ambiguities when $D(k)=0$ at a positive k, for sufficiently large k the phase shift for a nonlocal potential is uniquely defined by the asymptotic conditions[3]

$$\begin{aligned}&\delta\underset{\lim k\to\infty}{\to}0\ ,\\ &D(k)\underset{\lim k\to\infty}{\to}1\ .\end{aligned} \tag{3.10}$$

Also, as long as one is not at a zero of $D^{(\pm)}(k)$ or $D(k)$, the Jost function is well defined and one has[2,3]

$$\psi_k^{(+)}(r)=\frac{k\phi_k(r)}{f_+(k)}\ . \tag{3.11}$$

Consistent with Eqs. (3.3) and (3.11), one can either define δ conveniently by

$$\delta=-\text{phase of } f^{(+)}(k)=-\text{phase of }\left[\frac{D^{(+)}(k)}{D(k)}\right]\ , \tag{3.12}$$

or by

$$\delta=-\text{phase of } D^{(+)}(k)\ . \tag{3.13}$$

Using either definition, one constructs the same on shell t matrix $t(k)$. In the present work we shall stick to the definition (3.12) of δ. It is easy to realize that δ of (3.13) will experience a discontinuity of π at a continuum bound state $[D(k)=D^{(\pm)}(k)=0]$ and is continuous at a spurious state $[D(k)=0,\ D^{(\pm)}(k)\neq0]$, whereas δ of (3.12) is continuous at a continuum bound state and has a discontinuity of π at a spurious state.

It is known that the Jost solutions $f_k^{(\pm)}(r)$ do not exist at a spurious state. On the other hand, $\psi_k^{(\pm)}(r)$ exists at a spurious state. At a continuum bound state $\psi_k^{(\pm)}(r)$ and $\phi_k(r)$ always exist, while $f_k^{(\pm)}(r)$ may or may not exist.

The definition of a Jost function through Eqs. (3.8) and

(3.9) will break down when Jost solutions $f_k^{(\pm)}(r)$ do not exist. This is a problem in the case of spurious states, which would mean a failure of existence of Jost functions at any energy. It has been recently shown[3] that Eq. (3.9) holds even if Jost solution fails to exist. In this paper we shall take Eq. (3.9) as the defining equation for $f_{\pm}(k)$, which is valid, in general, and we propose a momentum space representation of the Jost function.

For $\gamma=1$ the denominator of the right-hand side of Eq. (2.4) is written as

$$\hat{f}_+(k)\equiv\left[1-\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\langle q\mid\Gamma(E)\mid k\rangle\right] \tag{3.14}$$

$$=\left[1+\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\langle q\mid t(E)\mid k\rangle\right]^{-1}. \tag{3.15}$$

The equivalence between forms (3.14) and (3.15) is easily established.[8] Using the formal solution of the Lippmann-Schwinger equation for $\psi_k^{(+)}$ in momentum space,

$$\langle q\mid\psi_k^{(+)}\rangle=\frac{\pi}{2}\frac{\delta(q-k)}{q^2}+(k^2-q^2+i\epsilon)^{-1}\langle q\mid t(E)\mid k\rangle\ ,$$

in Eq. (3.15) one has

$$\hat{f}_+(k)=\left[\frac{2}{\pi}\int q^2dq\langle q\mid\psi_k^{(+)}\rangle\right]^{-1}$$

$$=\left[\frac{2}{\pi}\int_0^\infty r^2dr\frac{1}{kr}\psi_k^{(+)}(r)\int q^2dq\, j_0(qr)\right]^{-1}. \tag{3.16}$$

Using the following representation of the δ function,

$$\frac{2}{\pi}\int_0^\infty q^2dq\, j_0(qr)=\frac{\delta(r)}{r^2}\ ,$$

and Eqs. (3.5) and (3.11), one has, from (3.16),

$$\hat{f}_+(k)=\lim_{r\to0}\left[\frac{1}{kr}\psi_k^{(+)}(r)\right]^{-1}=f_+(k)\ . \tag{3.17}$$

It is obvious[3] that $f_+(k)$ of (3.17) is the same as that of (3.9). Hence we show explicitly that in the case $\gamma=1$ the denominator of Eq. (2.4) is the Jost function $f_+(k)$ of (3.9). This is true for both local and nonlocal potentials.

The Jost function $f_+(k)$ of Eq. (3.9) for the symmetric nonlocal potential reduces to the usual Jost function for local potentials when $D(k)=1$. The Jost function for symmetric nonlocal potential satisfies all the properties listed in the Introduction. It may have infinities at certain real positive energies, however, because of the occurrence of spurious states where $D(k)=0$, $D^+(k)\neq0$. We shall study this possibility in Sec. IV. At a continuum bound state, $D(k)=D^{\pm}(k)=0$, and the Jost function is finite.

IV. JOST-LIKE FUNCTIONS

We shall show in this section that for a general class of γ the denominator of Eq. (2.4) satisfies all the properties of the Jost function for a nonlocal potential. So we have the possibility of the appearance of spurious states for local potentials and of the disappearance of spurious states for nonlocal potentials for special choices of γ. We present a unified description of a class of Jost-like functions for both local and nonlocal potentials, which have the properties of the Jost function.

This class of functions—which we call Jost-like functions—is conveniently defined by

$$F_+(k)\equiv\left[1-\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\times\langle q\mid\Gamma(E)\mid k\rangle\right] \tag{4.1}$$

$$=\left[1+\frac{2}{\pi}\int q^2dq(k^2-q^2+i\epsilon)^{-1}\gamma(k,q)\times\langle q\mid t(E)\mid k\rangle\right]^{-1}. \tag{4.2}$$

The equivalence between the forms (4.1) and (4.2) is easily established.[8] It is sometimes difficult to study the analytic properties of (4.1), because in this form the function $\gamma(k,q)$ implicitly appears in $\langle q\mid\Gamma(E)\mid k\rangle$. The alternate form (4.2) might be useful for studying analytic properties of $F_+(k)$. We shall try to find out the restrictions on γ such that $F_+(k)$ will have all the properties of the Jost function $f_+(k)$.

It is easy to see that $F_+(k)$ satisfies all the desired properties

(i) if we choose $\gamma(k,q)$ to be an analytic function of both k^2 and q^2, so that the integrals in Eqs. (4.1) and (4.2) converge and tend to zero as $\mid k\mid\to0$, and

(ii) if $\gamma(k,q)$ does not introduce endpoint singularities or pinch singularities in these integrals.

Also, $\gamma(k,q)$ has to satisfy Eq. (2.6). There are various ways to choose a γ which satisfies these properties.

Thus, all the properties mentioned in the Introduction can be satisfied by $F_+(k)$, except that $F_+(k)$ may have unwanted poles due to the vanishing of the quantity in the large square brackets in Eq. (4.2). Now we study the mechanism which leads to such poles. These poles should be called spurious states, in analogy with the case of nonlocal potentials with $\gamma=1$. We note from Eq. (4.2) that at such a pole we have

$$1+\frac{2}{\pi}\int q^2dq\,\gamma(k,q)(k^2-q^2+i\epsilon)^{-1}\langle q\mid t(E)\mid k\rangle=0\ , \tag{4.3}$$

and, consequently, the half-on-shell version of the Lippmann-Schwinger equation (2.1) reduces to

$$\langle p\mid t(E)\mid k\rangle=\frac{2}{\pi}\int q^2dq\langle p\mid A(E)\mid q\rangle(k^2-q^2+i\epsilon)^{-1}\times\langle q\mid t(E)\mid k\rangle\ . \tag{4.4}$$

Remembering that the half-on-shell t-matrix elements

of Eq. (4.4) have the elastic scattering phase given by[10]

$$\langle p \mid t(E) \mid k \rangle = e^{i\delta} \mathscr{H}(p,k) , \tag{4.5}$$

where $\mathscr{H}(p,k)$ is a real function of p and k, Eq. (4.4) becomes

$$\mathscr{H}(p,k) = \frac{2}{\pi} \int q^2 dq \langle p \mid A(E) \mid q \rangle (k^2 - q^2 + i\epsilon)^{-1} \mathscr{H}(q,k) . \tag{4.6}$$

Equation (4.6) is the half-on-shell version of the homogeneous equation (2.2) for $\Gamma(E)$. Defining

$$\Phi_k(q) = (k^2 - q^2 + i\epsilon)^{-1} \mathscr{H}(q,k) , \tag{4.7}$$

Eq. (4.6) becomes

$$\Phi_k(p) = \frac{2}{\pi} (k^2 - p^2 + i\epsilon)^{-1} \int q^2 dq \langle p \mid A(E) \mid q \rangle \Phi_k(q) . \tag{4.8}$$

Hence, at the pole of $F_+(k)$ the homogeneous version of Eq. (2.2) has a solution, as is clear from Eq. (4.6). In operator form, Eq. (4.8) becomes

$$[E - H_0 - A(E)] \Phi_k = 0 , \tag{4.9}$$

where H_0 is the kinetic energy operator. In operator form Eq. (2.2) becomes

$$\Gamma(E) = V + A(E)(E - H_0 + i\epsilon)^{-1} \Gamma(E) ,$$

which has the formal solution

$$\Gamma(E) = V + A(E)[E - H_0 - A(E) + i\epsilon]^{-1} V . \tag{4.10}$$

Making an eigenfunction expansion of the resolvent operator in Eq. (4.10) and using the functions Φ_k of Eq. (4.9) and their biorthogonal partners [note that $A(E)$ is non-Hermitian in general], it is easy to realize that, at the pole of $F_+(k)$, $\Gamma(E)$ develops a pole. This is very similar to the occurrence of poles in the t matrix at bound state energies which correspond to a discrete solution of the homogeneous t matrix equation. Hence, as in the case of the t matrix equation, in the neighborhood of these poles the iterative solution of Eq. (2.2) is expected to diverge due to the occurrence of an eigenvalue of the kernel of this equation greater than unity in magnitude. These poles should be called spurious states in analogy with spurious state of nonlocal potentials, where the Jost function develops a pole. Hence for an arbitrary γ both local and nonlocal potentials give rise to Jost-like functions $F_\pm(k)$ which may have a pole at a particular positive energy.

Once the conditions (i) and (ii) of this section are imposed on γ the matrix elements of the kernel of Eq. (2.2) will tend to zero as $\mid k \mid \to \infty$. Hence at high energies the iterative solution of Eq. (2.2) is expected to converge. The eigenvalues of the kernel of Eq. (2.2) are expected to be largest at low energies [as in the case of the kernel of the Lippmann-Schwinger equation (2.1)], where the iterative series solution of Eq. (2.2) will not easily converge. Hence if the iterative series solution of Eq. (2.2) converges at low energies, it should converge at all energies and consequently Γ or $F_+(k)$ is not expected to have any spurious state. This fact is not just of academic interest but has practical advantages also. Firstly, if for a γ satisfying properties (i) and (ii) of this section, Eq. (2.2) yields a rapidly convergent iterative series, then this will allow a simple and precise numerical computational method for the Jost-like function $F_+(k)$. At this point we recall that, for $\gamma(k,q) = 1$, $F_+(k)$ reduces to the Jost function $f_+(k)$. For a local potential this choice of γ will always yield a covergent iterative solution[7] of Eq. (2.2) but the convergence may not be rapid enough to be of practical use. Secondly, and most importantly, the main advantage lies in the case of nonlocal potentials, where the Jost-like function is easily calculable through the iterative solution of the auxiliary equation (2.2) for Γ. The iterative series solution of Eq. (2.2) may diverge for $\gamma = 1$ in this case. An example is the case of a separable potential, which we discuss in some detail below.

A one term separable potential is the simplest example of a nonlocal potential that may illustrate many of the above aspects. We consider the following nonlocal potential (s wave),

$$V(r,r') = \lambda g(r) g(r') , \tag{4.11}$$

with

$$g(r) = e^{-\alpha r} . \tag{4.12}$$

In this case the Jost function is analytically calculable and is given by[3]

$$f_+(k) = \frac{(k - i\alpha)[2\alpha k^2 + i4\alpha^2 k - (\lambda + 2\alpha^3)]}{(k + i\alpha)[2\alpha k^2 - (\lambda - 2\alpha^3)]} , \tag{4.13}$$

which is essentially given by Eq. (4.2) with $\gamma = 1$. It is easy to realize from Eq. (4.13) that, for $\lambda > 2\alpha^3$, $f_+(k)$ may have a pole for positive k^2, which corresponds to a spurious state. Also, in this case the iterative solution for Eq. (2.2) for Γ does not converge at low energies because of this spurious state. However, if we take the choice

$$\gamma(k,q) = \frac{k^2 + \alpha^2}{q^2 + \alpha^2} , \tag{4.14}$$

for γ, the kernel of Eq. (2.2) becomes identically equal to zero and the iterative series solution for Γ converges to yield

$$\langle p \mid \Gamma(E) \mid q \rangle = \langle p \mid V \mid q \rangle .$$

In this case the Jost-like function $F_+(k)$ becomes

$$F_+(k) = \frac{2\alpha k^2 + 4i\alpha^2 k - (\lambda + 2\alpha^3)}{2\alpha (k + i\alpha)^2} , \tag{4.15}$$

which has the same physical states as $f_+(k)$ of (4.13) but does not have a spurious state at positive energies. The convergence of the auxiliary equation (2.2) for Γ is clearly related to the nonexistence of spurious states in this case—the iterative solution of Eq. (2.2) does not converge for $\gamma = 1$ but converges for γ given by Eq. (4.14). The function $F_+(k)$ carries the scattering phase and also information about physical bound states, as does $f_+(k)$ of (4.13). From Eqs. (4.13) and (4.15) we easily verify that

$$F_+(k) = \frac{k^2 - (\lambda/2\alpha - \alpha^2)}{k^2 + \alpha^2} f_+(k) , \tag{4.16}$$

in agreement with Eq. (1.5).

So the appearance of a spurious state and its energy will depend on the choice of γ. For a local potential $\gamma=1$ leads to a convergent iterative solution of Eq. (2.2) for Γ, and does not generate any spurious state, whereas for another choice of γ the Jost-like function $F_+(k)$ for local potentials may have spurious states and Eq. (2.2) for Γ may not have a convergent iterative solution. The function $F_+(k)$ treats both local and nonlocal potentials in a unified way and hence should be considered as a generalization of the Jost function $f_+(k)$ for both local and nonlocal potentials.

Now, demanding that $F_\pm(k)$ satisfy Eq. (1.3) with $t(k)$ given by Eq. (2.4), one can easily introduce $F_-(k)$ in the upper half k plane by

$$F_-(k)=F_+(k)-2ik\Gamma(k)\ , \tag{4.17}$$

with $\Gamma(k)=\langle k\mid\Gamma(E)\mid k\rangle$. Note that $F_+(k)$ does not have any singularities (cuts) in the upper half k plane, but $F_-(k)$ has the singularities of $f_-(k)$ contained in $\Gamma(k)$. The analytic continuation to the entire k plane is now performed through the use of Eq. (1.4).

For local potentials, remembering that for $\gamma=1$ $F_+(k)$ reduces to the Jost function $f_+(k)$, using Eqs. (2.4) and (3.1) one easily obtains

$$F_+(k)=\frac{\Gamma_\gamma(k)}{\Gamma_1(k)}f_+(k)\ , \tag{4.18}$$

where $\Gamma_\gamma(k)$ is the on shell Γ calculated with a particular γ which leads to $F_+(k)$, and $\Gamma_1(k)$ is the on-shell Γ calculated with $\gamma=1$. From Eq. (1.5) one easily has $g(k^2)=\Gamma_\gamma(k)/\Gamma_1(k)$. The functions $\Gamma_\gamma(k)$ and $\Gamma_1(k)$ have the same left-hand cuts if we assume that $F_+(k)$ has the analytic properties of $f_+(k)$ and, consequently, $g(k^2)=\Gamma_\gamma/\Gamma_1$ is analytic in k^2 except for spurious poles.

Assuming that for a given problem we have succeeded in finding a suitable γ which leads to $F_\pm(k)$ possessing the properties of the Jost functions $f_\pm(k)$, then one can easily make the off-shell extension of these functions following Ref. 11, which will allow us to calculate the half off-shell t matrix via

$$\langle q\mid t(E)\mid k\rangle=\frac{F(k,q)-F(k,-q)}{2ikF_+(k)}\ ,$$

where

$$F_+(k)=F(k,k)=\mid F_+(k)\mid e^{-i\delta}$$

and

$$F(k,q)=1-\frac{2}{\pi}\int p^2dp(q^2-p^2+i\epsilon)^{-1}\gamma(k,p)\langle p\mid\Gamma(E)\mid k\rangle.$$

This last expression for the off-shell Jost-like function $F(k,q)$ is no more difficult to evaluate than the on-shell function $F_\pm(k)$.

V. NUMERICAL CALCULATION

In this section we present numerical results for the Jost functions $f_+(k)$ and $F_+(k)$ for the $^1S_0$ Reid soft core potential.[12] This potential is a linear superposition of Yukawa potentials. We employ an iterative solution of the equation for Γ for two choices of γ in order to obtain the Jost-like functions f_+ and F_+. The two choices of γ are

$$\text{Choice A: } \gamma(k,q)=1\ , \tag{5.1a}$$

$$\text{Choice B: } \gamma(k,q)=(k^2+\alpha^2)/(q^2+\alpha^2)\ . \tag{5.1b}$$

Choice A will lead to the usual Jost function $f_+(k)$ and choice B leads to a particular Jost-like function $F_+(k)$. As in Refs. 5 and 6, first we convert integrals from 0 to ∞ to integrals from -1 to $+1$ and use Gauss quadratures to change the integral equation for Γ to a matrix equation, which is then solved by iteration.

Convergence was easily achieved in the case of choice A ($\gamma=1$); in the case of choice B convergence was achieved for a wide choice of the parameter α. The parameter α was fixed in order in order to obtain good convergence, although no special search was made to obtain the best convergence. We used $\alpha=50\ \text{fm}^{-1}$.

The results are plotted in Fig. 1, where we exhibit the functions at different center of mass energies E. Because of wide variations of the functions and the center of mass energy, we employed an unusual scale; for both abscissa and ordinate between -1 and $+1$ we employ a linear scale and outside this region a logarithmic scale. It was pointed out by Oryu[13] in a different context that the equation for Γ can be solved and f_+ and F_+ calculated only up to $E=-\mu^2$ for a Yukawa potential $Ve^{-\mu r}/r$. A pinch singularity[13] prohibits the analytic continuation to be made for energy $E<-\mu^2$ when k—the point of subtraction in Sec. II—is taken to be the on-shell point. In the case of the Reid potential the analytic continuation can be made to $E=E_c=-\mu^2$, where μ is the smallest of the three μ's appearing in the Reid $^1S_0$ potential,[12] e.g., $\mu=0.7\ \text{fm}^{-1}$, which corresponds to $E_c=-20.32$ MeV. This point is indicated by an arrow in both cases. As bound states correspond to zeros of the Jost-like functions, the method cannot be used to find the bound state with binding energies greater than μ^2.

There is one natural continuation to all negative energies if we take k of Sec. II to be defined by

$$k^2=\mid E\mid\ , \tag{5.2}$$

where k is always real. With the choice (5.2) for k the equation for Γ of Sec. II can be solved at all energies and a physically interesting and useful continuation of the Jost-like functions can be made at all negative energies. The zero of this continuation of Jost-like functions can be proved to correspond to a bound state as in Sec. II. The proof of Sec. II outlined in the paragraph containing Eqs. (2.7)–(2.10) works equally for any real k at negative energies.

There are certain interesting aspects of the curves plotted in Fig. 1. For negative energies the functions are real and for positive energies they have a real and an imaginary part. The imaginary part goes to zero at $E=0$ and $E=\infty$. The real part is continuous at $E=0$ with both continuations at negative energies and goes to unity at $E=\pm\infty$. On the positive energy side, $\text{Re}f_+$ and $\text{Re}F_+$ have zeros at same energies, as is obvious from Eq. (4.18) if we recall that Γ_γ and Γ_1 are real functions. The same is true for $\text{Im}f_+$ and $\text{Im}F_+$. The functions oscillate quite

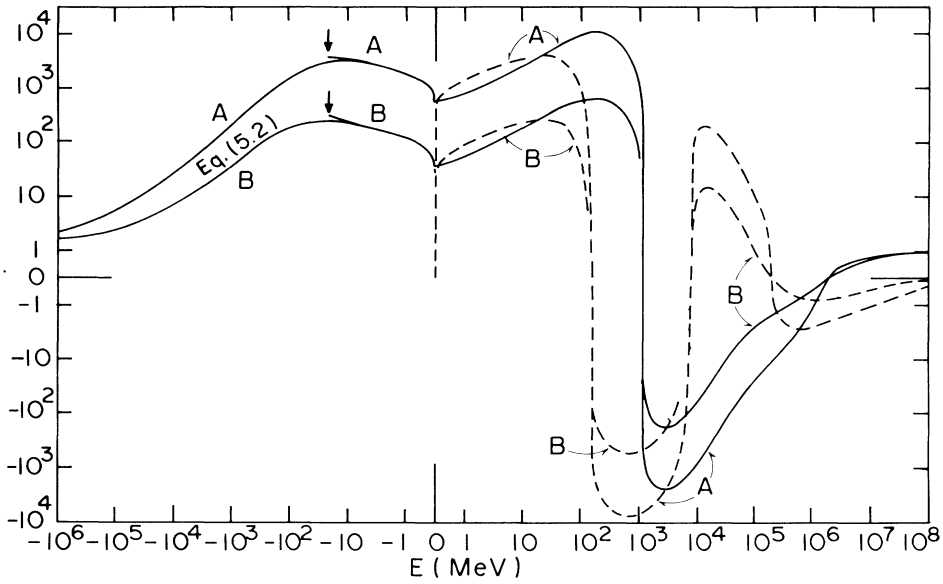

FIG. 1. Jost-like functions $F_+(k)$ for choices A and B of Eq. (5.1) for the Reid $^1S_0$ potential (for various center of mass energies). Choices A and B are indicated simply by A and B in the figure. Choice A corresponds to the usual Jost function f_+. The imaginary (real) parts are denoted by dashed (solid) lines. On the negative energy side the continuation that used Eq. (5.2) yields curves up to $E=-\infty$, whereas the continuation which uses $k^2=E$ yields curves up to a certain negative energy as explained in the text and indicated by an arrow in the figure.

widely to very large positive energies, although the underlying t matrix is a much smoother function of energy. Although the phase shift is expected to reduce to zero at $E=\infty$, in practice it (and also $\mathrm{Im}F_+$) does not do so until a very large energy is reached: $E\simeq 10^7$ MeV. We conclude this section by commenting that by studying the same function f_+ (or F_+) and its analytic continuation to negative energies, one can find the bound states and scattering phase shifts.

VI. SUMMARY AND DISCUSSION

We have shown how to introduce a set of functions which satisfy all the properties of the usual Jost function. The present set of functions is easily evaluated through the convergent iterative solution of a momentum space nonsingular integral equation. The present approach is applicable to both local and nonlocal potentials. We have also shown a simple way to make a natural off-shell extension of these functions. As an example we have numerically calculated in Sec. V these functions for the Reid $^1S_0$ potential.

This approach has an interesting application in calculating bound states, virtual states, and resonances. The on-shell S matrix can be written as

$$S(k)=\frac{F_-(k)}{F_+(k)}=\frac{F_+(k)-2ik\Gamma(k)}{F_+(k)} . \tag{6.1}$$

Bound states correspond to zeros of $F_+(k)$ in the upper half k plane and virtual states and resonances are the zeros of $F_-(k)\equiv[F_+(k)-2ik\Gamma(k)]$ in the upper half k plane. This allows for a simple and practical way of calculating virtual states and resonances. Instead of directly looking at the zeros of the numerator of Eq. (6.1) in order to calculate virtual states and resonances, we analytically continued Eq. (4.1) for $F_+(k)$ to the lower half k plane in Ref. 14 and looked for the zeros of the analytically continued $F_+(k)$. The analytic continuation yielded essentially the function $F_-(k)$, as it should. Hence the search for virtual states and resonances in Ref. 14 was essentially done through a search of zeros of $F_-(k)$. We conclude by noting that, using the solution of the equation for Γ at positive and negative energies, one can calculate—through the construction of the Jost-like function $F(k)$ in momentum space—the scattering phase shifts and bound, virtual, and resonant states. The present analysis unifies and generalizes the treatment of Ref. 5 for scattering states, that of Ref. 6 for bound states, and that of Ref. 14 for virtual states and resonances.

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