PHYSICAL REVIEW C VOLUME 35, NUMBER 5 MAY 1987

Relativistic Faddeev theory of the πNN system with application to πd scattering

Humberto Garcilazo*
Kernforschungszentrum Karlsruhe, Institut für Kernphsyik und Institut für Experimentelle Kernphysik der Universität Karlsruhe, D-7500 Karlsruhe, Federal Republic of Germany
(Received 24 March 1986)

We formulate a novel relativistic Faddeev theory of the πNN system using a variable-mass isobar ansatz for the πN and NN amplitudes and the requirement that the spectator particles be always on their mass shells. This theory takes into account the two possible time orderings in the propagator of the exchange particle and describes correctly the coupling between the two-body and three-body channels. As a first application of this theory, we study pion-deuteron elastic scattering by calculating all the observables in the region of the 3,3 resonance for which data exist.

I. INTRODUCTION

The πNN system is probably one of the most basic problems in nuclear physics. It is certainly more basic than the NN system, since in order for two nucleons to interact there needs first to be a pion. Moreover, if the energy of the nucleon-nucleon system is sufficiently large, a real pion will be produced, so that in order to have a consistent theory of the NN system which is valid both above and below the pion-production threshold it must be a three-body theory. Also, since the pion can be absorbed, one has transitions of the form NN$\leftrightarrows\pi$NN and NN$\leftrightarrows\pi$d which are possible only if the pion or at least one of the nucleons are allowed to be off their mass shells. Thus, a theory of the πNN system must also be relativistic in order to be able to describe intermediate states in which some of the particles are off the mass shell.

A candidate theory of the πNN system that has been proposed by Avishai and Mizutani[1] and independently by two other groups[2,3] is the coupled NN-πNN theory. There is considerable evidence however, from the measurements of the tensor polarization t_{20} in πd elastic scattering, that this theory may not be correct.[4–6] At the same time, it seems to be possible to describe most of the existing data by means of a theory which has much smaller effects from pion absorption as pointed out by the present author.[7] Thus, a complete exposition of such theory is needed in order to be able to compare it with other theories. Also, it is important to have the predictions of this theory for as many observables as possible so as to make a comparison with experiment more meaningful.

In the particular case of the πd elastic scattering reaction, data are available on the total cross section, differential cross section, vector analyzing power iT_{11}, tensor analyzing power T_{20}, and tensor polarization t_{20}. In addition, new experiments have been planned at SIN and TRIUMF to measure other tensor components, while an experiment is now underway[8] to measure the spin transfer coefficient it_{20}^{11} and possibly in the future more complicated spin-transfer coefficients.

There also exists a need for a set of background πd Faddeev amplitudes upon which, for example, a πd phase shift analysis can be attempted,[9,10] or effects that are not included in the Faddeev calculation can be studied by adding the remaining parts as a separate contribution to the scattering amplitude. Such a procedure has been carried out in order to search for possible evidence of dibaryon resonances in the πd system,[11,12] or to extract information on the short-range part of the intermediate delta-nucleon interaction.[13] Since such background amplitudes exist in the case of the coupled NN-πNN theory,[1–3] we like to provide also a set of amplitudes from the relativistic Faddeev theory that will be developed in this paper.

In the next section we describe the relativistic Faddeev theory. In Sec. III we present the predictions of this theory for πd elastic scattering and compare them with the available data in the region of the 3,3 resonance. Finally, we give our conclusions in Sec. IV.

II. THEORY

A. General remarks

The relativistic Faddeev equations proposed by Aaron, Amado, and Young[14] were first used by Kloet *et al.*[15] to describe nucleon-nucleon scattering below and above the pion-production threshold. They found that the nucleon-nucleon one-pion—exchange (OPE) potential in this theory has only one-half of the required strength, due to the fact that in the relativistic Faddeev propagator

$$G_0=\frac{1}{\omega_i\omega_j\omega_k}\frac{\omega_i+\omega_j+\omega_k}{S-(\omega_i+\omega_j+\omega_k)^2}, \tag{1}$$

proposed by Aaron, Amado, and Young,[14] only one of the two possible time orderings is included in the pion propagator.[15] This undercounting of the one-pion—exchange potential in the Aaron-Amado-Young theory has given rise to a great deal of confusion as well as to the development of the coupled NN-πNN theory,[1–3] which tried to compensate for it by introducing a new mechanism in which the pion is absorbed by one nucleon and emitted by the other one.

The relativistic Faddeev propagator (1) was derived by

Aaron, Amado, and Young by putting the three particles on their mass shells and performing a dispersion integral in the total energy squared of the system S which again takes the particles off their mass shells although not in an arbitrary way.[16] This theory, however, is not consistent since the propagator (1) is then used together with the requirement that the spectator particle be on its mass shell, which destroys conservation of total four-momentum. In order to restore conservation of total four-momentum to this theory, we have suggested before[17] that the propagator (1) must be replaced by

$$G_k=\frac{1}{\omega_i\omega_j}\frac{1}{(\sqrt{S}-\omega_i-\omega_j)^2-\omega_k^2}\ , \tag{2}$$

which is obtained by requiring that in a transition from a state in which particle i is the spectator and j,k the interacting pair to a state in which particle j is the spectator and i,k the interacting pair, both spectator particles must be on their mass shells. It is easy to see that

$$G_0^\dagger-G_0=G_k^\dagger-G_k=\pi i\frac{1}{\omega_i\omega_j\omega_k}\delta(\sqrt{S}-\omega_i-\omega_j-\omega_k)\ , \tag{3}$$

so that both propagators guarantee that the required three-body unitarity relation will be fulfilled. However, let us try to use the two theories to describe nucleon-nucleon scattering so that the propagators (1) and (2) represent the pion propagator. Then, if one considers the one-pion—exchange contribution to nucleon-nucleon scattering assuming that particles i and j are the two nucleons and particle k is the pion, one has that $\omega_i=\omega_j=\sqrt{S}/2$ and the propagators (1) and (2) become, respectively,

$$\begin{aligned}G_0&=-\frac{4}{S}\frac{\sqrt{S}+\omega_k}{2\sqrt{S}+\omega_k}\frac{1}{\omega_k^2}\\&\simeq-\frac{4}{S}\frac{1}{2}\frac{1}{\omega_k^2}\\&=-\frac{4}{S}\frac{1}{2}\frac{1}{m_\pi^2+(\mathbf{k}_i+\mathbf{k}_j')^2}\ ,\end{aligned} \tag{4}$$

$$G_k=-\frac{4}{S}\frac{1}{\omega_k^2}=-\frac{4}{S}\frac{1}{m_\pi^2+(\mathbf{k}_i+\mathbf{k}_j')^2}\ , \tag{5}$$

where $\mathbf{k}_i$ and $\mathbf{k}_j'$ are the momenta of particles i and j in the initial and final state, respectively. Equations (4) and (5) correspond to the well-known one-pion—exchange potential, except that Eq. (4) has only one-half of the required strength, while Eq. (5) has its full strength. Thus, as we see from Eqs. (4) and (5), the problem of having a one-pion—exchange potential with the correct strength has nothing to do with the new mechanism introduced by the NN-πNN theory, but it is just a question of having a consistent relativistic Faddeev propagator.

Since we use the isobar ansatz for the two-body amplitudes (see subsection C below), the three-body problem becomes an effective quasi-two-body problem involving all possible transitions between the various πN and NN isobars. These transition potentials are of two types which correspond to exchanging a pion between two πN isobars or exchanging a nucleon between a πN and a NN isobar. We show pictorially these two types of potentials in Fig. 1, where the crosses in the spectators indicate that they are on their mass shells. If in these potentials one of the πN isobars is the pion-nucleon P_{11} channel, then one has a state with the same quantum numbers of two nucleons, and one has to take into account the Pauli principle. First of all, one has to decompose the P_{11} amplitude into a pure nucleon part plus other contributions that do not reduce to a pure nucleon part; that is, the so-called pole and nonpole parts.[1–3] The Pauli principle then acts only if the spectator nucleon is taken together with the pole part, which means that only those channels that are consistent with the Pauli principle are allowed. If the spectator nucleon is taken together with the nonpole part, the Pauli principle does not act and all channels can exist. In the models of the P_{11} amplitude used in the NN-πNN theory, the pole and nonpole parts are both very large, although of opposite sign. Thus, when the pole and nonpole parts are taken together, as occurs in the Pauli allowed channels, they add up again to the full P_{11} amplitude, which is

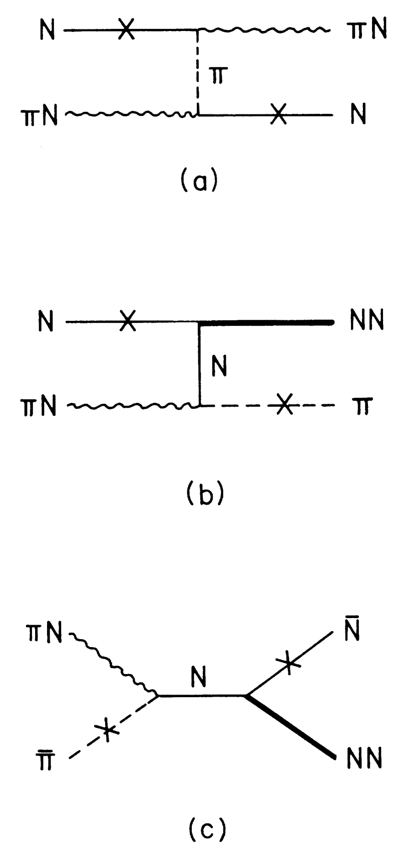

FIG. 1. Transition potentials of the πNN system, where the crosses in the particles mean that they are on the mass shell. (a) Transition from a πN isobar to another πN isobar by means of pion exchange. (b) Transition from a NN isobar to a πN isobar by means of nucleon exchange. (c) Another process which is also described by diagram (b).

very small; however, for the Pauli forbidden channels the nonpole part acts alone, so that there is no cancellation to give back the small P_{11} amplitude. Thus, they generate very large effects out of a very small amplitude, simply by writing it as the sum of two very large pieces and then applying the Pauli principle to only one of these pieces. It is by now clear that these large effects are spurious, since, as shown by Afnan and McLeod,[18] they give rise to large and positive values for the tensor polarization t_{20} of πd elastic scattering while, experimentally, t_{20} is large and negative.[3-6] In our case, where the decomposition of the P_{11} amplitude into pole and nonpole parts can be done unambiguously, these spurious effects do not appear, since, as shown in subsection H, both the pole and nonpole parts are small and, moreover, the nonpole part is much smaller than the pole part.

The basic property of the theory that we are discussing is the fact that the spectator particles are required to be always on their mass shells. This property determines all the important features of the theory. First of all, as already mentioned, it determines the relativistic Faddeev propagator (2) in which the exchanged particle has the two possible time orderings, which solves the problem of the insufficient strength of the one-pion—exchange potential. Similarly, the objection raised by the proponents of the NN-πNN theory, that in the standard Faddeev theory only one of the nucleons (the isobar) can emit the pion, does not apply to this theory due to the two time orderings. If the isobar emits a pion which is going forwards in time, that is the normal process included in the Aaron-Amado-Young theory; but, if the isobar emits a pion which is going backwards in time, that is the same as if the spectator had emitted a pion which is going forwards in time. Thus, both particles (the isobar and the nucleon) can emit the pion in this theory. In the case when we have a three-body final state where the pion and the two nucleons are all physical particles (that means they are all on their mass shells), the final pion must have been emitted only by the isobar, since as the spectator is always on its mass shell it cannot emit a physical pion and still remain on its mass shell. Thus, the mechanism introduced by the NN-πNN theory in which the pion is absorbed by one nucleon and emitted by the other one never enters in a theory where the spectators are kept always on their mass shells.

The relativistic Faddeev propagator given by Eq. (2) was first proposed by the author in connection with pion-deuteron elastic scattering.[17] Some of the consequences of this theory for the case of the one-pion—exchange potential were explored in a later work,[19] while numerical calculations with this three-body OPE potential have been performed recently by Mathelitsch and the author.[20] A theory based also on this idea has been proposed more recently by Gross.[21]

B. The relativistic Faddeev equations

Let us consider the Bethe-Salpeter equation for three particles such that it sums all processes in which two particles interact in all possible ways, while the third particle acts as spectator. The Bethe-Salpeter equation for this problem can be written in Faddeev form as

$$T_i^{jk}=t_i^{jk,jk}+t_i^{jk,jk}G_jG_kT_j^{ki}+t_i^{jk,jk}G_jG_kT_k^{ij}\ , \qquad i=1,2,3 \tag{6}$$

where G_j and G_k are the propagators for particles j and k, respectively, and $t_i^{jk,jk}$ is the scattering amplitude for particles j and k, where particle i is the spectator and the particles in both initial and final state are off their mass shells. In the particular case wherein one of the particles is on its mass shell, we will write the corresponding index with a capital letter. For example, if particle j is on its mass shell in the initial state and particle k is on its mass shell in the final state, the corresponding scattering amplitude will be $t_i^{jK,Jk}$. We can write Eq. (6) explicitly (assuming momentarily spinless particles) as

$$\langle \mathbf{k}_i\mathbf{k}_k \mid T_I^{jK} \mid \psi_{IJK}\rangle=\delta(\mathbf{k}_i-\mathbf{k}_{i0})\langle \mathbf{k}_k \mid t_I^{jK,JK}(q_i)\mid \psi_{JK}\rangle + \frac{i}{2\pi}\int d^4k_j'\langle \mathbf{k}_k \mid t_I^{jK,jk}(q_i)\mid k_j'\rangle G_j(k_j'^2)G_k(k_k'^2)\langle k_j'\mathbf{k}_i \mid T_j^{kI}\mid \psi_{IJK}\rangle+(T_j^{ki}\to T_k^{ij})\ , \tag{7}$$

where ψ_{IJK} is the initial state wave function of three free particles on their mass shells, and where in the final state we have put both the spectator particle i and particle k on their mass shells. The momenta q_i and k_k' in Eq. (7) are

$$q_i=K-k_i=[\sqrt{S}-(m_i^2+\mathbf{k}_i^2)^{1/2},\ -\mathbf{k}_i]\ , \tag{8}$$

$$k_k'=K-k_i-k_j'=[\sqrt{S}-(m_i^2+\mathbf{k}_i^2)^{1/2}-k_{j0}',\ -\mathbf{k}_i-\mathbf{k}_j']\ , \tag{9}$$

where $K=(\sqrt{S},\mathbf{0})$, is the total four-momentum of the three-body system. Using Eq. (9), we see that the propagators are given by

$$G_j(k_j'^2)=\frac{1}{k_{j0}'^2-\mathbf{k}_j'^2-m_j^2+i\epsilon}\ , \tag{10}$$

$$G_k(k_k'^2)=\frac{1}{[\sqrt{S}-(\mathbf{k}_i^2+m_i^2)^{1/2}-k_{j0}']^2-(\mathbf{k}_i+\mathbf{k}_j')^2-m_k^2+i\eta}\ . \tag{11}$$

We like first to perform the integration over dk_{j0}' in Eq. (7). If we close the contour of integration from below, we have to know all the poles of the integrand in the lower half k'_{j0} plane. For example, the propagator G_j given by Eq. (10) will contribute with the pole at

$$k'_{j0}=(\mathbf{k}_j'^2+m_j^2)^{1/2}-i\epsilon\equiv\omega'_j-i\epsilon \ , \tag{12}$$

and the propagator G_k given by Eq. (11) will contribute with the pole at

$$\begin{aligned}k'_{j0}&=\sqrt{S}-(\mathbf{k}_i^2+m_i^2)^{1/2}+[(\mathbf{k}_i+\mathbf{k}'_j)^2+m_k^2]^{1/2}-i\eta\\ &\equiv\sqrt{S}-\omega_i+\omega'_k-i\eta \ ,\end{aligned} \tag{13}$$

while, in general, the amplitudes $\langle k_k \mid t_I^{jK,jk}(q_i) \mid k'_j \rangle$ and $\langle k'_j\mathbf{k}_i \mid T_J^{kI} \mid \psi_{IJK} \rangle$ will bring in additional poles whose contribution, however, will be neglected. The theory discussed in the preceding subsection,[17,18–21] containing the relativistic Faddeev propagator (2), is obtained if one considers only the contribution of the spectator particle given by Eq. (12). This theory, however, also has problems since the propagator (2) gives rise not only to the physical cut generated by the delta function $\delta(\sqrt{S}-\omega_i-\omega_j-\omega_k)$, as shown in Eq. (3), but also to an unphysical cut generated by the delta function $\delta(\sqrt{S}-\omega_i-\omega_j+\omega_k)$. Thus, in order to eliminate this unphysical cut we will look into the contribution of the pole (13), so as to take into account a part of it which cancels the unphysical cut. If we take into account the contribution of the two poles (12) and (13) and neglect everything else, the relativistic Faddeev equations (7) become

$$\begin{aligned}\langle \mathbf{k}_i\mathbf{k}_k \mid T_I^{jK} \mid \psi_{IJK} \rangle &=\delta(\mathbf{k}_i-\mathbf{k}_{i0})\langle \mathbf{k}_k \mid t_I^{jK,JK}(q_i) \mid \psi_{JK} \rangle\\ &\quad+\int \frac{d\mathbf{k}'_j}{2\omega'_j}\langle \mathbf{k}_k \mid t_I^{jK,Jk}(q_i) \mid \mathbf{k}'_j \rangle G_k(k_k'^2)\langle \mathbf{k}'_j\mathbf{k}_i \mid T_J^{kI} \mid \psi_{IJK} \rangle\\ &\quad+\int \frac{d\mathbf{k}'_j}{2\omega'_k}\langle \mathbf{k}_k \mid t_I^{jK,JK}(q_i) \mid \mathbf{k}'_k \rangle G_j(k_j'^2)\langle \mathbf{k}'_k\mathbf{k}_i \mid T_J^{KI} \mid \psi_{IJK} \rangle+(T_J^{ki}\to T_K^{ij}) \ ,\end{aligned} \tag{14}$$

where, as we see, the first integral contains the amplitude T_J^{kI}, in which the spectator particle j is on its mass shell, while the second integral contains the function T_J^{kI}, where the spectator is off the mass shell. The propagators G_k and G_j that appear in Eq. (14) are given by

$$G_k(k_k'^2)=\frac{1}{2\omega'_k}\left[\frac{1}{\sqrt{S}-\omega_i-\omega'_j-\omega'_k+i\sqrt{\epsilon}+i\sqrt{\eta}}-\frac{1}{\sqrt{S}-\omega_i-\omega'_j+\omega'_k+i\sqrt{\epsilon}-i\sqrt{\eta}}\right] , \tag{15}$$

$$G_j(k_j'^2)=\frac{1}{2\omega'_j}\left[\frac{1}{\sqrt{S}-\omega_i-\omega'_j+\omega'_k+i\sqrt{\epsilon}-i\sqrt{\eta}}-\frac{1}{\sqrt{S}-\omega_i+\omega'_j+\omega'_k-i\sqrt{\epsilon}-i\sqrt{\eta}}\right] . \tag{16}$$

The first term in the propagator G_k diverges when $\sqrt{S}=\omega_i+\omega'_j+\omega'_k$, which gives rise to the unitarity cut associated with continuum states of three particles, while the second term diverges when $\sqrt{S}=\omega_i+\omega'_j-\omega'_k$, which gives rise to a completely unphysical cut. This unphysical cut, however, as we will see next, will be cancelled by the first term of the propagator G_j given by Eq. (16), while the second term in Eq. (16) can never diverge. In order to see how this cancellation takes place, we first notice that the momenta of the three particles in the first integral are

$$k_i=(\omega_i,\mathbf{k}_i) \ , \tag{17a}$$

$$k'_j=(\omega'_j,\mathbf{k}'_j) \ , \tag{17b}$$

$$k'_k=(\sqrt{S}-\omega_i-\omega'_j, \ -\mathbf{k}_i-\mathbf{k}'_j) \ , \tag{17c}$$

while in the second integral they are

$$k_i=(\omega_i,\mathbf{k}_i) \ , \tag{18a}$$

$$k'_j=(\sqrt{S}-\omega_i+\omega'_k, \ \mathbf{k}'_j) \ , \tag{18b}$$

$$k'_k=(-\omega'_k, \ -\mathbf{k}_i-\mathbf{k}'_j) \ . \tag{18c}$$

Thus, the amplitudes $t_I^{jk,jk}$ and T_J^{kI} appear at completely different kinematical regions in the first and second integrals. Only at the point that gives rise to the unphysical cut, which is determined by the condition $\sqrt{S}=\omega_i+\omega'_j-\omega'_k$, are the two sets of coordinates (17) and (18) identical. Thus, for this point the arguments of $t_I^{jk,jk}$ and T_J^{kI} are the same in both integrals so that the integral equations (14) are free of unphysical singularities. In particular, the delta function produced by the second term of Eq. (15) is cancelled by the corresponding delta function produced by the first term of Eq. (16).[22] If we now neglect the rest of the second integral, we obtain the final equations

$$\begin{aligned}\langle \mathbf{k}_i\mathbf{k}_k \mid T_I^{jK} \mid \psi_{IJK} \rangle &=\delta(\mathbf{k}_i-\mathbf{k}_{i0})\langle \mathbf{k}_k \mid t_I^{jK,JK}(q_i) \mid \psi_{JK} \rangle\\ &\quad+\int \frac{d\mathbf{k}'_j}{2\omega'_j}\langle \mathbf{k}_k \mid t_I^{jK,Jk}(q_i) \mid \mathbf{k}'_j \rangle G_k(\mathbf{k}_i,\mathbf{k}'_j)\langle \mathbf{k}'_j\mathbf{k}_i \mid T_J^{kI} \mid \psi_{IJK} \rangle+(T_J^{ki}\to T_K^{ij}) \ ,\end{aligned} \tag{19}$$

where the propagator G_k is now defined as

$$G_k(\mathbf{k}_i,\mathbf{k}_j')\equiv\frac{P}{(\sqrt{S}-\omega_i-\omega_j')^2-\omega_k'^2}-\frac{\pi i}{2\omega_k'}\delta(\sqrt{S}-\omega_i-\omega_j'-\omega_k') , \quad (20)$$

where P means the principal part, so that the propagator (20) gives rise in Eq. (19) only to the physical cut [the logaritmic singularities generated by the principal part of the second term of Eq. (15) are very weak and do not cause any trouble in the numerical solution of the πNN system]. It is very important to have a Faddeev propagator that has only the physical cut in order to satisfy unitarity, as we discuss in subsection D.

Let us now consider the case when the particles have spin. If particle j has spin s_j, then the propagator G_j becomes

$$G_j(k)=\frac{\Lambda_{s_j}(k;m_j)}{k^2-m_j^2+i\epsilon} , \quad (21)$$

where

$$\Lambda_0(k;m_j)=1 , \quad (22)$$

$$\Lambda_{1/2}(k;m_j)=\frac{\not k+m_j}{2m_j} , \quad (23)$$

$$\Lambda_1^{\mu\nu}(k;m_j)=g_{\mu\nu}+\frac{k_\mu k_\nu}{m_j^2} , \quad (24)$$

$$\Lambda_{3/2}^{\mu\nu}(k;m_j)=\frac{\not k+m_j}{2m_j}\left[g_{\mu\nu}-\frac{\gamma_\mu\gamma_\nu}{3}-\frac{2k_\mu k_\nu}{3m_j^2}+\frac{k_\mu\gamma_\nu-k_\nu\gamma_\mu}{3m_j}\right] , \quad (25)$$

etc., and similarly for the propagator G_k. Then, since we have performed the integration over k_{j0}' in Eq. (7) by keeping only the contributions that put particle j on the mass shell, we have that the factor $\Lambda_{s_j}(k_j';m_j)$ becomes a spinor projection operator for particle j; that is,

$$\Lambda_{s_j}(k_j';\, m_j=(k_j'^2)^{1/2})=\sum_{\nu_j}\phi_{\nu_j}^{s_j}(\mathbf{k}_j')\bar{\phi}_{\nu_j}^{s_j}(\mathbf{k}_j') , \quad (26)$$

where $\phi_{\nu_j}^{s_j}(\mathbf{k}_j')$ are on-shell spinors of momentum $\mathbf{k}_j'$, spin s_j, and helicity ν_j. Thus, in the case of particles with spin, Eq. (19) becomes

$$\langle\bar{\phi}_{\nu_i}^{s_i}(\mathbf{k}_i)\bar{\phi}_{\nu_k}^{s_k}(\mathbf{k}_k)\,|\,T_I^{jK}\,|\,\psi_{IJK}\rangle=\delta(\mathbf{k}_i-\mathbf{k}_{i0})\delta_{\nu_i\nu_{i0}}\langle\bar{\phi}_{\nu_k}^{s_k}(\mathbf{k}_k)\,|\,t_I^{jK,JK}(q_i)\,|\,\psi_{JK}\rangle$$
$$+\sum_{\nu_j}\int\frac{d\mathbf{k}_j'}{2\omega_j'}\langle\bar{\phi}_{\nu_k}^{s_k}(\mathbf{k}_k)\,|\,t_I^{jK,Jk}(q_i)\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\rangle G_k(\mathbf{k}_i,\mathbf{k}_j')$$
$$\times\Lambda_{s_k}(k_k';m_k)\langle\bar{\phi}_{\nu_j}^{s_j}(\mathbf{k}_j')\bar{\phi}_{\nu_i}^{s_i}(\mathbf{k}_i)\,|\,T_J^{kI}\,|\,\psi_{IJK}\rangle+(T_J^{ki}\rightarrow T_k^{ij}) . \quad (27)$$

The integral equation (27) contains all the possible contributions in which the spectator particle j is on its mass shell. If one considers contributions from other poles in the integration over k_{j0}' (such as, for example, the pole in T_j^{ki} arising when the two-body subsystem k,i has a bound state), they will always leave the spectator particles j off the mass shell, so that they do not contribute to the unitarity cuts. Thus, these equations are, in a sense, the minimal choice which is consistent with the requirements of Lorentz invariance and unitarity. With regard to the neglected contributions of the second integral, one can get an idea of how important these contributions are, by calculating them in the case of a single loop or so-called box diagram. Such type of calculations have been performed by Locher and colaborators,[23,24] who noticed that since in the second integral particle k propagates as an antiparticle, and if particle k is a nucleon which has a very large mass, this contribution is strongly suppressed by the propagators such that it is negligible. If particle k is a pion, on the other hand, due to its small mass the suppression by the propagators does not occur, and they found that the contribution of the second integral has an upper limit of about 20%. Thus, this theory will be much better suited for reactions that proceed through nucleon exchange, such as πd→πd or πd→πNN, whose dominant diagrams are shown in Figs. 2(a) and 2(b). The reactions NN→NN and NN→πNN, on the other hand, are driven by pion ex-

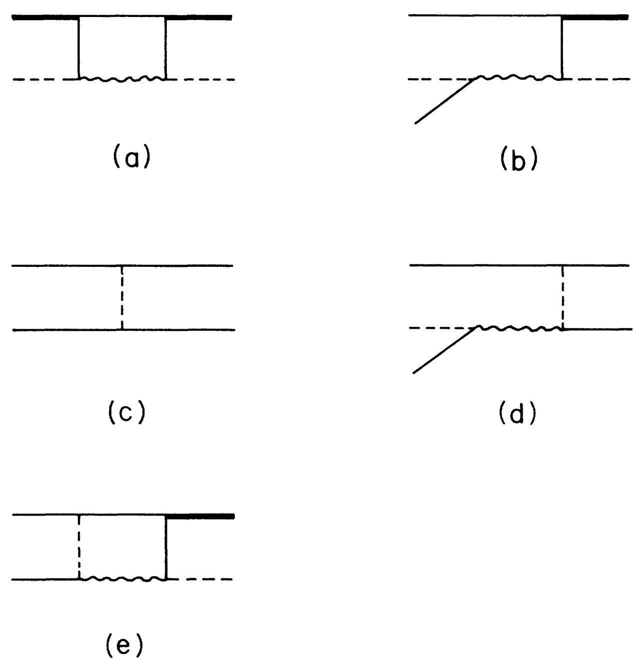

FIG. 2. The lowest-order diagrams for the processes (a) πd→πd, (b) πd→πNN, (c) NN→NN, (d) NN→πNN, and (e) πd→NN.

change, as shown in Figs. 2(c) and 2(d), while the reaction $\pi d \rightarrow NN$ is dominated by the process drawn in diagram 2(e), which proceeds halfway through nucleon exchange and halfway through pion exchange. Thus, the uncertainties from the neglect of the second integral will be larger for the last three reactions than for the first two.

C. The isobar ansatz

The two-body amplitudes $t_I^{jk,jk}(q_i)$ that enter in the relativistic Faddeev equation (27) obey the unitarity relation

$$t_I^{jk,jk^\dagger}(q_i)-t_I^{jk,jk}(q_i)=2\pi i\int d^4k_j\delta_+(k_j^2-m_j^2)\delta_+(k_k^2-m_k^2)t_I^{jk,JK^\dagger}(q_i)\Lambda_{s_j}(k_j;m_j=(k_j^2)^{1/2})\Lambda_{s_k}(k_k;m_k=(k_k^2)^{1/2})t_I^{JK,jk}(q_i)$$
$$+2\pi i\delta_+(q_i^2-q_\ell^2)f_\ell^2\Gamma_{j_\ell}^\dagger\Lambda_{j_\ell}(q_\ell;M_\ell=(q_\ell^2)^{1/2})\Gamma_{j_\ell}\ , \tag{28}$$

where the first term on the right-hand side (rhs) is the discontinuity associated with continuum states and the second term is the contribution of the process in which particles j and k are transformed into a stable isobar of spin j_ℓ and mass M_ℓ, where Γ_{j_ℓ} is the vertex operator that couples particles j and k with the stable isobar and f_ℓ is the corresponding coupling constant. In the πNN system there are two of these stable isobars; namely, the nucleon in the πN subsystem and the deuteron in the NN subsystem. Equation (28) can be written in the equivalent form

$$t_I^{jk,jk^\dagger}(q_i)-t_I^{jk,jk}(q_i)=2\pi i\sum_{\nu_j\nu_k}\int\frac{d\mathbf{k}_jd\mathbf{k}_k}{2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)t_I^{jk,JK^\dagger}(q_i)\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(\mathbf{k}_k)\rangle\langle\bar\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\bar\phi_{\nu_k}^{s_k}(\mathbf{k}_k)\,|\,t_I^{JK,jk}(q_i)$$
$$+2\pi i\sum_{m_\ell}\int\frac{d\mathbf{q}_\ell}{2\omega_\ell}\delta^4(K-k_i-q_\ell)f_\ell^2\Gamma_{j_\ell}^\dagger\,|\,\phi_{m_\ell}^{j_\ell}(\mathbf{q}_\ell)\rangle\langle\bar\phi_{m_\ell}^{j_\ell}(\mathbf{q}_\ell)\,|\,\Gamma_{j_\ell}\ . \tag{29}$$

We will now introduce the second main assumption of this theory. We will assume that the two-body amplitudes $t_I^{jk,jk}(q_i)$ can be written as a sum of isobars of variable mass $M_i=(q_i^2)^{1/2}$, so that Eq. (26) can be applied for each isobar; that is,

$$t_I^{jk,jk}(q_i)=\sum_{j_i}\Gamma_{j_i}^\dagger\tau_{j_i}(q_i^2)\Lambda_{j_i}(q_i;M_i=(q_i^2)^{1/2})\Gamma_{j_i}$$
$$=\sum_{j_im_i}\Gamma_{j_i}^\dagger\,|\,\phi_{m_i}^{j_i}(\mathbf{q}_i)\rangle\tau_{j_i}(q_i^2)\langle\bar\phi_{m_i}^{j_i}(\mathbf{q}_i)\,|\,\Gamma_{j_i}\ . \tag{30}$$

This assumption is somewhat unusual and perhaps even unjustified on physical grounds, since we know that real isobars have a constant mass; however, one can say in favor of it that it leads to integral equations in only one continuous variable as a result of the separable form of the two-body amplitudes (30), and that this separability has been achieved while maintaining Lorentz invariance at every stage of the derivation of the three-body equations. In Eq. (30) there may be more than one isobar for every value of the spin j_i (corresponding, for example, in the πN subsystem to the two possible values of the orbital angular momentum $l_{i\pm}=j_i\pm\frac{1}{2}$), and $\phi_{m_i}^{j_i}(\mathbf{q}_i)$ are isobar spinors of spin j_i, helicity m_i, three-momentum $\mathbf{q}_i$, and mass squared [see Eq. (8)],

$$q_i^2=[\sqrt{S}-(m_i^2+\mathbf{k}_i^2)^{1/2}]^2-\mathbf{k}_i^2\equiv M_i^2\ . \tag{31}$$

The on-shell two-body amplitude $t_I^{JK,JK}(q_i)$ corresponding to the case when all particles are on their mass shells is obtained from Eq. (30) by taking its matrix elements between on-shell spinors for all external particles; that is,

$$\langle\bar\phi_{\mu_j}^{s_j}(\mathbf{k}_j')\bar\phi_{\mu_k}^{s_k}(\mathbf{q}_i-\mathbf{k}_j')\,|\,t_I^{JK,JK}(q_i)\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(\mathbf{q}_i-\mathbf{k}_j)\rangle$$
$$=\sum_{j_im_i}\langle\bar\phi_{\mu_j}^{s_j}(\mathbf{k}_j')\bar\phi_{\mu_k}^{s_k}(\mathbf{q}_i-\mathbf{k}_i')\,|\,\Gamma_{j_i}^\dagger\,|\,\phi_{m_i}^{j_i}(\mathbf{q}_i)\rangle\tau_{j_i}(q_i^2)\langle\bar\phi_{m_i}^{j_i}(\mathbf{q}_i)\,|\,\Gamma_{j_i}\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(\mathbf{q}_i-\mathbf{k}_j)\rangle\ , \tag{32}$$

where the vertex functions in the rest frame of the pair j,k have the form

$$\langle\bar\phi_{m_i}^{j_i}(0)\,|\,\Gamma_{j_i}\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(-\mathbf{k}_j)\rangle=b_{j_i}^{\nu_j\nu_k}\,|\,\mathbf{k}_j\,|^{l_i}g_{j_i}(\,|\,\mathbf{k}_j\,|\,)\left[\frac{2j_i+1}{4\pi}\right]^{1/2}\mathscr{D}_{m_i,\nu_j-\nu_k}^{j_i}(\hat{\mathbf{k}}_j)\ , \tag{33}$$

where[25]

$$b_{j_i}^{\nu_j\nu_k}=\left[\frac{2l_i+1}{2j_i+1}\right]^{1/2}C_{0,\nu_j-\nu_k}^{l_iS_ij_i}C_{\nu_j,-\nu_k}^{s_js_kS_i}\ , \tag{34}$$

and the function $g_{j_i}(\,|\,\mathbf{k}_j\,|\,)$, which contains factors coming from the normalization of the spinors is determined by the vertex operator Γ_{j_i}. The vertex functions in an arbitrary frame that appear in Eq. (32) can be obtained from Eq. (33) by performing a Lorentz transformation[26] along the direction of $\mathbf{q}_i$, so as to get

$$\langle \bar{\phi}^{j_i}_{m_i}(\mathbf{q}_i) | \Gamma_{j_i} | \phi^{s_j}_{\nu_j}(\mathbf{k}_j)\phi^{s_k}_{\nu_k}(\mathbf{q}_i-\mathbf{k}_j)\rangle = \sum_{\lambda_j\lambda_k} d^{s_j}_{\nu_j\lambda_j}(\beta_j) d^{s_k}_{\nu_k\lambda_k}(\beta_k) b^{\lambda_j\lambda_k}_{j_i} |\mathbf{p}_i|^{l_i} g_{j_i}(|\mathbf{p}_i|) \left[\frac{2j_i+1}{4\pi}\right]^{1/2} \mathscr{D}^{j_i}_{m_i,\lambda_j-\lambda_k}(\hat{\mathbf{p}}_i) , \tag{35}$$

where λ_j, λ_k, and $\mathbf{p}_i$ are the helicities of the two particles, and the relative momentum between them as measured in the two-body c.m. frame; the functions $d^{s_j}_{\nu_j\lambda_j}(\beta_j)$ and $d^{s_k}_{\nu_k\lambda_k}(\beta_k)$ are the matrix elements of the unitary transformation that transforms the helicities between the two reference frames with β_j and β_k, the angles of the Wick triangle.[26] If we substitute Eq. (30) into Eq. (29) and use the expression for the vertex function given by Eq. (35), we find that the functions $\tau_{j_i}(q_i^2)$ obey the unitarity relation

$$\tau^\dagger_{j_i}(q_i^2)-\tau_{j_i}(q_i^2)=\frac{\pi i |\mathbf{p}_i|^{2l_i+1}}{2(q_i^2)^{1/2}} g^2_{j_i}(|\mathbf{p}_i|)\tau^\dagger_{j_i}(q_i^2)\tau_{j_i}(q_i^2)+2\pi i\delta_+(q_i^2-q_\ell^2)\delta_{j_i j_\ell} f_\ell^2 , \tag{36}$$

where

$$|\mathbf{p}_i|^2=\frac{[q_i^2-(m_j+m_k)^2][q_i^2-(m_j-m_k)^2]}{4q_i^2} , \tag{37}$$

so that in the case of uncoupled partial waves the functions $\tau_{j_i}(q_i^2)$ are related to the phase shifts as

$$\tau_{j_i}(q_i^2)=-\frac{4(q_i^2)^{1/2}}{\pi g^2_{j_i}(|\mathbf{p}_i|)}\frac{\sin\delta(q_i^2)e^{i\delta(q_i^2)}}{|\mathbf{p}_i|^{2l_i+1}} , \tag{38}$$

and a corresponding expression in the case of coupled waves. If $j_i=j_\ell$, then the two-body amplitude $\tau_{j_i}(q_i^2)$ has a pole at the mass squared of the stable isobar ℓ, so that near the pole it can be written as

$$\tau_{j_\ell}(q_i^2)\simeq\frac{f_\ell^2}{q_i^2-q_\ell^2+i\epsilon} . \tag{39}$$

D. Effective two-body equations

If we introduce the isobar ansatz (30) into Eq. (27) and make the substitution

$$\langle \bar{\phi}^{s_i}_{\nu_i}(\mathbf{k}_i)\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | T_I^{jK} | \psi_{IJK}\rangle = \delta_{\nu_i\nu_{i0}}\delta(\mathbf{k}_i-\mathbf{k}_{i0})\langle\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | t_I^{jK,JK}(q_i) | \psi_{JK}\rangle + \langle \bar{\phi}^{s_i}_{\nu_i}(\mathbf{k}_i)\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | U_I^{jK} | \psi_{IJK}\rangle , \tag{40}$$

then the functions U_I^{jK} will obey integral equations similar to Eq. (27), but with a new inhomogeneous term given by

$$\frac{1}{2\omega_{j0}}\langle\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | t_I^{jK,Jk}(q_i) | \phi^{s_j}_{\nu_{j0}}(\mathbf{k}_{j0})\rangle G_k(\mathbf{k}_i,\mathbf{k}_{j0})\Lambda_{s_k}(k_k;m_k)\langle\bar{\phi}^{s_i}_{\nu_i}(\mathbf{k}_i) | t_J^{kI,KI}(q_\ell) | \psi_{KI}\rangle + (t_J^{kI,KI}\to t_K^{Ij,IJ})$$

$$=\sum_{j\neq i}\sum_{j_i m_i}\sum_{j_j m_j}\frac{1}{2\omega_{j0}}\langle\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | \Gamma^\dagger_{j_i} | \phi^{j_i}_{m_i}(\mathbf{q}_i)\rangle\tau_{j_i}(q_i^2)\langle\bar{\phi}^{j_i}_{m_i}(\mathbf{q}_i) | \Gamma_{j_i} | \phi^{s_j}_{\nu_{j0}}(\mathbf{k}_{j0})\rangle$$

$$\times G_k(\mathbf{k}_i,\mathbf{k}_{j0})\Lambda_{s_k}(k_k;m_k)\langle\bar{\phi}^{s_i}_{\nu_i}(\mathbf{k}_i) | \Gamma^\dagger_{j_j} | \phi^{j_j}_{m_j}(\mathbf{q}_\ell)\rangle\tau_{j_j}(q_\ell^2)\langle\bar{\phi}^{j_j}_{m_j}(\mathbf{q}_\ell) | \Gamma_{j_j} | \psi_{KI}\rangle , \tag{41}$$

where we have taken $\mathbf{k}_{j0}=-\mathbf{q}_\ell$, with q_ℓ^2 the mass squared of one of the stable isobars, so that the inhomogeneous term (41) will be dominated by the contribution of the term with spin $j_j=j_\ell$, which is infinitely larger than the other ones and therefore we can neglect the sum over j and over j_j and set simply $j_j=j_\ell$. If we now introduce new amplitudes $F_{i\ell}^{j_i m_i\nu_i,j_\ell m_\ell\nu_\ell}(\mathbf{k}_i,\mathbf{k}_\ell)$ as

$$\langle\bar{\phi}^{s_i}_{\nu_i}(\mathbf{k}_i)\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | U_I^{jK} | \psi_{IJK}\rangle = \sum_{j_i m_i m_\ell}\frac{1}{2\omega_{j0}}\langle\bar{\phi}^{s_k}_{\nu_k}(\mathbf{k}_k) | \Gamma^\dagger_{j_i} | \phi^{j_i}_{m_i}(\mathbf{q}_i)\rangle\tau_{j_i}(q_i^2)$$

$$\times F_{i\ell}^{j_i m_i\nu_i,j_\ell m_\ell\nu_\ell}(\mathbf{k}_i,\mathbf{k}_\ell)\tau_{j_\ell}(q_\ell^2)\langle\bar{\phi}^{j_\ell}_{m_\ell}(\mathbf{q}_\ell) | \Gamma_{j_\ell} | \psi_{IJK}\rangle , \tag{42}$$

then these new amplitudes satisfy the set of multichannel coupled equations

$$F_{i\ell}^{j_i m_i\nu_i,j_\ell m_\ell\nu_\ell}(\mathbf{k}_i,\mathbf{k}_\ell)=V_{i\ell}^{j_i m_i\nu_i,j_\ell m_\ell\nu_\ell}(\mathbf{k}_i,\mathbf{k}_\ell)+\sum_{j\neq i}\sum_{j_j m_j\nu_j}\int\frac{d\mathbf{k}_j}{2\omega_j}V_{ij}^{j_i m_i\nu_i,j_j m_j\nu_j}(\mathbf{k}_i,\mathbf{k}_j)\tau_{j_j}(q_j^2)F_{j\ell}^{j_j m_j\nu_j,j_\ell m_\ell\nu_\ell}(\mathbf{k}_j,\mathbf{k}_\ell) , \tag{43}$$

where the transition potentials are

$$V_{ij}^{j_i m_i \nu_i, j_j m_j \nu_j}(\mathbf{k}_i,\mathbf{k}_j)=\langle \bar{\phi}\,^{j_i}_{m_i}(\mathbf{q}_i)\mid \Gamma_{j_i}\mid \phi^{s_j}_{\nu_j}(\mathbf{k}_j)\rangle G_k(\mathbf{k}_i,\mathbf{k}_j)\Lambda_{s_k}(k_k;m_k)\langle \bar{\phi}\,^{s_i}_{\nu_i}(\mathbf{k}_i)\mid \Gamma^\dagger_{j_j}\mid \phi^{j_j}_{m_j}(\mathbf{q}_j)\rangle \ . \tag{44}$$

If the particles also possess isospin, the potential (44) must be multiplied by the isospin transition coefficients

$$B_{ij}^{I_iI_j}=(-1)^{I_j+i_j-I_T}(2I_i+1)^{1/2}\times(2I_j+1)^{1/2}W(i_ji_kI_Ti_i;I_iI_j) \ , \tag{45}$$

where W is a Racah coefficient and i_i is the isospin of particle i, I_i is the isospin of the pair j,k, and I_T is the total isospin.

The solution of the integral equation (43) gives directly the transition amplitudes $F_{i\ell}^{j_i m_i \nu_i, j_\ell m_\ell \nu_\ell}(\mathbf{k}_i,\mathbf{k}_\ell)$ for the processes going from an initial state of the system where we have a stable isobar of spin j_ℓ and helicity m_ℓ to a final state of the system where we have an isobar of spin j_i and helicity m_i. If j_i is also one of the stable isobars, then these amplitudes correspond to two-body→two-body processes (such as πd→πd, NN→πd, or NN→NN), while if j_i is an unstable isobar, then the amplitudes F describe two-body→three-body processes (such as πd→πNN or NN→πNN).

E. Unitarity

The integral equation (43) is of the form

$$F_{i\ell}^{\beta\alpha}=V_{i\ell}^{\beta\alpha}+\sum_{j=1}^{3}\sum_{\gamma}V_{ij}^{\beta\gamma}\tau_j^\gamma F_{j\ell}^{\gamma\alpha}, \quad i=1,2,3 \tag{46}$$

where the transition potentials are such that

$$V_{ij}^{\beta\gamma}=0 \quad \text{if } i=j \ , \tag{47}$$

and $\alpha=\{j_\ell m_\ell \nu_\ell \mathbf{q}_\ell\}$ are the quantum numbers of the stable isobar, while $\beta=\{j_im_i\nu_i\mathbf{k}_i\}$ and $\gamma=\{j_jm_j\nu_j\mathbf{k}_j\}$, such that

$$\sum_\beta \to \sum_{j_im_i\nu_i}\int\frac{d\mathbf{k}_i}{2\omega_i} \ , \tag{48}$$

$$\sum_\gamma \to \sum_{j_jm_j\nu_j}\int\frac{d\mathbf{k}_j}{2\omega_j} \ . \tag{49}$$

Equation (46) obeys the discontinuity relation

$$F_{n\ell}^{\rho\alpha^\dagger}-F_{n\ell}^{\rho\alpha}=\sum_{\substack{ij\\ i\neq j}}\sum_{\beta\gamma}(\delta_{ni}\delta_{\rho\beta}+F_{ni}^{\rho\beta^\dagger}\tau_i^{\beta^\dagger})\times(V_{ij}^{\beta\gamma^\dagger}-V_{ij}^{\beta\gamma})(\delta_{j\ell}\delta_{\gamma\alpha}+\tau_j^\gamma F_{j\ell}^{\gamma\alpha})+\sum_i\sum_\beta F_{ni}^{\rho\beta^\dagger}(\tau_i^{\beta^\dagger}-\tau_i^\beta)F_{i\ell}^{\beta\alpha} \ , \tag{50}$$

where $\rho=\{j_nm_n\nu_n\mathbf{q}_n\}$ will be taken to be also one of the stable isobars. Thus, since the states on the right- and left-hand sides in Eq. (50) are both stable isobars, the potentials $V_{nj}^{\rho\gamma}$ and $V_{i\ell}^{\beta\alpha}$ are real, since with the momenta $\mathbf{q}_n$ or $\mathbf{q}_\ell$ the delta function in Eq. (20) cannot be satisfied. Thus, the terms

$$(V_{nj}^{\rho\gamma^\dagger}-V_{nj}^{\rho\gamma})(\delta_{j\ell}\delta_{\gamma\alpha}+\tau_j^\gamma F_{j\ell}^{\gamma\alpha})$$

and

$$(\delta_{ni}\delta_{\rho\beta}+F_{ni}^{\rho\beta^\dagger}\tau_i^{\beta^\dagger})(V_{i\ell}^{\beta\alpha^\dagger}-V_{i\ell}^{\beta\alpha})$$

both vanish in Eq. (50), so that dropping these terms and multiplying on right- and left-hand sides by the coupling constants f_ℓ and f_n, respectively, we obtain

$$f_nF_{n\ell}^{\rho\alpha^\dagger}f_\ell-f_nF_{n\ell}^{\rho\alpha}f_\ell=\sum_{\substack{ij\\ i\neq j}}\sum_{\beta\gamma}f_nF_{ni}^{\rho\beta^\dagger}\tau_i^{\beta^\dagger}(V_{ij}^{\beta\gamma^\dagger}-V_{ij}^{\beta\gamma})\tau_j^\gamma F_{j\ell}^{\gamma\alpha}f_\ell+\sum_i\sum_\beta f_nF_{ni}^{\rho\beta^\dagger}(\tau_i^{\beta^\dagger}-\tau_i^\beta)F_{i\ell}^{\beta\alpha}f_\ell \ . \tag{51}$$

Using Eqs. (44) and (20), we see that the discontinuity of the potentials is

$$\begin{aligned}V_{ij}^{\beta\gamma^\dagger}-V_{ij}^{\beta\gamma}&=2\pi i\langle \bar{\phi}\,^{j_i}_{m_i}(\mathbf{q}_i)\mid \Gamma_{j_i}\mid \phi^{s_j}_{\nu_j}(\mathbf{k}_j)\rangle\frac{1}{2\omega_k}\delta(\sqrt{S}-\omega_i-\omega_j-\omega_k)\Lambda_{s_k}(k_k;m_k)\langle \bar{\phi}\,^{s_i}_{\nu_i}(\mathbf{k}_i)\mid \Gamma^\dagger_{j_j}\mid \phi^{j_j}_{m_j}(\mathbf{q}_j)\rangle\\&=2\pi i\sum_{\nu_k}\int\frac{d\mathbf{k}_k}{2\omega_k}\delta^4(K-k_i-k_j-k_k)\langle \bar{\phi}\,^{j_i}_{m_i}(\mathbf{q}_i)\mid \Gamma_{j_i}\mid \phi^{s_j}_{\nu_j}(\mathbf{k}_j)\phi^{s_k}_{\nu_k}(\mathbf{k}_k)\rangle\langle \bar{\phi}\,^{s_i}_{\nu_i}(\mathbf{k}_i)\bar{\phi}\,^{s_k}_{\nu_k}(\mathbf{k}_k)\mid \Gamma^\dagger_{j_j}\mid \phi^{j_j}_{m_j}(\mathbf{q}_j)\rangle \ ,\end{aligned} \tag{52}$$

so that the first term on the rhs of Eq. (51) is

$$\begin{aligned}&\sum_{\substack{ij\\ i\neq j}}\sum_{\beta\gamma}f_nF_{ni}^{\rho\beta^\dagger}\tau_i^{\beta^\dagger}(V_{ij}^{\beta\gamma^\dagger}-V_{ij}^{\beta\gamma})\tau_j^\gamma F_{j\ell}^{\gamma\alpha}f_\ell\\&\quad=2\pi i\sum_{\substack{ij\\ i\neq j}}\sum_{\nu_i\nu_j\nu_k}\int\frac{d\mathbf{k}_id\mathbf{k}_jd\mathbf{k}_k}{2\omega_i2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)H_{ni}^{m_n\nu_n,\nu_i\nu_j\nu_k^\dagger}(\mathbf{q}_n,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)H_{\ell j}^{m_\ell\nu_\ell,\nu_i\nu_j\nu_k}(\mathbf{q}_\ell,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k) \ ,\end{aligned} \tag{53}$$

where we have used Eqs. (48) and (49), and where we have defined

$$H_{ni}^{m_n\nu_n,\,\nu_i\nu_j\nu_k^\dagger}(\mathbf{q}_n,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)=\sum_{j_im_i} f_n F_{ni}^{\rho\beta^\dagger}\tau_i^{\beta^\dagger}\langle\bar{\phi}_{m_i}^{j_i}(\mathbf{q}_i)\,|\,\Gamma_{j_i}\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(\mathbf{k}_k)\rangle\ . \tag{54}$$

Similarly, using Eqs. (29), (30), and (35), we find that the discontinuity of the functions τ_i^β is

$$\tau_i^{\beta^\dagger}-\tau_i^\beta=2\pi i\sum_{\nu_j\nu_k}\int\frac{d\mathbf{k}_jd\mathbf{k}_k}{2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)\tau_i^{\beta^\dagger}\langle\bar{\phi}_{m_i}^{j_i}(\mathbf{q}_i)\,|\,\Gamma_{j_i}\,|\,\phi_{\nu_j}^{s_j}(\mathbf{k}_j)\phi_{\nu_k}^{s_k}(\mathbf{k}_k)\rangle\langle\bar{\phi}_{\nu_j}^{s_j}(\mathbf{k}_j)\bar{\phi}_{\nu_k}^{s_k}(\mathbf{k}_k)\,|\,\Gamma_{j_i}^\dagger\,|\,\phi_{m_i}^{j_i}(\mathbf{q}_i)\rangle\tau_i^\beta$$
$$+2\pi i\delta_{j_ij_r}f_r^2\int\frac{d\mathbf{q}_r}{2\omega_r}\delta^4(K-k_i-q_r)\ , \tag{55}$$

so that the second term on the rhs of Eq. (51) is

$$\sum_i\sum_\beta f_nF_{ni}^{\rho\beta^\dagger}(\tau_i^{\beta^\dagger}-\tau_i^\beta)F_{i\ell}^{\beta\alpha}f_\ell=2\pi i\sum_i\sum_{\nu_i\nu_j\nu_k}\int\frac{d\mathbf{k}_id\mathbf{k}_jd\mathbf{k}_k}{2\omega_i2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)$$
$$\times H_{ni}^{m_n\nu_n,\,\nu_i\nu_j\nu_k^\dagger}(\mathbf{q}_n,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)H_{\ell i}^{m_\ell\nu_\ell,\,\nu_i\nu_j\nu_k}(\mathbf{q}_\ell,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)$$
$$+2\pi i\sum_{r=\ell,n}\sum_{m_i\nu_i}\int\frac{d\mathbf{k}_id\mathbf{q}_r}{2\omega_i2\omega_r}\delta^4(K-k_i-q_r)H_{nr}^{m_n\nu_n,\,m_i\nu_i^\dagger}(\mathbf{q}_n,\mathbf{q}_r)H_{\ell r}^{m_\ell\nu_\ell,\,m_i\nu_i}(\mathbf{q}_\ell,\mathbf{q}_r)\ , \tag{56}$$

where we have defined

$$H_{nr}^{m_n\nu_n,\,m_i\nu_i^\dagger}(\mathbf{q}_n,\mathbf{q}_r)=f_nF_{ni}^{\rho\beta^\dagger}f_r\,|_{\,j_i=j_r,\ q_i^2=q_r^2}\ . \tag{57}$$

Using Eqs. (53) and (56), Eq. (51) becomes

$$H_{n\ell}^{m_n\nu_n,\,m_\ell\nu_\ell^\dagger}(\mathbf{q}_n,\mathbf{q}_\ell)-H_{n\ell}^{m_n\nu_n,\,m_\ell\nu_\ell}(\mathbf{q}_n,\mathbf{q}_\ell)$$
$$=2\pi i\sum_{ij}\sum_{\nu_i\nu_j\nu_k}\int\frac{d\mathbf{k}_id\mathbf{k}_jd\mathbf{k}_k}{2\omega_i2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)H_{ni}^{m_n\nu_n,\,\nu_i\nu_j\nu_k^\dagger}(\mathbf{q}_n,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)H_{\ell j}^{m_\ell\nu_\ell,\,\nu_i\nu_j\nu_k}(\mathbf{q}_\ell,\,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)$$
$$+2\pi i\sum_{r=\ell,n}\sum_{m_i\nu_i}\int\frac{d\mathbf{k}_id\mathbf{q}_r}{2\omega_i2\omega_r}\delta^4(K-k_i-q_r)H_{nr}^{m_n\nu_n,\,m_i\nu_i^\dagger}(\mathbf{q}_n,\mathbf{q}_r)H_{\ell r}^{m_\ell\nu_\ell,\,m_i\nu_i}(\mathbf{q}_\ell,\mathbf{q}_r)\ , \tag{58}$$

which is the desired unitarity relation. Following similar steps, a corresponding unitarity relation can be obtained for the case when the final state is a three-body continuum state, in which case the term $(V_{nj}^{\rho\gamma^\dagger}-V_{nj}^{\rho\gamma})\tau_j^\gamma F_{j\ell}^{\gamma\alpha}$ in Eq. (50) also contributes.

Using Eq. (58) in the special case $n=\ell$, we get

$$-\frac{4\pi^3}{\sqrt{S}\,|\mathbf{q}_\ell|}\mathrm{Im}H_{\ell\ell}^{m_\ell\nu_\ell,\,m_\ell\nu_\ell}(\mathbf{q}_\ell,\mathbf{q}_\ell)=\frac{4\pi^4}{\sqrt{S}\,|\mathbf{q}_\ell|}\sum_{\nu_i\nu_j\nu_k}\int\frac{d\mathbf{k}_id\mathbf{k}_jd\mathbf{k}_k}{2\omega_i2\omega_j2\omega_k}\delta^4(K-k_i-k_j-k_k)\left|\sum_{i=1}^3H_{\ell i}^{m_\ell\nu_\ell,\,\nu_i\nu_j\nu_k}(\mathbf{q}_\ell,\,\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k)\right|^2$$
$$+\frac{4\pi^4}{\sqrt{S}\,|\mathbf{q}_\ell|}\sum_{r=\ell,n}\sum_{m_i\nu_i}\int\frac{d\mathbf{k}_id\mathbf{q}_r}{2\omega_i2\omega_r}\delta^4(K-k_i-q_r)\,|\,H_{\ell r}^{m_\ell\nu_\ell,\,m_i\nu_i}(\mathbf{q}_\ell,\mathbf{q}_r)\,|^2$$
$$=\sigma_{\ell\to ijk}^{m_\ell\nu_\ell}+\sigma_{\ell\to\ell}^{m_\ell\nu_\ell}+\sigma_{\ell\to n}^{m_\ell\nu_\ell}=\sigma_{\mathrm{tot}}^{m_\ell\nu_\ell}\ , \tag{59}$$

which is the optical theorem for the case when the projectile and target have definite helicities m_ℓ and ν_ℓ. If we average both sides of Eq. (59) over the helicities m_ℓ and ν_ℓ, we obtain the optical theorem for the case of unpolarized projectile and target.

F. Angular momentum decomposition

The transition potentials and amplitudes V_{ij} and F_{ij} given by Eqs. (43) and (44) describe transitions from a quasi-two-body state where the spins and helicities are $j_im_i,s_i\nu_i$ to a state where the spins and helicities are $j_jm_j,s_j\nu_j$. Thus, we can use the two-body helicity formalism of Jacob and Wick[25] to expand them in terms of angular momentum partial waves as

$$A_{ij}^{j_im_i\nu_i,j_jm_j\nu_j}(\mathbf{k}_i,\mathbf{k}_j)=\sum_{JM}\frac{2J+1}{4\pi}A_{ij,J}^{j_im_i\nu_i,j_jm_j\nu_j}(k_i,k_j)\mathscr{D}^{J*}_{M,m_i-\nu_i}(\hat{\mathbf{k}}_i)\mathscr{D}^{J}_{M,m_j-\nu_j}(\hat{\mathbf{k}}_j)\ , \tag{60}$$

where $A=\{F \text{ or } V\}$ and k_i and k_j mean, only for the rest of this subsection, the magnitudes of the three-momenta $\mathbf{k}_i$ and $\mathbf{k}_j$. The inverse transformation of Eq. (60) is

$$A_{ij,J}^{j_im_i\nu_i,j_jm_j\nu_j}(k_i,k_j)=\int_{-1}^{1}d\cos\theta_{ij}d^{J}_{m_j-\nu_j,m_i-\nu_i}(\theta_{ij})A_{ij}^{j_im_i\nu_i,j_jm_j\nu_j}(\mathbf{k}_i,\mathbf{k}_j)\ , \tag{61}$$

with

$$\cos\theta_{ij}=\hat{\mathbf{k}}_i\cdot\hat{\mathbf{k}}_j\ . \tag{62}$$

If we apply the partial wave expansion (60) into Eq. (43), we obtain the partial-wave integral equations

$$F_{i\ell,J}^{j_im_i\nu_i,j_\ell m_\ell\nu_\ell}(k_i,k_\ell)=V_{i\ell,J}^{j_im_i\nu_i,j_\ell m_\ell\nu_\ell}(k_i,k_\ell)+\sum_{j\neq i}\sum_{j_jm_j\nu_j}\int_0^\infty\frac{k_j^2dk_j}{2\omega_j}V_{ij,J}^{j_im_i\nu_i,j_jm_j\nu_j}(k_i,k_j)\tau_{j_j}(q_j^2)F_{j\ell,J}^{j_jm_j\nu_j,j_\ell m_\ell\nu_\ell}(k_j,k_\ell)\ . \tag{63}$$

These equations can be further reduced by taking into account the invariance of the strong interactions under the parity transformation, which implies that

$$A_{ij,J}^{j_im_i\nu_i,j_jm_j\nu_j}(k_i,k_j)=(-1)^{\eta_i+\eta_j}A_{ij,J}^{j_i-m_i-\nu_i,j_j-m_j-\nu_j}(k_i,k_j)\ , \tag{64}$$

with $A=\{F \text{ or } V\}$, and as indicated by Eqs. (34),

$$\eta_i=l_i-j_i+s_j+s_k\ . \tag{65}$$

Thus, if the particles have spins $s_j\leq\frac{1}{2}$, we can introduce the linear combination of amplitudes

$$A_{ij,J\pm}^{j_im_i,j_jm_j}(k_i,k_j)=(s_j+\tfrac{1}{2})[A_{ij,J}^{j_im_is_i,j_jm_js_j}(k_i,k_j)\pm(-1)^{\eta_j}A_{ij,J}^{j_im_is_i,j_j-m_j-s_j}(k_i,k_j)]\ ; \tag{66}$$

then, using Eqs. (64) and (66) in Eq. (63), we obtain the reduced equations

$$F_{i\ell,J\pm}^{j_im_i,j_\ell m_\ell}(k_i,k_\ell)=V_{i\ell,J\pm}^{j_im_i,j_\ell m_\ell}(k_i,k_\ell)+\sum_{j\neq i}\sum_{j_jm_j}\int_0^\infty\frac{k_j^2dk_j}{2\omega_j}V_{ij,J\pm}^{j_im_i,j_jm_j}(k_i,k_j)\tau_{j_j}(q_j^2)F_{j\ell,J\pm}^{j_jm_j,j_\ell m_\ell}(k_j,k_\ell)\ , \tag{67}$$

where the sum over the helicity ν_j of the spectator j has been eliminated and, similarly, the helicity ν_i of the spectator i has been set equal to s_i in Eq. (66). The amplitudes corresponding to $\nu_i=-s_i$ can be obtained by using the parity relation (64). Again, we emphasize that this complete elimination of the helicities ν_i and ν_j is possible in the case of the πNN system only because the spectator is either a pion or a nucleon, which have spins $s_j\leq\frac{1}{2}$.

Equation (67) can be further simplified in the case of the πNN system by taking into account the fact that the two nucleons are identical particles following exactly the same steps as in the nonrelativistic equations, as shown, for example, by Afnan and Thomas[27] or by Avishai and Mizutani.[1].

G. The πNN transition potentials

In the case of the πNN system as mentioned before, the transition potentials can be of two types which are shown pictorially in Figs. 1(a) and 1(b). The potential of Fig. 1(a), where a pion is exchanged between two πN isobars, is given by

$$V_{ij}^{j_im_i\nu_i,j_jm_j\nu_j}(\mathbf{k}_i,\mathbf{k}_j)=\bar{\phi}^{j_i}_{m_i}(\mathbf{q}_i)\Gamma_{j_i}u_{\nu_j}(\mathbf{k}_j)\times G_k(\mathbf{k}_i,\mathbf{k}_j)\bar{u}_{\nu_i}(\mathbf{k}_i)\Gamma^\dagger_{j_j}\phi^{j_j}_{m_j}(\mathbf{q}_j)\ , \tag{68}$$

where $u_{\nu_j}(\mathbf{k}_j)$ is a nucleon spinor of helicity ν_j. The vertex operator and isobar spinor for the case when the πN isobar has spin $j_j=\frac{1}{2}$ and orbital angular momentum $l_j=0$ (the S_{11} and S_{31} channels) are given by

$$\Gamma^\dagger_{j_j}\phi^{j_j}_{m_j}(\mathbf{q}_j)=f_\pi(q_j^2,k_k^2)u_{m_j}(\mathbf{q}_j)\ , \tag{69}$$

where $u_{m_j}(\mathbf{q}_j)$ is a spin $\frac{1}{2}$ spinor of helicity m_j and $f_\pi(q_j^2,k_k^2)$ is the pion form factor which is taken to be of the form

$$f_\pi(q_j^2,k_k^2)=\frac{\beta^2+\mathbf{p}_0^2}{\beta^2+\mathbf{p}^2}\ , \tag{70}$$

where $\mathbf{p}$ is the pion-nucleon relative three-momentum in the two-body c.m. frame, which is given by

$$\mathbf{p}^2=(q_j^2+m_{\rm N}^2-k_k^2)^2/4q_j^2-m_{\rm N}^2\ , \tag{71}$$

and $\mathbf{p}_0$ is the relative momentum when the pion is on the mass shell; that is,

$$\mathbf{p}_0^2=(q_j^2+m_N^2-m_\pi^2)^2/4q_j^2-m_N^2 \ . \tag{72}$$

The vertex operator and isobar spinor for the case when the πN isobar has spin $j_j=\frac{1}{2}$ and orbital angular momentum $l_j=1$ (the P_{11} and P_{31} channels) is given by

$$\Gamma_{j_j}^{\dagger}\phi_{m_j}^{j_j}(\mathbf{q}_j)=f_\pi(q_j^2,k_k^2)\not{k}_k\gamma_5 u_{m_j}(\mathbf{q}_j) \ , \tag{73}$$

while in the case when the πN isobar has spin $j_j=\frac{3}{2}$ and orbital angular momentum $l_j=1$ (the P_{33} and P_{13} channels) the vertex operator and isobar spinor are given by

$$\Gamma_{j_j}^{\dagger}\phi_{m_j}^{j_j}(\mathbf{q}_j)=f_\pi(q_j^2,k_k^2)k_{k\mu}W_{\mu m_j}(\mathbf{q}_j) \ , \tag{74}$$

where $W_{\mu m_j}(\mathbf{k}_j)$ is a Rarita-Schwinger spinor of helicity m_j. The momentum of the exchanged particle k_k in Eqs. (68)–(74) is given by

$$k_k=(\sqrt{S}-\omega_i-\omega_j,\ -\mathbf{k}_i-\mathbf{k}_j) \ , \tag{75}$$

while the mass in the isobar spinors u_{m_j} and $W_{\mu m_j}$ is $(q_j^2)^{1/2}$, where

$$q_j^2=(\sqrt{S}-\omega_j)^2-\mathbf{k}_j^2 \ . \tag{76}$$

The potentials of the type shown in Fig. 1(b), where a nucleon is exchanged between a πN and a NN isobar, are given by

$$V_{ij}^{j_i m_i \nu_i, j_j m_j \nu_j}(\mathbf{k}_i,\mathbf{k}_j)=\bar{v}_{\nu_j}(\mathbf{k}_j)\bar{\phi}_{m_i}^{j_i}(\mathbf{q}_i)\Gamma_{j_i} \times G_k(\mathbf{k}_i,\mathbf{k}_j)\frac{\not{k}_k+m_N}{2m_N}\Gamma_{j_j}^{\dagger}\phi_{m_j}^{j_j}(\mathbf{q}_j) \ , \tag{77}$$

where the πN vertex and isobar spinor $\Gamma_{j_j}^{\dagger}\phi_{m_j}^{j_j}(\mathbf{q}_j)$ will be the same as Eqs. (69)–(74) with the pion form factor f_π replaced by the nucleon form factor $f_N(k_k^2)$, and $\bar{v}_{\nu_j}(\mathbf{k}_j)$ is a charge conjugated spinor for the external nucleon. Notice that we have replaced the spinor $\phi_{\nu_j}^{s_j}(\mathbf{k}_j)$ that appears in Eq. (44) by the charge conjugated spinor $\bar{v}_{\nu_j}(\mathbf{k}_j)$ which has been moved to the extreme left on the rhs of Eq. (77). This corresponds to drawing the diagram of Fig. 1(b) as in Fig. 1(c). The NN isobar spinor and vertex operator in the case when the isobar has spin $j_i=1$ and orbital angular momentum $l_i=0$ or 2 (the $^3S_1$-$^3D_1$ channel) are given by

$$\bar{\phi}_{m_i}^{j_i}(\mathbf{q}_i)\Gamma_{j_i}=\bar{\epsilon}_{\mu m_i}(\mathbf{q}_i)[q_\mu A(q_i^2,k_k^2)+\gamma_\mu B(q_i^2,k_k^2)] \ , \tag{78}$$

while in the case when the isobar has spin $j_i=0$ and orbital angular momentum $l_i=0$ (the $^1S_0$ channel) they are given by

$$\bar{\phi}_{m_i}^{j_i}(\mathbf{q}_i)\Gamma_{j_i}=\gamma_5 C(q_i^2,k_k^2) \ , \tag{79}$$

where $\epsilon_{\mu m_i}(\mathbf{q}_i)$ is a spin-1 spinor of helicity m_i and the form factors A, B, and C are constructed in terms of the S- and D-wave components of the deuteron wave function, ψ_0 and ψ_2, and the wave function of the $^1S_0$ antibound state, ψ_a, as

$$A(q_i^2,k_k^2)=(\mathbf{p}^2+\alpha_d)\left[\psi_0(|\mathbf{p}|)+\frac{1}{\sqrt{2}}\psi_2(|\mathbf{p}|)\right] \ , \tag{80}$$

$$B(q_i^2,k_k^2)=(\mathbf{p}^2+\alpha_d)\left[-\frac{1}{\sqrt{2}\mathbf{p}^2}(2\omega_N+m_N)\psi_2(|\mathbf{p}|)+\frac{1}{\mathbf{p}^2}(\omega_N-m_N)\psi_0(|\mathbf{p}|)\right] \ , \tag{81}$$

$$C(q_i^2,k_k^2)=(\mathbf{p}^2+\alpha_a)\psi_a(|\mathbf{p}|) \ , \tag{82}$$

where

$$\mathbf{p}^2=(q_i^2+m_N^2-k_k^2)^2/4q_i^2-m_N^2 \ , \tag{83}$$

$$\omega_N=(m_N^2+\mathbf{p}^2)^{1/2} \ , \tag{84}$$

$$\alpha_i=B_i(m_N-B_i/4), i=\mathrm{d},a \tag{85}$$

and $\mathbf{p}$ is the relative three-momentum of the two nucleons in the nucleon-nucleon c.m. frame, while B_d and B_a are the binding energies of the deuteron and $^1S_0$ antibound states, respectively. The πN and NN vertex functions (69)–(85) are of the required form (33), which also defines the functions $g_{j_i}(|\mathbf{k}_j|)$ that determines the normalization of the on-shell partial-wave amplitudes (38). It should be noticed that the form factor $B(q_i^2,k_k^2)$ given by Eq. (81) is different from the one derived by Gourdin *et al.*,[28] who were interested only in the nonrelativistic limit of the NNd vertex and therefore used nonrelativistic two-component spinors to obtain the connection with the deuteron wave function. Equation (81), on the other hand, was obtained by taking the matrix elements of the NNd vertex (78) between u and $\bar{v}$ four-component helicity spinors and requiring that the NNd vertex function reduce to the form (33).

H. The intermediate NN states

When we have an intermediate state of a spectator nucleon and a πN isobar corresponding to the P_{11} channel, such an intermediate state has the same quantum numbers as a two-nucleon state. This, however, does not mean that it is a two-nucleon state, since in the pion-nucleon P_{11} amplitude there exist not only contributions from a pure nucleon state, but other contributions as well that do not reduce to a pure nucleon state. In order to see this more clearly, let us consider the diagrams of Fig. 3, where we have in Fig. 3(a) a true two-nucleon state in which one of the nucleons is on the mass shell and the other one is off the mass shell, and in Fig. 3(b) a state of a nucleon on the mass shell and a P_{11} isobar. If we would write explicitly the contribution of the diagram of Fig. 3(a), then we would have, from the off-shell nucleon, a vertex function at each vertex and a fermion propagator for the internal line; that is (ignoring the pion form factors),

$$G_{\pi NN}\not{p}_\pi\gamma_5 f_N(q_i^2)\frac{1}{q_i^2-m_N^2+i\epsilon}\frac{\not{q}_i+m_N}{2m_N}G_{\pi NN}\not{p}'_\pi\gamma_5 f_N(q_i^2) \ , \tag{86}$$

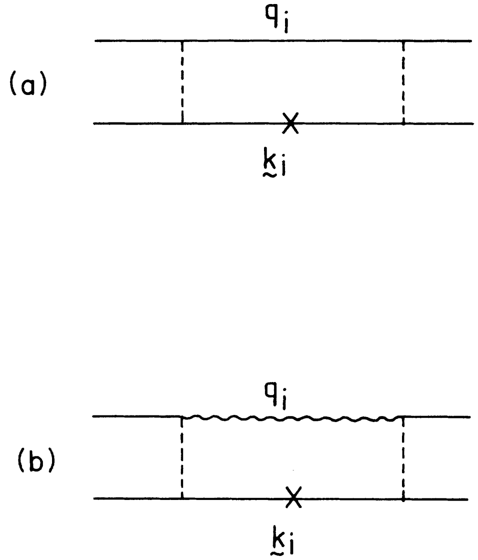

FIG. 3. (a) A process with an intermediate state of two nucleons. (b) A process with an intermediate state of a nucleon and a P_{11} isobar.

where $G_{\pi NN}$ is the πNN pseudovector coupling constant and

$$q_i^2=[\sqrt{S}-(\mathbf{k}_i^2+m_N^2)^{1/2}]^2-\mathbf{k}_i^2\equiv s \ . \tag{87}$$

Similarly, the contribution of the P_{11} isobar in Fig. 3(b) would be given using Eqs. (73), (30), and (23), and ignoring the pion form factors, as

$$\begin{aligned} t_{P_{11}}(q_i^2) &= \not{p}_\pi\gamma_5\tau_{P_{11}}(q_i^2)\Lambda_{1/2}(q_i;M_i=\sqrt{s}\,)\not{p}'_\pi\gamma_5 \\ &= \not{p}_\pi\gamma_5\tau_{P_{11}}(q_i^2)\frac{\not{q}_i+\sqrt{s}}{2\sqrt{s}}\not{p}'_\pi\gamma_5 \ , \end{aligned} \tag{88}$$

where the mass of the isobar is $\sqrt{s}$ as defined by Eq. (87). Equation (88) can also be written in the form

$$\begin{aligned} t_{P_{11}}(q_i) &= \not{p}_\pi\gamma_5\tau_{P_{11}}(q_i^2)\frac{\not{q}_i+m_N}{2\sqrt{s}}\not{p}'_\pi\gamma_5 \\ &\quad +\not{p}_\pi\gamma_5\tau_{P_{11}}(q_i^2)\frac{\sqrt{s}-m_N}{2\sqrt{s}}\not{p}'_\pi\gamma_5 \ , \end{aligned} \tag{89}$$

where we have added and substracted a term proportional to m_N. Since the function $\tau_{P_{11}}(q_i^2)$ has a pole when $q_i^2=m_N^2$ [see Eq. (39)], we see that the first term in Eq. (89) has very similar structure to Eq. (86). They can be made to be exactly identical by requiring that the nucleon form factor f_N be related to the function $\tau_{P_{11}}$ as

$$\frac{G_{\pi NN}^2 f_N^2(q_i^2)}{m_N}=\frac{q_i^2-m_N^2}{\sqrt{s}}\tau_{P_{11}}(q_i^2) \ . \tag{90}$$

Thus, with the help of Eq. (90) we see that we have decomposed in Eq. (89) the P_{11} amplitude into a pure nucleon term plus a remainder. These two terms can also be called by the more fashionable names of the pole and nonpole parts of the P_{11} amplitude, since in the second term of Eq. (89) the pole of $\tau_{P_{11}}(q_i^2)$ at $q_i^2=m_N^2$ is cancelled by the factor $\sqrt{s}-m_N$. We should also point out that the decomposition (89) is unique within a theory based in the isobar model for the P_{11} amplitude.

The problem of the decomposition of the P_{11} amplitude into a pole and a nonpole part has been the subject of great discussion, since when the pole part is taken together with another nucleon, only those channels that are consistent with the Pauli principle are allowed to exist, while if the nonpole part is taken together with another nucleon all channels can exist. Thus, as discussed before, when the pole and nonpole parts are both large, even though their sum is small, this generates large spurious effects, as encountered in the NN-πNN theory.[18] In the decomposition (89), on the other hand, both the pole and nonpole parts are small since they are both multiplied by the on-shell function $\tau_{P_{11}}(q_i^2)$, which is very small in the physical region $q_i^2\geq(m_N+m_\pi)^2$. In addition, not only are both pole and nonpole parts small, but the nonpole part is much smaller than the pole part due to the factor $(\sqrt{s}-m_N)/2\sqrt{s}$, which is approximately $\frac{1}{15}$ at the πN threshold $\sqrt{s}=m_N+m_\pi$. Thus, the contribution of the nonpole part of the P_{11} amplitude in those intermediate states where the pole part is not allowed by the Pauli principle is completely negligible in this theory.

III. RESULTS

In order to obtain the πNN transition potentials defined by Eqs. (68)—(85), we first constructed the isobar spinors for spin $\frac{1}{2}$, 1, and $\frac{3}{2}$ as shown in pp. 462—464 of Ref. 29, putting one isobar along the positive Z axis and the other in the XZ plane at an angle θ with respect to the Z axis, and after performing the Dirac algebra, carried out the partial-wave projection by using Eqs. (61) and (62). The partial-wave transition potentials were then checked against those obtained by using Wick's three-body helicity formalism[26,30] in the special case where the exchanged particle is on the mass shell, which also allowed us to fix the phases. The Wick formalism applies only for the case when all three particles are on the mass shell. We found, however, that if the exchanged particle is a pion, then our transition potentials are identical with those of the Wick formalism also when the pion is off the mass shell, provided we evaluate the full relativistic kinematics, using for the mass of the pion not the physical mass m_π, but the off-shell mass $(k_k^2)^{1/2}$. This result is, of course, a consequence of the fact that the pion has spin 0. In the case when the exchanged particle is a nucleon, our transition potentials are identical to those of the Wick formalism only on shell, but we found that if the nucleon is not very far from the mass shell, the Wick formalism provides also a very good approximation to the exact result.

Using Eq. (33) and the expressions for the isobar spinors and vertices given by Eqs. (69)—(85), we determine the connection between the functions $\tau_{j_i}(q_i^2)$ and the reduced on-shell amplitudes $e^{i\delta}\sin\delta/p^{2l+1}$. Thus, in the case of the pion-nucleon subsystem, the functions $\tau_{j_i}(q_i^2)$ for the six S and P-wave channels were constructed directly from the experimental pion-nucleon phase shifts for the physical region $q_i^2 \geq (m_N + m_\pi)^2$. For the unphysical region $(m_N - m_\pi)^2 \leq q_i^2 < (m_N + m_\pi)^2$, we used the partial-wave amplitudes obtained by Nielsen and Oades[31] from the application of fixed-t dispersion relations. In the region $0 < q_i^2 < (m_N - m_\pi)^2 = q_0^2$, we used the simple extrapolation formula

$$\tau_{j_i}(q_i^2) = \tau_{j_i}(q_0^2)/(2 - q_i^2/q_0^2)^n ,$$

where we have found that the results are completely insensitive to the value of the exponent n and therefore we have taken $n = 1$. In the case of the nucleon-nucleon subsystem, we constructed the functions $\tau_{j_i}(q_i^2)$ for the $^1S_0$ and $^3S_1$-$^3D_1$ channels by applying the unitary pole approximation to the $^1S_0$ antibound and $^3S_1$-$^3D_1$ bound-state wave functions of the Paris potential,[32] which are the same wave functions used in the vertices and form factors (78)–(82).

For the range parameter β of the pion form factor $f_\pi(q_j^2, k_k^2)$ given by Eq. (70), we used the value $\beta = 600$ MeV/c, which was obtained by making a rough fit of the total cross section of the reaction πd$\rightarrow$NN, which is very sensitive to the value of β. The reaction πd$\rightarrow\pi$d, on the other hand, is not sensitive to the value of β, since as shown in Fig. 2(a) it proceeds mainly by nucleon exchange. The nucleon form factor $f_N(q_i^2)$ in this theory is related to the reduced on-shell amplitude of the pion-nucleon P_{11} channel, $\tau_{P_{11}}(q_i^2)$, as shown in Eq. (90). Thus, we can obtain the reduced on-shell amplitude from this equation if we know the nucleon form factor or vice versa. Thus, in the region $q_i^2 \geq (m_N + m_\pi)^2$ we constructed the reduced amplitude $\tau_{P_{11}}(q_i^2)$ directly from the phase shift and used Eq. (90) to obtain the nucleon form factor in that region. In the region $q_i^2 \leq m_N^2$ we used a model for the nucleon form factor of the form

$$f_N(q_i^2) = \frac{\Lambda^2 - m_N^2}{\Lambda^2 - q_i^2} , \tag{91}$$

and applied Eq. (90) to obtain the reduced on-shell amplitude in that region. Finally, in the region $m_N^2 < q_i^2 < (m_N + m_\pi)^2$ we used the form

$$f_N(q_i^2) = \sum_{n=0}^{3} a_n (q_i^2)^n , \tag{92}$$

where the four constants a_n were obtained by requiring that $f_N(q_i^2)$ and its derivative be continuous at the two points $q_i^2 = m_N^2$ and $q_i^2 = (m_N + m_\pi)^2$. Since the reaction πd$\rightarrow\pi$d proceeds predominantly by nucleon exchange [see Fig. 2(a)], we expected that it would be sensitive to the value of the range Λ of the nucleon form factor (91). We show in Fig. 4 the pion-deuteron total cross section calculated using for the range Λ the values $\Lambda = 1300$ MeV/c (solid line) and $\Lambda = 1600$ MeV/c (dashed line). In the region below resonance the range 1600 MeV/c would seem to be better, although above the resonance the range 1300 MeV/c is favored. We show in Fig. 5 the differential cross section throughout the region of the 3,3 resonance, again using the same two values for Λ in the nucleon form factor. The range $\Lambda = 1300$ MeV/c seems to give the best agreement in the forward direction, where the

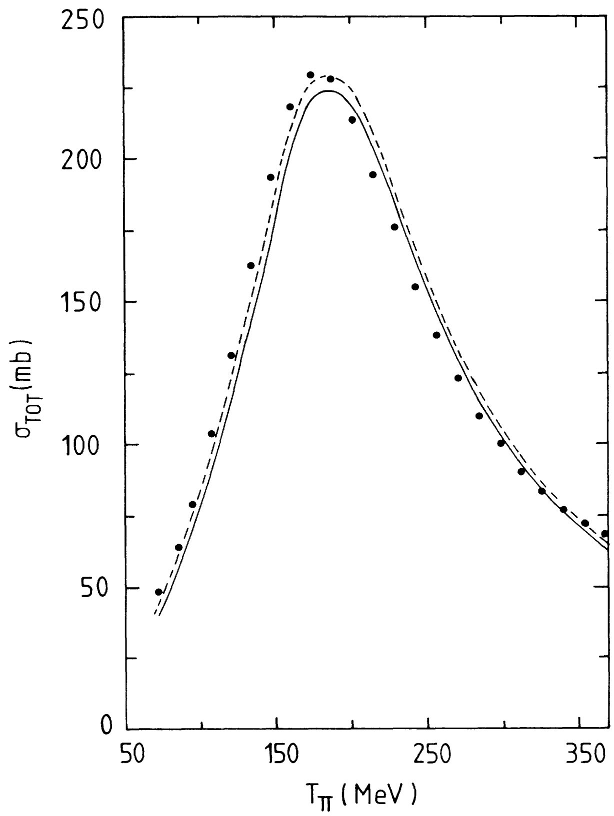

FIG. 4. The pion-deuteron total cross section calculated using for the range of the nucleon form factor Λ the values $\Lambda = 1300$ MeV/c (solid line) and $\Lambda = 1600$ MeV/c (dashed line). The experimental data are from Ref. 33.

cross section is large, although both models fail by about a factor of 2 in the region $\theta_{c.m.} > 90°$ at the higher energies. These discrepancies between theory and experiment have been interpreted by Ferreira and Dosch[13] as indicating a short-range delta-nucleon interaction which is not contained in the Faddeev amplitude and which they introduce phenomenologically and adjust so as to fit the data.

Since we found that the polarization observables are even less sensitive to the value of Λ than the total and differential cross sections, we will use from now on the value $\Lambda = 1300$ MeV/c. We show in Fig. 6 the vector analyzing power iT_{11} in the region of the 3,3 resonance, where we see that both the size of iT_{11} and its qualitative behavior as a function of energy and angle are reproduced by the theory, although some discrepancies still remain in some cases, particularly around $\theta_{c.m.} \simeq 70°$. In Figs. 7(a) and 7(b) we present the results for the tensor analyzing power T_{20} and tensor polarization t_{20}, respectively, where in the last case the results are calculated in the laboratory frame. As we see, there is quite good agreement with most of the data and with the fact that t_{20} and T_{20} are large and negative. A set of controversial measurements by Grüebler *et al.*[37] which gave values of t_{20} that were large and positive have now been contradicted by three different experimental groups,[4–6] while at the same time the calculations

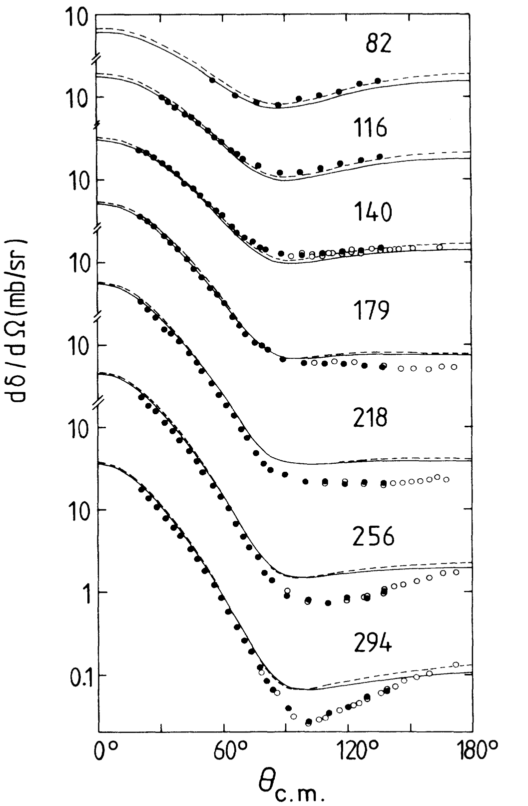

FIG. 5. The pion-deuteron differential cross section for seven different laboratory kinetic energies of the pion (in MeV). The solid and dashed lines have the same meaning as in Fig. 4. The experimental data are from Refs. 34 (solid circles) and 35 (open circles).

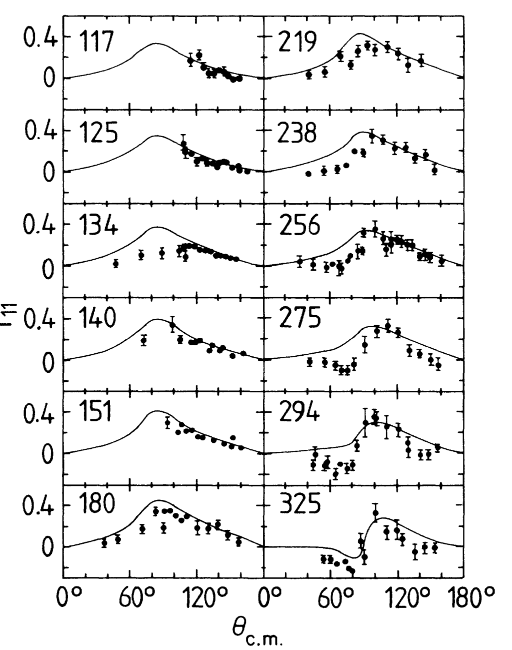

FIG. 6. The pion-deuteron vector analyzing power iT_{11} for 12 different laboratory kinetic energies of the pion (in MeV). The experimental data are from Ref. 36.

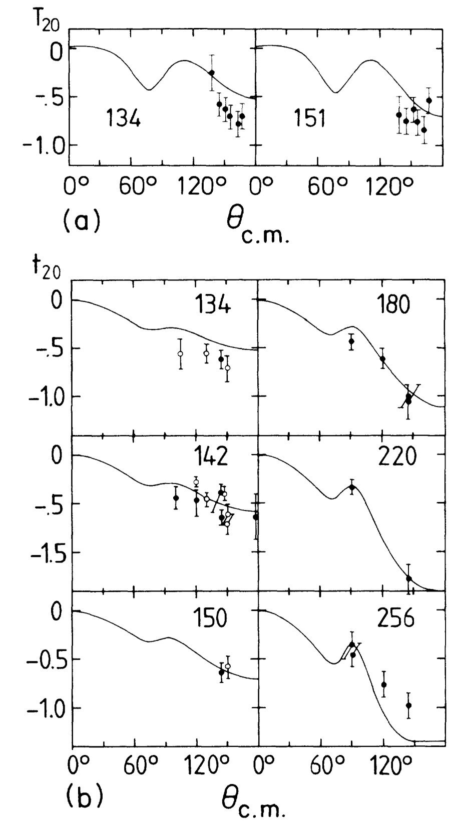

FIG. 7. (a) The pion-deuteron tensor analyzing power T_{20} for two different kinetic energies of the pion (in MeV). The experimental data are from Ref. 6. (b) The pion-deuteron tensor polarization t_{20} for six different kinetic energies of the pion (in MeV). The experimental data are from Refs. 4 (solid circles) and 5 (open circles).

of Afnan and McLeod[18] that predicted large and positive values of t_{20} were the result of the spurious effects introduced by the application of the Pauli principle in intermediate NN states as explained before. In Fig. 8 we show some predictions at three different energies for the tensor analyzing powers T_{20}, T_{21}, and T_{22} and the spin-transfer coefficients it_{20}^{11}, it_{21}^{11}, and it_{22}^{11}, where the last three are calculated in the laboratory system. Of particular interest is the spin-transfer coefficient it_{20}^{11} that will be measured in the near future[8] and which shows a strong energy dependence, particularly around $\theta_{\rm c.m.} \simeq 90°$.

IV. CONCLUSIONS

We have formulated a novel relativistic Faddeev theory of the πNN system by applying a variable-mass isobar ansatz to the πN and NN amplitudes and requiring that the spectator particles be always on their mass shells. This theory takes into account the two possible time orderings in the propagator of the exchanged particle, which solves the problem of the lack of strength of the one-pion—exchange potential and eliminates the need to introduce artificial processes in which the pion is absorbed by

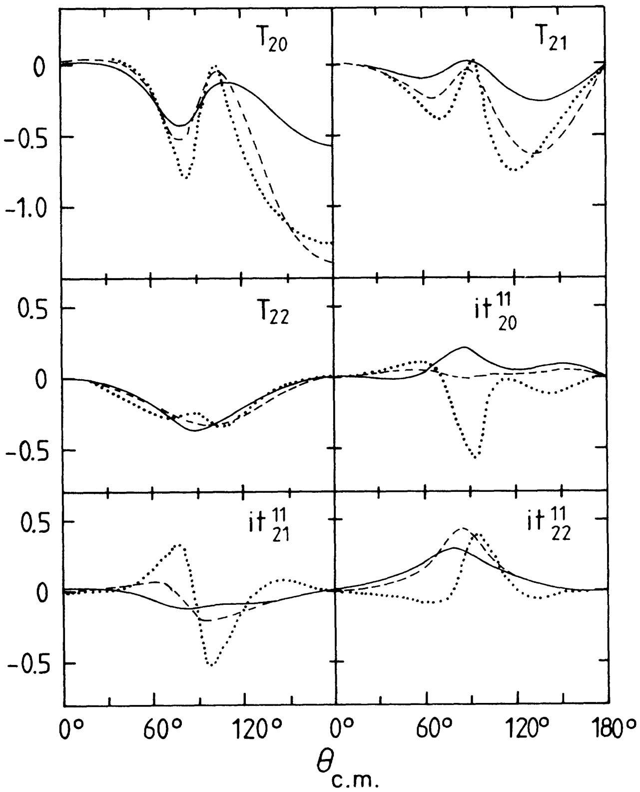

FIG. 8. Some tensor analyzing powers and spin-transfer coefficients calculated at laboratory pion kinetic energies of 140 MeV (solid lines), 220 MeV (dashed lines), and 325 MeV (dotted lines).

one nucleon and emitted by the other one. If we decompose the pion-nucleon P_{11} amplitude into pole and nonpole parts, this does not generate large spurious effects as a result of the application of the Pauli principle in intermediate NN states. As a first application of this theory, we have studied pion-deuteron elastic scattering in the region of the 3,3 resonance by calculating all the observables for which data exists, finding reasonably good agreement over the entire set. Some predictions for other polarization observables that will be measured in the near future have been presented. Future applications of this theory will be the simultaneous description of all five reactions that can take place in the πNN system; that is, $\pi d \to \pi d$, $\pi d \to \pi NN$, $\pi d \to NN$, $NN \to NN$, and $NN \to \pi NN$. This hopefully may help us to understand some unknown aspects of the πNN system, such as the nature of the coupling between a pion and a nucleon; that is, whether it is pseudovector (as we have assumed) or pseudoscalar. Similarly, as already pointed out by Ferreira and Dosch,[13] the reactions of the πNN system can be used to extract information on the short-range part of the nucleon-nucleon and nucleon-delta interaction. Finally, by treating all the reactions of the πNN system simultaneously, one may perhaps even be able to learn something about the subnuclear degrees of freedom of the pion and the nucleon, by going into kinematical regions where the standard description based in the exchange of mesons and baryons is not sufficient to give an adequate description of the experimental data.

ACKNOWLEDGMENTS

I have benefited greatly during the development of this work from the discussions with my colleagues B. Blankleider, E. Boschitz, E. Ferreira, M. Locher, T. Mizutani, and P. U. Sauer. This work was supported by the Bundesministerium für Forschung und Technologie of the Federal Republic of Germany.

---

*On leave from Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, México 14 Distrito Federal, México.

[1]Y. Avishai and T. Mizutani, Nucl. Phys. **A326**, 352 (1979).
[2]B. Blankleider and I. R. Afnan, Phys. Rev. C **24**, 1572 (1981).
[3]A. S. Rinat, Y. Starkand, and E. Hammel, Nucl. Phys. **A364**, 486 (1981).
[4]E. Ungricht *et al.*, Phys. Rev. Lett. **52**, 333 (1984); Phys. Rev. C **31**, 934 (1985).
[5]Y. M. Shin *et al.*, Phys. Rev. Lett. **55**, 2672 (1985).
[6]G. R. Smith *et al.*, Phys. Rev. Lett. **57**, 803 (1986).
[7]H. Garcilazo, Phys. Rev. Lett. **53**, 652 (1984).
[8]Y. M. Shin, private communication.
[9]J. Arvieux and A. S. Rinat, Nucl. Phys. **A350**, 205 (1980).
[10]N. Hiroshige *et al.*, Progr. Theor. Phys. **72**, 1282 (1984).
[11]W. Grein and M. P. Locher, J. Phys. G **7**, 1355 (1981).
[12]M. P. Locher and M. E. Sainio, Phys. Lett. **121B**, 227 (1983).
[13]E. Ferreira and H. G. Dosch (unpublished).
[14]R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. **174**, 2022 (1968).
[15]W. M. Kloet *et al.*, Phys. Rev. Lett. **39**, 1643 (1977); Nucl. Phys. **A338**, 281 (1980); *ibid.* **A338**, 317 (1980).
[16]R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
[17]H. Garcilazo, Phys. Rev. Lett. **45**, 780 (1980).
[18]I. R. Afnan and R. J. McLeod, Phys. Rev. C **31**, 1821 (1985).
[19]H. Garcilazo, Phys. Lett. **99B**, 195 (1981).
[20]L. Mathelitsch and H. Garcilazo, Phys. Rev. C **33**, 2075 (1986).
[21]F. Gross, Phys. Rev. C **26**, 2226 (1982).
[22]This cancelation of the unphysical cut was communicated to me by T. Mizutani, private communication.
[23]W. Grein *et al.*, Ann. Phys. (N.Y.) **153**, 301 (1984).
[24]M. P. Locher and A. Švarc, J. Phys. G **11**, 183 (1985).
[25]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).
[26]G. C. Wick, Ann. Phys. (N.Y.) **18**, 65 (1962).
[27]I. R. Afnan and A. W. Thomas, Phys. Rev. C **10**, 109 (1974).
[28]M. Gourdin *et al.*, Nuovo Cimento **37**, 524 (1965).
[29]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), pp. 462–464.
[30]H. Garcilazo, Nucl. Phys. **A360**, 411 (1981); J. Math. Phys.

27, 2576 (1986).
[31]H. Nielsen and G. C. Oades, Nucl. Phys. **B49**, 573 (1972).
[32]M. Lacombe *et al.*, Phys. Rev. C **21**, 861 (1980).
[33]E. Pedroni *et al.*, Nucl. Phys. **A300**, 321 (1978).
[34]K. Gabathuler *et al.*, Nucl. Phys. **A350**, 253 (1980).
[35]C. R. Ottermann *et al.*, Phys. Rev. C **32**, 928 (1985).
[36]G. R. Smith *et al.*, Phys. Rev. C **29**, 2206 (1984).
[37]W. Grüebler *et al.*, Phys. Rev. Lett. **49**, 444 (1982); **52**, 333 (1984).