Coulomb sum rules in the relativistic Fermi gas model

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Coulomb sum rules are studied in the framework of the Fermi gas model. A distinction is made between mathematical and observable sum rules. Differences between nonrelativistic and relativistic Fermi gas predictions are stressed. A method to deduce a Coulomb response function from the longitudinal response is proposed and tested numerically. This method is applied to the ^{40}Ca data to obtain the experimental Coulomb sum rule as a function of momentum transfer.

I. INTRODUCTION

The Coulomb sum rule (CSR) in nuclei had first been evoked as a means of studying the two-body correlations.[1] In the recent years, measurements of separated longitudinal responses in (e,e′) experiments[2–4] have motivated a renewed interest[5] in this subject. The attractive feature of the classical, i.e., nonrelativistic, CSR is that it has a simple and model-independent limit when the three-momentum transfer q becomes very large. This limit is just the nuclear charge number Z. Then, the deviation of the classical CSR from Z at finite q gives information on the nuclear correlation function.

An apparently surprising result lies in the fact that (e,e′) experiments lead to integrated Coulomb responses which are significantly smaller than Z (Refs. 6 and 7) even at q values as large as 550 MeV/c. Would this mean that correlation effects are so important, or is there a sizable amount of strength which lies outside the measured energy region? To answer this question, one must keep in mind that the classical CSR suffers from two defects which may affect its usefulness. First, it is a *mathematical* sum rule where the Coulomb response is integrated up to infinity over the energy variable ω, whereas the *physical* domain permitted by the kinematics of (e,e′) reactions is restricted to ω smaller than q. Even at large but finite values of q, the mathematical CSR does not strictly represent a measurable quantity. Second, a nonrelativistic description of nuclear structure may become questionable when q and w are large as compared to the nucleon mass.

Quite generally, one starts from the Coulomb nuclear response:[1,8]

$$R_C(\mathbf{q},\omega)\equiv\sum_{n\neq 0}|\langle n|\hat{\rho}(\mathbf{q})|0\rangle|^2\delta(\omega-\omega_n)\ , \qquad (1)$$

where $\hat{\rho}(\mathbf{q})$ is the nucleon point-charge density operator and $|n\rangle$ is a nuclear state with excitation energy ω_n. One can then define a mathematical CSR,

$$\Sigma_C(\mathbf{q})\equiv\int_0^\infty R_C(\mathbf{q},\omega)d\omega\ , \qquad (2)$$

and a physical (i.e., observable) CSR,

$$S_C(\mathbf{q})\equiv\int_0^q R_C(\mathbf{q},\omega)d\omega\ . \qquad (3)$$

In this work, we discuss in some detail the properties of $\Sigma_C(\mathbf{q})$ and $S_C(\mathbf{q})$ in the framework of a relativistic model of nuclear matter. These sum rules have already been studied in the same relativistic framework by various authors,[5,9] sometimes with errors which did not help to clarify the issue. We shall stress the difference between the relativistic sum rules and their nonrelativistic counterparts. In particular, we wish to dissipate the prejudice that the high-q limit of the mathematical CSR must be the charge number Z as a result of charge conservation. We also propose a method for extracting an experimental Coulomb response $R_C(\mathbf{q},\omega)$ from the measured longitudinal response $R_L(\mathbf{q},\omega)$. This allows one to construct an integrated experimental quantity which can be directly compared to the sum rule (3) predicted by any model.

In Sec. II we briefly review for completeness the nonrelativistic CSR. The relativistic model of nuclear matter is used to study the properties of the relativistic CSR in Sec. III. Effects of nucleon form factors and the link between the longitudinal and Coulomb responses are examined in Sec. IV. A pseudoexperiment is used to test the accuracy of the procedure of extracting the CSR from the longitudinal response. In Sec. V the CSR extracted from the measured longitudinal response in ^{40}Ca is compared to predictions of the relativistic model.

II. NONRELATIVISTIC SUM RULES

In a nonrelativistic (NR) approach, the nucleon field operators $\psi_\alpha(\mathbf{x})$ and $\psi_\alpha^\dagger(\mathbf{x})$ are two-component fields which, respectively, annihilate and create a nucleon at point $\mathbf{x}$ in a spin-isospin state α. The Fourier transform of the point-charge density operator is

$$\hat{\rho}(\mathbf{q})=\int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}\sum_{\alpha}\psi_\alpha^\dagger(\mathbf{x})\hat{Q}\psi_\alpha(\mathbf{x}) \ , \tag{4}$$

where $\hat{Q}=(1+\tau_3)/2$.

It is straightforward to derive the mathematical sum rule (2) for a nucleus containing Z protons and $N=A-Z$ neutrons. Using a closure relation for the nuclear states and fermion anticommutation relations for the fields ψ and $\psi^\dagger$,

$$\{\psi_\alpha(\mathbf{x}),\psi_\beta^\dagger(\mathbf{x}')\}=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \ , \tag{5}$$

one obtains

$$\Sigma_C^{\rm NR}(\mathbf{q})=Z+C(\mathbf{q}) \ , \tag{6}$$

where $C(\mathbf{q})$ is the Fourier transform of the proton correlation function:

$$C(\mathbf{q})=\int d^3x\, d^3x'\, e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\sum_{\alpha\beta}[\langle 0|\psi_\alpha^\dagger(\mathbf{x})\hat{Q}\psi_\beta^\dagger(\mathbf{x}')\psi_\beta(\mathbf{x}')\hat{Q}\psi_\alpha(\mathbf{x})|0\rangle-\langle 0|\psi_\alpha^\dagger(\mathbf{x})\hat{Q}\psi_\alpha(\mathbf{x})|0\rangle\langle 0|\psi_\beta^\dagger(\mathbf{x}')\hat{Q}\psi_\beta(\mathbf{x}')|0\rangle] \ . \tag{7}$$

Here, $|0\rangle$ is the exact nuclear ground state. One expects the correlation function to be well behaved in coordinate space, and therefore $C(\mathbf{q})$ should tend to zero as q becomes very large. Thus, one obtains the familiar, model-independent result that $\Sigma_C^{\rm NR}(\mathbf{q})$ goes to Z asymptotically.

However, the mathematical sum rule does not have a direct bearing on measured quantities in (e,e′) reactions, as we have argued in the preceding section. Let us now examine the observable sum rule (3). It is no longer possible to eliminate all excited states by a closure relation, and hence this sum rule can only be studied within specific models. For a general orientation on the behavior of $S_C^{\rm NR}(\mathbf{q})$ at large q values, we shall just look at the Fermi gas model. The Coulomb response is

$$R_C^{\rm NR}(\mathbf{q},\omega)=2\sum_{\mathbf{k}}n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})\delta\left[\frac{(\mathbf{k}+\mathbf{q})^2}{2M}-\frac{k^2}{2M}-\omega\right] \ , \tag{8}$$

where the factor 2 accounts for spin degeneracy, and $n_{\mathbf{k}}=\Theta(k_F-k)$, with k_F denoting the Fermi momentum. It is easy to see that $R_C^{\rm NR}(\mathbf{q},\omega)$ is nonzero only when ω belongs to the interval $[\omega_-,\omega_+]$, where

$$\omega_-={\rm Sup}\left[0,\frac{q^2}{2M}\left[1-\frac{2k_F}{q}\right]\right] \ , \qquad \omega_+=\frac{q^2}{2M}\left[1+\frac{2k_F}{q}\right] \ . \tag{9}$$

When q is larger than $2(M+k_F)$, ω_- is larger than q and then the physical sumrule $S_C^{\rm NR}(\mathbf{q})$ becomes identically zero. This result follows from a strict interpretation of the expression (8), where the energy conservation is written in terms of the NR kinetic energies. As an *ad hoc* remedy, these may be replaced by total relativistic energies, in which case the response function is nonvanishing for any q. The example shows, however, that one should be aware of the physical constraint $\omega<q$, which will have its full effect in a relativistic calculation, as we shall see in the next section.

The mathematical sum rule $\Sigma_C^{\rm NR}(\mathbf{q})$ could still be a useful quantity to consider in the range of q values up to $(2-3)k_F$. However, this would be sound only if a NR description of the response function remains valid in this energy-momentum region. Several authors have shown[10–13] that the longitudinal response function is significantly affected by relativistic nuclear dynamics. We shall now examine the two sum rules (2) and (3) when the nuclear structure is described relativistically. For simplicity, we consider the case of infinite nuclear matter treated in the mean field approximation.[14]

III. COULOMB SUM RULES IN RELATIVISTIC NUCLEAR MATTER

A. The Coulomb response function

We start from an effective Lagrangian containing nucleons coupled to isoscalar mesons, σ and ω. In the mean field approximation, or Hartree approximation, the fermion field operators obey the Dirac equation:

$$(i\partial\!\!\!/-M-V_s-\gamma_0V_0)\psi(x)=0 \ , \tag{10}$$

where $x=(t,\mathbf{x})$; V_s and V_0 are the scalar and timelike vector self-energies originating, respectively, from σ and ω exchange. Since we are in a homogeneous medium, V_s and V_0 have no space-time dependence. Equation (10) is a linear equation quite similar to that of a free particle, and its solution can be found in the same way.[15] First, we define an effective mass M^* and effective energies $E_{\mathbf{k}}^*$ by

$$M^*=M+V_s \ , \tag{11a}$$

$$E_{\mathbf{k}}^*=(M^{*2}+k^2)^{1/2} \ . \tag{11b}$$

We also introduce the spinors $w^r(\mathbf{k})$ for a particle in the medium, as solutions of

$$(\gamma_0E_{\mathbf{k}}^*-\boldsymbol{\gamma}\cdot\mathbf{k}-\epsilon_rM^*)w^r(\mathbf{k})=0 \ , \tag{12}$$

where $r=1,2$ for positive energy solutions ($\epsilon_r=+1$) and $r=3,4$ for negative energy ones ($\epsilon_r=-1$). These medium-modified spinors are formally identical to the free ones with the free mass M replaced by the effective mass M^*. We normalize them according to[15]

$$\bar{w}^r(\mathbf{k})w^{r'}(\mathbf{k})=\delta_{rr'}\epsilon_r \ . \tag{13}$$

It is easy to verify that the field,

$$\psi(x)=\frac{1}{\sqrt{\Omega}}\sum_{r=1}^{4}\sum_{\mathbf{k}}\left[\frac{M^*}{E_{\mathbf{k}}^*}\right]^{1/2}e^{-i(E_{\mathbf{k}}^{r}t-\epsilon_r\mathbf{k}\cdot\mathbf{x})}w^r(\mathbf{k})a_{r\mathbf{k}} \ , \tag{14}$$

satisfies Eq. (10). In (14), Ω is a normalization volume, and $a_{r\mathbf{k}}$ is an annihilation operator for a nucleon in a state $(r,\mathbf{k})$ with energy $E_{\mathbf{k}}^{r}=\epsilon_r E_{\mathbf{k}}^{*}+V_0$. As a consequence of the anticommutation relations for the fermion operators $a_{r\mathbf{k}}$, the components of the field (14) satisfy the usual equal-time anticommutation relations:

$$\{\psi_\alpha(t,\mathbf{x}),\psi_\beta(t,\mathbf{x}')\}=\{\psi_\alpha^\dagger(t,\mathbf{x}),\psi_\beta^\dagger(t,\mathbf{x}')\}=0\ ,$$
$$\{\psi_\alpha(t,\mathbf{x}),\psi_\beta^\dagger(t,\mathbf{x}')\}=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')\ . \quad (15)$$

In the present model of infinite homogeneous medium, the nucleon current $j^\mu(x)\equiv\bar{\psi}(x)\gamma^\mu\psi(x)$ obeys the continuity equation just as in the free particle case. If we denote by $\psi(\mathbf{x})$ the fermion field at time $t=0$, the charge density operator is again given by Eq. (4), where the α sum now runs over the four spinor components in addition to the isospin components.

The nuclear matter ground state is

$$|0\rangle=\left[\prod_{\substack{r=1,2\\ k\le k_F}} a_{r\mathbf{k}}^\dagger\right]\left[\prod_{\substack{r'=3,4\\ \text{all } k'}} a_{r'\mathbf{k}'}^\dagger\right]|\ \rangle\ , \quad (16)$$

where $|\ \rangle$ denotes the vacuum. An excited state $|n\rangle$ is obtained by promoting a nucleon to an unoccupied state with momentum $\mathbf{k}+\mathbf{q}$, leaving a hole with momentum $\mathbf{k}$ either in the Fermi sea (type F) or in the Dirac sea (type D). These two types of excitations are illustrated in Fig. 1. For an excited state of type F, the excitation energy is

$$\omega_n^F=(E_{\mathbf{k}+\mathbf{q}}^*+V_0)-(E_{\mathbf{k}}^*+V_0)=E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*\ , \quad (17a)$$

whereas for an excitation of type D we have

$$w_n^D=(E_{\mathbf{k}+\mathbf{q}}^*+V_0)-(-E_{\mathbf{k}}^*+V_0)=E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*\ . \quad (17b)$$

It is important to note that

$$\omega_n^F\le q,\quad \omega_n^D\ge q\ . \quad (18)$$

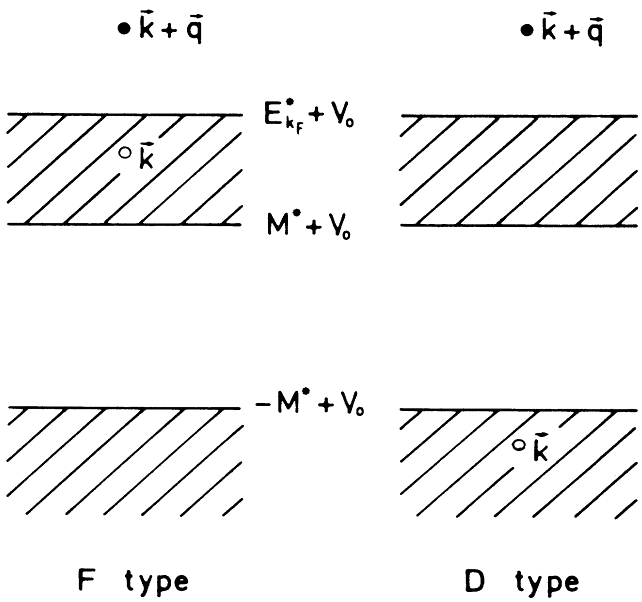

FIG. 1. F-type and D-type particle-hole excitations in nuclear matter.

These inequalities are easily obtained by using the relation (11b). The inequalities (18) show that in the present model only F-type excitations contribute to the observable sum rule (3), whereas both types contribute to the mathematical sum rule (2). Of course, $\Sigma_C(\mathbf{q})$ will be infinite and one has to define a procedure to remove its infinities.

The Coulomb response function (1) can be calculated directly. By introducing projection operators on positive or negative energy states,

$$\Lambda\pm(\mathbf{k})=\frac{\pm\not{k}+M^*}{2M^*}\ , \quad (19)$$

and using standard trace techniques, one obtains

$$R_C(\mathbf{q},\omega)=R_C^F(\mathbf{q},w)+R_C^D(\mathbf{q},\omega)\ , \quad (20)$$

where

$$R_C^F(\mathbf{q},\omega)=\sum_{\mathbf{k}} n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})\frac{[(E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)^2-q^2]}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*}\times\delta(\omega-E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)\ , \quad (21a)$$

$$R_C^D(\mathbf{q},\omega)=-\sum_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})\frac{[(E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*)^2-q^2]}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*}\times\delta(\omega-E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*)\ . \quad (21b)$$

The δ functions in (21a) and (21b) and the relations (17) and (18) clearly show that the domains where $R_C^F(\mathbf{q},\omega)$ and $R_C^D(\mathbf{q},\omega)$ are nonzero do not overlap. For $\omega<q$, only R_C^F is nonvanishing, whereas for $\omega>q$ only R_C^D contributes to Eq. (20).

B. Relativistic sum rules

From the above expressions for $R_C(\mathbf{q},\omega)$, and keeping in mind the inequalities (18), we can calculate the sum rule (3):

$$S_C(\mathbf{q})=Z-\sum_{\mathbf{k}} n_{\mathbf{k}}n_{\mathbf{k}+\mathbf{q}}\frac{(E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*}+\sum_{\mathbf{k}} n_{\mathbf{k}}\frac{(E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*)^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*}\ . \quad (22)$$

This result just corresponds to the spacelike (i.e., $\omega<q$) energy integral of the Coulomb response function (21a) calculated by Matsui.[9] As we have mentioned in the preceding subsection, this observable sum rule only involves F-type excitations.

The limit of $S_C(\mathbf{q})$ as q goes to infinity can be seen easily. The second term of Eq. (22) tends to zero, whereas the third term tends to $-Z/2$, so that $S_C(\mathbf{q})$ goes asymptotically to $Z/2$ when q becomes very large. This behavior is quite different from that of $S_C^{\mathrm{NR}}(\mathbf{q})$ studied in Sec. II. The nonrelativistic observable sum rule is zero for $q>2(M+k_F)$, but this is due to the use of nonrelativistic kinematics in a domain where this is not valid. Clearly, $S_C^{\mathrm{NR}}(\mathbf{q})$ becomes meaningless when q/M is not small compared to unity. Shown in Fig. 2 is $S_C(\mathbf{q})$ calculated for the two cases $M^*=M$ and $M^*=0.56M$. The observable CSR decreases somewhat with M^* and approaches the asymptotic value $Z/2$ faster as M^* is smaller.

Let us now examine the mathematical CSR $\Sigma_C(\mathbf{q})$ defined by Eq. (2). It differs from $S_C(\mathbf{q})$ by the integral from q to infinity of the response function (21b), where only D-type excitations are involved. We have

$$\Sigma_C(\mathbf{q})=S_C(\mathbf{q})+S_C^D(\mathbf{q}) , \tag{23}$$

where

$$S_C^D(\mathbf{q})=-\sum_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})\frac{(E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*)^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*} . \tag{24}$$

This quantity $S_C^D(\mathbf{q})$ is the timelike part of the mathematical CSR found by Matsui.[9] We notice that $\Sigma_C(\mathbf{q})$ was also studied by Walecka[5] in the same framework as the present relativistic model, but with a quite different result. The origin of this discrepancy lies in the fact that, in Ref. 5, $\Sigma_C(\mathbf{q})$ is calculated by casting it into a sum of one-body and two-body contributions. While the one-body term is correctly treated (i.e., both F- and D-type contributions are present), the D-type excitations are discarded in calculating the two-body correlation function. Because of this inconsistency, the quantity calculated in Ref. 5 corresponds neither to $S_C(\mathbf{q})$ nor to $\Sigma_C(\mathbf{q})$. Actually, if one does consistently separate $\Sigma_C(\mathbf{q})$ into one-body and two-body contributions using the complete anticommutation relations (15) for the four-component fermion fields, one gets

$$\Sigma_C(\mathbf{q})=\Sigma_C^{(1)}+C(\mathbf{q}) . \tag{25}$$

Now, the anticommutation relations (15) automatically involve negative as well as positive frequency components, i.e., both Fermi and Dirac sea contributions appear. Thus, the one-body part of $\Sigma_C(\mathbf{q})$ is

$$\Sigma_C^{(1)}=Z+2\sum_{\mathbf{k}}1 , \tag{26}$$

where the second (infinite) term is the total proton number in the Dirac sea. Similarly, the correlation function $C(\mathbf{q})$, whose definition is given by Eq. (7), is the sum of a Fermi sea contribution,

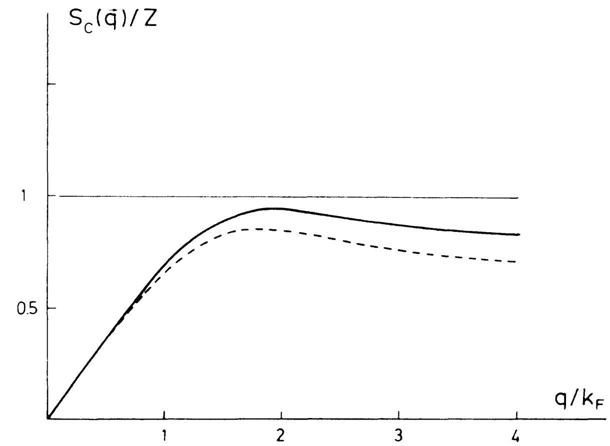

FIG. 2. The observable sum rule $S_C(\mathbf{q})$ calculated with $M^*/M=1$ (solid curve) and 0.56 (dashed curve) for $k_F=1.42$ fm^{-1}.

$$C^F(\mathbf{q})=-\sum_{\mathbf{k}}n_{\mathbf{k}}n_{\mathbf{k}+\mathbf{q}}\frac{(E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*} , \tag{27}$$

and a Dirac sea contribution,

$$C^D(\mathbf{q})=\sum_{\mathbf{k}}n_{\mathbf{k}}\frac{(E_{\mathbf{k}+\mathbf{q}}^*-E_{\mathbf{k}}^*)^2-q^2}{E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*} -\sum_{\mathbf{k}}\frac{(E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}^*E_{\mathbf{k}}^*} . \tag{28}$$

The results (25)–(28) are now identical to the relations (22)–(24).

The field-theoretic method using anticommutation relations can also be used for calculating the observable sum rule $S_C(\mathbf{q})$. In this case, only positive frequency components $\psi_\alpha^{(+)}$ of the fermion fields are involved because of the constraint $\omega<q$, as we have mentioned before. Therefore, one must use the projected anticommutation relations,

$$\{\psi_\alpha^{(+)}(\mathbf{x}),\psi_\beta^{(+)\dagger}(\mathbf{x}')\}=\frac{1}{\Omega}\sum_{\mathbf{k}}\left[\frac{M^*}{E_{\mathbf{k}}^*}\right]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}[\Lambda_+(\mathbf{k})\gamma^0]_{\alpha\beta} , \tag{29}$$

to obtain

$$S_C(\mathbf{q})=\frac{1}{\Omega}\sum_{\mathbf{k}}\left[\frac{M^*}{E_{\mathbf{k}}^*}\right]\int d^3x\,d^3x'\,e^{i(\mathbf{k}+\mathbf{q})\cdot(\mathbf{x}'-\mathbf{x})}\langle 0|\bar{\psi}^{(+)}(\mathbf{x})\hat{Q}\Lambda_+(\mathbf{k})\psi^{(+)}(\mathbf{x}')|0\rangle+C^{(+)}(\mathbf{q}) , \tag{30}$$

where $C^{(+)}(\mathbf{q})$ has the same form as Eq. (7) with all the fields ψ replaced by $\psi^{(+)}$. The calculation of Eq. (30) for relativistic nuclear matter again gives the result (22), as it should.

As they stand, Eqs. (23) and (24) give an infinite value for $\Sigma_C(\mathbf{q})$ because of the Dirac sea contribution. The removal of divergences is a difficult task and leads to the problem of renormalization of the starting Lagrangian.[14] Keeping in mind that the remaining finite part of $\Sigma_C(\mathbf{q})$ may depend on the particular subtraction procedure adopted, we just examine here the simple prescription of subtracting out the vacuum contributions. In the case of a free relativistic Fermi gas, i.e., when the self-consistent self-energies V_s and V_0 are omitted, these contributions are what remains from Eq. (24) when we set $n_{\mathbf{k}+\mathbf{q}}=0$ and replace all energies $E_{\mathbf{k}}^*$ by free particle energies $E_{\mathbf{k}}=(k^2+M^2)^{1/2}$. We thus obtain the finite part of the mathematical CSR:

$$\Sigma_C^{FG}(\mathbf{q}) = \sum_{\mathbf{k}} n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})\frac{(E_{\mathbf{k}+\mathbf{q}}+E_{\mathbf{k}})^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}E_{\mathbf{k}}} + \sum_{\mathbf{k}} n_{\mathbf{k}}\frac{(E_{\mathbf{k}+\mathbf{q}}-E_{\mathbf{k}})^2-q^2}{2E_{\mathbf{k}+\mathbf{q}}E_{\mathbf{k}}} . \quad (31)$$

In contrast to the nonrelativistic CSR which tends to Z at large q, the relativistic mathematical CSR tends to zero, as one can see from Eq. (31). Obviously, it is just a mathematical object and one should not try to give it any physical interpretation. On the other hand, if we use the same subtraction procedure for nuclear matter, i.e., when $M^* \neq M$, the remaining part of $\Sigma_C(\mathbf{q})$ is finite at finite q but increases linearly with q for $M^* < M$.

IV. FROM THE EXPERIMENTAL DATA TO THE COULOMB SUM RULE

Experiment gives us the longitudinal response function $R_L^{\text{expt}}(\mathbf{q},\omega)$ and, consequently, the longitudinal sum $S_L(\mathbf{q})$, but not directly the Coulomb sum $S_C(\mathbf{q})$. In view of the particular high-q limit of the latter, which may be used as a test of nuclear models and theories, one would like to have a formula by which it could be obtained from the experimental data. First, let us examine what the experimental results are supposed to represent by looking at the relativistic Fermi gas model.

A. The relativistic longitudinal response function

In contradistinction to the Coulomb response function for pointlike nucleons, which is a mathematical object, the measured function corresponds to real nucleons having a finite size (form factors) as well as an anomalous magnetic moment. These features are contained in the nuclear current, which is assumed of the conventional form

$$J^\mu(q) = F_1(q^2)\gamma^\mu + i\kappa\frac{F_2(q^2)}{2M}\sigma^{\mu\nu}q_\nu , \quad (32)$$

where F_1 and F_2 are the nucleon form factors and κ is the anomalous magnetic moment ($\kappa_p = 1.79$ for a proton and $\kappa_n = -1.91$ for a neutron). In terms of the Sachs charge and magnetic form factors, with $\mu_n = \kappa_n$ and $\mu_p = 1+\kappa_p$ being the total magnetic moments of the nucleons, F_1 and F_2 are given by[15]

$$F_1(q^2) + \frac{\kappa q^2}{4M^2}F_2(q^2) = G_E(q^2) ,$$
$$F_1(q^2) + \kappa F_2(q^2) = G_M(q^2) . \quad (33)$$

There is some uncertainty in the form factor $G_E^n(q^2)$. The parametrizations suggested by Preston and Bhaduri[16] and by Hohler *et al.*[17] give comparable values of G_E^n over a wide range of q^2, but G_E^n becomes negative when $-q^2 \gtrsim 4$ GeV^2 for the parametrizations of Ref. 17. On the other hand, the often used form

$$G_E^n(q^2) = \kappa_n(q^2/4M^2)G_E^p(q^2)$$

gives much larger values. In this work, we adopt the parametrization of Ref. 16:

$$G_E^p(q^2) = [1-q^2/(0.71\ \text{GeV}^2)]^{-2} ,$$
$$G_M^{n,p}(q^2) = \mu_{n,p}G_E^p(q^2) , \quad (34)$$
$$G_E^n(q^2) = \kappa_n[(\mathbf{q}^2/4M^2)/(1-5.6q^2/4M^2)]G_E^p(q^2) .$$

The longitudinal response function can be calculated straightforwardly as in Sec. III. It involves only particle-hole excitations from the Fermi sea and takes the form[12]

$$R_L(\mathbf{q},\omega) = \sum_{\mathbf{k},\tau}\frac{n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})}{E_{\mathbf{k}}^*(E_{\mathbf{k}}^*+\omega)}\delta(\omega - E_{\mathbf{k}+\mathbf{q}}^* + E_{\mathbf{k}}^*) \times \left[2\left[E_{\mathbf{k}}^* + \frac{\omega}{2}\right]^2 T_2 - \frac{\mathbf{q}^2}{2}T_1\right] , \quad (35)$$

where

$$T_1 = \left[F_1 + \frac{M^*}{M}\kappa F_2\right]^2 ,$$
$$T_2 = \left[F_1^2 - \frac{q^2}{4M^2}\kappa^2 F_2^2\right] . \quad (36)$$

The sum on τ in Eq. (35) involves both protons and neutrons.

A realistic calculation of the longitudinal response (35) must be done using the form factors F_1 and F_2 deduced from Eqs. (33) and (34). Notice that F_1 and F_2 are not equal and their difference increases with q^2. However, just for the purpose of comparing our results with those of Refs. 5 and 9, let us assume for the moment that $F_1^p = F_2^p = F_2^n \equiv F$ and $F_1^n = 0$, and define a simplified response function where these factors are divided out:

$$\overline{R}_L(\mathbf{q},\omega) \equiv R_L(\mathbf{q},\omega)/|F|^2 . \quad (37)$$

We note that if one calculates the sum rule $\overline{S}_L(\mathbf{q})$ corresponding to $\overline{R}_L(\mathbf{q},\omega)$, its asymptotic value for $|\mathbf{q}| \to \infty$ is only half of that given in Ref. 5. The origin of the discrepancy is, of course, the same as we have seen earlier for the CSR, namely the inclusion in Ref. 5 of undue contributions coming from $|\mathbf{q}| < \omega$ excitations in the one-body term as the result of the use of complete anticommutation relations (15) for field operators.

Figure 3 shows the numerical results for the simplified sum rule $\overline{S}_L(\mathbf{q})/Z$ for $M^*/M = 1$ and 0.56, and the nonrelativistic limit $M^*/M = \infty$. The following remarks can be made:

(i) The longitudinal sum rule depends on M^*. This dependence is, however, rather weak in the physical region of interest ($q < 1$ GeV) because of a compensating effect between the charge and magnetic contributions: When M^* decreases, the charge contributions (dominant at small q) decrease, whereas the magnetic contributions (dominant at large q) increase.

(ii) In Fig. 3 we also show the results of Walecka[5] and Matsui,[9] for $M^*/M = 1$. For Walecka's result, we have already seen that the very strong increase of $\overline{S}_L$ with q is due to the inclusion of pair contributions. Matsui's numerical result is more surprising since he used the same formulae as we do. We have checked that his result comes from the use of the *wrong sign* for the proton anomalous magnetic moment.

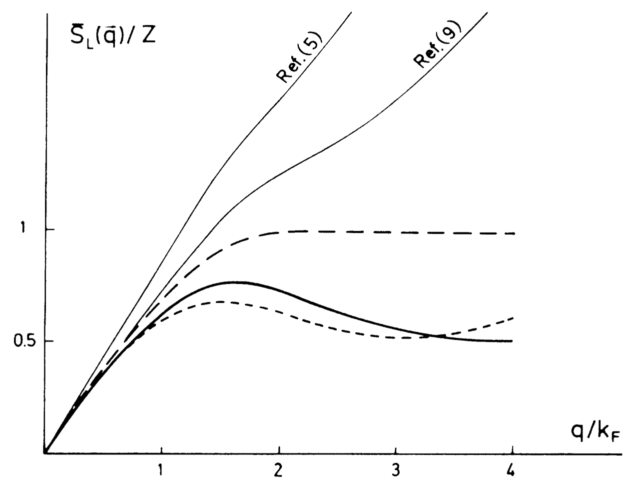

FIG. 3. The simplified sum rule $\bar{S}_L(\mathbf{q})/Z$ for $M^*/M=1$ (solid curve) and 0.56 (dashed curve). The long-dashed line shows the nonrelativistic limit ($M^*/M=\infty$). The results of Refs. 5 and 9 for $M^*/M=1$ are also indicated. All curves correspond to $k_F=1.42$ fm^{-1}.

Figure 4 shows the results obtained by integrating the longitudinal response function (35) for ^{40}Ca, together with the Saclay experimental data.[7] This case is representative of what one gets for other nuclei. The curves represent the complete longitudinal sum rule $S_L(\mathbf{q})$, where the form factors (33) and (34) are included. The Fermi momentum is $k_F=1.42$ fm^{-1}. As already noticed above, $S_L(\mathbf{q})$ depends only slightly on M^*. Clearly, a fit to the data is not possible with the Fermi gas model, unless some drastic assumption is made about the form factors.[11,18]

B. The Coulomb sum rule from experiment

Experiments give us $R_L^{\text{expt}}(\mathbf{q},\omega)$, i.e., for each q, a set of data points over a range of ω that is experimentally accessible. Here, we propose a formula by which the CSR $S_C^{\text{expt}}(\mathbf{q})$ may be obtained from the data.

Let us write the Coulomb response function (21a) in the following form:

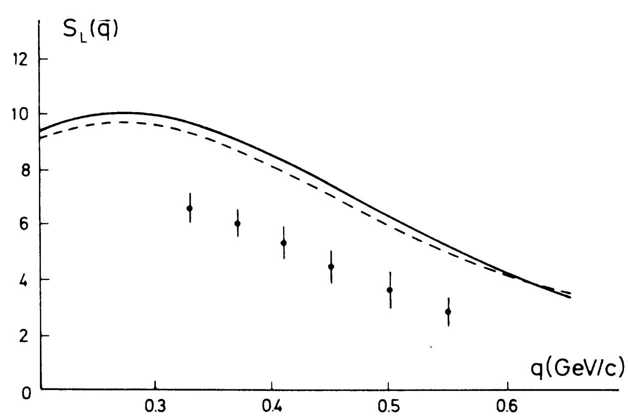

FIG. 4. The longitudinal sum rule $S_L(\mathbf{q})$ in ^{40}Ca, calculated with $M^*/M=1$ (solid curve) and 0.56 (dashed curve) for $k_F=1.42$ fm^{-1}. The experimental data are from Ref. 7.

$$R_C(\mathbf{q},\omega)=\sum_{\mathbf{k}}\frac{n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}})}{E_{\mathbf{k}}^*(E_{\mathbf{k}}^*+\omega)}\delta(\omega-E_{\mathbf{k}+\mathbf{q}}^*+E_{\mathbf{k}}^*)\times 2\left[E_{\mathbf{k}}^*+\frac{\omega}{2}\right]^2\left[1-\mathbf{q}^2/4\left[E_{\mathbf{k}}^*+\frac{\omega}{2}\right]^2\right]. \tag{38}$$

To proceed further, let us assume that the quantity in the large square brackets could be replaced by some constant independent of the momentum $\mathbf{k}$, namely that $E_{\mathbf{k}}^*=(M^{*2}+\mathbf{k}^2)^{1/2}$ could be replaced by some average value $\bar{E}^*=(M^{*2}+\bar{k}^2)^{1/2}$, the precise value of which will be discussed later. The main reason for such an approximation is that, as $k\le k_F \ll M^*$, any value of $\bar{k}$ would make little difference. With this approximation, the Coulomb response function becomes

$$\bar{R}_C(\mathbf{q},\omega)\simeq\bar{R}^{\,\text{p}}(\mathbf{q},\omega)\left[1-\mathbf{q}^2/4\left[\bar{E}^*+\frac{\omega}{2}\right]^2\right], \tag{39}$$

where $\bar{R}^{\,\text{p}}(\mathbf{q},\omega)$ can be identified from Eq. (38). The superscript p for protons actually denotes only the dependence on the proton Fermi momentum. One may also define similarly a function $\bar{R}^{\,\text{n}}(\mathbf{q},\omega)$ for neutrons. For a nucleus with Z protons and N neutrons occupying the same volume, a reasonable approximation is $\bar{R}^{\,\text{n}}=(N/Z)\bar{R}^{\,\text{p}}\equiv\lambda\bar{R}^{\,\text{p}}$. Using this approximation, the longitudinal response function (35) becomes

$$R_L(\mathbf{q},\omega)\simeq\bar{R}^{\,\text{p}}(\mathbf{q},\omega)\left[(T_2^{\text{p}}+\lambda T_2^{\text{n}})-(T_1^{\text{p}}+\lambda T_1^{\text{n}})\mathbf{q}^2/4\left[\bar{E}^*+\frac{\omega}{2}\right]^2\right], \tag{40}$$

where T^i ($i=$p,n) are given by Eqs. (36).

Equations (39) and (40) clearly indicate the procedure to be followed to extract the experimental CSR: Starting from the experimental response function $R_L^{\text{expt}}(\mathbf{q},\omega)$, one should define the following quantity at each energy point:

$$R_C^{\text{expt}}(\mathbf{q},\omega)\equiv R_L^{\text{expt}}(\mathbf{q},\omega)/\mathcal{F}(\mathbf{q},\omega)\,, \tag{41}$$

$$\mathcal{F}(\mathbf{q},\omega)=\left[(T_2^{\text{p}}+\lambda T_2^{\text{n}})-(T_1^{\text{p}}+\lambda T_1^{\text{n}})\mathbf{q}^2/4\left[\bar{E}^*+\frac{\omega}{2}\right]^2\right]\times\left[1-\mathbf{q}^2/4\left[\bar{E}^*+\frac{\omega}{2}\right]^2\right]^{-1}.$$

R_C^{expt} should be interpreted as the experimental Coulomb response function from which the CSR is obtained by integration over the available range of ω.

Obviously, such a procedure is based on the assumption that the excitation spectrum results from a quasielastic process with a current given by (32). Whether this is or is not the case is another question which is outside the scope of this paper and which may be answered only by the comparison of theory and experiment.

Returning to the approximation that has been made to obtain Eqs. (39) and (40), let us make the following remarks:

(i) Concerning the choice of the average energy $\bar{E}^*$, we know that $0 \leq \bar{k} \leq k_F$. The two limits of $\bar{k}$ define the boundaries of a zone in which $S_C^{\text{expt}}(\mathbf{q})$ is supposed to lie. A possible choice for $\bar{k}$ may be $\bar{k}^2 = 2M\epsilon_B$, ϵ_B being the average binding energy of the nucleons ($\simeq 16$ MeV).

(ii) The factor $\mathscr{F}$ that divides R_L^{expt} in Eq. (41) depends on M^*, which is not known *a priori*. However, once a value of M^* is chosen to make the extraction, then the result must be compared with the theoretical result obtained in a model having the same value of M^*.

C. A pseudoexperimental test

As a test of the accuracy of the extraction procedure, let us consider the pseudoexperimental results obtained in the following way. We assume that the "experimental" excitation spectrum, and the longitudinal response function obtained from it, is the result of a quasielastic process induced by the current of Eq. (32). We assume, furthermore, that it is accurately described by the relativistic Fermi gas model. Under these assumptions, for each value of the momentum transfer $\mathbf{q}$, we have at our disposal the following.

(i) A set of "pseudoexperimental" data points for the longitudinal response function $R_L^{\text{expt}}(\mathbf{q},\omega)$ in the range $0 < \omega < |\mathbf{q}|$. They are given by Eq. (35).

(ii) A set of values for the "exact" Coulomb response function which may be obtained from Eq. (38).

The test then consists of using the proposed extraction procedure to deduce the "pseudoexperimental" CSR from the "pseudoexperimental" longitudinal response function of (i) and comparing it with the "exact" one obtained from (ii).

Figure 5 gives, using again the parameters of ^{40}Ca with $M^* = M$, the results of the extraction procedure corresponding to the different approximations for the average momentum: the two limiting dashed curves ($\bar{k} = 0$ and $\bar{k} = k_F$) and the solid curve corresponding to the optimum choice $\bar{k}^2 = 2M\epsilon_B$. On the scale of the figure, the "exact" Coulomb sum rule practically coincides with this solid curve. This clearly shows the accuracy of the proposed extraction procedure, even at relatively large values of the momentum transfer. This conclusion remains valid for $M^* \neq M$ and/or $Z \neq N$.

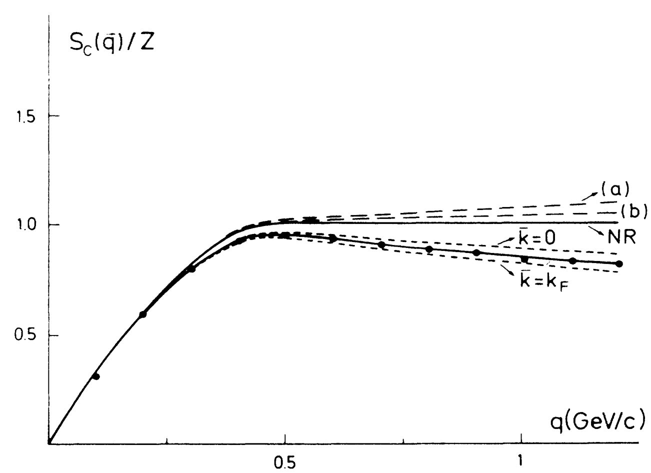

FIG. 5. The extracted Coulomb sum rule for various choices of $\bar{k}$. The solid circles show the exact values. Curves (a) and (b) are obtained by using the dividing factors (43) and (45), respectively. The curve (NR) shows the nonrelativistic CSR.

The above "pseudoexperimental" test also provides an opportunity to examine the validity of other extraction procedures. As the CSR is related to point-charge particles, all the extraction procedures consist of finding a factor, similar to the factor $\mathscr{F}(\mathbf{q},\omega)$ of Eq. (41), by which the experimental response function has to be divided out in order to eliminate, as completely as possible, the nucleon form factors. The two methods that are closest in spirit to ours are those of De Forest[19] and Friar.[20] This may be seen as follows. The longitudinal response function is essentially given by the squared matrix element:

$$|M|^2 = \text{Tr}\,|\bar{u}(\mathbf{k}+\mathbf{q})\mathscr{J}^0 u(\mathbf{k})|^2 = \frac{(E_\mathbf{k}+\omega/2)^2}{E_\mathbf{k}(E_\mathbf{k}+\omega)}\left[T_2 - T_1\mathbf{q}^2/4\left[E_\mathbf{k}+\frac{\omega}{2}\right]^2\right], \quad (42)$$

where the u's are nucleon spinors (assuming $M^* = M$) and use is made of the energy conservation relation $\omega = E_{\mathbf{k}+\mathbf{q}} - E_\mathbf{k}$. Our method consists of making an approximation for the large-square-bracketed term in the above equation, so that it can be divided out, but keeping the energy- dependent factor that is characteristic of the relativistic kinematics. This allows us to recover the relativistic Coulomb response function (38), where this kinematical factor is present.

De Forest, on the other hand, preferred to recover the *nonrelativistic* result and consequently proposed to divide out the whole factor (42). Making the approximation $k = 0$, it can be shown that

$$|M|^2_{k=0} = G_E^{p2}(q^2)(1-q^2/4M^2)/(1-q^2/2M^2)\,. \quad (43)$$

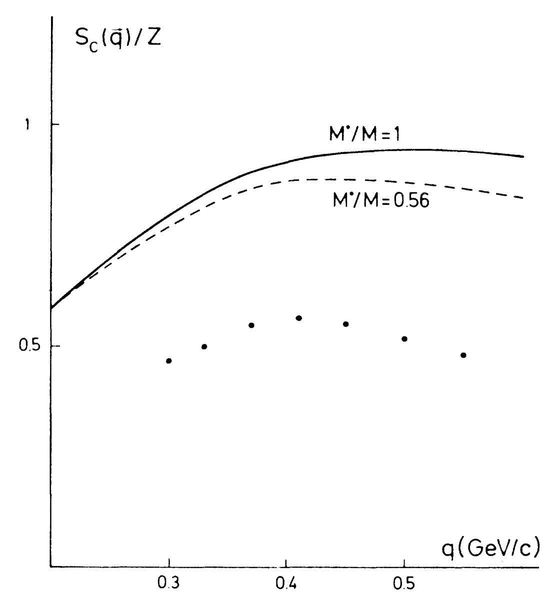

FIG. 6. The CSR in ^{40}Ca extracted from the data of Ref. 7 (solid circles). The curves show the calculated CSR for two values of M^*/M.

Obviously, within the same approximation, when this factor is divided out from Eq. (35), the result is just the nonrelativistic one, namely (neglecting neutron contributions)

$$R_L(\mathbf{q},\omega)/|M|^2=2\sum_{\mathbf{k}}n_{\mathbf{k}}(1-n_{\mathbf{k+q}})\delta(\omega-E_{\mathbf{k+q}}+E_{\mathbf{k}})\ , \tag{44}$$

with the understanding that the energy conserving condition be replaced by its nonrelativistic limit. Friar's prescription may be understood in the same spirit, with the difference that, instead of the laboratory frame ($\mathbf{k}=0$), he chose the Breit frame of the nucleon by taking $\mathbf{k}=-\mathbf{q}/2$ in Eq. (42).

Concerning the above prescriptions to extract the CSR, let us make the following remarks:

(i) From a mathematical point of view, there is no reason why one of the two sum rules derived from Eqs. (38) and (44) should be preferred over the other. However, if the sum rule is to be physically interpreted as the energy integral of the strength of the charge operator, then only the relativistic result (38) will be valid for any momentum transfer q.

(ii) By including the kinematical factor whose k dependence is correct only in the Fermi gas model, De Forest's procedure may be more model dependent.

(ii) Strictly speaking, the dividing factor (43) is correct only for protons. Since the experimental data include contributions from neutrons, one should also add to (43) the neutron contributions. With the parametrization (34), the dividing factor becomes

$$|M|^2_{k=0,\text{ protons}}+|M|^2_{k=0,\text{ neutrons}}=G_E^{p2}(q^2)[1+\kappa_n^2(q^2/4M^2)^2/(1-5.6q^2/4M^2)^2](1-q^2/4M^2)/(1-q^2/2M^2)\ . \tag{45}$$

Clearly, the effect of the neutrons on the dividing factor is small for all values of q^2. In Fig. 5 are shown the results obtained by dividing the longitudinal response (35) by the factor (43) [curve (a)] or (45) [curve (b)] and integrating over ω. These results should not be compared with the relativistic CSR (solid-circles), but with the nonrelativistic one, which the procedure is intended to reproduce. The slight deviation of curve (b) from the nonrelativistic result may be traced to the approximation $k=0$ in reducing Eq. (42).

V. EXPERIMENTAL RESULTS AND DISCUSSION

Now that we have at our disposal a procedure to extract the CSR that is accurate enough, at least in the Fermi gas model, let us apply it to the experimental data. We divide the experimental longitudinal response function at each point $(\mathbf{q},\omega)$ by the factor $\mathcal{F}(\mathbf{q},\omega)$ of Eq. (41) to produce an experimentally extracted Coulomb response function. The CSR is then obtained by integration over ω.

Figure 6 shows the results obtained using the Saclay experimental data[7] for ^{40}Ca. On the same figure are given the two curves representing the Fermi gas results for $M^*/M=1$ and 0.56. It is clear that, within the range of acceptable values of the effective mass M^*, there is no way to reproduce the experimental results with the Fermi gas model.

A number of possible explanations of this discrepancy have been proposed, but up to now none has been completely satisfactory. Aside from effects of long as well as short range correlations, which may reduce the theoretical result by 5—10%, the two candidates that seem at first most promising, as they may give quite sizable strength reduction, are the following:

(i) The hypothesis that nucleons become larger in the nuclear medium,[11,18] with the subsequent change of the form factors. This certainly is an exciting eventuality, as it would mean that one starts to see already quark effects in the nucleus. A serious shortcoming of the model resides in the fact that by reducing the form factors, the transverse response function is proportionally reduced and would strongly deviate from the experimental results.

(ii) The relativistic mean-field approach for finite nuclei. Here, the reduction of the longitudinal strength is due to the $\mathbf{r}$-dependent effective mass, leading to a Perey effect.[21] The model has the virtue that it leaves the transverse function almost unchanged, as required by experiments. The Perey effect, however, is not sufficient to reproduce the measured longitudinal response function.[22,23]

It is obvious that more work, both theoretical and experimental, is necessary in order to have a clear-cut idea about the origin of the missing strength. In this paper we have laid down the basis for a comparison between theory and experiment through the use of the CSR.

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[1]K. W. Mc Voy and L. Van Hove, Phys. Rev. **125**, 1034 (1962).
[2]R. Altemus, A. Caffolla, D. Day, J. S McCarthy, R. R. Whitney, and J. E. Wise, Phys. Rev. Lett. **44**, 965 (1980).
[3]P. Barreau *et al.*, Nucl. Phys. **A358**, 287 (1981).
[4]W. Bertozzi, Nucl. Phys. **A374**, 109 (1982).
[5]J. D. Walecka, Nucl. Phys. **A399**, 387 (1983).
[6]Z. E. Meziani *et al.*, Phys. Rev. Lett. **52**, 2130 (1984).
[7]J. Morgenstern, Nucl. Phys. **A446**, 315c (1985).
[8]T. De Forest, Jr. and J. D. Walecka, Adv. Phys. **15**, 1 (1966).
[9]T. Matsui, Phys. Lett. **132B**, 260 (1983).
[10]R. Rosenfelder, Ann. Phys. (N.Y.) **128**, 188 (1980).
[11]J. V. Noble, Phys. Rev. Lett. **46**, 412 (1981).

[12]G. Do Dang and Pham Van Thieu, Phys. Rev. C **28**, 1845 (1983).

[13]J. W. Van Orden, Ph. D. dissertation, Stanford University, 1978 (unpublished).

[14]B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. **16** (1986).

[15]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[16]M. A. Preston and R. K. Bhaduri, *The Structure of the Nucleus* (Addison-Wesley, Reading, Mass., 1975).

[17]G. Hohler *et al.*, Nucl. Phys. **B114**, 505 (1976).

[18]L. S. Celenza, A. Rosenthal, and C. M. Shakin, Phys. Rev. C **31**, 232 (1985).

[19]T. De Forest, Jr., Nucl. Phys. **A414**, 347 (1984); in *Proceedings of the Bates Users Theory Group Workshop,* edited by J. Dubach and F. Gross (M.I.T., Cambridge, Mass., 1985).

[20]J. L. Friar, Ann. Phys. (N.Y.) **81**, 332 (1973); see also Ref. 19.

[21]G. Do Dang and Nguyen Van Giai, Phys. Rev. C **30**, 731 (1984).

[22]S. Nishizaki, H. Kurasawa, and T. Suzuki, Phys. Lett. **171B**, 1 (1986).

[23]A. Picklesimer, J. W. Van Orden, and S. J. Wallace, Phys. Rev. C **32**, 1312 (1985).