Boson representations of fermion systems

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The correspondence between the *SD* fermion and *sd* boson spaces is investigated. A method is discussed for generating boson images of fermion Hamiltonians. The boson Hamiltonians obtained are at most two-body and, in general, non-Hermitian. Applications to a quadrupole-quadrupole interaction are presented. $E2$ operators are also studied. Fermion spectra and $E2$ matrix elements are well reproduced in the boson space.

I. INTRODUCTION

In recent years, considerable interest has been devoted to the description of low-energy nuclear spectra in subspaces of the full shell model space built in terms of collective pairs. A number of investigations[1–11] have been carried out to test the effects of the truncation of the model space on physical observables such as energies, $B(E2)$'s, quadrupole moments, etc. Working mostly within a pairing-plus-quadrupole model, it has been observed[7–11] that drastic truncations to subspaces including only pairs of low angular momentum ($L \leq 4$) can still keep the values of these observables (referring to the low-lying states) quite close to those obtained in the full space.

A description of the nuclear properties in such restricted subspaces is more transparent and appealing than the one that a straightforward application of the shell model could give. Even in these subspaces, however, a realistic microscopic description of nuclei with several active nucleons in different j shells is still a difficult task. This has to be ascribed to the great complexity of the commutation relations of the fermion operators.

Mapping the above fermion subspaces onto corresponding boson spaces in which bosons replace collective pairs offers a way to escape these problems provided that resulting boson operators are easy to handle. Since the introduction of the interacting boson model[12] (IBM), much effort has gone into trying to understand the mechanism of such a mapping. Up to now, however, especially the treatment of deformed systems does not appear fully satisfactory and calls for further investigations. This work is devoted to this subject.

In this paper I will concern myself with the correspondence between the fermion *SD* and the boson *sd* spaces, i.e., spaces formed by $L = 0$ and 2 fermion pairs and bosons, respectively. I will discuss a mapping procedure between these spaces and show detailed applications of the procedure for systems of nucleons interacting with a quadrupole-quadrupole (QQ) residual interaction. The effectiveness of the method will be tested by comparing calculations performed in the fermion and in the boson spaces.

All the applications which will be presented in this paper will refer to the case of nucleons moving in a single j orbit. This has made it possible to perform exact fermion calculations without running into serious computational problems. Extensions of the procedure to the case of many j orbits as well as to larger spaces (for instance, *SDG* and *sdg* spaces) appear, however, possible.

The paper is organized as follows. In Sec. II, I will present a method to generate boson images of fermion Hamiltonians. In Sec. III, fermion spectra will be compared to the corresponding boson ones. In Sec. IV, I will study electromagnetic transitions between fermion and boson eigenstates. Finally, in Sec. V, I will compare the procedure discussed in this paper with previous approaches and draw some conclusions.

An account of this work has already been presented in Ref. 13.

II. THE MAPPING PROCEDURE

I begin this section by recalling some results of a work of Ginocchio and Talmi.[14]

Let the $L = 0$ and 2 pair creation operators be defined in single j orbit as

$$\begin{aligned} S^\dagger &= \frac{1}{\sqrt{2}}[a_j^\dagger a_j^\dagger]^0 , \\ D_\mu^\dagger &= \frac{1}{\sqrt{2}}[a_j^\dagger a_j^\dagger]_\mu^2 , \end{aligned} \tag{1}$$

and let H_F be a fermion Hamiltonian (at most two body) having the following commutation relations in the *SD* subspace,

$$\begin{aligned} &[H_F,S^\dagger]\,|\,0\rangle = \epsilon_s S^\dagger\,|\,0\rangle , \\ &[H_F,D_\mu^\dagger]\,|\,0\rangle = \epsilon_d D_\mu^\dagger\,|\,0\rangle , \\ &[[H_F,S^\dagger],S^\dagger] = a_1 S^\dagger S^\dagger + a_3 (D^\dagger \cdot D^\dagger) , \\ &[[H_F,S^\dagger],D_\mu^\dagger] = a_2 S^\dagger D_\mu^\dagger + a_6 [D^\dagger D^\dagger]_\mu^2 \\ &[[H_F,D_\mu^\dagger],D_{\mu'}^\dagger] = (-1)^\mu \delta_{\mu',-\mu} a_4 S^\dagger S^\dagger \\ &\qquad + 2(2\mu\, 2\mu' \,|\, 2\, \mu+\mu') a_5 S^\dagger D_{\mu+\mu'}^\dagger \\ &\qquad + \sum_{J=0,2,4} (2\mu\, 2\mu' \,|\, J\, \mu+\mu') \\ &\qquad\qquad \times a_7^J [D^\dagger D^\dagger]_{\mu+\mu'}^J . \end{aligned} \tag{2}$$

These commutation relations guarantee that H_F does not couple the SD subspace to the rest of the fermion space. This Hamiltonian has, then, a class of eigenstates which belong to this subspace, i.e., which are some linear combination of states of the type

$$(S^\dagger)^{N_s}(D^\dagger)^{N_d}_{\gamma JM}\,|0\rangle\ , \tag{3}$$

where J is the total angular momentum, M is its projection, and γ is an additional quantum number which identifies unambiguously the states.

A boson Hamiltonian can be constructed which satisfies the same commutation relations in the sd boson subspace. This Hamiltonian has the form

$$\begin{aligned} H_B = {} & \epsilon_s\hat{n}_s+\epsilon_d\hat{n}_d+\tfrac{1}{2}a_1\hat{n}_s(\hat{n}_s-1)+a_2\hat{n}_s\hat{n}_d \\ & +\tfrac{1}{2}[a_3(d^\dagger\cdot d^\dagger)ss+a_4s^\dagger s^\dagger(\tilde{d}\cdot\tilde{d})] \\ & +[a_5s^\dagger d^\dagger\cdot[\tilde{d}\tilde{d}]^2+a_6[d^\dagger d^\dagger]^2\cdot\tilde{d}s] \\ & +\tfrac{1}{2}\sum_{J=0,2,4}a_7^J[d^\dagger d^\dagger]^J\cdot[\tilde{d}\,\tilde{d}]^J\ . \end{aligned} \tag{4}$$

Here, $\hat{n}_s=s^\dagger s$, $\hat{n}_d=\sum_\mu d^\dagger_\mu d_\mu$, and $\tilde{d}_\mu=(-1)^\mu d_{-\mu}$.

It follows from the commutation relations of H_B that this Hamiltonian has the same eigenvalues as H_F in the SD subspace, as well as a set of eigenstates which are the same linear combinations (a part from a normalization factor) of the boson states

$$(s^\dagger)^{N_s}(d^\dagger)^{N_d}_{\gamma JM}\,|0) \tag{5}$$

as the eigenstates of H_F are of the states (3).

A complete analysis of this correspondence can be found in Ref. 14. Here, I only stress the following properties of H_B: (i) H_B contains at most two-body interactions and, (ii) it is, in general, non-Hermitian. As a consequence of the latter property, the correspondence between fermion and boson eigenstates is such not to preserve orthogonality.

The derivation of the Hamiltonian (4) clearly relies upon the existence of commutation relations of the type (2). Such relations are quite unlikely to occur, in general, and only special Hamiltonians have been found which satisfy them.[15] In normal cases, then, the procedure just discussed is of no utility.

Let us suppose, however, that in correspondence to a given H_F one can derive a new fermion Hamiltonian $\mathscr{H}_F$ such that (a) it has commutation relations of the form (2), and (b) it is "equivalent" to H_F in SD, i.e., it has the same eigenvalues and eigenstates as H_F in SD. Starting from $\mathscr{H}_F$ (and all one needs to know, in practice, are the commutation relations of $\mathscr{H}_F$ in SD), one can now apply the simple procedure of Ginocchio and Talmi to generate a boson image of H_F in the sd boson subspace. Searching for $\mathscr{H}_F$ is therefore equivalent to searching for the boson image of H_F in sd. This is the way that has been followed to derive this Hamiltonian.

As a first step toward the construction of $\mathscr{H}_F$, let us consider a projected Hamiltonian H_F' whose commutators in SD are defined as follows

$$\begin{aligned} &[H_F',\Gamma^\dagger_{\lambda\mu}]\,|0\rangle=[H_F,\Gamma^\dagger_{\lambda\mu}]\,|0\rangle\ , \\ &[[H_F',\Gamma^\dagger_{\lambda\mu}],\Gamma^\dagger_{\lambda'\mu'}]\,|0\rangle=P_2^{(SD)}[[H_F,\Gamma^\dagger_{\lambda\mu}],\Gamma^\dagger_{\lambda'\mu'}]\,|0\rangle\ . \end{aligned} \tag{6}$$

Here, $\Gamma^\dagger_{\lambda\mu}$ are either $S^\dagger$ ($\lambda=0$) or $D^\dagger_\mu$ ($\lambda=2$) pair creation operators and $P_2^{(SD)}$ is an operator which projects into the SD subspace when acting on states with two pairs of nucleons. We notice that in single j orbit

$$[H_F,\Gamma^\dagger_{\lambda\mu}]\,|0\rangle=\epsilon_\lambda\Gamma^\dagger_{\lambda\mu}\,|0\rangle\ , \tag{7}$$

so that no explicit projection into the SD subspace has been introduced for this commutator. Commutators (6) are of the form (2) and thus a boson Hamiltonian can immediately be associated to H_F'. However, in order to understand the meaning of this Hamiltonian, one needs to investigate the effects of the projection into SD.

The simplest case to study is that of a system of $N=2$ pairs. One can now verify that matrix elements of H_F and H_F' between SD states are identical. As a consequence, both these Hamiltonians have the same eigenvalues and eigenstates in this subspace. In this case, the projected Hamiltonian defined in Eq. (6) has the properties (a) and (b) discussed above ($\mathscr{H}_F\equiv H_F'$) and thus generates an exact boson image of H_F.

The general expression for the boson Hamiltonian which one constructs in correspondence to a QQ Hamiltonian

$$H_F=-Q\cdot Q,\quad Q=[a_j^\dagger\tilde{a}_j]^2 \tag{8}$$

is rather complicated and is given in the Appendix. For clarity, I show here its explicit expression for $j=\frac{23}{2}$,

$$\begin{aligned} H_B^{QQ}= {} & -0.833\hat{n}_s-0.807\hat{n}_d-0.144\hat{n}_s\hat{n}_d \\ & -0.167(d^\dagger\cdot d^\dagger)ss-0.105s^\dagger s^\dagger(\tilde{d}\cdot\tilde{d}) \\ & +0.250s^\dagger d^\dagger\cdot[\tilde{d}\,\tilde{d}]^2+0.286[d^\dagger d^\dagger]^2\cdot\tilde{d}s \\ & +0.041[d^\dagger d^\dagger]^0\cdot[\tilde{d}\,\tilde{d}]^0+0.135[d^\dagger d^\dagger]^2\cdot[\tilde{d}\,\tilde{d}]^2 \\ & +0.035[d^\dagger d^\dagger]^4\cdot[\tilde{d}\,\tilde{d}]^4\ . \end{aligned} \tag{9}$$

For $N>2$ nothing can be said, *a priori*, about the equivalence of H_F and H_F' and, consequently, the boson Hamiltonian (9) is not necessarily a good boson image of H_F. It is indeed plausible that neglecting non-SD components in the commutators of H_F can have important consequences. In such cases, in order to satisfy the requirement (b), one should be able to take into account the effect of these components by suitably "renormalizing" H_F'. A possible way to do that is suggested by the generator-coordinate method[16] (GCM).

This method consists of searching for solutions of the Schrödinger equation which are of the form

$$|\psi\rangle=\int d\alpha\, f(\alpha)\,|\phi(\alpha)\rangle\ , \tag{10}$$

and therefore solving the Hill-Wheeler equation

$$\int[\langle\phi(\alpha)\,|\,H_F\,|\,\phi(\alpha')\rangle-E\langle\phi(\alpha)\,|\,\phi(\alpha')\rangle]f(\alpha')d\alpha'=0\ . \tag{11}$$

Depending on the choice of the generating states $|\phi(\alpha)\rangle$, the states (10) may eventually contain the exact solutions. If we suppose that this is our case, $|\phi(\alpha)\rangle$ being SD states, it follows that the property (b) of $\mathscr{H}_F$ is certainly satisfied if the equality

$$\langle\phi(\alpha)|H_F|\phi(\alpha')\rangle=\langle\phi(\alpha)|\mathscr{H}_F|\phi(\alpha')\rangle \tag{12}$$

holds for any choice of α and α'. If non-SD components of the commutators of H_F play a significant role, one should find that

$$\langle\phi(\alpha)|H_F|\phi(\alpha')\rangle\neq\langle\phi(\alpha)|H_F'|\phi(\alpha')\rangle \ . \tag{13}$$

Equation (12) suggests a way to renormalize H_F' in order to take into account the effect of these components. What one should find is an "extra" interaction W possessing the property (a) and such that

$$\langle\phi(\alpha)|H_F|\phi(\alpha')\rangle=\langle\phi(\alpha)|H_F'+W|\phi(\alpha')\rangle \tag{14}$$

for any choice of α and α'. How I have been looking for this term, in the case of a QQ interaction, is described in the next section.

III. COMPARISON OF FERMION AND BOSON SPECTRA

A straightforward way of investigating the role of the non-SD components of the commutators of (8) is that of comparing the spectra of H_F and H_F'. Two examples of this comparison can be seen in Figs. 1 and 2, for $j=\frac{23}{2}$ and $N=3$ and 4, respectively. Clearly these components play a very important role, especially with increasing N.

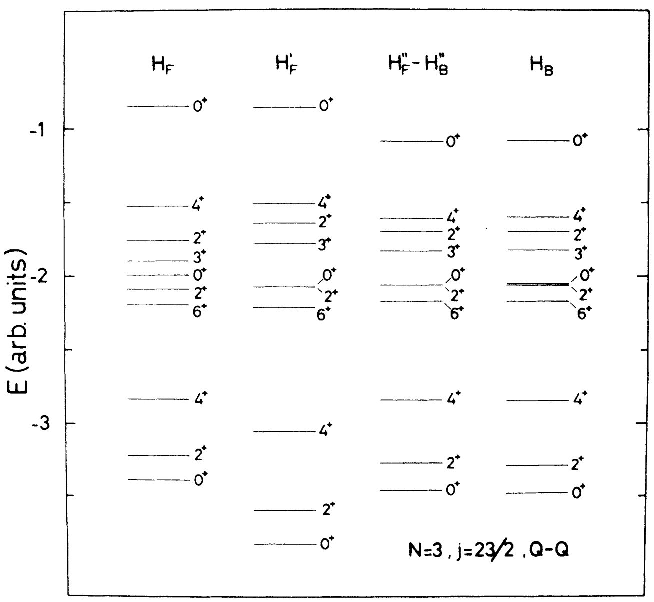

FIG. 1. Spectra of fermion and boson Hamiltonians generated in the SD and sd spaces ($j=\frac{23}{2}$, $N=3$). H_F is the Hamiltonian of Eq. (8); H_F' is the projected Hamiltonian defined in Eq. (6); H_F'' is the renormalized Hamiltonian defined in Eq. (17); H_B'' is the boson image of H_F''; H_B is the Hermitian part of H_B''. Spectra are given in absolute values.

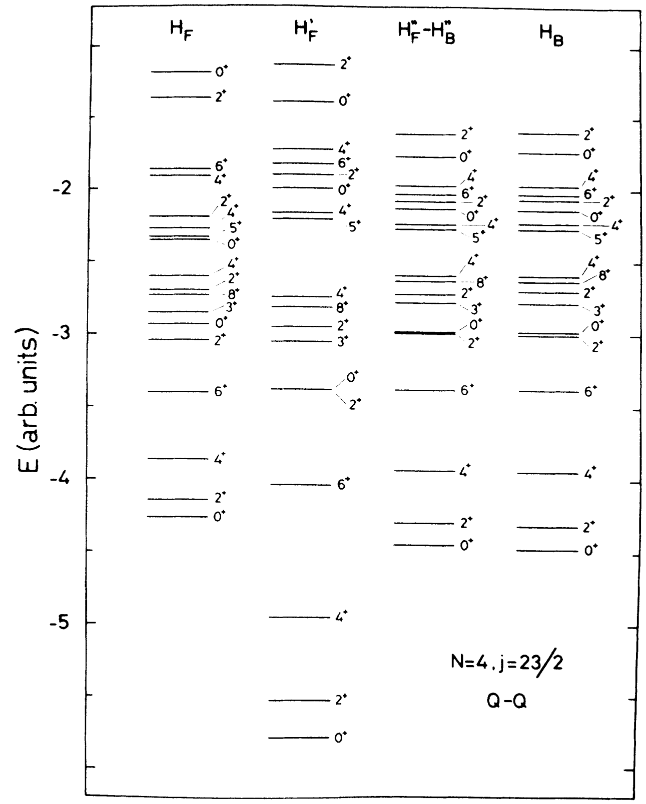

FIG. 2. See caption of Fig. 1, but for $N=4$.

In particular, the projection (6) has the effect of lowering the ground state energy and reducing the moment of inertia of the system.

In order to apply the renormalization procedure discussed in the preceding section, we have first to choose the generating states $|\phi(\alpha)\rangle$. My choice has been

$$|\phi(\alpha)\rangle=\frac{1}{\sqrt{\mathscr{N}(\alpha)}}(S^\dagger+\alpha D_0^\dagger)^N|0\rangle \ . \tag{15}$$

In spite of the simplicity of (15), it has been shown[7] that a state of this form provides a good description of the intrinsic state associated with the ground state band of a system of nucleons interacting with a QQ interaction. Matrix elements of $H_F=-Q\cdot Q$ and corresponding H_F' are shown in Figs. 3 and 4 (diagonal ones at $N=3$ and 4) and in Figs. 5 and 6 (examples of nondiagonal ones at $N=3$ and 4). Differences are very evident also in this case.

By looking more carefully at the difference between the matrix elements of H_F and those of H_F', we can notice a very interesting fact: this difference behaves quite similarly to the matrix elements of H_F'. This is illustrated in Figs. 7 and 8 (diagonal matrix elements, $N=3$ and 4, $j=\frac{23}{2}$). Similar results are found also for different j orbits. This suggests, then, that an expression of the type

$$W=(a-1)H_F'+b \ , \tag{16}$$

with a and b real coefficients, could be appropriate. I have assumed this expression for W and I have fixed the

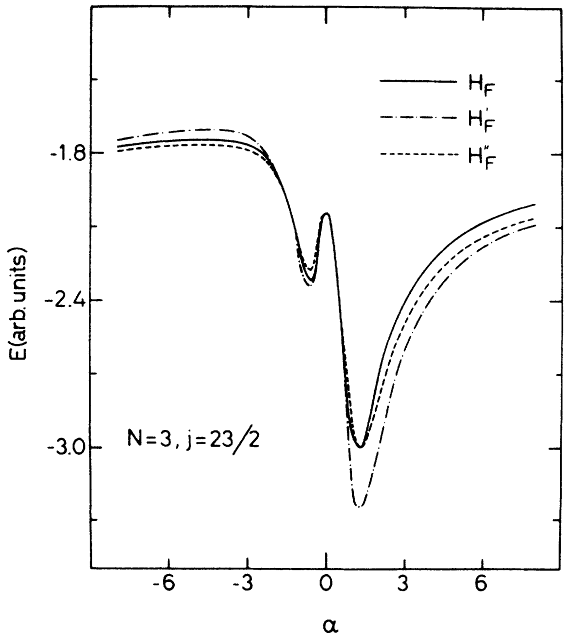

FIG. 3. Diagonal matrix elements of the fermion Hamiltonians of Fig. 1 between intrinsic states (15); $N=3$, $j=\frac{23}{2}$.

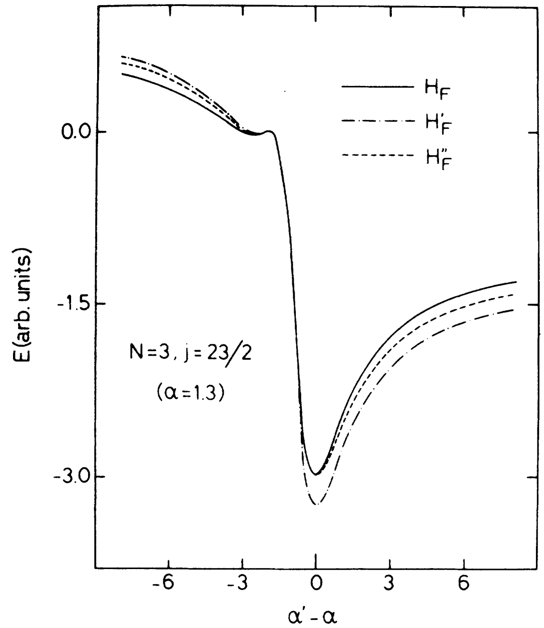

FIG. 5. An example of nondiagonal matrix elements of the fermion Hamiltonians of Fig. 1 between intrinsic states (15); $N=3$, $j=\frac{23}{2}$, $\alpha=1.3$.

coefficients a and b by requiring that the equality (14) be satisfied for the Hamiltonian

$$H_F'' \equiv H_F' + W = aH_F' + b \ , \tag{17}$$

particularly in the region near the minimum of the energy curve. The values found for these coefficients are shown in Table I for $j=\frac{23}{2}$ and $\frac{41}{2}$. Their dependence on j and N is rather smooth. That of the coefficients $a=a_j(N)$, in particular, is shown in Fig. 9 and is rather well reproduced by the analytical expression

$$a_j(N) = \frac{2}{2-\Omega_j}N - \frac{2+\Omega_j}{2-\Omega_j}, \quad \Omega_j = j + \tfrac{1}{2} \ . \tag{18}$$

The matrix elements of (17) can be observed in Figs. 3–6. A much better agreement is now found with the corresponding matrix elements of H_F. As one expects,

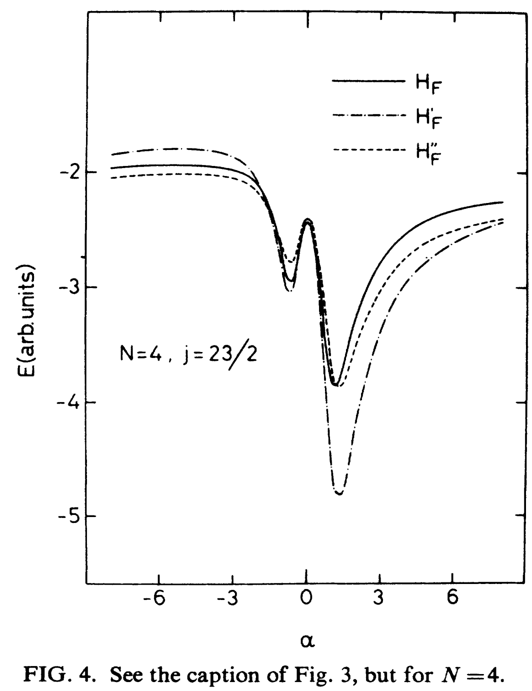

FIG. 4. See the caption of Fig. 3, but for $N=4$.

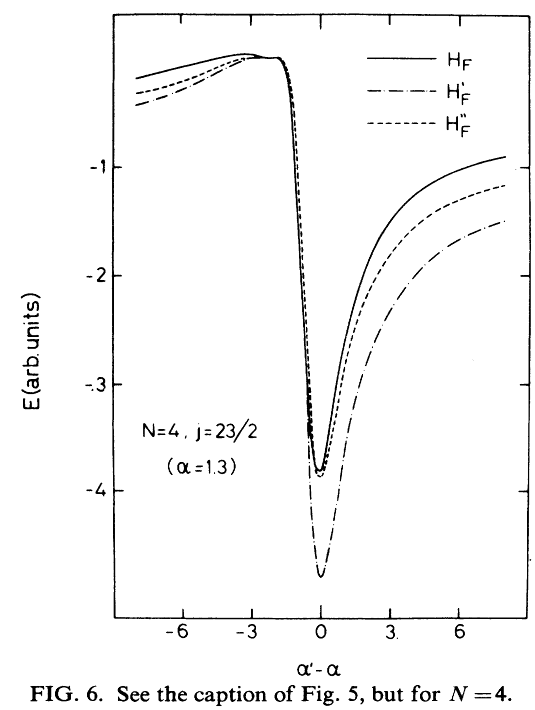

FIG. 6. See the caption of Fig. 5, but for $N=4$.

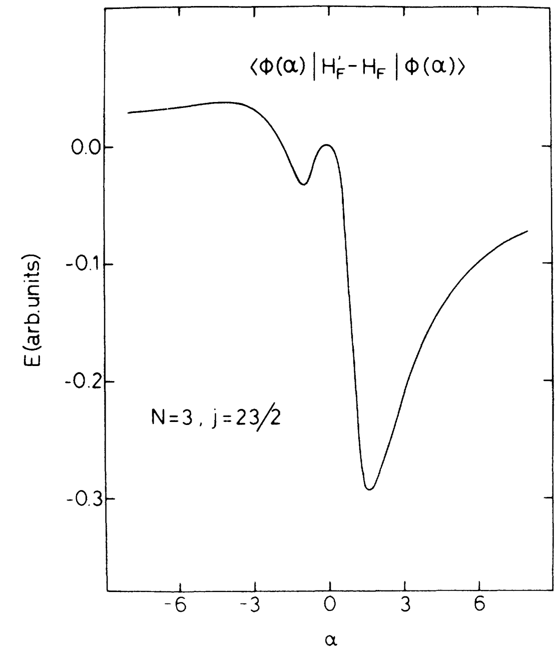

FIG. 7. Diagonal matrix elements between intrinsic states (15) of $H'_F - H_F$, these Hamiltonians being those discussed in Fig. 1; $N=3, j=\frac{23}{2}$.

the spectra of H''_F also compare much better with those of H_F (see Figs. 1 and 2). The Hamiltonian (17) fulfills, by definition, requirement (a) of Sec. II and, as we have just seen, requirement (b) is satisfied also, at least to a good extent, so that $H''_F \approx \mathscr{H}_F$. This Hamiltonian is therefore ready to be mapped onto the boson space and, by doing this, one gets

$$H''_B = a_j(N) H_B^{QQ} + b_j(N) , \tag{19}$$

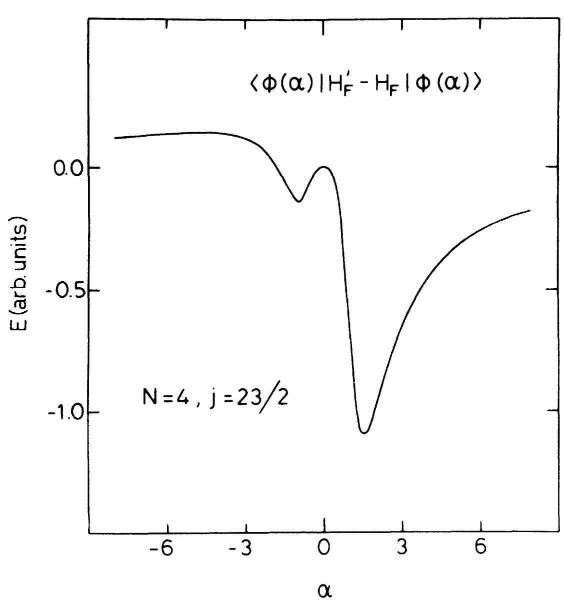

FIG. 8. See the caption of Fig. 7, but for $N=4$.

TABLE I. The coefficients $a=a_j(N)$ and $b=b_j(N)$ of Eq. (17) obtained as described in the text for different values of j and N.

		N 2	3	4
$j=\frac{23}{2}$	a	1.00	0.80	0.61
	b	0.00	-0.40	-0.92
$j=\frac{41}{2}$	a	1.00	0.90	0.82
	b	0.00	-0.13	-0.26

where H_B^{QQ} is given in Eq. (9). As result of the mapping procedure, then, one finds a boson Hamiltonian whose basic structure is that fixed at $N=2$ [Eq. (9)], but which has to be rescaled at larger N according to (19).

IV. $E2$ OPERATORS

Having constructed fermion and boson spectra, we now turn to electromagnetic transitions. In particular, I wish to investigate whether, in analogy to what is observed for the Hamiltonians, where the boson zeroth order approximation[20] has been found to work well, the $E2$ one-body fermion operator

$$T_F^{E2} = [a_j^\dagger \tilde{a}_j]^2 \tag{20}$$

can be effectively mapped onto a one-body boson operator. The testing for this operator will be the comparison of $E2$ matrix elements between corresponding fermion and boson eigenstates.

A preliminary problem which one has to face is that related to the nonhermiticity of the boson Hamiltonian (19). Already, a glance at the coefficients of (9) suggests, however, that the importance of this nonhermiticity should be limited. A simple way of verifying that consists of expressing (19) as a sum of its Hermitian part and its anti-Hermitian part,

$$H''_B = \frac{H''_B + (H''_B)^\dagger}{2} + \frac{H''_B - (H''_B)^\dagger}{2} , \tag{21}$$

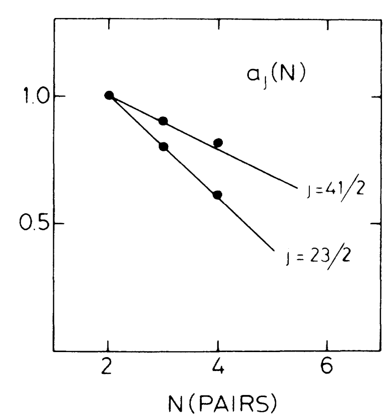

FIG. 9. The coefficients $a=a_j(N)$ of Eq. (17) obtained as described in the text, for different values of j and N (dots). The lines are given by Eq. (18).

and of comparing spectra of the full Hamiltonian and of its Hermitian part ($\equiv H_B$). Examples of this comparison can be seen in Figs. 1 and 2. Differences are hardly noticeable. In view of these results, I have taken the Hermitian part, H_B, of H_B'' as the final boson image of the fermion Hamiltonian (8) and have evaluated $E2$ transitions between its eigenstates.

The most general one-body, Hermitian, rank-2 tensor operator which one can form in the *sd* space is the IBM operator

$$T_B^{E2}=\alpha[d^\dagger s+s^\dagger\tilde{d}]+\beta[d^\dagger\tilde{d}]^2 . \tag{22}$$

The approach I have followed to fix the coefficients α and β has been that of equating the two matrix elements corresponding to the transitions 0_1^+-2_1^+, 2_1^+-4_1^+ in the fermion and boson spaces. The values of α and β obtained in this way, for different values of j and N, are shown (dots) in Fig. 10. In correspondence to these values, one calculates the $E2$ matrix elements shown in Figs. 11—13 ($N=2,3,4$ and $j=\frac{23}{2}$). For completeness I also show, in Fig. 14, a calculation referring to the case $N=4$, $j=\frac{41}{2}$. A good agreement is found between fermion and boson matrix elements.

The dependence of the coefficients α and β derived on j and N is well described by the analytical expressions

$$\begin{aligned}\alpha_j(N)&=\frac{1}{\Omega_j}\left[\frac{2}{\Omega_j}\right]^{1/2}N-\left[1+\frac{1}{\Omega_j}\right]\left[\frac{2}{\Omega_j}\right]^{1/2},\\ \beta_j(N)&=-10\begin{Bmatrix}2&2&2\\ j&j&j\end{Bmatrix}\left[\frac{\Omega_j}{2}\right]^{1/2}\alpha_j(N)\end{aligned} \tag{23}$$

(lines in Fig. 10). An interesting feature of this behavior is that the ratio

$$\chi_j=\beta_j(N)/\alpha_j(N) \tag{24}$$

is independent of N.

For $N=1$, Eqs. (23) define an $E2$ boson operator whose matrix elements in *sd* are identical to the corresponding ones in *SD*. At larger N, similar to what has been observed for the Hamiltonian (19), only a rescaling of this operator by means of an appropriate coefficient $\alpha_j(N)$ has been found necessary. The boson $E2$ operator which emerges from this analysis is therefore different from that which would be obtained in terms of the procedure of Otsuka, Arima, and Iachello (OAI).[20]

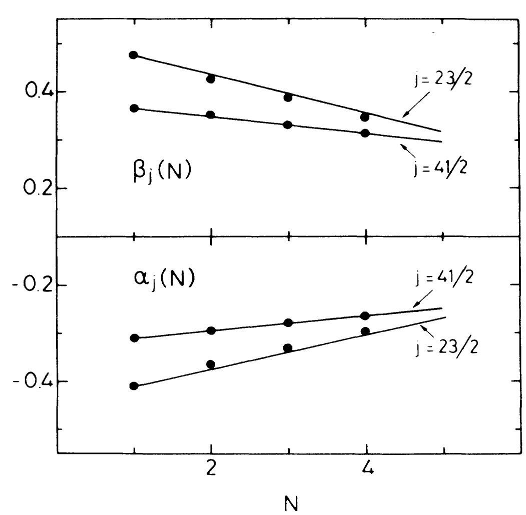

FIG. 10. The coefficients α and β of the $E2$ boson operator [Eq. (22)] obtained as described in the text.

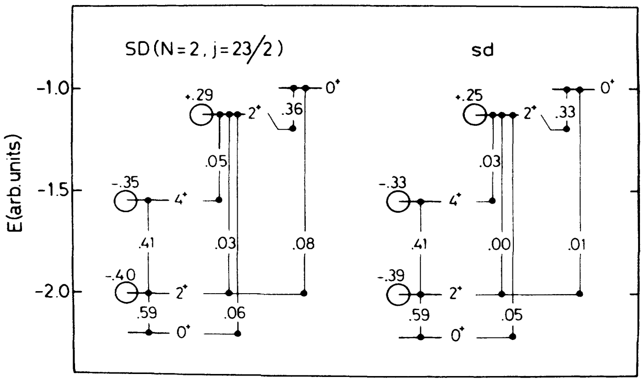

FIG. 11. The spectra of $H_F=-Q\cdot Q$ and of the corresponding boson Hamiltonian H_B for $N=2$, $j=\frac{23}{2}$. Also shown are $E2$ matrix elements between their eigenstates (states are constructed with angular momentum projection $M=0$). When no sign is indicated, the absolute value is given.

V. DISCUSSION AND CONCLUSIONS

In this paper we have dealt with the problem of relating the description of a fermion system in subspaces of the full shell model space built in terms of collective pairs to a description in corresponding boson spaces. In particular, I have discussed the case of the *SD* fermion and the *sd* boson spaces.

I have explored a procedure for deriving boson images of fermion Hamiltonians. The basic idea of this procedure has consisted of searching, in correspondence to a given fermion Hamiltonian H_F, for a new fermion Hamiltonian $\mathscr{H}_F$ equivalent to H_F in the *SD* fermion subspace, such as not to couple this subspace to the rest of the fermion space. Then, an *sd* boson Hamiltonian has been constructed having the same commutation relations as $\mathscr{H}_F$ in *SD*.

I have shown applications of the procedure for a *QQ* interaction. I have derived a boson Hamiltonian that is, at most, two body and non-Hermitian. I have also verified that its degree of nonhermiticity is not relevant. Spectra of this Hamiltonian have been found in good agreement with the fermion ones.

A number of procedures can be found in the literature which deal with the problem discussed in this paper.[17–30] Here, I mention only few of them. With respect to the OAI procedure, important differences can be found in the way the correspondence between the fermion and boson states is established (the OAI correspondence preserves the orthogonality between corresponding states) and in the way the boson Hamiltonian is derived (the OAI method

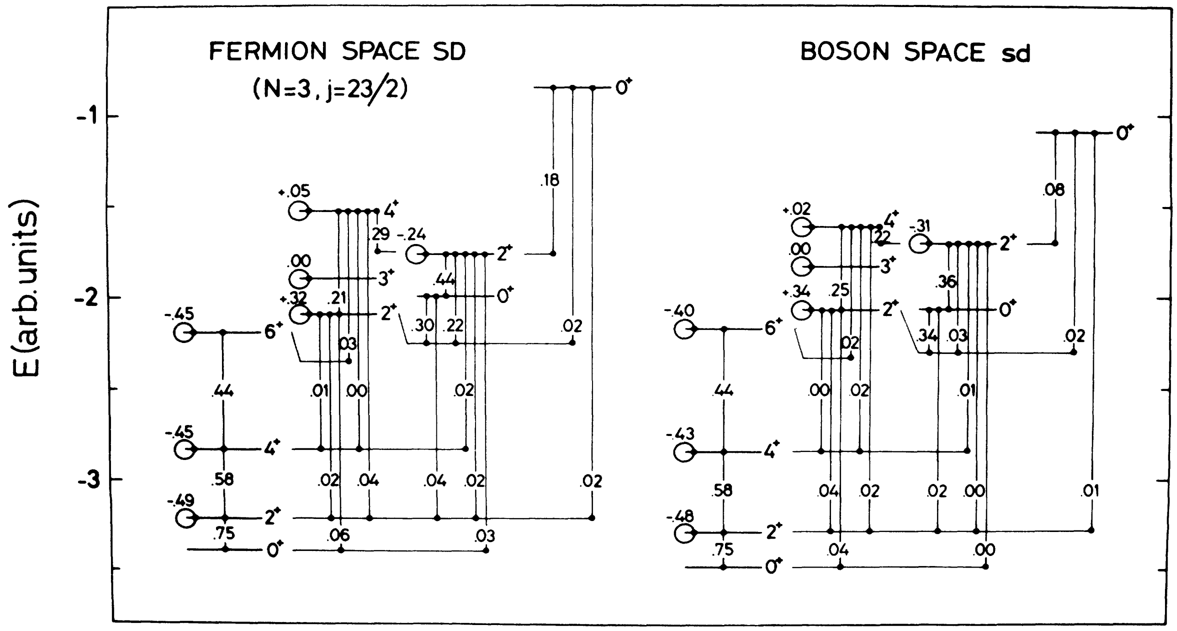

FIG. 12. See caption of Fig. 11, but for $N=3$.

requires that matrix elements of the fermion Hamiltonian between some given states match the matrix elements of the boson Hamiltonian between corresponding states).

A closer link can be found with the approach of Zirnbauer and Brink.[22] These authors, in fact, after establishing a correspondence between coherent fermion and boson states in terms of the Dyson method, end by mapping onto the *sd* boson subspace a fermion Hamiltonian whose commutators are defined exactly as in Eqs. (6). Apart from the different formalism used, then, the major difference between their approach and the one discussed in this paper is embodied in the renormalization of the projected

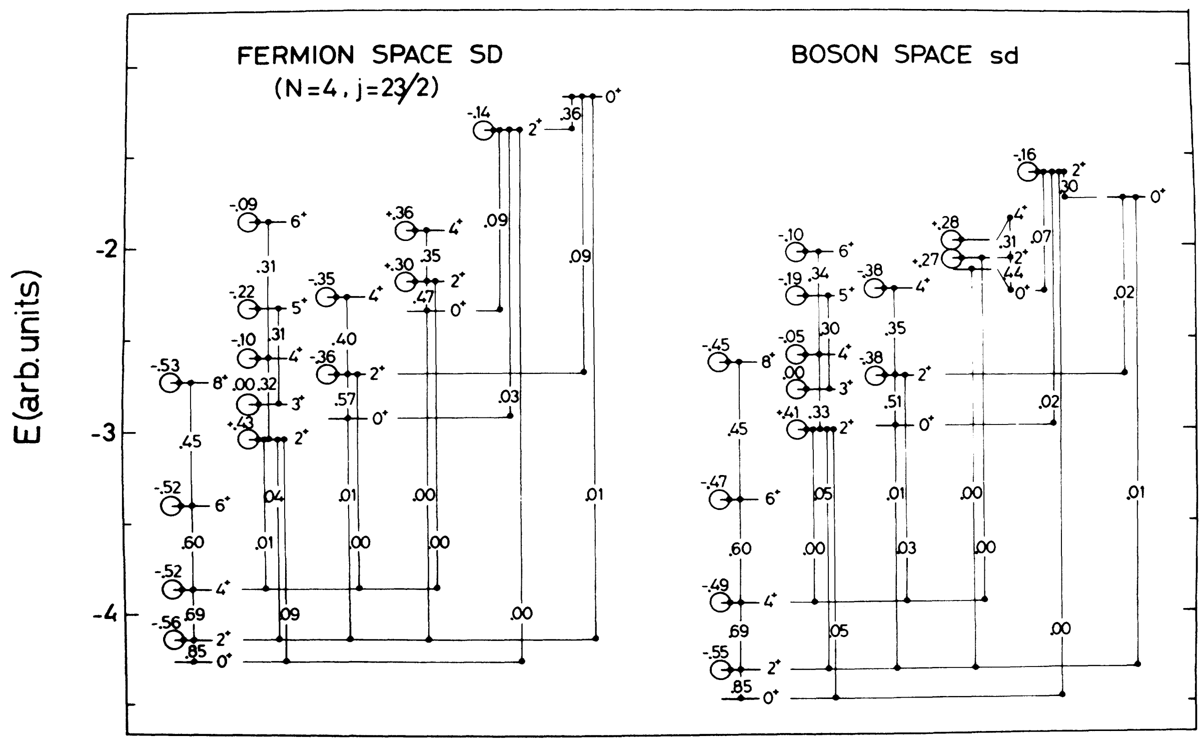

FIG. 13. See caption of Fig. 11, but for $N=4$.

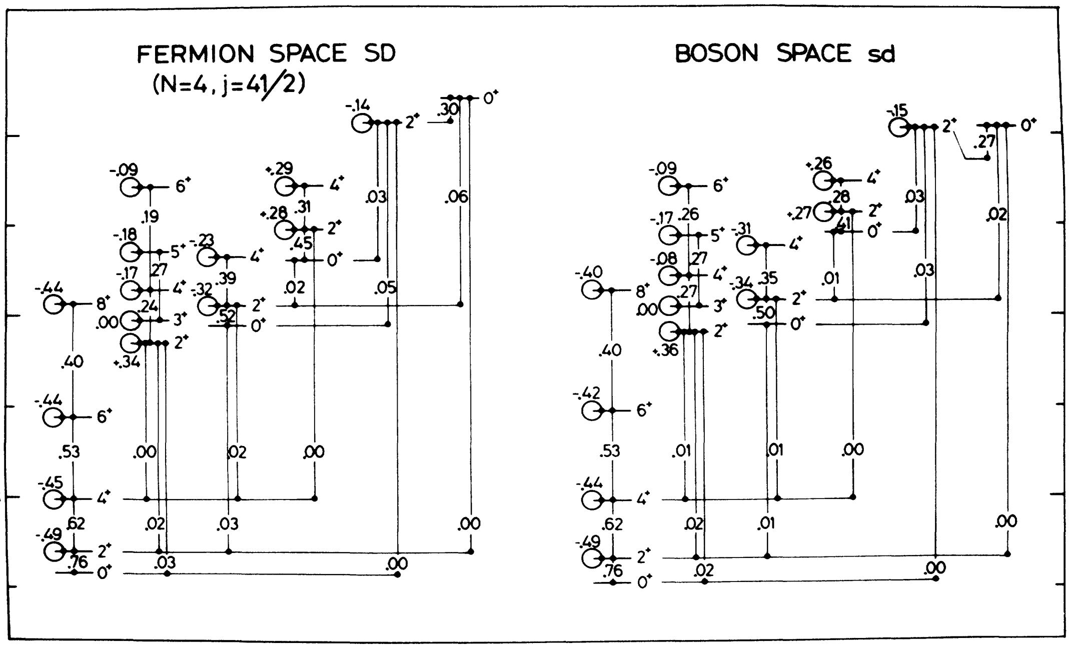

FIG. 14. See caption of Fig. 11, but for $N=4, j=\frac{41}{2}$.

Hamiltonian H_F' which may be needed to restore its equivalence to the original H_F. The case of a QQ Hamiltonian studied here is indeed a case in which this renormalization plays a crucial role. A case in which this does not happen is, instead, that of a pairing Hamiltonian. Spectra generated by the projected Hamiltonian are now in excellent agreement with the exact ones. This case has been treated in Ref. 25.

Some similarities can also be found with the approach of Ref. 29. Also in this work, in fact, arguments based on the GCM are used to map fermion Hamiltonians. There one gets an equation of the type (12), but where the right hand side matrix element refers to the boson space. A similar equation is introduced also for the overlaps. A nontrivial problem which one has to face is, however, that of relating the fermion and boson variables of the generating functions.

As a final point, I have studied $E2$ transitions. I have shown that a one-body quadrupole operator can be successfully mapped onto a one-body boson operator. Besides providing us with information about the $E2$ operator, this analysis represents a severe test for the boson wave functions and thus for the boson Hamiltonians which have been derived. The good agreement found gives further support to these Hamiltonians.

The calculations discussed in this paper have been confined to systems of identical particles. An extension to proton-neutron systems has already been undertaken and will be discussed in a subsequent publication.

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APPENDIX

I give the general expressions of the coefficients of the boson Hamiltonian (4) obtained in correspondence to the fermion Hamiltonian (8):

$$\epsilon_s=-\frac{10}{\Omega_j}\,,$$

$$\epsilon_d=-\frac{5}{\Omega_j}+10\begin{Bmatrix} j & j & 2\\ j & j & 2\end{Bmatrix}\,,$$

$$a_1=0\,,$$

$$a_2=-\frac{4}{\Omega_j}+\frac{120}{\sqrt{10\Omega_j}}\begin{Bmatrix} 2 & 2 & 4\\ j & j & j\end{Bmatrix}R_2(24,02,22)\,,$$

$$a_3=-\frac{4}{\Omega_j}\,,$$

$$a_4=-\frac{4}{\Omega_j}-120\begin{Bmatrix} 2 & 2 & 4\\ j & j & j\end{Bmatrix}^2 R_0(44,00,22)\,,$$

$$a_5=+\frac{40}{\sqrt{2\Omega_j}}\begin{Bmatrix} 2 & 2 & 2\\ j & j & j\end{Bmatrix}$$
$$-\tfrac{240}{7}\sqrt{5}\begin{Bmatrix} 2 & 2 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix} 2 & 2 & 4\\ j & j & j\end{Bmatrix}R_2(24,02,22)$$
$$-\tfrac{300}{7}\sqrt{11/2}\begin{Bmatrix} 2 & 2 & 4\\ j & j & j\end{Bmatrix}^2 R_2(44,02,22)\,,$$

$$a_6=\frac{40}{\sqrt{2\Omega_j}}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}+\frac{120}{\sqrt{10\Omega_j}}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}R_2(24,22,02),$$

$$a_7^0=-200\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2-\frac{600}{\sqrt{5}}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}^2 R_0(44,22,00)\ ,$$

$$a_7^2=\frac{300}{7}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2 -\tfrac{480}{7}\sqrt{5}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}R_2(24,22,02) -\tfrac{600}{7}\sqrt{11/2}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}^2 R_2(44,22,02)\ ,$$

$$a_7^4=-\frac{400}{7}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}^2 -\tfrac{400}{7}\sqrt{55/2}\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}S(2,4) -\tfrac{100}{7}\sqrt{143/5}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}^2 S(4,4) +\frac{80}{\sqrt{2\Omega_j}}\begin{Bmatrix}2 & 2 & 4\\ j & j & j\end{Bmatrix}S(0,4)\ ,$$

where

$$R_J(\lambda_a\lambda_a',\lambda_1\lambda_1',\lambda_2\lambda_2')=(\langle\lambda_2\lambda_2',J\mid\lambda_2\lambda_2',J\rangle\langle\lambda_1\lambda_1',J\mid\lambda_a\lambda_a',J\rangle-\langle\lambda_1\lambda_1',J\mid\lambda_2\lambda_2',J\rangle\langle\lambda_2\lambda_2',J\mid\lambda_a\lambda_a',J\rangle)$$
$$\times(\langle\lambda_1\lambda_1',J\mid\lambda_1\lambda_1',J\rangle\langle\lambda_2\lambda_2',J\mid\lambda_2\lambda_2',J\rangle-\langle\lambda_1\lambda_1',J\mid\lambda_2\lambda_2',J\rangle^2)^{-1}\ ,$$

$$S(\lambda_a,\lambda_a')=\frac{\langle 22,4\mid\lambda_a\lambda_a',4\rangle}{\langle 22,4\mid 22,4\rangle}\ ,$$

with

$$|\lambda\lambda',J\rangle=[A_\lambda^\dagger A_{\lambda'}^\dagger]^J|0\rangle\ ,$$
$$A_\lambda^\dagger=\frac{1}{\sqrt{2}}[a_j^\dagger a_j^\dagger]^\lambda\ .$$

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