Deriving an interacting boson model from the fermion SO(8) model

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(Received 14 October 1986)

To see how closely an interacting-boson-model-like model can be approached, we systematically develop a similarity transformation that unitarizes the simple correspondence version of the Dyson boson mapping of the Ginocchio SO(8) model. For a large region of Hamiltonian parameter space, a one-body plus two-body Hermitian boson Hamiltonian of the interacting-boson-model form is achieved. The similarity transformation consists of a dynamical part and a normalizing part. The dynamical transformation results in a one-body plus two-body Hermitian Hamiltonian and simple transition operators. The normalizing transformation commutes with the dynamically transformed Hamiltonian. This enables the transition rates to be calculated without introducing dual bases. However, the resulting boson model differs from the phenomenological interacting boson model in that transition rates are not calculable from the absolute squares of matrix elements. Dynamically transformed operators are given explicitly, and normalizing transformations for SO(6) and seniority limits are obtained in closed form.

I. INTRODUCTION

The phenomenological interacting boson model[1] (IBM) has been very successful in describing nuclear collective motion. One approach to establishing a microscopic foundation for this model involves isomorphically mapping the collective bifermion operators into an algebra of boson operators. The result is a boson Hamiltonian and observables that incorporate the physics of the fermion system. The exactly solvable SO(8) fermion model[2] due to Ginocchio provides useful tests for methods of deriving an IBM Hamiltonian from a fermion Hamiltonian. In the SO(8) model a fermion collective "S-D" space constructed from monopole and quadrupole pairs is an exact invariant subspace of the Hamiltonian; thus it exactly realizes the dynamical assumption from which the Otsuka, Arima, and Iachello (OAI) method[3] begins. Accordingly, the SO(8) model is exceptionally favorable to IBM development, in that dynamical difficulties are largely precluded. However, even within this invariant subspace, the interesting problem of putting the SO(8) model in IBM form remains. If this can be done exactly, we will have learned that the domain of applicability of the IBM is not a null set, and we can proceed to probe its full extent. If the SO(8) model cannot be put into IBM form, we would like to know where the sticking points are.

Our purpose, then, is to convert the SO(8) model into an IBM equivalent that satisfies (as far as possible) the rules of the phenomenological IBM:

(a) only s and d bosons are involved;

(b) the Hamiltonian is a Hermitian one- plus two-body operator that conserves the total boson number;

(c) collective transition operators are one-body boson operators;

(d) the boson version of each pair-transfer operator is a boson creation operator with a cutoff factor that depends on the numbers of s and d bosons;

(e) transition and pair-transfer rates are calculated from absolute squares of boson matrix elements;

(f) the parameters of the boson Hamiltonian and of the observables in (d) and (e) depend at most on the number of fermions.

Of these rules, it is (b) that most essentially characterizes the IBM, so we give it first priority. Our general strategy is to defer the onset of complications as long as possible. Therefore we choose a boson mapping of the Dyson type, because such mappings (unlike Holstein-Primakoff mappings) automatically give a simple one- plus two-body Hamiltonian. Unfortunately, the Dyson mappings, in general, lead to non-Hermitian boson Hamiltonians,[4] in violation of rule (b). Thus our immediate challenge is to hermitize the boson Hamiltonian that results from Dyson mapping, without destroying its one- plus two-body nature.

In Sec. II we outline the SO(8) model. In Sec. III we describe the particular version of Dyson mapping that we use, namely the simple correspondence (SC) method due to Talmi and Ginocchio.[5] In Sec. IV we try to find, within the class of unitarily equivalent Hermitian boson Hamiltonians, one that is one- plus two-body in nature. We show that if the SC boson Hamiltonian $\widetilde{H}$ can be hermitized while preserving its one- plus two-body nature, a further transformation can always be found that *unitarizes* the SC mapping *without destroying the hermiticity and one- plus two-body nature of the hermitized* $\widetilde{H}$. We call the resulting method the "unitarized SC mapping." We then exhibit the boson images of the Hamiltonian and the transition and pair-transfer operators, and discuss the extent to which these results fit into an IBM scheme. A concluding discussion is given in Sec. V.

II. REVIEW OF THE SO(8) MODEL

We give a brief review of the model closely following the notations used in Ref. 2. Throughout this work, fermion operators will be designated by carets, and operators without carets will be boson operators.

The set of single-particle angular momenta j must be such that

$$j=k+\tfrac{3}{2},k+\tfrac{1}{2},\ldots,|k-\tfrac{3}{2}|\ , \tag{2.1}$$

where k is a positive integer. Let $a_{jm}^{\dagger}$ be a nucleon creation operator for a single-particle state with angular momentum j and projection m. We introduce a spherical-tensor annihilation operator

$$\tilde{a}_{jm}=(-1)^{j+m}a_{j-m}\ . \tag{2.2}$$

Let, e.g., $(a_j^{\dagger}a_{j'}^{\dagger})_M^J$ denote angular momentum coupling. With

$$\Omega\equiv\tfrac{1}{2}\sum_j(2j+1)=2(2k+1) \tag{2.3}$$

and

$$\alpha(jj'rk)=[(2j+1)(2j'+1)]^{1/2}\begin{Bmatrix} j & j' & 2\\ \frac{3}{2} & \frac{3}{2} & k\end{Bmatrix}\times(-1)^{r+k+j+3/2}\ , \tag{2.4}$$

the collective bifermion operators of the model are

$$\hat{S}^{\dagger}=(2\Omega)^{-1/2}\sum_j(2j+1)^{1/2}(a_j^{\dagger}a_j^{\dagger})_0^0\ , \tag{2.5a}$$

$$\hat{D}_{\mu}^{\dagger}=\Omega^{-1/2}\sum_{jj'}\alpha(jj'2k)(a_j^{\dagger}a_{j'}^{\dagger})_{\mu}^2\ , \tag{2.5b}$$

$$\hat{P}_{\mu}^{r}=2\sum_{jj'}\alpha(jj'rk)(a_j^{\dagger}\tilde{a}_{j'})_{\mu}^{r}\quad(r=0,1,2,3)\ , \tag{2.5c}$$

and the Hermitian conjugates $\hat{S}$ and $\hat{D}$. The bifermion operators in equations (2.5) make up the 28 generators of the group SO(8). From the Lie algebra of these operators, three subgroup chains can be found:

$$\mathrm{SO}(8)\begin{array}{c}\nearrow\ \mathrm{SU}(2)\otimes\mathrm{SO}(5)\ \searrow\\ \rightarrow\ \mathrm{SO}(6)\ \rightarrow\\ \searrow\ \mathrm{SO}(7)\ \nearrow\end{array}\mathrm{SO}(5)\rightarrow\mathrm{SO}(3)\ . \tag{2.6}$$

The collective fermion subspace, called L_{SD}, is the space spanned by states made up of only the monopole and quadrupole pairs $\hat{S}^{\dagger}$ and $\hat{D}^{\dagger}$. Following Arima *et al.*,[6] we use a simplified version of the SO(8) Hamiltonian. It differs from Ginocchio's original Hamiltonian by a linear combination of Casimir operators of SO(5) and SO(3). We take

$$\hat{H}=\hat{H}(g)\equiv G_0\Omega\hat{S}^{\dagger}\hat{S}+G_2\Omega\hat{D}^{\dagger}\cdot\tilde{\hat{D}}+\tfrac{1}{4}b_2\hat{P}^2\cdot\hat{P}^2\ . \tag{2.7}$$

The dots in this equation denote scalar products of spherical tensors. We denote the parameters of $\hat{H}$ collectively by

$$g\equiv(G_0,G_2,b_2)\ . \tag{2.8}$$

The Hamiltonian $\hat{H}$ does not couple $\hat{L}_{SD}$ to the rest of fermion space. $\hat{H}$ is so constructed as to be diagonal in τ, the irreducible-representation label for SO(5). We can therefore denote its orthonormal eigenstates by

$$|N\kappa\tau n_{\Delta}JM\rangle=\phi_{N\kappa\tau n_{\Delta}JM}(\hat{S}^{\dagger},\hat{D}_{\mu}^{\dagger})\,|0\rangle\ . \tag{2.9}$$

Here, N is the total number of pairs, while n_{Δ} is the number of triplets of quadrupole pairs coupled to angular momentum zero, and J and M are the angular momentum quantum numbers. The additional quantum number κ, on which the energy may depend, is needed to distinguish different orthogonal states of the same $N\tau n_{\Delta}JM$. In later discussion we frequently abbreviate the state labels by the notation $x=(n_{\Delta}JM)$. Depending on the values of g, $\hat{H}(g)$ may have one of three possible symmetries. The label κ for the eigenstates of $\hat{H}(g)$ is chosen differently in each of these symmetry classifications:

$$\text{(A)}\ \ \mathrm{SU}(2)\otimes\mathrm{SO}(5):\ G_2=b_2,\ \ \kappa\leftrightarrow v/2\ , \tag{2.10a}$$

$$\text{(B)}\ \ \mathrm{SO}(6):\ G_0=G_2,\ \ \kappa\leftrightarrow\sigma\ , \tag{2.10b}$$

$$\text{(C)}\ \ \mathrm{SO}(7):\ G_0=b_2,\ \ \kappa\leftrightarrow w/2\ . \tag{2.10c}$$

The SU(2) representation label v is the seniority. σ labels the irreducible representation of SO(6), while the SO(7) label w is the number of nucleons not paired to angular momentum 2. The SO(6) irreducible representations σ allowed for N nucleon pairs are

$$\sigma=N,N-2,\ldots,0\text{ or }1\quad(N\leq\Omega/2)\ , \tag{2.11a}$$

$$\sigma=\Omega-N,\Omega-N-2,\ldots,0\text{ or }1\quad(N>\Omega/2)\ . \tag{2.11b}$$

III. SIMPLE CORRESPONDENCE

The simple correspondence proposed by Ginocchio and Talmi[5] can be formulated in terms of a linear transformation T that maps the collective Fermion space onto a model boson space. SC can be viewed as a kind of Dyson boson mapping, with the noteworthy feature that under T the Pauli-correction terms are attached to the boson images of annihilation operators rather than to the creation operators. A more rigorous and complete account of this formulation will be given elsewhere.

We introduce ideal boson creation and annihilation operators $s^{\dagger},s$ with angular momentum 0, and $d_{\mu}^{\dagger},d_{\mu}$ with angular momentum 2. These operators have the usual boson commutation relations. The space spanned by products of any number of boson creation operators $s^{\dagger},d_{\mu}^{\dagger}$ acting on the boson vacuum $|0)$ is called L_{sd}. The spherical-tensor annihilation operator is

$$\tilde{d}_{\mu}\equiv(-1)^{\mu}d_{-\mu}\ . \tag{3.1}$$

Ginocchio[2] has shown that there is one-to-one correspondence between fermion states in $\hat{L}_{SD}$ and boson states in L_{sd} for $N\leqq\Omega/2$. The SC boson image of a fermion operator $\hat{X}$ is denoted by

$$X=T\hat{X}T^{-1}\ , \tag{3.2}$$

and the notation $\hat{X}\xrightarrow{T}X$ is often used. The SC images of the collective fermion operators in the SO(8) model are then given by

$$\hat{S}^{\dagger} \xrightarrow{T} s^{\dagger} , \tag{3.3a}$$

$$\hat{S} \xrightarrow{T} S \equiv s - \Omega^{-1}(2Ns - s^{\dagger}I) , \tag{3.3b}$$

$$\hat{D}^{\dagger}_{\mu} \xrightarrow{T} d^{\dagger}_{\mu} , \tag{3.3c}$$

$$\hat{D}_{\mu} \xrightarrow{T} D_{\mu} \equiv d_{\mu} - \Omega^{-1}[2Nd_{\mu} + (-1)^{\mu} d^{\dagger}_{-\mu} I] , \tag{3.3d}$$

$$\hat{P}^{0}_{0} \xrightarrow{T} 2N , \tag{3.3e}$$

$$\hat{P}^{r}_{\mu} \xrightarrow{T} P^{r}_{\mu} \equiv 2\sqrt{2} i^{r-1} (d^{\dagger}\tilde{d})^{r}_{\mu} \quad (r=1,3) , \tag{3.3f}$$

$$\hat{P}^{2}_{\mu} \xrightarrow{T} P^{2}_{\mu} \equiv 2(s^{\dagger}\tilde{d}_{\mu} + d^{\dagger}_{\mu}s) , \tag{3.3g}$$

where

$$I \equiv ss - \tilde{d}\cdot\tilde{d} , \tag{3.4}$$

$$N \equiv s^{\dagger}s + d^{\dagger}\cdot\tilde{d} . \tag{3.5}$$

Because T is not unitary, it is necessary to specify separately the transformation properties of kets and bras:

$$\phi(\hat{S}^{\dagger},\hat{D}^{\dagger}_{\mu}) | 0 \rangle \rightarrow T\phi(\hat{S}^{\dagger},\hat{D}^{\dagger}_{\mu}) | 0 \rangle = \phi(s^{\dagger}, d^{\dagger}_{\mu}) | 0) , \tag{3.6}$$

$$\langle 0 | \phi(\hat{S},\hat{D}_{\mu}) \rightarrow \langle 0 | \phi(\hat{S},\hat{D}_{\mu}) T^{-1} = (0 | \phi(S,D_{\mu}) . \tag{3.7}$$

Note that fermion states are always represented by kets $| \;\rangle$ and bras $\langle \;|$, while boson states will always be represented by kets $| \;)$ and bras $(\;|$. T does not, in general, preserve the orthogonality or normalization of kets. However, (3.6) and (3.7) do imply *biorthogonality* of the bra and ket boson images of originally orthogonal fermion states.

There always exists a special orthogonal basis in the collective fermion space with the property that its orthogonality is preserved under T. We call this the "invariantly-orthogonal (*i-o*) basis for T"; it is, in fact, simply the eigenbasis of the purely kinematical boson operator

$$K = TT^{\dagger} . \tag{3.8}$$

Note that T does *not* preserve the normalizations of the *i-o* basis states.

SC, being of the form (3.2), preserves all algebraic relations. Therefore the boson images (3.3) of the fermion SO(8) generators constitute generators of a boson realization of SO(8), and the same holds for each subgroup of SO(8) in the chains (2.5). In the Appendix K is shown to be a (boson) SO(6) scalar, so that

$$[P^{r}_{\mu}, K] = 0 \quad (r=1,2,3) . \tag{3.9}$$

Let us transform this equation by T^{-1}, and introduce

$$\hat{K} \equiv T^{\dagger}T = T^{-1}KT \tag{3.10}$$

as a fermion analog of K. We easily find

$$[\hat{K}, \hat{P}^{r}_{\mu}] = 0 \quad (r=1,2,3) , \tag{3.11}$$

since $T^{-1}P^{r}_{\mu}T = \hat{P}^{r}_{\mu}$. Thus $\hat{K}$ is a (fermion) SO(6) scalar. An important consequence of this is that the SC images

$$| N\sigma\tau n_{\Delta} JM) = T \,| N\sigma\tau n_{\Delta} JM \rangle \tag{3.12}$$

of the orthogonal basis states $| N\sigma\tau n_{\Delta} JM \rangle$ are again orthogonal in the SO(6) representation label σ, and in the labels $\tau n_{\Delta} JM$ that label the rows of the SO(6) representation, because

$$\begin{aligned}(N\sigma\tau x \,|\, N'\sigma'\tau'x') &= \langle N\sigma\tau x \,|\, T^{\dagger}T \,|\, N'\sigma'\tau'x' \rangle \\ &= \mathcal{M}(N,\sigma)\delta_{NN'}\delta_{\sigma\sigma'}\delta_{\tau\tau'}\delta_{xx'}\end{aligned} \tag{3.13}$$

by Schur's Lemma. In fact *the SO(6) basis is the i-o basis for T.*

The operator K has further interesting properties. One can show that

$$KS^{\dagger}K^{-1} = s^{\dagger} , \tag{3.14a}$$

$$KD^{\dagger}_{\mu}K^{-1} = d^{\dagger}_{\mu} . \tag{3.14b}$$

For example, taking the Hermitian conjugate of (3.2) and applying (3.3b) gives

$$T^{\dagger}S^{\dagger}(T^{\dagger})^{-1} = \hat{S}^{\dagger} , \tag{3.15}$$

so that

$$KS^{\dagger}K^{-1} = TT^{\dagger}S^{\dagger}(T^{\dagger})^{-1}T^{-1} = T\hat{S}^{\dagger}T^{-1} = s^{\dagger} . \tag{3.16}$$

In the Appendix, Eqs. (3.14) are used to determine K for the SO(8) model. The result is

$$\begin{aligned}K &= K(N,\sigma) \\ &= \Omega^{N}(\Omega - N - \sigma)!!(\Omega - N + \sigma + 4)!!/\Omega!!(\Omega+4)!!\end{aligned} \tag{3.17}$$

for $N \leqq \Omega/2$ and $\sigma = N, N-2, \ldots$. This should be interpreted as a function of two commuting observables N and σ, where

$$\sigma(\sigma+4) = C_6 = \tfrac{1}{4}(P^1\cdot P^1 + P^2\cdot P^2 + P^3\cdot P^3) \tag{3.18}$$

is the quadratic Casimir operator of SO(6).

Under SC the fermion Hamiltonian in Eq. (2.7) is mapped as follows:

$$\begin{aligned}\hat{H}(g) \rightarrow \tilde{H}(g) = {} & G_0[n_s(\Omega - N - n_d + 1) - s^{\dagger}s^{\dagger}\tilde{d}\cdot\tilde{d}] \\ & + G_2[\Omega n_d - 2n_d n_s - d^{\dagger}\cdot d^{\dagger}ss + 3(d^{\dagger}d^{\dagger})^0\cdot(\tilde{d}\tilde{d})^0 - 2(d^{\dagger}d^{\dagger})^2\cdot(\tilde{d}\tilde{d})^2 - 2(d^{\dagger}d^{\dagger})^4\cdot(\tilde{d}\tilde{d})^4] \\ & + b_2(5n_s + n_d + 2n_s n_d + d^{\dagger}\cdot d^{\dagger}ss + s^{\dagger}s^{\dagger}\tilde{d}\cdot\tilde{d}) .\end{aligned} \tag{3.19}$$

This result agrees with the Dyson mapping result given in Ref. 6, except that $d^\dagger \cdot d^\dagger ss$ and $s^\dagger s^\dagger \tilde{d}\cdot\tilde{d}$ are interchanged. This difference arises because while SC attaches the Pauli corrections to the annihilation operators, the Dyson mapping of Ref. 6 attaches them to the creation operators.

A Hermitian fermion $\hat{H}$ that is diagonal in the i-o basis will always have a simple-correspondence image $\tilde{H}$ that is *Hermitian*, simply because $\tilde{H}$ is then guaranteed to have an orthogonal eigenbasis and real eigenvalues. When $\hat{H}(g)$ has SO(6) symmetry, it will be diagonal in the i-o basis, which is just the basis classified by SO(6). This is the underlying reason for the relation (noted by Ginocchio and Talmi[5]) between the symmetry of H and the hermiticity of its SC boson image. We note that this result is confirmed by Eq. (3.19), which shows that $H(g)$ is Hermitian in the SO(6) limit, $G_0=G_2$.

IV. SIMILARITY TRANSFORMATIONS

A. Unitarizing the SC transformation

The boson Hamiltonian $\tilde{H}(g)$ of (3.19) has one advantage and one disadvantage. It is a one- plus two-body operator; however, it is, in general, non-Hermitian. We now ask if we can transform $\tilde{H}(g)$ so as to remove its disadvantage while preserving its advantage.

We can certainly find a similarity transformation A_g such that

$$H \equiv A_g \tilde{H} A_g^{-1} \tag{4.1}$$

is Hermitian. Indeed this requirement by itself would leave A_g undetermined up to an arbitrary *unitary* transformation. Suppose now we can, in addition, exploit this arbitrariness to make H a Hermitian one- plus two-body operator; what then remains to be done? The answer is that we would like the boson image of *every* Hermitian operator to be Hermitian, and this requires that we replace A_g by a transformation A such that AT is a *unitary* transformation from the collective fermion space to the model boson space, and $A\hat{H}A^{-1}$ is a one- plus two-body Hermitian operator. We call the resulting transformation AT a *unitarized SC transformation.*

We now show that such an operator A can always be constructed, provided A_g exists. Define

$$\xi \equiv A_g K A_g^\dagger \tag{4.2}$$

and note that $\xi=\xi^\dagger$. ξ commutes with

$$H = A_g \tilde{H} A_g^{-1} \tag{4.3}$$

because, by using the definition $K \equiv TT^\dagger$ [Eq. (3.8)],

$$\begin{aligned}\xi H = \xi H^\dagger &= A_g TT^\dagger A_g^\dagger (A_g^\dagger)^{-1}(T^{-1})^\dagger \hat{H} T^\dagger A_g^\dagger \\ &= A_g T\hat{H}T^\dagger A_g^\dagger ,\end{aligned} \tag{4.4}$$

while

$$H\xi = (\xi H^\dagger)^\dagger = A_g T\hat{H}T^\dagger A_g^\dagger . \tag{4.5}$$

Hence,

$$\xi H = H\xi . \tag{4.6}$$

ξ is positive-definite because every expectation value of ξ is positive. (Of course, the restriction $N \leqq \Omega/2$ that forces L_{sd} to have the same dimensionality as $\hat{L}_{SD}$ is needed to rule out the possibility that ξ might have zero as an eigenvalue.) One can therefore unambiguously define a positive-definite operator η by

$$\eta = \xi^{-1/2} = (A_g K A_g^\dagger)^{-1/2} . \tag{4.7}$$

Now take

$$A = \eta A_g \tag{4.8}$$

and consider the composite transformation

$$U = AT , \tag{4.9}$$

which satisfies

$$UU^\dagger = \eta A_g TT^\dagger A_g^\dagger \eta = \xi^{-1/2}\xi\xi^{-1/2} = 1 , \tag{4.10}$$

and

$$\begin{aligned}U\hat{H}U^{-1} &= \eta A_g T\hat{H}T^{-1}A_g^{-1}\eta^{-1} = \eta A_g \tilde{H} A_g^{-1}\eta^{-1} \\ &= \eta H \eta^{-1} = H ,\end{aligned} \tag{4.11}$$

because, by (4.6) and (4.7), η commutes with H. It is now clear that AT has both the desired properties: it is unitary, and it transforms $\hat{H}$ into a one- plus two-body boson operator, which is, of course, Hermitian.

B. The dynamical transformation operator

We now look for A_g for various values of the strength parameters in $\tilde{H}(g)$. We separate $\tilde{H}(g)$ of Eq. (3.19) into an n_d conserving part $H_0(g)$ and the remainder,

$$\tilde{H}(g) = H_0(g) + (b_2 - G_0) s^\dagger s^\dagger \tilde{d}\cdot\tilde{d} + (b_2 - G_2) d^\dagger \cdot d^\dagger ss , \tag{4.12}$$

where

$$\begin{aligned}H_0(g) = {} & G_0 n_s(\Omega - N - n_d + 1) \\ & + G_2[\Omega n_d - 2n_s n_d + 3(d^\dagger d^\dagger)^0\cdot(\tilde{d}\tilde{d})^0 - 2(d^\dagger d^\dagger)^2\cdot(\tilde{d}\tilde{d})^2 - 2(d^\dagger d^\dagger)^4\cdot(\tilde{d}\tilde{d})^4] + b_2(5n_s + n_d + 2n_s n_d) .\end{aligned} \tag{4.13}$$

Obviously, any similarity transformation generated by a one-body operator will convert a one- plus two-body operator into a new one- plus two-body operator, because its effect is simply to transform to new boson creation operators (and annihilation operators) that are linear combinations of the old ones. We therefore first assume that A_g can be generated by a one-body operator. If we require A_g to be also a rotational scalar, its generators can only be $n_s = s^\dagger s$ and $n_d = d^\dagger \cdot \tilde{d}$. Since $n_s + n_d$ commutes with H, it is sufficient to consider n_d. We easily verify that

$$A_g = \lambda^{-n_d} \quad \text{with } \lambda \equiv [(b_2 - G_2)/(b_2 - G_0)]^{1/4} \tag{4.14}$$

has the desired properties, provided that the parameters lie in the "regular" region

$$(b_2 - G_2)(b_2 - G_0) \geq 0 \; . \tag{4.15}$$

The transformed Hamiltonian is

$$\begin{aligned} H &= A_g \tilde{H} A_g^{-1} \\ &= H_0(g) + \sqrt{(b_2 - G_2)(b_2 - G_0)} (s^\dagger s^\dagger \tilde{d} \cdot \tilde{d} + d^\dagger \cdot d^\dagger ss) \; . \end{aligned} \tag{4.16}$$

These results have been given by Arima[7] for a slightly more restricted region. The regular region (4.15) is composed of the first and third quadrants in Fig. 1. Its boundaries are formed by the seniority limit $G_2 = b_2$, and the SO(7) limit $G_0 = b_2$. The SO(6) limit lies in the middle of the regular region. Accordingly, Eqs. (4.14) and (4.16) apply to the SO(6) limit if one sets $G_2 = G_0$, which gives $A_g = 1$.

It is of some interest that the obvious method of constructing A_g fails in the problematic region

$$(b_2 - G_2)(b_2 - G_0) < 0 \; . \tag{4.17}$$

A transformation generated by n_d can be constructed to convert any problematic $H(g)$ into one standard form:

$$H_P = H_0(g) + \sqrt{|(b_2 - G_2)(b_2 - G_0)|} (s^\dagger s^\dagger \tilde{d} \cdot \tilde{d} - d^\dagger \cdot d^\dagger ss) \; , \tag{4.18}$$

in which the n_d-changing term is *anti*-Hermitian. In spite of this anti-Hermitian term, H_P must, in fact, have only real eigenvalues, because of its relation to the fermion Hamiltonian $\hat{H}$ and the fact that there are no spurious eigenvalues for $N \leqq \Omega/2$. Therefore, if one can transform the particular Hamiltonian H_P into a Hermitian operator, the problem of constructing A_g in the problematic region is solved. The fact that transformations generated by n_d cannot do the job, of course, does not prove that it cannot be done. We shall see examples of this in the seniority limit and the SO(7) limit.

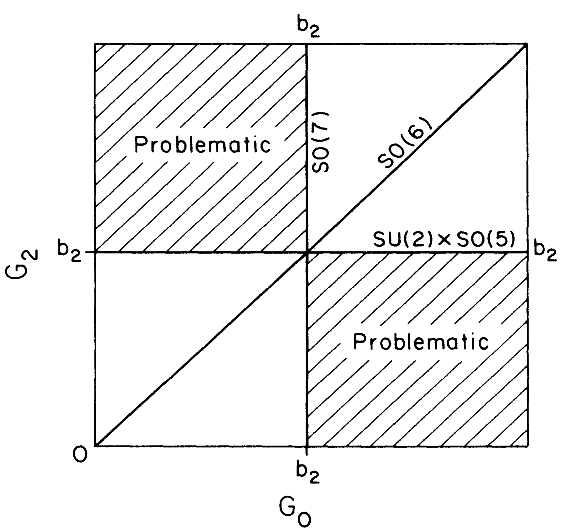

FIG. 1. Hamiltonian parameter space. The unitarized SC is shown to exist in the unshaded region and on its boundaries, which correspond to symmetry limits.

For the seniority ($G_2 = b_2$) and SO(7) ($G_0 = b_2$) symmetry limits, A_g in (4.14) becomes singular, so that a fresh start is needed. For these limits, the matrix of $\tilde{H}(g)$ in an $| N\, n_d \tau x)$ basis is either upper or lower triangular, so the eigenvalues of $\tilde{H}(g)$ are the same as those of $H_0(g)$. In both cases we can therefore find A_g such that

$$H(g) = H_0(g) \; . \tag{4.19}$$

For the seniority limit, this transformation is achieved by

$$A_g = \exp(-\tfrac{1}{2} s^\dagger s^\dagger \tilde{d} \cdot \tilde{d})(\Omega - n_d - \overleftarrow{n}_d + 1)!! \; . \tag{4.20}$$

For the SO(7) limit,

$$A_g = \exp(-\tfrac{1}{2} d^\dagger \cdot d^\dagger ss)(\Omega - n_s - \overleftarrow{n}_s + 5)!! \; . \tag{4.21}$$

In these two equations $\overleftarrow{n}_d$ and $\overleftarrow{n}_s$ act to the immediate left of the exponential. The origin of these positional operators is in the fact that the $A_g T$ mapping identifies $v/2 \leftrightarrow n_d$ (seniority) and $w/2 \leftrightarrow n_s$ [SO(7)]. The structures of (4.20) and (4.21) are similar because of the analogous structure of $\tilde{H}(g)$ in the two limits. The seniority limit result is actually the same as one given in Refs. 5 and 8, although it is written in a different form.

C. A_g transformation of observables

We now examine the effect of applying the similarity transformation A_g to the SC images of pair-transfer and transition operators. To avoid needless repetition, we note that the results are always particularly simple for the SO(5) generators:

$$A_g P^r_\mu A_g^{-1} = P^r_\mu \quad (r = 1,3) \; . \tag{4.22}$$

This is because for all cases in which we have succeeded in finding it, A_g is an SO(5) scalar.

For the regular case $(b_2 - G_2)(b_2 - G_0) > 0$, we find that A_g from Eq. (4.14) gives

$$A_g s^\dagger A_g^{-1} = s^\dagger \; , \tag{4.23a}$$

$$A_g d^\dagger_\mu A_g^{-1} = \lambda^{-1} d^\dagger_\mu \; , \tag{4.23b}$$

$$A_g S A_g^{-1} = \frac{1}{\Omega} [(\Omega - 2N + n_s)s - \lambda^2 s^\dagger \tilde{d} \cdot \tilde{d}] \; , \tag{4.23c}$$

$$\begin{aligned} A_g D_\mu A_g^{-1} = \frac{1}{\Omega} [& (\Omega - 2N)\lambda d_\mu \\ & - (-1)^\mu d^\dagger_{-\mu} (\lambda^{-1} ss - \lambda \tilde{d} \cdot \tilde{d})] \; , \end{aligned} \tag{4.23d}$$

$$A_g P^2_\mu A_g^{-1} = 2(\lambda s^\dagger \tilde{d}_\mu + \lambda^{-1} d^\dagger_\mu s) \; . \tag{4.23e}$$

It is interesting that these consist of simply renormalized operators.

In the SO(6) limit, $A_g=1$, so, of course, all the pair-transfer and transition operators are invariant.

In the SU(2) seniority limit, A_g from (4.20) gives, for example,

$$A_g s^\dagger A_g^{-1}=s^\dagger \ , \tag{4.24a}$$

$$A_g d_\mu^\dagger A_g^{-1}=\frac{1}{\Omega-2n_d+3}(d_\mu^\dagger)-\frac{1}{\Omega-2n_d-1}[(\Omega-2n_d+1)(s^\dagger s^\dagger \tilde{d}_\mu)+s^\dagger s^\dagger d_\mu^\dagger \tilde{d}\cdot\tilde{d}] \ . \tag{4.24b}$$

The results of type (4.24) are different from those obtained by Arima *et al.*[6] in the Dyson scheme by the OAI method. However, the two results are related by Hermitian conjugation followed by a similarity transformation which is diagonal in the $U(1)\times U(5)$ basis.

In the SO(7) limit, with A_g given by (4.21), the results are analogous to the SU(2) results, e.g.,

$$A_g S A_g^{-1}=\Omega^{-1}[(\Omega-2N)(\Omega-2n_s+5)(s)-(\Omega-2n_s+7)^{-1}(s^\dagger\tilde{d}\cdot\tilde{d})+(\Omega-n_s+5)(\Omega-2n_s+3)^{-1}(d^\dagger\cdot d^\dagger\tilde{d}\cdot\tilde{d}s)] \ . \tag{4.25}$$

Complete versions of (4.24) and (4.25) are available on request.

D. η transformation of observables

Even though K and A_g are known, η is hard to construct because it involves a square root. *A fortiori* it is not easy to express the general result of the η transformation. However, K is an SO(6) scalar, so that by Eq. (4.22)

$$\eta P_\mu^r \eta^{-1}=P_\mu^r \quad (r=1,3) \ . \tag{4.26}$$

The difficulty of handling the η transformation is greatly eased by the fact that η commutes with H, so that one may discuss all the physics in terms of matrix elements in a simultaneous eigenbasis of H and η. Because η commutes with H, nondegenerate eigenvectors of H will be eigenvectors of η, and even if there are accidental degeneracies for some g values, eigenvectors of an H slightly perturbed to remove the degeneracy will still be eigenvectors of η. The systematic degeneracy with respect to the SO(5) row labels x does not affect this argument, because η is an SO(5) scalar. Degeneracies in the conserved quantities τ and N would also cause no difficulty, as long as the eigenvectors of H are chosen to have definite τ and N. There is no systematic degeneracy in κ. Let us then examine the form of the matrix elements of some η transformed operator θ in an orthonormal simultaneous eigenbasis $|E\alpha)$ such that

$$H\,|E\alpha)=E\,|E\alpha), \quad \eta\,|E\alpha)=\eta_\alpha\,|E\alpha) \ . \tag{4.27}$$

We find

$$(E\alpha\,|\,\eta\theta\eta^{-1}\,|\,E'\alpha')=\eta_\alpha(E\alpha\,|\,\theta\,|\,E'\alpha')\eta_{\alpha'}^{-1} \ . \tag{4.28}$$

However, physical results will depend on the *absolute square* of the matrix element of $\hat{\theta}$ between normalized fermion states $|E\alpha\rangle$ related to $|E\alpha)$ by the unitary transformation $U=\eta A_g T$; thus,

$$\begin{aligned}|E\alpha\rangle&=U^{-1}\,|E\alpha)=T^{-1}A_g^{-1}\eta^{-1}\,|E\alpha) \ ,\\ \langle E\alpha\,|&=(E\alpha\,|\,(U^\dagger)^{-1}=(E\alpha\,|\,U=(E\alpha\,|\,\eta A_g T \ .\end{aligned} \tag{4.29}$$

We have

$$\begin{aligned}|\langle E\alpha\,|\,\hat{\theta}\,|\,E'\alpha'\rangle|^2&=\langle E\alpha\,|\,\hat{\theta}\,|\,E'\alpha'\rangle\langle E'\alpha'\,|\,\hat{\theta}^\dagger\,|\,E\alpha\rangle\\ &=(E\alpha\,|\,\eta A_g T\hat{\theta}T^{-1}A_g^{-1}\eta^{-1}\,|\,E'\alpha')\\ &\quad\times(E'\alpha'\,|\,\eta A_g T\hat{\theta}^\dagger T^{-1}A_g^{-1}\eta^{-1}\,|\,E\alpha)\\ &=(E\alpha\,|\,(\hat{\theta})_{A_gT}\,|\,E'\alpha')\\ &\quad\times(E'\alpha'\,|\,(\hat{\theta}^\dagger)_{A_gT}\,|\,E\alpha) \ ,\end{aligned} \tag{4.30}$$

because all the η_α and $\eta_{\alpha'}$ factors from (4.28) cancel out. We use a notation in which

$$(\hat{\theta})_{A_gT}\equiv A_g T\hat{\theta}T^{-1}A_g^{-1} \ . \tag{4.31}$$

Note that the matrix elements in the final equation (4.30) are not complex conjugates, because $A_g T$ does not preserve Hermitian conjugation. Thus, IBM requirement (e) is violated.

Equation (4.30) gives a general method of obtaining physical results without explicitly introducing η, at the price of calculating twice as many boson matrix elements as would ordinarily be required. Because we have carried out the A_g transformation, we do not need the dual eigenvectors suggested by Ring and Schuck.[9]

Some methods are available for calculating η_α in particular limits, and are discussed in subsection 3 of the Appendix. In the SO(6) limit, we find

$$\eta_\alpha=[K(N,\sigma)]^{-1/2} \quad (G_0=G_2) \ . \tag{4.32}$$

since K is diagonal in the SO(6) basis. In the SU(2) limit we obtain

$$\eta=\left[\frac{\Omega!!(\Omega+4)!!}{\Omega^N(\Omega-N-n_d)!(\Omega-2n_d+1)(\Omega-n_d+\tau+4)!!(\Omega-n_d-\tau+1)!!}\right]^{1/2} . \tag{4.33}$$

Combined with Eq. (4.24a), this gives

$$(\hat{S}^\dagger)_{AT}=\eta s^\dagger\eta^{-1}=s^\dagger[(\Omega-N-n_d)/\Omega]^{1/2} \ . \tag{4.34}$$

Use has been made of the fact that $s^\dagger$ commutes with n_d and τ. The result agrees exactly with that obtained by the OAI method[6] when allowance is made for our fermion pairs being $\Omega^{1/2}$ times smaller. For the SO(7) limit one can obtain a similar result.

V. DISCUSSION

We have shown that, at least in the regular region and on its SU(2) and SO(7) boundaries, there exists a transformation A_g that hermitizes the SC boson Hamiltonian and preserves its one- plus two-body nature. This makes it unnecessary to find both the right and left eigenvectors of the boson Hamiltonian. Whenever A_g exists, there always exists a further transformation η that makes the full transformation $U=\eta A_g T$ unitary, so that U preserves the relationship of *all* operators under Hermitian conjugation. An interesting new result is that η can always be chosen so that it also preserves the one- plus two-body nature of the Hamiltonian.

We have obtained the results of the A_g transformation over a large part of the parameter space, and of the η transformation over a more restricted region. We can now ask how well the boson models that result from the hermitized SC and the unitarized SC exemplify the phenomenological IBM rules, (a)—(f), given in the Introduction.

We begin with the hermitized SC in the regular region, where the A_g transformation is simplest. Here, A_g converts the boson creation and annihilation operators into linear combinations of the original operators. Since the A_g transformation preserves algebraic relations, this implies that the transition operators, the Hamiltonian, and the pair-transfer operators must all remain as simple under A_g as their SC forms were. The only complications are λ-dependent modifications of the coefficients in the operators. Because $A_g T$ is not unitary, transition rates have to be calculated with allowance for the normalizing transformation η. However, this can be done by means of Eq. (4.30), without construction of or explicit reference to η. Thus one obtains a scheme which is tantalizingly close to the phenomenological IBM, and which, in fact, violates only rule (e). It is even possible to calculate with this scheme within the machinery of a standard IBM program such as PHINT,[11] provided that it can handle such transfer operators as $\tilde{d}\cdot\tilde{d}\,d^\dagger$. The failure of our attempt to justify rule (e) in the context of the other rules—even for SO(8), the best candidate for success—should perhaps serve as motivation for a phenomenological reexamination of the validity of rule (e). The *quantitative* importance of the normalization effects also remains to be studied.

In the SU(2) and SO(7) limits the A_g transformation again exists, but increases the complexity of the boson creation and annihilation operators. It is only by special cancellations of the extra pieces that the Hamiltonian manages to retain its one- plus two-body form. Not surprisingly, the transition operators become complicated beyond the standard of the IBM because the "coefficients" depend (nonpolynomially) on the operator n_d.

In the problematic region of parameter space, we do not know how to hermitize the SC boson Hamiltonian without introducing three- and more-body interactions. If it is possible at all, it requires a transformation A_g radically different in structure from Eq. (4.14). Perhaps this is an indication that the SO(8) model in the problematic region really will not fit into the IBM mold. This question is a challenge for future work.

Among previous authors, Bonatsos and Klein[12] have attacked the problem of unitarizing the Dyson mapping, but without attempting to maintain the one- plus two-body nature of the Hamiltonian. Their similarity transformation is given in a complicated algebraic form which is meaningful only in the SO(6) representation. Geyer and Lee[13] give some qualitative discussion of similarity transformations. Arima,[7] Ginocchio,[8] and Ginocchio and Talmi[5] all studied similarity transformations analogous to A_g, which hermitize the boson Hamiltonian and preserve its one- plus two-body form, but these works do not discuss unitarization.

There is also a recent treatment of the SO(8) model by Li, Pedrocchi, and Tamura.[14] These authors use the SO(8) model for numerical tests of several approximate boson methods. The most accurate of these is found to be NCQP + BET; that is, the number-conserving quasiparticle theory[15] followed by third-order boson expansion[16] of the bifermion operators. The resulting transformation preserves the hermiticity of the Hamiltonian as well as the relations of all transition operators under Hermitian conjugation. The boson Hamiltonian is not strictly of one- plus two-body form, but only because it contains coefficients that depend on n_d, and this does not seem to violate the spirit of the IBM. In principle, if the expansion is carried out to higher order, the boson Hamiltonian will contain nontrivial three-body, four-body, etc., operators. Nevertheless, the good numerical accuracy of the third order NCQP + BET method does suggest that the three-body and more complicated parts of the boson Hamiltonian may be small.

This work was supported in part by the National Science Foundation under Grant No. PHY-82-13597.

APPENDIX

1. Proof that K is an SO(6) scalar

We must show that, for $N\leq\Omega/2$,

$$[P^r_\mu,K]=0 \quad (r=1,2,3)\ . \tag{A1}$$

First, we write

$$S^\dagger=s^\dagger+R^\dagger_s,\quad D^\dagger_\mu=d^\dagger_\mu+R^\dagger_{d_\mu}\ , \tag{A2}$$

where

$$\begin{aligned}R^\dagger_s&=-\frac{1}{\Omega}[2s^\dagger N-I^\dagger s]\ ,\\ R^\dagger_{d_\mu}&=-\frac{1}{\Omega}[2d^\dagger_\mu N+I^\dagger\tilde{d}_\mu]\ .\end{aligned} \tag{A3}$$

The property of K expressed by Eqs. (3.14) can be written as

$$[s^\dagger,K]=KR^\dagger_s,\quad [d^\dagger_\mu,K]=KR^\dagger_{d_\mu}\ . \tag{A4}$$

By Hermitian conjugation of Eq. (A4) we obtain (using $K^\dagger=K$)

$$[s,K]=-R_sK,\quad [d_\mu,K]=-R_{d_\mu}K\ , \tag{A5}$$

where R_s and R_{d_μ} are the Hermitian conjugates of (A3). We give a proof of (A1) for $r=1$. Similar arguments hold for $r=2$ and 3. Consider the commutator

$$[P^1_\mu,K]=[2\sqrt{2}(d^\dagger\tilde{d})^1_\mu,K]$$
$$=2\sqrt{2}\{([d^\dagger,K]\tilde{d})^1_\mu+(d^\dagger[\tilde{d},K])^1_\mu\} \ . \quad (A6)$$

Using the commutation relations (A4) and (A5) in (A6), we obtain

$$[P^1_\mu,K]=2\sqrt{2}\{K(R^\dagger_d\tilde{d})^1_\mu-(d^\dagger\tilde{R}_d)^1_\mu K\} \ . \quad (A7)$$

One can easily obtain

$$(R^\dagger_d\tilde{d})^1_\mu=-\frac{1}{\Omega}(d^\dagger\tilde{d})^1_\mu(2N-2) \ ,$$
$$(d^\dagger\tilde{R}_d)^1_\mu=-\frac{1}{\Omega}(d^\dagger\tilde{d})^1_\mu(2N-2) \ . \quad (A8)$$

Application of (A8) to (A7) results in

$$\left[\frac{\Omega-2N+2}{\Omega}\right][P^1_\mu,K]=0 \ . \quad (A9)$$

Note that hole formalism is used for $N\geqq\Omega/2+1$. Thus K commutes with P^1_μ in the subspace of our concern, i.e., $N\leqq\Omega/2$. The singular behavior of the commutator at $N=\Omega/2+1$ reflects the onset of spurious boson states.

2. Construction of K

We use a recursive method based on the shift operator I defined in Eq. (3.4). Its Hermitian conjugate $I^\dagger$ is an SO(6) scalar operator and satisfies

$$[N,I^\dagger]=2I^\dagger \ . \quad (A10)$$

It follows that $I^\dagger$ probes the N dependence of K:

$$[I^\dagger,K(N,\sigma)]=\{K(N-2,\sigma)-K(N,\sigma)\}I^\dagger \ . \quad (A11)$$

After somewhat lengthy but straightforward algebraic manipulation using the basic commutation relations (A4) and (A5), one obtains

$$[I^\dagger,K(N,\sigma)]=\left[\frac{1}{\Omega^2}\right]K(N,\sigma)\Lambda I^\dagger \ , \quad (A12)$$

where

$$\Lambda=II^\dagger-16N-2\Omega(N-4) \ . \quad (A13)$$

In terms of a quadratic Casimir operator C_6 of SO(6),

$$II^\dagger=(N+2)(N+6)-C_6 \ . \quad (A14)$$

The definition of C_6 is given in Eq. (3.18). From Eqs. (A11)–(A14), we obtain

$$K(N-2,\sigma)=K(N,\sigma)\left[\frac{(\Omega-N-\sigma+2)(\Omega-N+\sigma+6)}{\Omega^2}\right] . \quad (A15)$$

The general solution K of Eq. (A15) is

$$K(N,\sigma)=\Omega^N(\Omega-N-\sigma)!!(\Omega-N+\sigma+4)!!f(\sigma) \ , \quad (A16)$$

where $f(\sigma)$ is an undetermined function of σ. To determine $f(\sigma)$, one can use an operator Σ that probes the σ dependence of K. Or one can determine $A(N=\sigma,\sigma)$ explicitly from the SO(6) eigenstates given in Refs. 2 and 9. A straightforward comparison between two eigenstates, as done in the wave-function-norm method of Eq. (4.39), gives

$$K(N=\sigma,\sigma)=\left[\frac{\Omega^\sigma(\Omega-2\sigma)!!}{\Omega!!}\right] . \quad (A17)$$

From Eqs. (A16) and (A17), Eq. (3.17) for $K(N,\sigma)$ is obtained. Note that K in Eq. (3.17) has the correct boundary value, unity, at the vacuum state ($N=\sigma=0$), as expected.

3. Methods for normalization eigenvalues

The normalization eigenvalue η_α can be inferred from the expectation of η^{-2} in its eigenstate:

$$\eta_\alpha^{-2}=(E\alpha\,|\,\eta^{-2}\,|\,E\alpha)=(E\alpha\,|\,A_gKA_g^\dagger\,|\,E\alpha) \ . \quad (A18)$$

Though K is known from (3.17), it is diagonal only in the SO(6) representation, so no closed-form evaluation of this matrix element is available, except in the SO(6) limit, where $A_g=1$ and $|\,E\alpha)$ is an eigenvector of σ and τ. The result is then

$$\eta_\alpha=[K(N,\sigma)]^{-1/2} \quad (G_0=G_2) \ . \quad (A19)$$

In the seniority and SO(7) limits η_α can be obtained by methods similar to those used in this Appendix to obtain K. In the two limits η_α will be a function of N, n_d, and τ since η is an SO(5) scalar. An alternative method reduces the problem to calculating the norm of a wave function. Note that η^{-2} has the same eigenvalues as

$$\hat{\eta}^{-2}\equiv T^\dagger A_g^\dagger A_g T \ ,$$

its fermion analog, because A_gT converts the eigenvectors of η into eigenvectors of $\hat{\eta}$. Therefore,

$$\eta_\alpha^{-2}=\langle E\alpha\,|\,\hat{\eta}^{-2}\,|\,E\alpha\rangle=\langle E\alpha\,|\,T^\dagger A_g^\dagger A_gT\,|\,E\alpha\rangle \ . \quad (A20)$$

Thus, from the normalized fermion eigenfunction $|\,E\alpha\rangle$, one first calculates a boson wave function $A_gT\,|\,E\alpha\rangle$ and then its norm. The need to know the *fermion* eigenfunction emphasizes that this is not a method for routine practical use.

The wave-function-norm method can be applied in the SU(2) symmetry limit, where the wave functions analogous to $|\,E\alpha\rangle$ and $|\,E\alpha)$ are $|\,Nv\tau x\rangle$ and $|\,N\,n_d=v/2\tau x)$, which are known in the form of Eq. (2.9) from the work of Refs. 2 and 10. Since these wave functions are already normalized, it is only necessary to compute $A_gT\,|\,Nv\tau x\rangle$ and see by what factor (η_α) it differs from $|\,N\,n_d=v/2\,\tau x)$. We obtain Eq. (4.33) of the text.

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