Heavy ion collisions and anisotropic hydrodynamics

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The study of the isotropization of momentum is important in heavy ion collisions. To do this we construct a generalized hydrodynamical equation system, in which the anisotropy of the momentum distribution is added as a new variable. These equations are derived from the moment equations of the relativistic Boltzmann equation where the closure of the set is achieved by assuming a particular class of initial conditions. The equations are then explicitly solved for two uniform interpenetrating hadron streams. The collision cross sections are the bare hadron cross sections; the presence of the other hadrons can be simulated by the use of a density- and energy-density-dependent temperature and mass, taken over from self-consistent calculations. The results are compared with other theoretical results. We find that the isotropization occurs sufficiently rapidly for medium energy head-on collisions to reach local thermal equilibrium.

I. INTRODUCTION

A. The setting of the problem

A popular description of heavy ion collisions utilizes conventional hydrodynamical[1,2] or thermodynamical[3] models. This presupposes that local thermal equilibrium reestablishes itself after a few hadron collisions, during the short time 4–8 fm/c. Only then can we hope that a complicated dynamical system can be characterized (as in these models) by a few macroscopic variables, such as the momentum density, density, energy density, and entropy density, coupled with each other through the local thermodynamical relations and the equations of motion. A particular, and experimentally testable, feature of local thermalization is the isotropy of the momentum distribution. In fact, experiments indicate that the end result of head-on collisions is a nearly isotropic momentum distribution.[4–6] In the present essay we study on a relativistic kinetic model the evolution of the anisotropy of the momentum distribution. (The previous studies[7–10] were nonrelativistic.)

Our aim is to construct a model which is (a) relativistic, (b) accounts for pair collisions explicitly (these being the most important for altering the anisotropy), and (c) may include the presence of the other particles as an external average field of force.

We shall base our considerations on the Boltzmann equation. The emergence of a hydrodynamical description from the Boltzmann equation has an immense literature.[11] We mention here briefly only the salient facts. Boltzmann himself has shown that the first five moment equations of the distribution function correspond to the five conservation laws, to wit, the conservation of mass, momentum, and energy. However, these equations are not yet closed because an equation describing a moment of a given order contains moments of higher order, which depend functionally on the distribution function. If local thermal equilibrium has been established, this distribution function becomes locally Maxwellian, but contains five largely arbitrary functions which (because of this arbitrariness) can be identified with the first five moments. The celebrated Chapman-Enskog expansion scheme is based on this observation, by assuming that the distribution function already has this form. In turn, Hilbert has shown, that in a dilute gas, for a large class of interaction forces, practically all initial distribution functions relax into a local Maxwellian one which depends on five arbitrary functions. Once the distribution function is of this form, the equation system formed by the first five moments closes (since the distribution function depends only on the five moments and has a given form). In addition, the form of the distribution function is such that the local thermodynamical relations hold true. This, then is the general situation. What else can happen? There are only three places where novelty can enter. We may study how particular initial conditions thermalize; we may change the expansion parameter used by Hilbert; we may change the forces of interaction. In the present study we shall concentrate on the first aspect, and study the evolution of distribution functions which depend functionally on the first ten moments of the distribution function. We lack a corresponding Hilbertian theorem to show that many initial distribution functions relax into this form; and we do not expect that there is, in general, such a theorem. However, we expect that for relativistic collisions, and for such initial situations which play a role in heavy ion collisions, this is the case.

B. The choice of initial data and additional macroscopic variables

We may then consider initial data which deviate from local thermal equilibrium only through a small term expressing a small anisotropy, and we will do this. Then, however, the following objection can be raised to our plan. It is well known that the Chapman-Enskog procedure leads to a series of approximate hydrodynamical equations; to first order, these are the ideal fluid equations with no dissipation; to second order we find the usual Navier-Stokes equations, where the presence of the anisotropy is buried in the sheer viscosity, bulk viscosity, and thermal conductivity. Consequently, it seems that the problem of a hydrodynamics with anisotropy in the momentum distribution has already been solved. This, however, is not the case. First, this setting of the problem does not answer directly the question about the decay of anisotropy; at most, it shows that the measure of anisotropy should be expressible by the coefficients associated with irreversible processes. Second, it is also known that the next higher order approximation (the so called Burnett equation) gives a different answer from the one obtained by the moment method of Maxwell and Grad. Thus, the question still remains as to the actual equation system which explicitly expresses the thermalization of the momentum anisotropy and makes use of only the usual macroscopic variables explicitly augmented by variables describing this anisotropy. Maxwell's moment equations, or Grad's 13 moment equations (or their relativistic extension using 14 moments) can be thought of as accomplishing this task. However, there no explicit anisotropy variables are introduced, but rather the heat flow and the viscous pressure tensor are used implicitly as the new state variables (added to the usual hydrodynamical variables) to form a closed equation system. This can indeed be done, relatively simply for the Maxwell case, and with some labor for the Grad case, while for relativistic motions only approximate expressions are known.

Thus, it seems to us that a direct derivation of these equations is worthwhile, introducing from the beginning the natural variables to describe the anisotropy, without prejudicing their relations to the irreversible transport coefficients.

This is then our plan. We accept that the equations describing the moments of the momentum distribution correspond to the generalized hydrodynamical equations, if the equation system is closed. The closure of this equation system outside local thermal equilibrium is obtained by restricting ourselves to special initial data for the distribution function, these being slightly anisotropic in the momentum distribution.

If we now introduce an anisotropy field as a new variable, and make use of the smallness assumption, we obtain a closed extended hydrodynamical equation system.

Finally, we apply this model to study numerically the relation of two uniform streams of nuclear matter which interpenetrate each other. The forces between the nucleons are either considered as simple pair interactions, or as average forces, following Walecka's mean field theory.[12] The initial data chosen correspond to momentum distributions in the parameter range relevant to relativistic heavy ion collisions, the bombarding energies per nucleon ranging from 0.2 to 1 GeV.

II. THE RELATIVISTIC EQUATION SYSTEM

A. The general equations

We follow de Groot *et al.*[13] and introduce the relativistic Boltzmann equation [using the following conventions: $g_{\mu\nu}$ diagonal $(+1,-1,-1,-1)$; $\mu=0,1,2,3$; $c=1$; Boltzmann's constant equal to 1].

Accordingly, Boltzmann's equation is given by

$$\partial_\nu p^\nu f(x,p)=C^1(f,f)\ . \tag{1}$$

Here, x^i,p^i are the position and momentum three-vectors; p^ν is the momentum four-vector ($p_\nu p^\nu=m^2$), and the invariant volume element in the three-momentum space is d^3p/p_0; $f(x,p)$ is a scalar, p_0 is a function of p^1,p^2,p^3 via $p_\nu p^\nu=m$. The moments of the distribution function are defined as averages of powers of p given by

$$M^{\alpha\beta\gamma}\cdots=\int(d^3p/p_0)f(x,p)p^\alpha p^\beta p^\gamma\ldots\ . \tag{2}$$

C^1 is the relativistic collision integral. Its detailed properties do not concern us at the moment, save the conditions

$$\begin{aligned}&\int(d^3p/p_0)C^1f=0\ ,\\&\int(d^3p/p_0)C^1p^\nu f=0\ ,\end{aligned} \tag{3}$$

which are required by the conservation of particle number, conservation of energy, and conservation of momentum in a pair collision.

We now multiply (1) by 1, p^ν, p^μ, and p^ν in succession and integrate. This gives the equations

$$\partial_\mu N^\mu=0\ , \tag{4}$$

$$\partial_\mu T^{\mu\nu}=0\ , \tag{5}$$

$$\partial_\mu A^{\mu\nu\tau}=\int(d^3p/p_0)p^\nu p^\tau C^1\ , \tag{6}$$

where

$$N^\mu=\int(d^3p/p_0)fp^\mu\ , \tag{7}$$

$$N^{\mu\nu}=\int(d^3p/p_0)fp^\mu p^\nu\ , \tag{8}$$

$$A^{\mu\nu\tau}=\int(d^3p/p_0)fp^\mu p^\nu p^\tau\ , \tag{9}$$

are the first few moments of f. The first equation expresses the conservation of the particle four-current; the second expresses the conservation of the four-momentum. [The right hand sides are zero, by using (2).] The last equation gives the evolution of $A^{\mu\nu\tau}$. The latter will be related to the anisotropy of the momentum distribution. In general, these equations do not form a closed system. Conventional relativistic hydrodynamics arises from them when we are able to relate $T^{\mu\nu}$ and N^μ either by assuming local thermal equilibrium, or by obtaining these relations via a Chapman-Enskog—type expansion.

Our present aim is to establish a closed equation system for N^μ, $T^{\mu\nu}$, and $A^{\mu\nu\tau}$ for a special class of initial conditions. To express this condition and the resulting equations in a physically more transparent way, we must introduce new quantities with more obvious meanings. This is done for the general case in Appendix A, leading to formidable expressions. For a specially homogeneous system the problem simplifies, and the general formalism can be obviated.

B. The spatially homogeneous, cylindrical case

Introduce

$$U^{\nu}=N^{\nu}/(N_{\alpha}N^{\alpha})^{1/2}\ , \tag{10}$$

a unit vector along the four-current N^{ν}, which defines Eckart's hydrodynamical four-velocity (using thus, for this purpose, the conserved baryon number). If the situation is spatially homogeneous, N^{μ}, $T^{\mu\nu}$, and $A^{\mu\nu\tau}$ simplify since there exists a global instantaneous rest frame, in which U^{ν} is (1,0,0,0).

According to our scheme we close this set of equations for a particular choice of initial data, by linearizing the collision integral.

We put at $t=0$

$$f=f_0(1+\Phi)\ , \tag{11}$$

with

$$f_0=[1/(2\pi\hbar)^3]\exp[(\mu-p^{\nu}U_{\nu})/T]\ , \tag{12}$$

$$\Phi=C_{\overline{\mu\nu}}p^{\overline{\mu}}p^{\overline{\nu}}/T^2 \ \text{(small to first order)}\ . \tag{13}$$

Here, T,μ, are as yet auxiliary variables, which will turn out to be the temperature and chemical potential. (The latter can be ultimately eliminated, and thus it no longer concerns us.) $\overline{p^{\mu}p^{\nu}}$ is the trace free part of p^ip^j in the local rest frame (Appendix A). $C_{\overline{\mu\nu}}$ regulates the anisotropy of f; since $\overline{p^{\mu}p^{\nu}}$ has only space-like components, Φ is a quadratic form in p^1,p^2,p^3, with $C_{\overline{\mu\nu}}$ being the coefficients.

Considering the momentum space symmetry of the system, we get

$$N^{\nu}=nU^{\nu}\ , \tag{14}$$

with n the number density in the rest frame;

$$T^{\mu\nu}=\mathscr{E}U^{\nu}U^{\mu}+P^{\nu\mu}\ , \tag{15}$$

with $\mathscr{E}$ the energy density in the rest frame and $P^{\nu\mu}$ the pressure tensor (which will not be needed);

$$A^{\nu\mu\tau}=Q^{\nu\mu}U^{\tau}+Q^{\nu\tau}U^{\mu}+Q^{\mu\tau}U^{\nu}+QU^{\mu}U^{\nu}U^{\tau}\ . \tag{16}$$

$Q^{\nu\mu}$ has only spatial components in the rest frame and describes the anisotropy of the momentum distribution; Q is defined by (A13) (but will not be needed here).

Choosing then a frame with $U^{\mu}=(1,0,0,0)$, we immediately get

$$\frac{\partial n}{\partial t}=0\ , \tag{17}$$

$$\frac{\partial \mathscr{E}}{\partial t}=0\ ; \tag{18}$$

projecting for the space-like, traceless part of Eq. (6), we obtain

$$\frac{\partial Q^{\overline{\mu\nu}}}{\partial t}=\int (d^3p/p_0)\overline{p^{\mu}p^{\nu}}C^1\ , \tag{19}$$

where C^1 is the relativistic collision integral. We notice that $p^{\mu\nu}$ and Q decouple from the set and can be determined afterward from (A22) and (A25).

Use (8) and (9) with (17), and equate the result to (12) and (13). A straightforward but tedious evaluation gives

$$\mathscr{E}=nT(a_2/a_1)\ , \tag{20}$$

$$\int (d^3p/p_0)\overline{p^{\mu}p^{\nu}}C^1=-\eta nQ^{\overline{\mu\nu}}\ . \tag{21}$$

Here, a_1, a_2, and n are complicated functions of T; η is also a function of the differential cross section (see Appendix B). (We notice that the equation system is closed because the integral involving C^1 can be expressed in terms of $Q_{\mu\nu}$.)

The initial value of $C_{\overline{\mu\nu}}$ determines the initial value of $Q^{\overline{\mu\nu}}$ and vice versa. Given $\mathscr{E}$, n, and $C_{\mu\nu}$ at time $t=0$, we implicitly give T at $t=0$ as well, and thus obtain the full set of initial data required to integrate the equations.

We find

$$n=n_0\ , \tag{22}$$

$$\mathscr{E}=\mathscr{E}_0\ , \tag{23}$$

$$Q^{\mu\nu}(t)=Q(\tau=0)\exp(-\eta nt)\ \ [\text{with } U^{\nu}=(1,0,0,0)]\ . \tag{24}$$

We recovered Maxwell's famous result: the anisotropy relaxes exponentially with a relaxation time $(n\eta)^{-1}$.[11]

III. THE EVALUATION OF THE RELAXATION TIME AND THE ANISOTROPY CHANGE

The results are determined if we give the collision cross section and the initial data.

A. The scattering cross section

The nucleon-nucleon cross section in the center of mass system is specified by $\sigma(|p_{\text{c.m.}}|,\theta_{\text{c.m.}})$, where $p_{\text{c.m.}}$ is the three-momentum and $\theta_{\text{c.m.}}$ is the angle in the center of mass frame; σ is taken from Ref. 14.

The presence of the other particles may be taken into account by a self-consistent calculation.[12,15–18] The crudest approximation is obtained if we retain σ as given, and alter the temperature and mass using the relations obtained by Walecka, which gives

$$T=T^*(n,\mathscr{E})\ ,\quad m=m^*(n,\mathscr{E})\ .$$

Since n and $\mathscr{E}$ are time independent, it is sufficient to adjust T and m at the initial time according to these relations.

B. The initial data

At $t=0$ we observe two interpenetrating, homogeneous streams of nuclear matter with equal and opposite momenta. Thus, we have chosen the center of momentum frame. This implies a distribution function f which is independent of the position; the momenta are distributed in momentum space within two Fermi spheres whose centers

TABLE I. Ellipsoidal approximation to the two-Fermi-sphere distribution. E_b/A is the bombarding energy per nucleon, n_0 the initial density, $\mathscr{E}_0$ the initial energy density, $Q_0^{\overline{zz}}$ the relevant component of the tensor Q, C^{zz} the relevant component of the tensor C, and T the temperature.

Ideal gas					
E_b/A (GeV)	n_0 (1/fm^3)	$\mathscr{E}_0$ (GeV/fm^3)	$Q_0^{\overline{zz}}$ (GeV^2/fm^3)	T (GeV)	C^{zz}
0.2	0.3997	0.4054	0.0276	0.0478	0.0292
0.3	0.4093	0.4252	0.0425	0.0625	0.0401
0.4	0.4186	0.4450	0.0579	0.0764	0.0494
0.5	0.4277	0.4648	0.0739	0.0896	0.0572
0.6	0.4366	0.4847	0.0906	0.1023	0.0634
0.7	0.4453	0.5045	0.1078	0.1145	0.0697
0.8	0.4538	0.5243	0.1255	0.1263	0.0748
0.9	0.4622	0.5441	0.1438	0.1376	0.0792
1	0.4705	0.5639	0.1627	0.1486	0.0831

Self-consistent approximation						
E_b/A (GeV)	n_0 (1/fm^3)	$\mathscr{E}_0$ (GeV/fm^3)	$Q_0^{\overline{zz}}$ (GeV^2/fm^3)	T (GeV)	m^*/m[a]	C^{zz}
0.2	0.3997	0.3877	0.0276	0.011	0.722	0.0685
0.3	0.4093	0.4064	0.0425	0.031	0.730	0.0849
0.4	0.4186	0.4251	0.0579	0.045	0.737	0.0994
0.5	0.4277	0.4437	0.0739	0.057	0.744	0.1113
0.6	0.4366	0.4624	0.0906	0.069	0.750	0.1204
0.7	0.4453	0.4811	0.1078	0.080	0.757	0.1277
0.8	0.4538	0.4998	0.1255	0.090	0.763	0.1343
0.9	0.4622	0.5184	0.1438	0.099	0.768	0.1406
1	0.4705	0.5371	0.1627	0.107	0.773	0.1467

[a]This is the effective mass.

are symmetrically displaced from the origin in the flow direction. The separation of the Fermi spheres is a function of the bombarding energy.

One can evaluate the moments of this distribution function and approximate it with the moments of an ellipsoidal distribution implied through the choice $\Phi=C_{\overline{\mu\nu}}p^{\mu}p^{\nu}$ in (17). This, then, determines the initial data for n, $\mathscr{E}$, T, Q^{zz}, and C^{zz} as a function of the bombarding energy per nucleon (Table I). The question, however, arises as to how far the ellipsoidal distribution can adequately represent our assumed distribution function at $t=0$. After all, two displaced spheres of equal size are hardly an ellipsoid! The error can be estimated by comparing the first and second moments computed for the displaced spheres distribution, and the ellipsoidal distribution. These results are found in Table II. In fact, we find that these deviations increase as the bombarding energy increases, finally reaching 25% in the anisotropy tensor. Since the equations are linear, we do not expect a larger change in the results. This does not affect the conclusions.

C. The coefficient η

η appears as a coefficient in an expression specifying a moment of the collision integral C^1. Formally, C^1 con-

TABLE II. The first two moments of the Fermi-sphere and ellipsoidal distribution functions compared (the moments of the latter carry the index zero).

E_b/A (GeV)	Q (GeV^2/fm^3)	Q_0 (GeV^2/fm^3)	P (GeV/fm^3)	P_0 (GeV/fm^3)	$P^{\overline{zz}}$ (GeV^2/fm^3)	$P_0^{\overline{zz}}$ (GeV^2/fm^3)
0.2	0.4118	0.4126	0.0193	0.0191	0.0264	0.0243
0.3	0.4428	0.4445	0.0259	0.0255	0.0396	0.0362
0.4	0.4746	0.4775	0.0325	0.0319	0.0529	0.0471
0.5	0.5071	0.5115	0.0391	0.0383	0.0661	0.0575
0.6	0.5403	0.5465	0.0457	0.0446	0.0793	0.0676
0.7	0.5742	0.5825	0.0523	0.0510	0.0925	0.0774
0.8	0.6088	0.6195	0.0590	0.0573	0.1057	0.0869
0.9	0.6440	0.6575	0.0656	0.0636	0.1189	0.0962
1	0.6799	0.6964	0.0722	0.0699	0.1321	0.1053

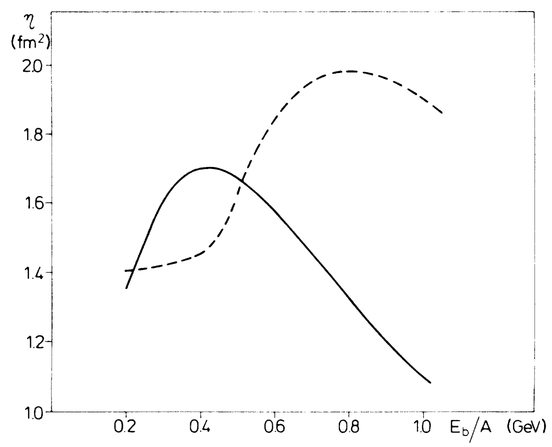

FIG. 1. The relaxation coefficient for anisotropy, η in nuclear matter as a function of E_b/A, the bombarding energy per nucleon. (The dashed is for pure pair collisions, the solid line for pair collisions in an average field of force.)

tains momentum integrations over the initial momentum of each of a pair of colliding particles (two times three integrations) and an integration over the final momenta of one of the particles in the colliding pair (three integrations); taking moments requires another three integrations. Thus η will be defined by a twelvefold integral. The resulting expression can be simplified to the form given in Appendix B.

The results exhibit a maximum, showing the competition between the increase of the collision rate and the decrease of the cross section as we increase the bombarding energy (see Fig. 1).

D. The anisotropy parameter q

The quantity

$$q = 2\langle p_{\parallel}^2 \rangle / \langle p_{\perp}^2 \rangle - 1$$

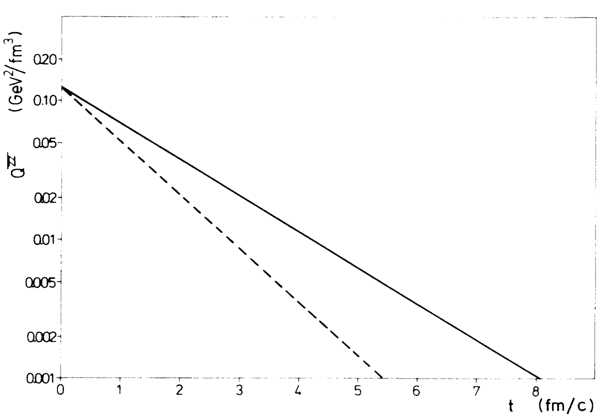

FIG. 2. The zz component of the anisotropy tensor, $Q^{\overline{zz}}$, for nuclear matter as a function of time. The bombarding energy per nucleon is 800 MeV. (The dashed line is for pure pair collisions, the solid line for pair collisions in an average field of force.)

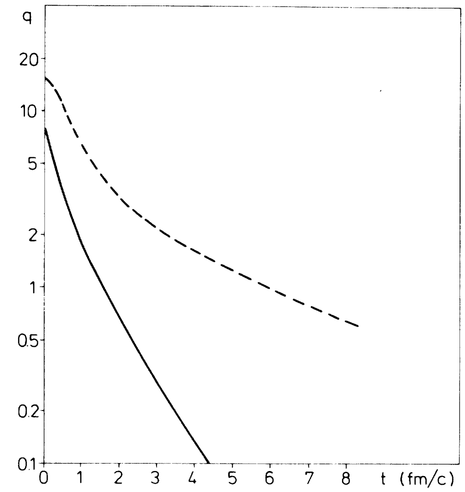

FIG. 3. The anisotropy coefficient $q = 2\langle p_{\parallel} \rangle / \langle p_{\perp} \rangle - 1$ as a function of time. The bombarding energy per nucleon is 400 MeV. (The solid line is for pure pair collisions, the dashed shows Danielewicz's results.)

is often used to characterize the anisotropy of the momentum distribution ($p_{\parallel}, p_{\perp}$ are the parallel and perpendicular components of the momentum relative to the collision axis; $\langle\ \rangle$ denotes averaging). Using the relation (A25) between $P^{\mu\nu}$ and $Q^{\mu\nu}$, this can be expressed as

$$q = 3Q^{\overline{zz}} / (2TP/B - Q^{\overline{zz}}) ,$$

with B given by (B5). Figure 2 shows $Q^{\overline{zz}}$, the relevant component of the traceless anisotropy tensor. Figures 3 and 4 compare our results with those of Randrup[9] and Danielewicz.[10] Randrup solves the Uehling-Uhlenbeck equation numerically (i.e., the Boltzmann equation with Pauli blocking in the collision term). His results show a more rapid loss of anisotropy. Danielewicz integrates numerically the equations satisfied by the time dependent Green functions in the Born approximation, using a

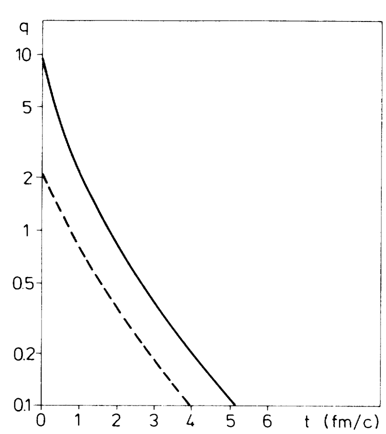

FIG. 4. The anisotropy coefficient q as a function of time. The bombarding energy per nucleon is 800 MeV. (The solid line is for pure pair collisions, the dashed shows Randrup's results.)

Gaussian potential for the nucleon-nucleon interaction. His results indicate a less rapid loss of anisotropy.

All these calculations give the same order of magnitude; for $q=\frac{1}{2}$, the relaxation times are between 1.5 and 7 fm/c.

This suggests that thermalization (for these bombarding energies) already occurs in the initial phases of a heavy ion reaction. One may object by arguing that this rapid thermalization is a consequence of assuming an unrealistically small initial anisotropy. However, this is not so.

In Table II we have computed the initial size of the anisotropy regulating parameter C^{zz}, as a function of the bombarding energy (without assuming that C is small) and found that for the maximum bombarding energy used (1 GeV per nucleon) it is only 0.12. Thus, the actual starting anisotropy is *de facto* small, and consequently the thermalization rate is computed for realistic initial data.

IV. CONCLUSIONS

We have constructed generalized hydrodynamical equations in which local thermal equilibrium no longer is assumed. The equations are the moment equations of a relativistic Boltzmann equation; they become a closed set of equations by restricting ourselves to that class of particular solutions for which the initial data imply that (a) the lack of local thermal equilibrium is expressible through the anisotropy of the local momentum distribution, and (b) that initial anisotropy is small.

Explicit solutions are given for two uniform interpenetrating hadron streams. We find that for medium energy collisions isotropization occurs in times short enough to real local thermal equilibrium during times characteristic of heavy ion collisions.

Our results are compared with other calculations using different approaches and we find that the orders of magnitude are the same, while some of the details differ.

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APPENDIX A: AUXILIARY VARIABLES AND GENERAL EQUATIONS

The baryon four-current is given by N^ν. This defines a unit U^ν at every point of space time where $N_\nu N^\nu \neq 0$ as

$$U^\nu = N^\nu/(N_\alpha N^\alpha)^{1/2} . \tag{A1}$$

Given a vector field, F^ν, $U^\nu(U_\alpha F^\alpha)$ is the component of F^ν along U^ν; the component perpendicular to U^ν is given by

$$F^\nu - U^\nu(U_\alpha F^\alpha) = (g^{\nu\mu} - U^\nu U^\mu) F_\mu \equiv \Delta^{\nu\mu} F_\mu = \hat{F}^\nu .$$

(The four-components $\hat{F}^\nu$ are not independent since $U_\nu \hat{F}^\nu = 0$.) Introducing the scalar $f = U_\alpha F^\alpha$, we can express the four independent functions F^ν in terms of the five quantities $f, \hat{F}^\nu$ (with three $\hat{F}^\nu$ independent) as

$$F^\alpha = f U^\alpha + \hat{F}^\alpha . \tag{A2}$$

Since in relativity the physical interpretations of the symbols are usually done in the local instantaneous rest frame [where at that point U^ν is (1,0,0,0)], the meaning of $f, \hat{F}^\alpha$ is more immediate.

The gradient operator ∂_ν similarly decomposes as

$$\partial_\nu = D + \nabla^\nu , \quad D = U_\alpha \partial^\alpha \text{ and } \nabla^\nu = \Delta^\nu_\alpha \partial^\alpha . \tag{A3}$$

Tensors are decomposed for each tensorial index in succession. A general tensor of rank r has 4^r independent components; indeed, this decomposition uses (5−1) independent functions. (Scalars cannot be decomposed this way; nor is this needed since they are unchanged if we change frames.)

The present problem requires the decompositions of N^ν, $T^{\mu\nu}$, and $A^{\mu\nu\tau}$. They are

$$N^\nu = n U^\nu + \nabla^\nu , \tag{A4}$$

$$T^{\mu\nu} = \mathscr{E} U^\mu U^\nu + W^\mu U^\nu + W^\nu U^\mu + \Pi^{\mu\nu} , \tag{A5}$$

$$A^{\mu\nu\tau} = Q U^\mu U^\nu U^\tau + Q^\mu U^\nu U^\tau + Q^\tau U^\mu U^\nu + Q^{\mu\nu} U^\tau + Q^{\mu\tau} U^\nu + Q^{\nu\tau} U^\mu + Q^{\mu\nu\tau} . \tag{A6}$$

If we introduce (temporarily) a local rest frame in which U^ν is (1,0,0,0), we can easily interpret the new functions introduced. These are, for the particle flow N,

$$n = N_\nu U^\nu , \tag{A7}$$

the particle density,

$$V^\nu = \Delta^\nu_\mu N^\mu \tag{A8}$$

(the particle current), for the energy momentum tensor $T^{\mu\nu}$,

$$\mathscr{E} = T^{\mu\nu} U_\mu U_\nu , \tag{A9}$$

the energy density,

$$W^\nu = \Delta^\nu_\mu U_\alpha T^{\mu\alpha} , \tag{A10}$$

the momentum current,

$$P^{\mu\nu} = \Delta^\mu_\alpha \Delta^\nu_\beta T^{\alpha\beta} , \tag{A11}$$

the pressure tensor (this is further decomposed into P, the scalar pressure, and the difference $\bar{P}_{\mu\nu}$),

$$P = -\tfrac{1}{3} \Delta^{\mu\nu} P_{\mu\nu} , \qquad \bar{P}_{\mu\nu} = P_{\mu\nu} + \Delta_{\mu\nu} P , \tag{A12}$$

and, for the asymmetry tensor $A^{\mu\nu\tau}$,

$$Q = A^{\mu\nu\tau} U^\mu U^\nu U^\tau , \tag{A13}$$

$$Q^\nu = \Delta^\nu_\tau U_\sigma U_\mu A^{\tau\sigma\mu} , \tag{A14}$$

$$Q^{\mu\nu} = \Delta^\mu_\tau \Delta^\nu_\sigma U_\rho A^{\tau\sigma\rho} , \tag{A15}$$

$$Q^{\mu\nu\tau} = \Delta^\mu_\alpha \Delta^\nu_\beta \Delta^\tau_\nu A^{\alpha\beta\nu} . \tag{A16}$$

Of these, $Q^{\mu\nu}$ is the most useful for us. In the local rest frame it has only spatial components. The associated quadratic form describes the anisotropy in this frame; the eigenvectors point along the anisotropy directions, while the differences among the eigenvalues specify the extent of the anisotropy.

The decomposition of the equations

$$\partial_\nu N^\nu=0\ ,\quad \partial_\nu T^{\mu\nu}=0$$

yields the equivalent sets

$$Dn+n\Delta_\nu U^\nu+U_\mu DV^\mu+\nabla_\mu V^\mu=0\ , \tag{A17}$$

$$D\mathscr{E}+(\mathscr{E}+P)\nabla_\mu U^\mu-2W^\mu DU^\mu +\nabla_\mu W^\mu-P^{\overline{\mu\nu}}\nabla_\mu U_\nu=0\ , \tag{A18}$$

$$(\mathscr{E}+P)DU^\alpha+\Delta^\alpha_\nu DW^\nu+W^\alpha\Delta_\nu U^\nu + W^\nu D_\nu U^\alpha-P^{\overline{\alpha\mu}}DU_\mu+\Delta^\alpha_\nu\nabla_\mu P^{\overline{\nu\mu}}+\nabla^\alpha P=0\ . \tag{A19}$$

The anisotropy equation is now expressed as

$$DQ^{\overline{\nu\mu}}+(Q^{\overline{\nu\mu}}\Delta_\tau U^\tau+Q^{\nu\tau}U^\mu Q^{\mu\tau}U^\nu)-\tfrac{2}{3}\Delta^{\nu\mu}Q^{\overline{\tau\alpha}}\nabla_\tau U_\alpha+Q^\nu DU^\mu+Q^\mu DU^\nu-\tfrac{2}{3}\Delta^{\nu\mu}Q^\tau DU$$
$$-(Q^{\nu\mu\tau}-\tfrac{1}{3}\Delta^{\nu\mu}Q^{\tau\alpha}_{\alpha})DU_\tau+\nabla_\tau Q^{\tau\nu\mu}-\tfrac{1}{3}\Delta^{\nu\mu}_\tau\Delta Q^{\tau\alpha}_\alpha+(Q^{\nu\tau\alpha}U^\mu+Q^{\mu\tau\alpha}U^\nu)\nabla_\tau U_\alpha$$
$$+\tfrac{1}{3}\Delta^{\alpha\tau}Q_{\alpha\tau}(\nabla^\nu U^\mu+\nabla^\mu U^\nu-\tfrac{2}{3}\Delta^{\nu\mu}\Delta_\tau U^\tau)=+\int(d^3p/p_0)p^{\overline{\nu}}p^{\overline{\mu}}C^1(f,f)\ . \tag{A20}$$

At this point we shall not analyze the equations in greater detail, but we focus immediately on the case where it is homogeneous and has cylindrical symmetry in the center of mass frame.

We find that V^i, W^i, Q^i, Q^{ijk}, and Q^{ij} vanish by symmetry ($i=1,2,3$) since f is invariant under reflection of the spatial components of the momenta, while the quantities listed are averages of an odd power of these components.

The surviving quantities can be written as

$$\mathscr{E}=(a_2/a_I)nT\ , \tag{A21}$$

$$Q=(a_3/a_1)\ , \tag{A22}$$

$$p=-\tfrac{1}{3}(z^2a_0-a_2/a_1)nT\ , \tag{A23}$$

$$Q=(z^2a_1-a_3/a_1)nT\ , \tag{A24}$$

$$P^{\overline{\mu\nu}}=[(z^4a_0-2z^2a_2+a_4)(z^4a_1-2z^2a_3+a_5)]T^{-1}Q^{\overline{\mu\nu}}\ , \tag{A25}$$

with

$$a_n=\int_0^\infty dy(y+z)^n(y^2+2yz)^{1/2}\exp(-y)\ , \tag{A26}$$

$$z=m/T\ . \tag{A27}$$

The surviving equations are (14)–(16).

If the system is not homogeneous but we use the special initial conditions (11)–(13), we get the following equation system:

$$Dn+n\nabla_\mu U^\mu=0\ , \tag{A28}$$

$$D\mathscr{E}+(\mathscr{E}+P)\nabla_\mu U^\mu-P^{\nu\mu}\nabla_\nu U^\mu=0\ , \tag{A29}$$

$$(\mathscr{E}+P)DU^\nu+\nabla^\nu P-P^{\overline{\nu\mu}}DU_\mu+\Delta^\nu_\tau\nabla_\nu P^{\overline{\tau\mu}}=0\ , \tag{A30}$$

$$DQ^{\nu\mu}+Q^{\nu\mu}\nabla_\tau U^\tau+(Q^{\nu\tau}U^\mu+Q^{\mu\tau}U^\nu)DU_\tau+Q^{\nu\tau}\nabla_\tau U^\mu+Q^{\mu\tau}\nabla_\tau U^\nu-\tfrac{2}{3}Q^{\sigma\tau}\Delta^{\nu\mu}\nabla_\sigma T_\tau$$
$$+\tfrac{1}{3}Q(\nabla^\nu U^\mu+\nabla^\mu U^\nu-\tfrac{2}{3}\Delta^{\nu\mu}\nabla_\tau U^\tau)=-\eta nQ\ . \tag{A31}$$

APPENDIX B: THE ANISOTROPY COEFFICIENT

The coefficient η in Eq. (21) can be written as

$$\eta=(3\pi/2)[\exp(2z)/a_1(z^4a_1-2z^2a_3+a_5)]K\ , \tag{B1}$$

with

$$z=m/T\ ,$$
$$a_n(z)=\int_0^\infty dy(y+z)^n(y+2yz)^{1/2}\exp(-y)\ , \tag{B2}$$

$$K=-2z^4J(0,0,0,0,0)+z^2J(1,0,0,0,0)+\tfrac{5}{6}z^2J(0,0,2,0,0)+\tfrac{1}{6}z^2J(0,0,0,2,0)$$
$$-\tfrac{1}{8}J(2,0,0,0,0)-\tfrac{1}{4}J(1,0,2,0,0)+\tfrac{1}{8}J(0,0,0,0,2)$$
$$-\tfrac{1}{4}J(0,0,1,1,1)+\tfrac{1}{12}J(0,2,0,2,0)-\tfrac{1}{12}J(0,2,2,0,0)$$
$$-\tfrac{1}{12}J(0,0,4,0,0)+\tfrac{1}{12}J(0,0,2,2,0)\ , \qquad \text{(B3)}$$

where

$$J(a,b,d,e,f)=\int_{2z}^{\infty}ds\int_{s}^{\infty}dt\, e^{-t}(t^2-s^2)^{(d+e+1)/2}s^{2a-d-e-1}(s^2-4z^2)^{1+f+(d+e)/2}\sum\sigma^{fl}(s)K(d,e,l)\ ,$$
$$\sigma^{fl}(s)=[2l+1)/2]\int dx^f x\,\sigma(s,x)P_1(x) \qquad \text{(B4)}$$

[$\sigma(s,x)$ is the scattering cross section with $x=\cos\theta$ and s is the square of the center of mass energy],

$$K(d,e,l)=\begin{cases}(d!e!)/[(d-l)!!(d+l+l)!!(e-l)!!(e+l+1)!!] \\ (\text{if } l<\min\{d,e\} \text{ and has same parity as } d \text{ and } e)\ , \\ 0 \ \text{ otherwise}\ .\end{cases}$$

The coefficient B in the anisotropy parameter q is given by

$$B=(z^4a_0-2z^2a_2+a_4)/(z^4a-2z^2a_3+a_5)\ . \qquad \text{(B5)}$$

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