

Complex scaling in the description of nuclear resonances

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A powerful approximation method based on the separable expansion of the potential has been used to solve the Schrödinger equation for a complex rotated Hamiltonian with nuclear short-ranged and nuclear short-ranged plus Coulomb potentials. It is shown that for potentials analytic in a proper domain of the complex r plane (e.g., the Woods-Saxon potential up to its first pole) solving the rotated Schrödinger equation on the real harmonic oscillator wave function basis is equivalent to solving the original equation on the complex harmonic oscillator wave function basis. For nonanalytic potentials (e.g., the nuclear charged-sphere Coulomb potential which is discontinuous in the complex r plane) the equivalence does not hold and it is the latter version that gives the correct solution. It is demonstrated that the method leads to an accurate determination of the resonance energy and (for not very broad resonances) of the wave function as well.

I. INTRODUCTION

In nuclear physics, as in almost all branches of physics, the description of resonances is a most important task. A resonance can be viewed and approached from two different angles: as a delay connected with an enhanced phase shift in a scattering process or as a long-lived but decaying state of a compound system. The main observable characteristics of a resonance are the position and the width. In the first picture they are to be determined from the phase shift (or cross section) as a function of energy. In the second picture the long-lived state is regarded as an extension of the concept of a bound state in that it is a solution to the Schrödinger equation with purely outgoing asymptotics belonging to complex energy. The real and imaginary parts of the energy give the position and width of the resonance, respectively. The two aspects are connected by the fact that the complex eigenenergy of the Schrödinger equation with purely outgoing asymptotics is a complex pole of the scattering function or S matrix.

There are problems in the case of which the second viewpoint offers some advantages over the first one. The well-elaborated bound state methods can easily be generalized so as to cover resonances, and can provide us not only with the position and width of the resonance but also with the resonant wave function. Although this wave function, called the Gamow or Siegert function, does not belong to the Hilbert space, there are rules worked out to make its use possible in calculations. So nuclear reactions leading to unbound states can be described with standard theories by substituting for the bound-state wave functions with the resonant-state wave functions instead of considering a continuum of scattering states. Also, in structure calculations profit can be derived from the use of the concept of a resonant state, e.g., in the description of unbound members of rotational bands or rotational bands based on unbound states, in representing isobar analog resonances, etc. Moreover, the resonant states may be members of complete sets consisting of the bound states,

selected resonant states, and appropriate continuum states; by including several resonant states in an expansion, which can be done on an equal footing with the bound states, a part of the continuum can be taken into account. The advantage lying in the concept of the resonant state can be exploited also in the determination of many-body resonances, as many calculations, carried out mainly in atomic and molecular physics, testify.

All of these explain the huge amount of work invested in studying the ways and means of describing the resonant state. Recently it was demonstrated¹ that the powerful approximation method based on a separable expansion of the potential (local or nonlocal) and used in a great variety of bound-state problems, can be extended so as to produce the resonant states as generalized bound states both in coordinate and momentum representation. Those problems ranged from one particle moving in a spherical or deformed short-ranged potential^{2,3} to two neutral particles moving a short-ranged potential (¹⁸O),⁴ and two neutral particles and one charged particle interacting pairwise via a realistic local potential,⁵ etc. It is noteworthy that the potential separable expansion (PSE) method proved to be superior to wave-function expansion methods, especially in representing the wave function in the asymptotic region on a much smaller basis and in generalizability. E.g., the otherwise inaccessible Gamow states of deformed potentials or potentials containing a tensor term can easily be generated with the PSE method; similarly the bound and resonant solutions of Saito's orthogonality condition model, which is a successful substitute for the microscopic cluster model, can efficiently be solved with it.⁶ The inclusion of long-ranged interactions into the scope of the method is a most natural demand.

In Sec. II we briefly review the PSE method as it was successfully applied to short-ranged potential problems and display how the direct application to the long-ranged Coulomb potential fails. The aim of this paper is to show that the complex scaling method borrowed from atomic physics helps to manufacture the energies and wave func-

tions of the resonances developed in a short-ranged plus long-ranged potential with essentially the same machinery that was used for the short-ranged potential.

The complex coordinate scaling, which is a well-founded, mathematically sound, and widely used method of generating single- and many-electron resonances, is described in Sec. III as it is applied to neutron resonances. With what can be learned from the neutron case we attack the proton problem in Sec. IV. As a numerical example we have chosen the commonplace Saxon-Woods plus charged-sphere Coulomb potential case that can numerically be integrated, in order to have a basis for comparison. Our conclusions and remarks will be gathered in Sec. V.

II. THE PSE METHOD

This method amounts to finding the solution to the problem

$$H|\psi\rangle \equiv (T+V)|\psi\rangle = E|\psi\rangle \quad (1)$$

as the $N \rightarrow \infty$ limit of the series of intermediate problems

$$H^N|\psi^N\rangle \equiv (T+V^N)|\psi^N\rangle = E^N|\psi^N\rangle, \quad (2)$$

where

$$V^N \equiv \sum_{i,j=0}^N |i\rangle V_{ij} \langle j|, \quad (3)$$

$$V_{ij} = \langle i|V|j\rangle,$$

and $|i\rangle$ and $|j\rangle$ are members of an orthonormalized complete set $\{|i\rangle\}$. The operator V^N is Hermitian if V is Hermitian.

Limiting ourselves to a purely outgoing (bound or resonant) solution of Eq. (1), after some straightforward rearrangement we obtain the approximate state vector

$$|\psi_n^N\rangle = \sum_{i,j=0}^N c_{nj}^N V_{ij} G_0^{(+)}(E_n^N) |i\rangle, \quad (4)$$

where $c_{nj}^N = \langle j|\psi_n^N\rangle$, and $G_0^{(+)}$ is the outgoing free Green's operator. Multiplying Eq. (4) by $\langle k|$ we arrive at the system of algebraic equations

$$\sum_{j=0}^N \left[\delta_{kj} - \sum_{i=0}^N \langle k|G_0^{(+)}(E_n^N)|i\rangle V_{ij} \right] c_{nj}^N = 0, \quad k=0,1,\dots,N \quad (1')$$

for the coefficients c_{nj}^N . The necessary and sufficient condition of the solvability of Eq. (1'), namely

$$D(E) \equiv \det \left\{ \delta_{kj} - \sum_{i=0}^N \langle k|G_0^{(+)}(E)|i\rangle V_{ij} \right\} = 0,$$

provides a transcendent equation for the approximate eigenenergy. The negative real roots yield the bound-state energies; the complex ones with

$$(2mE_n^N/\hbar^2)^{1/2} \equiv k_n^N = \kappa_n^N - i\gamma_n^N, \quad \kappa_n^N > \gamma_n^N > 0$$

yield the resonant-(Gamow) state energies. The quantum mechanical inner product is taken here in a generalized fashion so as to be sensible even if one or both of the fac-

tors are Gamow vectors and to lead to the usual result if the factors are bound-state vectors. From among the more or less equivalent prescriptions we have chosen the ones given by Berggren⁷⁻⁹

$$\langle g|h\rangle = \lim_{\epsilon \rightarrow 0} \int_0^\infty \exp[-\epsilon r^2] g(r) h(r) dr \quad (5a)$$

$$= \int_{L^+} \hat{g}(k) \hat{h}(k) dk. \quad (5b)$$

Here $\hat{h}(k)$ is the generalized Fourier transform of $h(r)$,

$$\hat{h}(k) = \lim_{\epsilon \rightarrow 0} \left[\frac{2}{\pi} \right]^{1/2} \int_0^\infty \exp[-\epsilon r^2] h(r) J_l(kr) dr, \quad (6a)$$

and *vice versa*

$$h(r) = \left[\frac{2}{\pi} \right]^{1/2} \int_{L^+} \hat{h}(k) J_l(kr) dk. \quad (6b)$$

$J_l(kr) = kr j_l(kr)$ are the free-particle wave functions, $j_l(kr)$ standing for the spherical Bessel functions. L^+ is the $\text{Re}k > 0$ part of a curve L of the complex k plane characterized by the following properties: if k is on L so is $-k$; $|\text{Re}k| > |\text{Im}k|$ on L ; and for large values of $|\text{Re}k|$ L coincides with the real axis. The functions $g(r)$ and $h(r)$ are bound, Gamow or scattering wave functions taking their asymptotic k values from that part of the complex k plane which is above L . Based on Eqs. (5) and (6) a code has been written¹⁰ for the solution of (1'); in Ref. 10 all formulas and numerical procedures used can be found. As a basis set the harmonic oscillator wave functions (HOWF) were used because they are equally well manageable in coordinate and momentum representation and make the computation of the matrix elements, necessary in (1'), possible in whatever representation is more favorable. Encouraged by our really good results with short-ranged and screened Coulomb potentials,^{1-3,9} we tackled the nuclear Coulomb problem. We searched for the bound and resonant states in a potential

$$V = v_N + v_c,$$

$$v_N = V_0 f(r) + V_{\text{so}} \frac{1}{r} \frac{df(r)}{dr} l\sigma, \quad (7)$$

$$f(r) = \frac{1}{1 + \exp[(r - r_0 A^{1/3})/a]},$$

$$v_c = (e^2 Z_1 Z_2 / 2R_c)(3 - r^2/R_c^2) \quad r < R_c$$

$$= e^2 Z_1 Z_2 / r \quad R_c < r.$$

We use here the commonplace notation. In Fig. 1, the convergence of the complex eigenvalues E^N to the results obtained with direct numerical integration (DNI) of Eq. (1), with the aid of the code GAMOW (Ref. 11), is displayed for a selection of states. A most confusing picture emerges: The bound and pseudobound (positive energy, zero width) cases show rapid convergence to the DNI results, the energies of the narrow resonance fluctuate around it, but those of the broad resonance just depart. The failure reminds one of the doubts cast on the very existence of any basis on which the separable expansion of the Coulomb potential might converge;¹² it certainly does

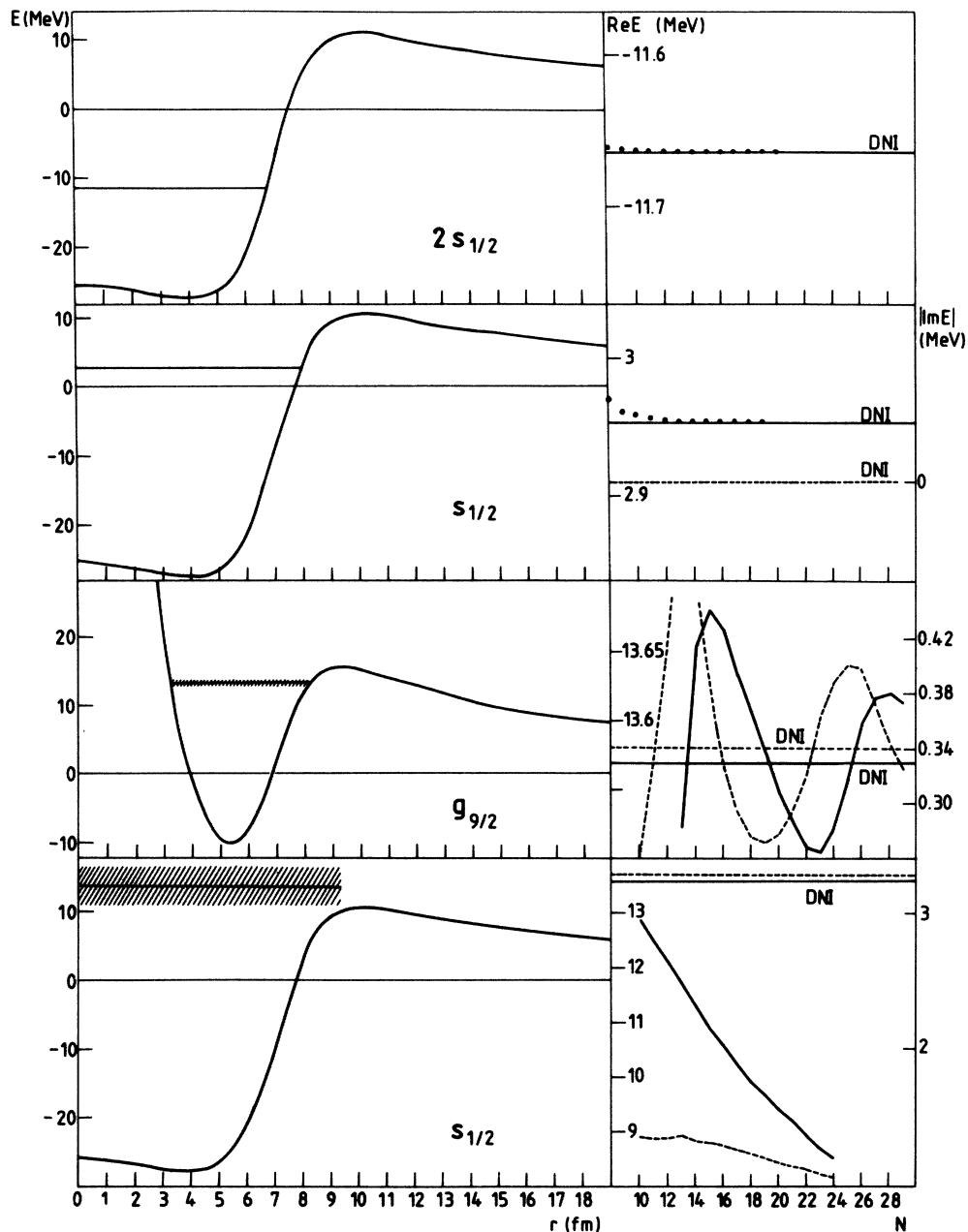


FIG. 1. Left-hand side (lhs): the potential (7) with parameters indicated in Table II plus the centrifugal barrier and the energies of the $2s_{1/2}$ bound, the first and second unbound s , and the first unbound $g_{9/2}$ levels as obtained by DNI. Right-hand side (rhs): the PSE energies as functions of the basis size; solid lines, left-hand scales for the real parts; dashed lines, right-hand scales for the imaginary parts.

not converge on the HOWF basis, at least with a tolerable basis size. This is a consequence of the long-range character of the potential. Still, this alone cannot be the reason for the lack of convergence of the energy, since there is the very same (nonconvergent) potential present in all of our examples. It is the wave function that differs strikingly: In the convergent cases it is square-integrable, in the nonconvergent cases it is not. One is tempted to conclude that the poor behavior of the potential can be over-

balanced by the good behavior of the wave function. If it were so we could resort to the Aguilar-Balslev-Combes (ABC) theorem¹³ standing a fair chance. This theorem states that under a special $U(\theta)$ transformation the Schrödinger equation may transform so as to keep its bound and resonant eigenvalues unchanged while the respective eigenfunctions become square-integrable. The next section will be devoted to the ABC theorem and its combination with the PSE method.

III. COMPLEX SCALING

The $U(\theta)$ transformation of the ABC theorem is defined by

$$(U(\theta)f)(\mathbf{r}) = e^{3/2\theta} f(\mathbf{r}e^\theta) \equiv f_\theta(\mathbf{r}), \quad (8a)$$

$$(U(\theta)\hat{f})(\mathbf{p}) = e^{-3/2\theta} \hat{f}(\mathbf{p}e^{-\theta}) \equiv \hat{f}_\theta(\mathbf{p}), \quad (8b)$$

where \mathbf{r} and \mathbf{p} are the space and momentum variables, respectively, and θ is complex with the restriction that $|\operatorname{Im}\theta| > \arctan(\gamma/\kappa)$. The strongly restrictive *sufficient conditions* of the ABC theorem are given with mathematical rigor in Ref. 14; loosely speaking they amount to the requirement that all quantities in the Schrödinger equation be dilation analytic. This means that there exists a finite region of $|\operatorname{Im}\theta|$ in which their transforms obtained by the application of $U(\theta)$ are analytic. For the solution of the transformed Schrödinger equation

$$\begin{aligned} H(\theta)\psi_\theta &= E\psi_\theta, \\ H(\theta) &= U(\theta)HU(\theta)^{-1}, \end{aligned} \quad (9)$$

the whole arsenal of methods developed for the bound-state problem is at our disposal. The obtained value of E is the wanted eigenenergy; the wave function ψ comes from ψ_θ as

$$\psi = U(\theta)^{-1}\psi_\theta. \quad (10)$$

The resultant E and ψ are exact or approximate depending on whether the method of solving (9) is exact or approximate.

Let us now use the PSE method as described in Sec. II as an approximation method to (9). It is known (see, e.g., Ref. 14) that the transformed kinetic energy $T(\theta) = e^{-2\theta}T$, so the equation to be approximated with the PSE method reads

$$[E^{-2\theta}T + U(\theta)VU(\theta)^{-1}]|\psi_\theta\rangle = E|\psi_\theta\rangle$$

or (11)

$$[T + e^{2\theta}U(\theta)VU(\theta)^{-1}]|\psi_\theta\rangle = e^{2\theta}E|\psi_\theta\rangle,$$

for which the analogs of (1') and (4) are

$$\sum_{j=0}^N \left\{ \delta_{kj} - e^{2\theta} \sum_{i=0}^N \langle k | G_0^{(+)}(e^{2\theta}E_n^N) | i \rangle \langle i | U(\theta)VU(\theta)^{-1} | j \rangle \right\} c_{nj,\theta}^N = 0, \quad k=0,1,\dots,N, \quad (11')$$

$$|\psi_\theta^N\rangle = e^{2\theta} \sum_{i,j=0}^N c_{nj,\theta}^N \langle i | U(\theta)VU(\theta)^{-1} | j \rangle G_0^{(+)}(e^{2\theta}E_n^N) | i \rangle. \quad (12)$$

For the sake of the following examples let us confine ourselves to local potentials $V \equiv V(\mathbf{r})$ and calculate their matrix elements in coordinate representation; the matrix element of the Green's operator is easier to calculate in momentum representation.

The matrix elements of the rotated potential

$$V(\mathbf{r},\theta) = U(\theta) \left[\sum_{lm} V_{lm}(r) Y_{lm}(\Omega_{\mathbf{r}}) \right] U(\theta)^{-1} \quad (13)$$

on the HOWF basis $\{|i\rangle\} = \{|nljm\rangle\}$ are, in obvious notation,

$$\begin{aligned} \langle nljm | V | n'l'j'm' \rangle &= \lim_{\epsilon \rightarrow 0} \sum_{l''m''} \int \exp[-\epsilon r^2] R_{nl}(r,b) \mathcal{Y}_{ljm}^*(\Omega_{\mathbf{r}},s_z) \\ &\quad \times U(\theta) V_{l''m''}(r) Y_{l''m''}(\Omega_{\mathbf{r}}) U(\theta)^{-1} R_{n'l'}(r,b) \mathcal{Y}_{l'j'm'}(\Omega_{\mathbf{r}},s_z) r^2 dr d\Omega_{\mathbf{r}}, \end{aligned} \quad (14)$$

with

$$\langle \mathbf{r} | nljm \rangle = R_{nl}(r,b) \mathcal{Y}_{ljm}(\Omega_{\mathbf{r}},s_z), \quad (15)$$

where b is the real oscillator parameter.

Due to the square integrable HOWF, the integral exists also for $\epsilon=0$. Because of the well-known dependence of Y_{lm} on $\mathbf{r} \equiv \{x,y,z\}$ (Ref. 15) the application of (8a) leads to

$$\langle nljm | V | n'l'j'm' \rangle = \sum_{l''m''} \int R_{nl}(r,b) \mathcal{Y}_{ljm}^*(\Omega_{\mathbf{r}},s_z) V_{l''m''}(re^\theta) Y_{l''m''}(\Omega_{\mathbf{r}}) R_{n'l'}(r,b) \mathcal{Y}_{l'j'm'}(\Omega_{\mathbf{r}},s_z) r^2 dr d\Omega_{\mathbf{r}}.$$

The integration over the angular variables and the summation over the spin degrees of freedom can easily be performed in each term; the radial part $V_{nl,n'l'}$ with the transformation

$$t = re^\theta \quad (16)$$

takes the form

$$V_{nl,n'l'} = e^{-3\theta} \int_c R_{nl}(te^{-\theta},b) V(t) R_{n'l'}(te^{-\theta},b) t^2 dt; \quad (17)$$

the equation of c is (16). If the potential is dilation analytic, by virtue of the Cauchy theorem,

$$\begin{aligned} V_{nl,n'l'} &= e^{-3\theta} \int_0^\infty R_{nl}(re^{-\theta},b) V(r) \\ &\quad \times R_{n'l'}(re^{-\theta},b) r^2 dr \end{aligned} \quad (18)$$

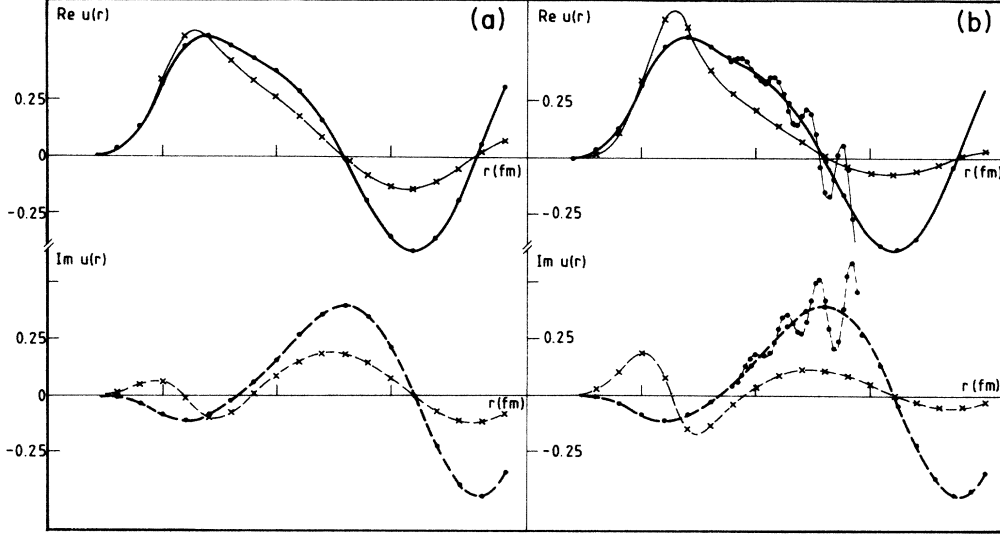


FIG. 2. The real and imaginary part of the radial wave functions of the resonance displayed in Table I. (a) Heavy solid and dashed lines are obtained with DNI for $\text{Re}\theta=\text{Im}\theta=0$; light solid and dashed lines for $\text{Re}\theta=0, \text{Im}\theta=0.12$. The crosses indicate the PSE results to the rotated problem; the dots show the rotated-back wave functions. (b) The same for $\text{Im}\theta=0.20$.

is obtained. If, however, the potential possesses discontinuity or singularity, a contribution to (18) may arise. The radial HOWF has the property

$$R_{nl}(re^{-\theta}, b) = e^{3/2\theta} R_{nl}(r, be^{\theta}); \quad (19a)$$

therefore,

$$V_{nl, n'l'} = \int_0^{\infty} R_{nl}(r, be^{\theta}) V(r) R_{n'l'}(r, be^{\theta}) r^2 dr. \quad (20)$$

Also for the matrix element of the free Green's operator occurring in (11'),

$$\langle n'ljm | U(\theta) U(\theta)^{-1} G_0^{(+)}(e^{2\theta} E) | n'l'j'm' \rangle,$$

a simple expression is derived, by simple manipulation and by using the property of the HOWF analogous to (19a) for the momentum representation

$$\hat{R}_{nl}(pe^{\theta}, b) = e^{-3/2\theta} \hat{R}_{nl}(p, be^{\theta}), \quad (19b)$$

namely

$$(G_0^{(+)}(e^{2\theta} E))_{nl, n'l'} = e^{-2\theta} \int_{L^+} \hat{R}_{nl}(p, be^{\theta}) \frac{1}{E - p^2/2m} \times \hat{R}_{n'l'}(p, be^{\theta}) p^2 dp. \quad (21)$$

Equations (20) and (21) inserted in (11') and (12) produce the unrotated PSE equations (1'), the only difference being that the real oscillator parameter b has to be substituted for by the complex quantity $\bar{b} = be^{\theta} = \bar{b}e^{i\varphi}$.

It is the equivalence of two distinct procedures that is stated here for the case of dilation analytic potentials: the approximation of the rotated Schrödinger equation with the PSE method based on HOWF with a *real* size parameter (RPSE), and the approximation of the original (unrotated) Schrödinger equation with the PSE method based

on HOWF with a *complex* size parameter (CPSE).

As a simple example let us consider a resonance in the potential v_N of Eq. (7). It is dilation analytic up to its first pole in the complex r plane; the position of this pole imposes an upper bound on $\text{Im}\theta$,

$$\text{Im}\theta < \arctan \frac{\pi a}{r_0 A^{1/3}}. \quad (22)$$

In Table I are displayed the eigenenergies corresponding to an $l=6, j=\frac{11}{2}$ resonance as obtained for various values of the purely imaginary θ by the direct numerical integration of Eq. (9), by the real-basis-function PSE approximation to Eq. (9) (RPSE), and by the complex-basis-function PSE approximation (CPSE) to Eq. (1).

In Fig. 2 the intermediate wave functions and the rotated-back wave functions are compared with the ones obtained by DNI for two different values of $\text{Im}\theta=\varphi$. The energies show the independence from θ and the equivalence of the RPSE and CPSE methods. The figure demonstrates that the quality of the approximate inter-

TABLE I. An $l=6, j=\frac{11}{2}$ neutron resonance (in MeV) in a Woods-Saxon potential.^a

$\text{Im}\theta$	DNI	RPSE=CPSE
0.00	9.596 - i0.916	9.592 - i0.915
0.08	9.596 - i0.916	9.592 - i0.915
0.12	9.596 - i0.916	9.592 - i0.915
0.16	9.596 - i0.916	9.592 - i0.915
0.20	9.596 - i0.916	9.592 - i0.915

^aThe parameters of the potential are $V_0=30$ MeV, $V_{s0}=5.8$ MeV, $r_0=1.19$ fm, $a=0.75$ fm, and $A=208$. Thirty basis functions were used; $b=2.4$ fm, $\text{Re}\theta=0$.

TABLE II. An $l=4, j=\frac{9}{2}$ proton resonance (in MeV) in potential (7).^a

$\text{Im}\theta \equiv \varphi$	DNI	RPSE	CPSE
0.00	13.5701 - i0.3394	b	b
0.08	13.5684 - i0.3302	13.5743 - i0.3336	13.5751 - i0.3402
0.12	13.5620 - i0.3127	13.5646 - i0.3155	13.5724 - i0.3402
0.16	13.5374 - i0.2648	13.5384 - i0.2653	13.5716 - i0.3399
0.20	13.4673 - i0.1425	13.4702 - i0.1466	13.5713 - i0.3398

^aThe parameters of the potential are as follows: $V_0=50.9$ MeV, $V_{s0}=5.8$ MeV, $r_0=1.19$ fm, $a=0.75$ fm, $A=208$, $Z_1=82$, $Z_2=1$, and $R_c=r_0A^{1/3}$. Thirty basis functions were used; $\bar{b}=2.4$ fm.

^bNot converged.

mediate wave functions is independent of θ and is satisfactory (at both φ values the crosses obtained with the PSE method lie on the curves yielded by the DNI), but the operation $U(\theta)^{-1}$ on the intermediate wave function amplifies the error, especially in the asymptotic region (the dots from the PSE method follow the curve of the DNI in case a but deviate, especially in the asymptotic region, in case b). The larger θ is, the stronger the amplification. This means that (i) whenever it is possible [e.g., for computing the matrix elements of an operator O whose $U(\theta)OU(\theta)^{-1}$ transform is known] we had better use the intermediate wave functions; (ii) if we need the rotated-back wave function we should take the one belonging to the smallest φ value in the interval of θ independence; (iii)

as the complex-basis-function PSE method automatically yields the rotated-back wave function it may be restricted, from the point of view of the wave function, to the region of the narrow resonances.

IV. PROTON RESONANCES

As we have seen in Fig. 1, the PSE method does not converge at $\theta=0$ for resonances in the full potential (7). So this problem would be a nontrivial testing ground of the complex rotation technique. Unfortunately the v_c term of the potential is not continuous in the complex r plane. This discontinuity adds a nonvanishing, θ -dependent contribution to the matrix element (18)

$$e^2 Z_1 Z_2 e^{-3\theta} \int_c R_n(te^{-\theta}, b) [(3-t^2/R_c^2)/2R_c - 1/t] R_{n'}(te^{-\theta}, b) t^2. \quad (23)$$

The curve c connects the complex points R_c and $R_c e^\theta$. This is a deviation from the dilation analyticity, the sufficient condition of the ABC theorem. To decide whether

the ABC theorem remains valid in spite of this particular deviation, i.e., that the energy of Eq. (9) is really θ independent, we solved the rotated Schrödinger equation

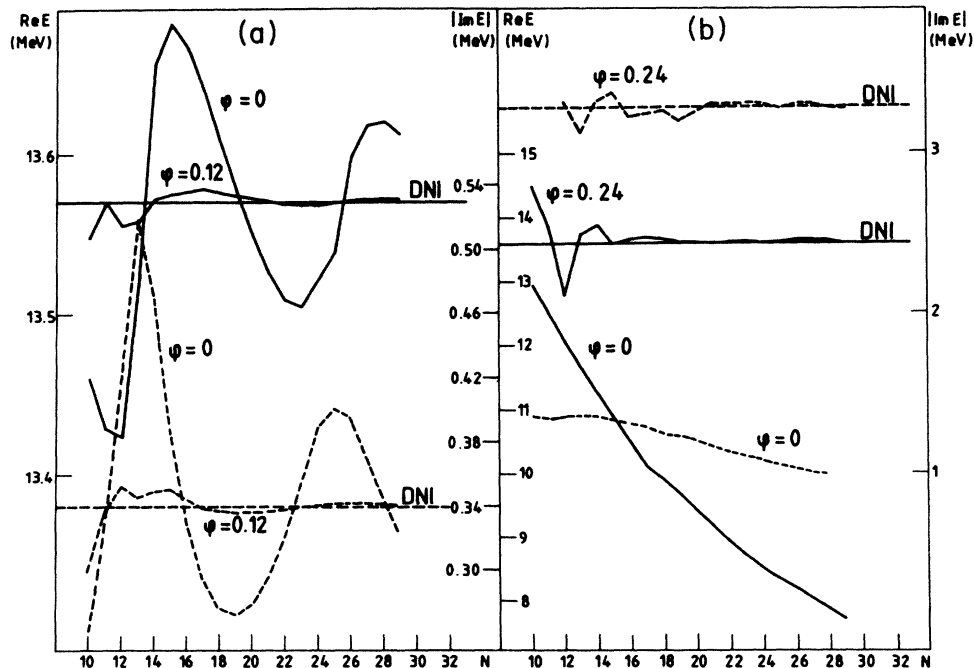


FIG. 3. The energies, as functions of the basis size, of the $g_{9/2}$ (a) and the broad $s_{1/2}$ (b) resonances of Fig. 1. Solid lines, lhs scales for the real parts; dashed lines, rhs scales for the imaginary parts; $\bar{b}=2.4$ fm.

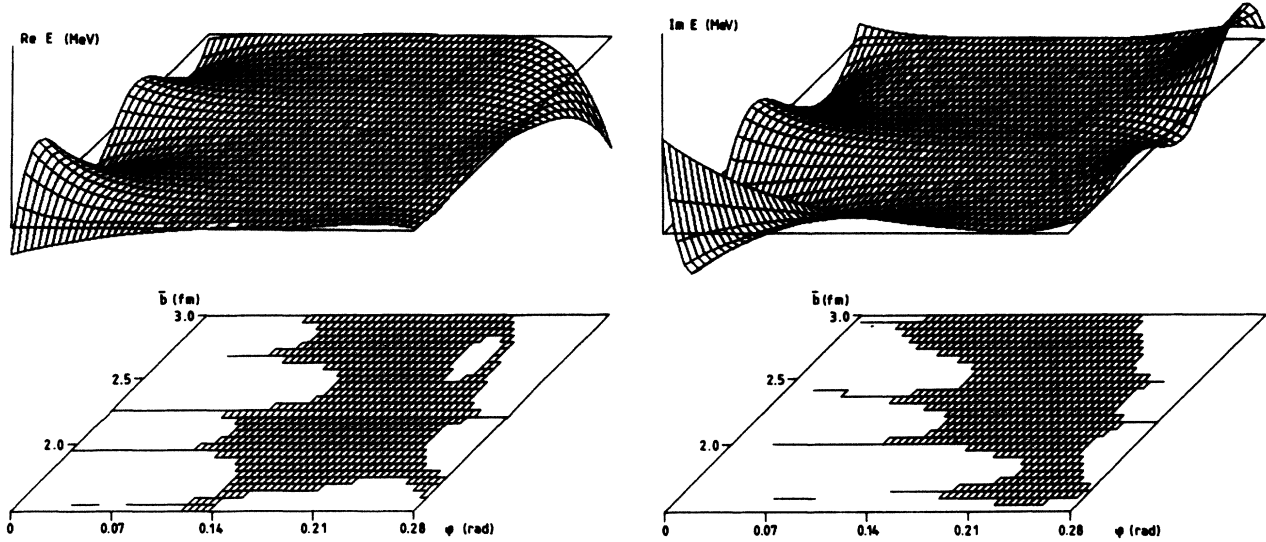


FIG. 4. The surfaces of the real and imaginary energies of the $g_{9/2}$ level of Fig. 1 over the complex $\tilde{b} = \bar{b}e^{i\varphi}$ oscillator parameter as obtained by cubic spline interpolation (upper part). Lower part: the $13.57 - 0.002 < \text{Re}E < 13.57 + 0.002$ and the $0.339 - 0.002 < \text{Im}E < 0.339 + 0.002$ values (in MeV) projected onto the lhs plane and rhs plane, respectively, are intended to show the region of parameter independence.

“exactly,” i.e., with DNI. What we have found is displayed in Table II: The energy is θ dependent; the ABC theorem is not valid. The RPSE, which is an approximation to the rotated problem, faithfully exhibits the same θ dependence. This same deviation destroys the equivalence between the RPSE and the CPSE, which is an approximation to the unrotated problem. The third column of Table II contains the remarkable result that the CPSE is convergent and yields θ -independent energy that approximates the DNI energy of the unrotated problem ($\varphi=0$) well.

In Fig. 3 the energies of the two resonances of Fig. 1 are shown as functions of the basis size N at fixed (complex) θ . Figure 4 displays the energy surface over the complex $\tilde{b} = \bar{b}e^{i\varphi}$ and the region of \bar{b} independence at a fixed basis size. In Table III the regions of φ independence at various basis sizes are presented for the narrower resonance. Figure 5 displays the wave function compared to the one obtained by DNI. As is to be expected, the wave function of the wide resonance follows the exact one only up to ~ 10 fm at a φ value which makes the procedure convergent. On the basis of the results presented, the separable expansion of the potential on complex basis

functions appears as a viable method for describing (not very broad) proton resonances, even if it is lacking mathematical pedigree, having lost the connection with the complex rotation of the Hamiltonian.

V. CONCLUSIONS

The complex scaling is a well-developed technique in atomic and molecular physics which has been thoroughly investigated both from the conceptual and computational points of view.¹⁶ For no obvious reason it does not belong to the commonly used means of nuclear physics. We hope to have demonstrated that some profit can be derived from its application to nuclear problems.

Its combination with the HOWF-based PSE method led to a fair approximation to the nuclear Coulomb problem notwithstanding that our numerical example was the extreme case of a Pb-like problem. Also, it appeared to be an interesting example of the complex-basis-function method working without being substantiated by an underlying complex rotation of the Hamiltonian. This seems to be a counter-example to the common belief.¹⁷ Being as it

TABLE III. An $l=4, j = \frac{9}{2}$ proton resonance (in MeV) in potential (7). (For the parameters see the caption to Table II.)

φ	0.04	0.08	0.12	0.16	0.20	0.24
18	13.606 - i0.326	13.587 - i0.337	13.579 - i0.340	13.575 - i0.339	13.574 - i0.333	13.558 - i0.299
21	13.564 - i0.314	13.570 - i0.329	13.572 - i0.336	13.572 - i0.339	13.573 - i0.341	13.588 - i0.352
24	13.549 - i0.344	13.564 - i0.340	13.570 - i0.339	13.571 - i0.339	13.571 - i0.339	13.563 - i0.337
27	13.575 - i0.357	13.571 - i0.345	13.571 - i0.341	13.571 - i0.340	13.572 - i0.340	13.576 - i0.340
30	13.584 - i0.338	13.575 - i0.340	13.572 - i0.340	13.572 - i0.340	13.571 - i0.340	13.569 - i0.340

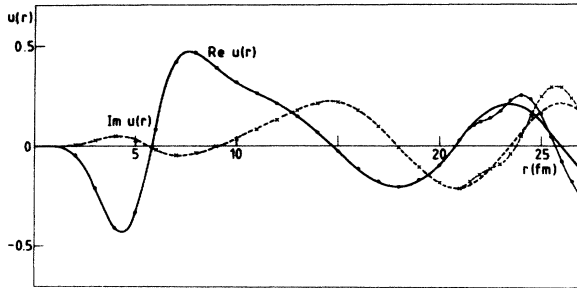


FIG. 5. The real and imaginary parts of the radial wave function of the $g_{9/2}$ resonance of Fig. 1 as obtained by DNI (heavy solid and dashed lines, respectively) and by rotating back to $\varphi=0$ the PSE function belonging to $\bar{b}=2.4$ fm, $\varphi=0.12$ rad (dots).

appears here, a method detached from the complex coordinate rotation, the complex-basis-function technique deserves some further investigation which may lead to an improvement of the wave function for broad resonances. In determining the position and width in general, and also the wave function for narrow resonances, it is successful in its present form.

We conclude by remarking that the complex scaling technique is obviously not restricted to the PSE method; it can be associated with any (single-particle or many-body) approximation method. We have promising preliminary results concerning resonances in six-nucleon systems in the framework of the generator coordinate method.

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