Potential model calculation of proton-proton bremsstrahlung using the Paris potential

R. L. Workman and Harold W. Fearing
TRIUMF, Vancouver, British Columbia, Canada V6T 2A3
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Cross sections and analyzing powers are calculated for proton-proton bremsstrahlung (ppγ) using the Paris and extended Reid soft-core potentials. One pion exchange contributions have been added for partial waves with $J \geq 6$. Relativistic spin corrections, certain other relativistic corrections, and some Coulomb corrections have also been added. The calculation utilizes fully relativistic kinematics and is performed in the center of mass frame in order to put to zero the leading contribution of the double scattering terms. This is the first ppγ calculation to use one of the modern theoretically based nucleon-nucleon potentials. The results of this calculation are compared with experimental data, as well as older potential model and soft-photon approximation calculations. The off-shell behaviors of the Paris and extended Reid potentials are also investigated. We find that the analyzing powers are sensitive to details of the calculation. Asymmetry data are expected from a new TRIUMF ppγ experiment which will provide the first opportunity for a detailed comparison of such data with a modern potential model calculation.

I. INTRODUCTION

Proton-proton bremsstrahlung (ppγ),

$$p+p \rightarrow p+p+\gamma ,$$

has been studied both theoretically and experimentally by a number[1-6] of authors in the hope that off-shell aspects of the nucleon-nucleon (N-N) interaction could be understood. To date, however, little has been learned about these off-shell effects. For some of the older experiments, where standard calculations agree with data, and at lower energies, it is now known that there should be less sensitivity[1,5] to off-shell effects than for some of the more recent experiments. However, in an experiment where previous potential model calculations indicate that off-shell effects are enhanced, namely the 200 MeV TRIUMF experiment, it appears that a soft-photon approximation (SPA) calculation fits the data as well as previous potential model calculations.[2] This is particularly disturbing since SPA calculations require only on-shell information. Even more puzzling is the poor fit by existing potential model calculations to some of the 42 MeV data.[6] At such a low energy, one would expect relativistic corrections to be small and the potential model calculation to be essentially exact.

Motivated by this poor agreement between theory and experiment and the possibility that it may be simply due to unsatisfactory potential model calculations of the ppγ observables, we have undertaken a new and more modern potential model calculation of ppγ cross sections and asymmetries. Our calculation utilizes for the first time a modern theoretically based potential, namely the Paris potential[7] and, for comparison purposes, an extended version of the Reid soft-core potential.[8] Neither of these potentials have been used in previous ppγ calculations. In addition, this calculation combines several contributions which have been examined individually in previous potential model calculations but have not been combined in a single calculation. In particular, we have included the so called relativistic spin contributions, some Coulomb corrections to the N-N amplitudes, one pion exchange for the high partial waves, and appropriate frame transformations for the N-N amplitudes. All kinematics have been calculated relativistically and we have, in addition, included some other relativistic corrections not previously examined.

Our basic approach is that of a nonrelativistic potential model, which originates from the two-potential formalism, with the electromagnetic interaction taken just to first order. The half-off-shell N-N amplitudes are calculated by solving a T-matrix equation in momentum space, to obtain the on-shell amplitudes and the half-off-shell extension functions for a given potential.[9,10] These amplitudes are then combined with electromagnetic vertex and propagator factors to produce the full ppγ amplitude. This is basically the same general approach utilized by Drechsel and Maximon[11] and differs from the alternative, but in principle equivalent, approach used by, for example, Heller and Rich,[12] in which the full wave functions are calculated first and used to evaluate matrix elements of the electromagnetic current operator numerically. The calculation is done at present in the center of mass frame so as to make zero the leading contribution from double scattering terms, which have not yet been included explicitly.

Our attempt to improve the status of ppγ calculations has been motivated also by a new ppγ experiment at TRIUMF.[13] The new experiment has obtained data at 280 MeV, just below the threshold for pion production, and will eventually have results for both cross sections and analyzing powers. The measurement of ppγ analyzing powers is an important feature, as such data are virtually nonexistent. We will thus be afforded a unique opportunity for the comparison of analyzing powers from experiment with our potential model calculations.

In the next section we give the total T matrix for ppγ and show how the electromagnetic vertex operator is calculated and how relativistic corrections are introduced. The gauge invariance of this calculation is also discussed. Section III deals with the half-off-shell N-N T matrices, the associated half-off-shell functions, Coulomb corrections, and the one pion exchange contributions. In Sec. IV we briefly discuss the partial wave Paris potential and in Sec. V we construct the cross section and analyzing powers. Results are given in Sec. VI where our potential model calculations confront experimental data and SPA calculations. The significance of our results is also considered there. Some preliminary results of these calculations have been given in Ref. 14.

II. THE T-MATRIX FORMALISM

The total T matrix for the ppγ process is obtained using the two-potential formalism of Gell-Mann and Goldberger.[15] This method splits the total proton-proton interaction into a strong N-N interaction, which is treated "exactly," and a weak electromagnetic interaction, which is taken to first order only. Drechsel and Maximon have given a detailed derivation[11] of this T matrix as appropriate for ppγ.

The total T-matrix operator, to first order in $V_{\rm em}$, is then

$$T = t(E_f)(E_f - H_0 + i\epsilon)^{-1} V_{\rm em} + V_{\rm em}(E_i - H_0 + i\epsilon)^{-1} t(E_i) + t(E_f)(E_f - H_0 + i\epsilon)^{-1} V_{\rm em}(E_i - H_0 + i\epsilon)^{-1} t(E_i) \, . \tag{2.1}$$

Here, $t(E)$ is the N-N T matrix, H_0 is the free two proton Hamiltonian, and $V_{\rm em}$ is the electromagnetic interaction operator. E_i and E_f are the initial and final energies of the two protons. Figure 1 shows the diagrams arising from this T matrix.

We have included in our calculation the first two terms of Eq. (2.1) corresponding to the "single scattering" terms in which a photon is emitted either before or after the N-N interaction. Our calculation currently neglects the final or "double scattering" term in Eq. (2.1). In order to minimize the contribution of double scattering terms, however, the calculation has been carried out in the center of mass frame, where it can be shown that the leading term of the double scattering piece is proportional to the total center of mass momentum, and thus vanishes.[16] The result for the T matrix is eventually multiplied by the appropriate factors to make it an invariant, so that the cross section can be easily evaluated in the lab, as appropriate for comparison with experiment.

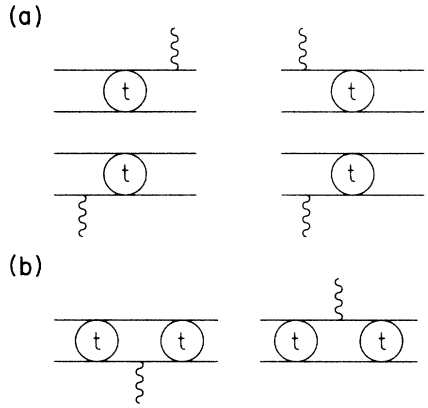

FIG. 1. Diagrammatic expansion of the ppγ T matrix: (a) single scattering terms; (b) double scattering terms.

The total ppγ T-matrix operator is next evaluated between the states $|\mathbf{p},SM\rangle$ and $|\mathbf{k}\rangle$, where $\mathbf{p}$, S, and M are, respectively, the relative momentum, total spin, and spin projection of the two protons, and $\mathbf{k}$ is the photon momentum. The momenta in this process are kinematically constrained by

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu + k^\mu \, ,$$

where $p_i^\mu = (E_{\mathbf{p}_i}, \mathbf{p}_i)$ and k^μ are, respectively, the proton and photon four-momenta and $E_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$ with m the proton mass. Using these momenta we have, for the single scattering contribution to the T matrix,

$$\langle T \rangle_{\rm single} = \sum_{S''M''} \langle S'M', (\mathbf{p}_3 - \mathbf{p}_4)/2 \,|\, t(E_f) \,|\, (\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{k})/2, S''M'' \rangle (E_f - E_{\mathbf{p}_1 - \mathbf{k}} - E_{\mathbf{p}_2})^{-1} \langle S''M'' \,|\, V_{\rm em}^{(1)} \,|\, SM \rangle + \text{terms for radiation from protons 2, 3, and 4} \, . \tag{2.2}$$

The quantization axis for all spins has been taken along the beam axis and the intermediate spin variables S'' and M'' are summed over all allowed values.

The $V_{\rm em}^{(i)}$ factors represent matrix elements of the electromagnetic interaction operator corresponding to the ith radiating proton. The electromagnetic interaction operator is obtained through a Foldy-Wouthuysen reduction[17] of the Dirac Hamiltonian describing a proton in an external electromagnetic field. Liou and Sobel have explicitly written out this operator to $O(m^{-2})$.[18] The usual nonrelativistic electromagnetic interaction

$$V_{\rm em}^{(i)} = \frac{-e}{2\pi m\sqrt{k}} \left[\mathbf{p}_i \cdot \boldsymbol{\epsilon}_\lambda - \frac{i}{2} \mu_{\rm p} \boldsymbol{\sigma}^{(i)} \cdot \mathbf{k} \times \boldsymbol{\epsilon}_\lambda \right] \tag{2.3}$$

is obtained if one retains only the $O(m^{-1})$ terms, while the additional pieces of $O(m^{-2})$ are called the relativistic spin correction terms.

We find, for the spin matrix elements of $V_{\rm em}^{(i)}$,

$$\langle S'M' \mid V_{\rm em}^{(i)} \mid SM \rangle = \frac{-e}{2\pi m\sqrt{k}} \left[\mathbf{A}^{(i,j)} \cdot \boldsymbol{\epsilon}_\lambda \delta_{SS'} \delta_{MM'} + 3\mu_{\rm p} \cdot [(2S+1)(2S'+1)]^{1/2} \right.$$
$$\left. \times \begin{Bmatrix} S' & 1 & S \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} (-1)^{M'} \sum_{m_1 m_2 m_3 m_4} (\boldsymbol{\epsilon}_\lambda)_{m_3} \mathbf{B}^{(i,j)}_{m_2 m_4 m_1} \begin{pmatrix} S' & 1 & S \\ M' & -m_1 & -M \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ m_4 & m_2 & m_3 \end{pmatrix} \right], \tag{2.4}$$

where $\mu_{\rm p}=2.793$ is the proton total magnetic moment, $\boldsymbol{\epsilon}_\lambda$ is the photon polarization vector, $j \neq i$ refers to the other of the two protons, and the m_l subscripts label spherical components. In the static limit, the factor $\mathbf{B}^{(i,j)}_{m_2 m_4 m_1} = \mathbf{k}_{m_2} \delta_{m_4 m_1}$ and $A^{(i,j)}=1$. The full expressions for A and B containing the relativistic spin correction terms which we have included in our calculation are

$$\mathbf{A}^{(i,j)} = \left[1 - \frac{2p_i^2 + 2\delta_i \mathbf{p}_i \cdot \mathbf{k} + k^2}{4m^2} \right] \mathbf{p}_i + \frac{k}{8m} (\mathbf{p}_i + \mathbf{p}_j)$$

and (2.5)

$$\mathbf{B}^{(i,j)}_{m_2 m_4 m_1} = \delta_{m_4 m_1} \left\{ \mathbf{k} - \frac{k}{m} \left[1 - \frac{1}{2\mu_{\rm p}} \right] \left[\mathbf{p}_i + \delta_i \frac{\mathbf{k}}{2} \right] - \frac{\mathbf{k}}{4m^2 \mu_{\rm p}} \left[2p_i^2 + 2\delta_i \mathbf{k} \cdot \mathbf{p}_i + k^2 - (\mu_{\rm p} - 1) \left[\mathbf{k} \cdot \mathbf{p}_i + k^2 \frac{\delta_i + 1}{2} \right] \right] \right.$$
$$\left. - \frac{(\mu_{\rm p}-1)k^2}{4m^2 \mu_{\rm p}} \mathbf{p}_i - \frac{k}{4m\mu_{\rm p}} [1 - (-1)^{S+S'}] \mathbf{p}_j \right\}_{m_2} + \frac{\mu_{\rm p} - 1}{2m^2 \mu_{\rm p}} \left[(-1)^{m_1} \mathbf{k}_{m_4} (\mathbf{p}_i)_{m_2} \left[\mathbf{p}_i + \frac{(\delta_i + 1)\mathbf{k}}{2} \right]_{-m_1} \right],$$

where $\delta_i = 1$ for $i = 3,4$ and $\delta_i = -1$ for $i = 1,2$, j is the member of the pair (1,2) or (3,4) different from i, and $B^{(2,1)}$ and $B^{(4,3)}$ are to be multiplied by an additional phase $(-1)^{S+S'}$.

We have chosen the incident beam direction for our axis of quantization in order to facilitate the calculation of analyzing powers and to make the axis the same in the laboratory and center of mass frames. The form of our relation for the spin matrix elements of $V_{\rm em}$ thus differs somewhat from the results of Ref. 11, in which the axis of quantization is defined by the photon momentum.

The first relativistic correction to $A^{(i,j)}$ is due to a term of $O(m^{-3})$ in the Foldy-Wouthuysen expansion. Liou and Sobel[18] have added this term in their calculation, arguing that it is comparable in size to the $O(m^{-2})$ terms. We have included in addition a number of other $O(m^{-3})$ terms. We find that the $O(m^{-3})$ terms contribute normally at the few percent level, though in a few instances can change the analyzing power by 10% or so. The $\mathbf{p}_j$ term in $B^{(i,j)}$ is the correction term required when one sums the electromagnetic interactions for the two individual protons.[18–21]

Now consider the important question of gauge invariance in the potential model calculation. For a local potential, the sum of single and double scattering terms is, in general, gauge invariant to all orders in k. Furthermore, the leading $O(k^0)$ pieces of the double scattering terms are just those needed to make the single scattering terms gauge invariant to $O(k^0)$.[16] Since we have not yet included the double scattering term, something must be done to make the result gauge invariant. Note first that these $O(k^0)$ double scattering terms are both proportional to the momentum of the center of mass and unique, being determined by the soft photon approach. Thus they vanish in the c.m. frame. Since we work in the center of mass, our amplitude is uniquely gauge invariant to $O(k^0)$. In addition, Liou and Cho[22] have shown that one can make the ppγ amplitude gauge invariant to all orders in k by adding to the single scattering terms a piece which is proportional to the c.m. momentum. Therefore, by working in the c.m. frame, our result is also gauge invariant to all orders in k. However, the $O(k)$ and higher terms are not uniquely gauge invariant and, in fact, the Liou-Cho gauge term has the wrong analytic structure since there is a $k \cdot p$ term in the denominator which can come only from the external radiation (single scattering) terms, not from the double scattering term.[23,24] To have a complete and unique amplitude, one must add the higher order pieces of the double scattering terms, as was done using different methods by Brown[25] and Heller and Rich.[12]

One must, in principle, also add gauge terms arising from the momentum dependence of the N-N potentials. At present, no one has included these effects in a ppγ potential model calculation[26] and the correct form of such terms is, in fact, not agreed upon.

In Eq. (2.2) we use the same form of propagator as Drechsel and Maximon.[11] It should be noted that the covariant propagators used by Liou and Sobel[18] are not strictly equivalent to ours. They differ by some higher order terms in the nonrelativistic reduction, and if used, would change the numerical results at the few percent level.

III. THE N-N T MATRICES

In order to evaluate Eq. (2.2) for the total ppγ T matrix, we require half-off-shell N-N T matrices of the form $\langle S'M', \mathbf{p}' \mid t(E') \mid \mathbf{p}, SM \rangle$ and $\langle S'M', \mathbf{p}' \mid t(E) \mid \mathbf{p}, SM \rangle$.

We need only calculate one of these, say the latter, obtaining the former through the symmetry relation

$$\langle S'M',\mathbf{p}' \mid t(E') \mid \mathbf{p},SM \rangle = \langle SM,\mathbf{p} \mid t(E') \mid \mathbf{p}',S'M' \rangle , \tag{3.1}$$

which follows from parity and time-reversal invariance. An added complication is due to the fact that for Eq. (2.1) we require the N-N T matrix in the frame in which the ppγ T matrix is evaluated (the overall center of mass frame) and with spin quantized along the beam axis, whereas it is most conveniently calculated in the individual N-N center of mass (which differs for different diagrams) with spins quantized along the initial relative momentum. The transformation between frames is handled most easily by multiplying the N-N T matrix by the square roots of all the initial and final energies so as to convert it to an invariant which can be calculated in the N-N center of mass and then used in whatever frame is desired. The on- and off-shell momenta used for evaluating the N-N T matrix were also obtained by Lorentz transforming from the overall center of mass frame to the center of mass of the N-N system. The transformation to a common axis of quantization is obtained through a rotation so that the N-N T matrix, quantized along the common beam axis, is given in terms of that quantized along the initial relative momentum or $\mathbf{p}$ axis in the N-N center of mass by

$$\begin{aligned}\langle 1\bar{M},\mathbf{p}' \mid t(E) \mid \mathbf{p},1M \rangle = \sum_{M'M''} & D^{(1)}_{\bar{M}M'}(\alpha\beta\gamma) \\ & \times \langle 1M',\mathbf{p}' \mid t(E) \mid \mathbf{p},1M'' \rangle \\ & \times [D^{(1)}(\alpha\beta\gamma)]^{\dagger}_{M''M} ,\end{aligned} \tag{3.2}$$

where $D^{(1)}_{M'M}$ are the usual rotation matrices[27] and where the angles are given by

$$\begin{aligned} &\cos\beta=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}', \quad \sin\alpha=\frac{\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}'}{\sin\beta} , \\ &\cos\alpha=\frac{\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}'}{\sin\beta}, \quad \sin\gamma=\frac{\hat{\mathbf{z}}\cdot\hat{\mathbf{y}}'}{\sin\beta} , \\ &\cos\gamma=\frac{-\hat{\mathbf{z}}\cdot\hat{\mathbf{x}}'}{\sin\beta} , \end{aligned} \tag{3.3}$$

with the beam along $\hat{\mathbf{z}}$ in the xyz frame and with the primed frame defined by $\hat{\mathbf{z}}'=\hat{p}$ and $\hat{y}'=\mathbf{p}\times\mathbf{p}'/|\mathbf{p}\times\mathbf{p}'|$.

Our N-N matrices are related to Stapp's scattering amplitudes[28] through the relation[29]

$$M_{ij}=-(2\pi)^2 p \left[\frac{dE}{dp}\right]^{-1} \langle S'M',\mathbf{p}' \mid t(E) \mid \mathbf{p},SM \rangle , \tag{3.4}$$

wherein

$$p\left[\frac{dE}{dp}\right]^{-1} = \begin{cases} m_r & \text{nonrelativistic} , \\ (\mathbf{p}^2+m_r^2)^{1/2} & \text{relativistic} , \end{cases} \tag{3.5}$$

and m_r is the reduced mass of the two-proton system. In Stapp's notation, M_{ss} is the singlet p-p scattering amplitude and, for the triplet scattering amplitudes, i and j label the initial and final total spin projection. Stapp has given the relation between the full p-p scattering amplitude and the partial wave scattering amplitudes.

We calculate the partial wave N-N T matrices through a Lippmann-Schwinger (LS) equation and sum them using the formulas of Stapp to obtain the full N-N T matrix. The LS equation is[10]

$$t(E,q',q)=V(q',q)+\int d^3q''V(q',q'')G(E,q'')t(E,q'',q) , \tag{3.6}$$

in which the following notation is used,

$$\begin{aligned} &t(E,q',q)=\langle \mathbf{q}' \mid t(E) \mid \mathbf{q} \rangle , \\ &G(E,q')=(E-E_{\mathbf{q}'}+i\epsilon)^{-1} , \\ &V(q',q)=\langle \mathbf{q}' \mid V \mid \mathbf{q} \rangle , \end{aligned} \tag{3.7}$$

where $|\hat{\mathbf{q}}\rangle$ is a plane wave state of center of mass momentum q, and V is the N-N potential.

We next expand $V(q',q)$, $t(E,q',q)$, and $t(E,q'',q)$ in terms of partial waves, using the general relation

$$V(q',q)=\sum_{LM} V_L(q',q)Y_{LM}(\hat{\mathbf{q}})Y^*_{LM}(\hat{\mathbf{q}}') , \tag{3.8}$$

with an analogous expansion for $t(E,q',q)$. The amplitudes are actually evaluated for states of spin S and orbital angular momentum L coupled to total angular momentum J, $|LSJ\rangle$. Throughout, these matrix elements will be implicit with the eigenvalues labeled by superscripts or subscripts. For uncoupled waves we label only the orbital angular momentum state—leaving S and J implicit. This will not be possible for coupled waves. We therefore begin the analysis with a treatment of uncoupled waves. The equations for uncoupled waves are easily generalized to the coupled case and are much less cumbersome. Equation (3.6) for uncoupled waves then becomes

$$\begin{aligned} t_L(E,q',q)=V_L(q',q)+\int_0^{\infty} & dq''q''^2V_L(q',q'') \\ & \times G(E,q'')t_L(E,q'',q) . \end{aligned} \tag{3.9}$$

A problem arises in Eq. (3.9), for q'' such that $E_{\mathbf{q}''}=E$. Here $G(E,q'')^{-1}\to 0$, which could cause problems in a numerical solution. Kowalski[9] has shown a simple way of avoiding this singularity. By adding linear combinations of Eq. (3.9) for $t_L(E,q',q)$ and $t_L(E,q,q)$, one obtains

$$t_L(E,q',q)=\frac{V_L(q',q)}{V_L(q,q)}+\int_0^{\infty} dq''q''^2G(E,q'')\left[V_L(q',q)-V_L(q',q)\frac{V_L(q,q'')}{V_L(q,q)}\right]t_L(E,q'',q) . \tag{3.10}$$

It has been shown[9] that $t_L(E,q',q)$ can be written as

$$t_L(E,q',q)=f_L(q',q)t_L(E,q,q) , \tag{3.11}$$

where $f_L(q',q)$ is a real function, the so called "half-off-shell extension function." Substitution of Eq. (3.11) into (3.10) yields, for $f_L(q',q)$,

$$f_L(q',q)=\frac{V_L(q',q)}{V_L(q,q)}+\int_0^\infty dq''q''^2G(E,q'')\left[V_L(q',q'')-\frac{V_L(q',q)}{V_L(q,q)}V_L(q,q'')\right]f_L(q'',q)\ . \tag{3.12}$$

Now, in order to solve for $f_L(q',q)$ numerically, the integral in Eq. (3.12) is approximated by a sum. The methods of Gaussian integration are used to pick integration points and weighting factors. Having solved for $f_L(q',q)$, we may obtain the on-shell N-N T matrix, $t_L(E,q,q)$, by substituting Eq. (3.11) into (3.9) and using the condition $f_L(q,q)=1$ to obtain

$$t_L(E,q,q)=V_L(q,q)\left[1-\int_0^\infty dq'q'^2G(E,q')V_L(q,q')f_L(q',q)\right]^{-1}\ . \tag{3.13}$$

Having evaluated Eqs. (3.12) and (3.13), we have the on-shell N-N T matrix $t_L(E,q,q)$ and the half-off-shell function evaluated at N values of q_i. In order to obtain the half-off-shell T matrix $t_L(E,q',q)$, where q' is some arbitrary value of momentum, we use Eq. (3.11). The appropriate value of $f_L(q',q)$ is found using Lagrange interpolation on the grid of $f_L(q_i,q)$ values. Since the on-shell T matrix is calculated from the half-off-shell extension function, the whole procedure can be checked by calculating the phase shifts from the on-shell amplitudes and comparing them with those tabulated for the potential used or those calculated by solving the Schrödinger equation or the on-shell LS equation directly.

Generally, we must include coupled waves made necessary by the tensor part of the N-N interaction which couples different L values to the same J. The coupled LS equation analogous to Eq. (3.9) is

$$t^J_{L'L}(E,q',q)=V^J_{L'L}(q',q)+\sum_{L''=J-1}^{J+1}\int_0^\infty dq''q''^2V^J_{L'L''}(q',q'')\times G(E,q'')t^J_{L''L}(E,q'',q)\ . \tag{3.14}$$

For coupled waves, the scalar multiplication in Eq. (3.11) is replaced by matrix multiplication, and we have

$$t^J_{L'L}(E,q',q)=\sum_{L''=J-1}^{J+1}f^J_{L'L''}(q',q)t^J_{L''L}(E,q,q)\ . \tag{3.15}$$

In order to obtain the matrix of off-shell extension functions, we numerically solve the matrix equation

$$\sum_{L''=J-1}^{J+1}\Lambda^J_{L_1L''}(i,j)f^J_{L''L_2}(j)=C^J_{L_1L_2}(i)\ , \tag{3.16}$$

with

$$C^J_{L'L}(i)=\sum_{L''=J-1}^{J+1}V^J_{L'L''}(q_i,q)(V^J_{L''L})^{-1}(q,q) \tag{3.17}$$

and

$$\Lambda^J_{L_1L''}(i,j)=\delta_{ij}\delta_{L_1L''}-w_jq^2 2m_r G(E,q_j)[V^J_{L_1L''}(q_i,q_j)-V^J_{L_1L}(q_i,q)(V^J_{LL'})^{-1}(q,q)V^J_{L'L''}(q,q_j)]\ . \tag{3.18}$$

Note that all our summations on intermediate angular momenta include $L=J-1$ and $L=J+1$ but not $L=J$. Throughout, V^{-1} denotes the matrix inverse of V. In Eq. (3.18), the intermediate angular momenta L and L' are summed as in Eq. (3.16). The factor w_j is a Gaussian weight.

The on-shell N-N T matrix is now calculated via

$$t^J_{L_1L_2}(E,q,q)=(D^J_{L_1L})^{-1}(E,q,q)V^J_{LL_2}(q,q)\ , \tag{3.19}$$

wherein D^{-1} means the matrix inverse of

$$D^J_{L_1L_2}(E,q,q)=\delta_{L_1L_2}+i\pi m_r qV^J_{L_1L_2}(q,q)-\sum_{j=1}^{N}w_j[q^2V^J_{L_1L}(q_j,q)f^J_{LL_2}(q_j,q)-q^2V^J_{L_1L_2}(q,q)]2m_r(q^2-q_j^2)^{-1}\ . \tag{3.20}$$

In both (3.19) and (3.20), L is summed.

With the above relations and a particular choice for the potential, we can calculate the N-N T matrices for any given partial wave. We have generally calculated from a potential all partial waves up to and including $J=5$. The higher partial waves as obtained from one pion exchange amplitudes have also been included, unlike most previous $pp\gamma$ calculations. These have been conveniently calculated in Ref. 30. Note that we have used an on-shell approximation to the true one pion exchange amplitudes.

These partial wave amplitudes are now summed to give the full half-off-shell N-N T matrices, in the $|\,SM\,\rangle$ basis. At this point, Coulomb corrections are added, in essentially the same way as done by Stapp, i.e., Coulomb scattering amplitudes are added to our N-N amplitudes, which are multiplied by the appropriate phase in order to produce interference between the two interactions. Thus, in this approximation the Coulomb corrections are also on shell.

This method of making Coulomb corrections does not make the overall ppγ calculation consistent to second order in the electromagnetic interaction. It does, however, account for those Coulomb corrections to the N-N amplitudes which can be obtained via adding a nonrelativistic Coulomb potential to the strong interaction, and it does go one step further than many previous ppγ calculations.

IV. THE PARTIAL WAVE PARIS N-N POTENTIAL

Most previous calculations of ppγ have used simple phenomenological potentials such as the Hamada-Johnston[31] or Reid soft-core[32] potentials. Since these calculations, however, there have been major advances in the development of theoretically based potentials such as those of the Bonn[33] and Paris[7] groups. One of the main aims of this calculation has been to use one of these potentials, the Paris potential, for the first time in a ppγ calculation. As a check, and an alternative for comparison, we have also used an extended Reid soft-core potential.[8] This is phenomenological like the original Reid soft-core and differs from it by the inclusion of several higher partial waves. At a later date we plan to perform the calculation with the Bonn potential as well.

A word of caution is in order, however. It becomes possible to use the Paris potential in a calculation such as this only because of the existence of a simple parametrized form.[7] This form was obtained by fitting the on-shell predictions of the full Paris potential results. Thus, there is no real guarantee *a priori* that the off-shell behavior of the N-N amplitudes obtained with this parametrization will agree in detail with that which would be obtained in a full off-shell calculation starting with the same physical picture. It should be approximately correct, however, since the physical picture of the one boson exchange does lead to a potential of the Yukawa form.

To evaluate the expressions of the preceeding section for any potential we must obtain the partial wave matrix element $V^J_{LS}(q',q)$. The method used to decompose a potential into partial waves is well known. Since the Paris potential has been parametrized in terms of a series of Yukawa potentials, it may be partial waved, as is the familiar Reid soft-core (RSC) potential. Most of the necessary results are tabulated in Ref. 34. The RSC potential, however, does not have an energy dependent central potential, nor does it have a quadratic spin-orbit part. An energy dependence presents no difficulties in momentum space but, the quadratic spin-orbit (SO2) potential requires extra attention.

In order to evaluate $V^J_{LS}(q',q)$ for the SO2 potential, we must evaluate

$$V^J_{L'S',LS}(q',q)\equiv\langle q',L'S'J\,|\,V_{\rm SO2}(r)\Omega_{\rm SO2}\,|\,q,LSJ\rangle=\frac{2}{\pi}\int_0^\infty r^2dr\,j_{L'}(q'r)j_L(qr)V_{\rm SO2}(r)\langle L'S'J\,|\,\Omega_{\rm SO2}\,|\,LSJ\rangle\ , \tag{4.1}$$

wherein

$$V_{\rm SO2}(r)=\sum_{j=1}^{12}\frac{g_j^{\rm SO2}}{(m_jr)^2}\left[1+\frac{3}{m_jr}+\frac{3}{(m_jr)^2}\right]\frac{e^{-m_jr}}{m_jr}\ , \tag{4.2}$$

$$\Omega_{\rm SO2}\,|\,LSJ\rangle=\{\tfrac{1}{2}[J(J+1)-L(L+1)-S(S+1)]^2-L(L+1)\}\,|\,LSJ\rangle\ . \tag{4.3}$$

The parameters g_i and m_j are tabulated in Ref. 7 and $j_L(pr)$ is a spherical Bessel function. Since the N-N potential conserves the total spin S, and the SO2 potential does not couple angular momentum states, L and S are good quantum numbers and we may write $V^J_{L'S',LS}$ simply as V^J_{LS}. For $L=0$ states, the SO2 potential is zero, while for $L\geq 2$ the integral in Eq. (4.1) is easily evaluated to give

$$\sum_{J=1}^{12}\frac{g_j^{\rm SO2}}{\pi m_j^3(2L+1)^2}\left[\left[1+\frac{q^2+q'^2}{m_j^2}\right][Q_{L+1}(Z_j)+Q_{L-1}(Z_j)]-\frac{4qq'}{m_j^2}Q_L(Z_j)+\frac{3qq'}{m_j^2(2L+3)}[Q_L(Z_j)-Q_{L+2}(Z_j)]-\frac{3qq'}{m_j^2(2L-1)}[Q_L(Z_j)-Q_{L-2}(Z_j)]\right]\langle\Omega_{\rm SO2}\rangle\ , \tag{4.4}$$

where

$$Z_j=\frac{q^2+q'^2+m_j^2}{2qq'}\ . \tag{4.5}$$

For the $L=1$ case, however, the integral in Eq. (4.1) is divergent. By using the constraints[7]

$$\sum_j\frac{g_j^{\rm SO2}}{m_j}=\sum_j\frac{g_j^{\rm SO2}}{m_j^3}=\sum_j\frac{g_j^{\rm SO2}}{m_j^5}=0 \tag{4.6}$$

for the SO2 parameters, the divergent part of Eq. (4.1) can be removed as follows. The troublesome term is

$$\int_0^\infty r^2 dr\, j_0(q'r)\frac{1}{(m_j r)^3}e^{-m_j r}j_0(qr)\ . \tag{4.7}$$

By using the substitution

$$\frac{1}{(mr)^3}=\lim_{\beta\to 0}\frac{mr}{[(mr)^2+\beta^2]^2} \tag{4.8}$$

and evaluating the integral before the limit $\beta\to 0$, the logarithmic divergence can be separated out and is proportional to $\sum_j (g_j^{SO2}/m_j^3)$, which vanishes by virtue of the constraint in Eq. (4.6). Thus, taking the finite part into account gives for states with $L=1$ the result that the last term in square brackets in Eq. (4.4), i.e., the one proportional to $Q_L - Q_{L-2}$, should be replaced by

$$3a\tan^{-1}\left[\frac{2a'}{1+a^2-a'^2}\right]+3a'\tan^{-1}\left[\frac{2a}{1+a'^2-a^2}\right]+\tfrac{3}{4}\{[1-(a'+a)^2]\ln[1+(a'+a)^2]-[1-(a'-a)^2]\ln[1+(a'-a)^2]$$
$$-4(a'+a)\tan^{-1}(a'+a)+4(a'-a)\tan^{-1}(a'-a)\}$$
$$+\tfrac{3}{2}\ln\frac{1+(a'-a)^2}{1+(a'+a)^2}\ , \tag{4.9}$$

with the abbreviations $a_j=q/m_j$ and $a_j'=q'/m_j$.

Having calculated the necessary components of our total ppγ T matrix, we now proceed to construct the observables which have been calculated.

V. THE ppγ OBSERVABLES

We use the following relation for the ppγ cross section:

$$d\sigma=(2\pi)^4[(\mathbf{p}_1\cdot\mathbf{p}_2-E_1E_2)^2-m^4]^{-1/2}d^3p_3d^3p_4d^3k(E_3E_4k)^{-1}\delta^4(p_1+p_2-p_3-p_4-k)\overline{\sum_{\text{spins}}}\sum_\lambda|(E_3E_4k)^{1/2}T(E_1E_2)^{1/2}|^2\ , \tag{5.1}$$

in which the flux, phase space, and amplitude factors are individually invariant. In the invariant amplitude, we evaluate the sum on photon spins,

$$\overline{\sum_{\text{spins}}}\sum_\lambda|T|^2=\tfrac{1}{4}\sum_\lambda\sum_{SM}\sum_{S'M'}\langle S'M'|T|SM\rangle^\dagger \times\langle S'M'|T|SM\rangle\ , \tag{5.2}$$

and sum the photon polarization, in the transverse gauge, using

$$\sum_\lambda|T|^2=\mathbf{M}\cdot\mathbf{M}^*-\hat{\mathbf{k}}\cdot\mathbf{M}\hat{\mathbf{k}}\cdot\mathbf{M}^*\ , \tag{5.3}$$

where $\hat{\mathbf{k}}$ is a unit vector along the photon momentum and $T=\boldsymbol{\epsilon}\cdot\mathbf{M}$.

In order to calculate the analyzing powers, we must change the spin basis using

$$|m_1m_2\rangle=\sum_{SM}|SM\rangle\langle SM|m_1m_2\rangle\ , \tag{5.4}$$

where m_1 and m_2 label the spin projections of the individual protons.

The analyzing powers can then be calculated from

$$A_i=\frac{\mathrm{Tr}(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}_i TT^\dagger)}{\mathrm{Tr}(TT^\dagger)}\ , \tag{5.5}$$

where $\hat{\mathbf{n}}_i$ is a unit vector along either the x, y, or z axis. Explicitly, we have

$$A_x=(T_{12}^2+T_{21}^2)/(T_{11}^2+T_{22}^2)\ ,$$
$$A_y=i(T_{12}^2-T_{21}^2)/(T_{11}^2+T_{22}^2)\ , \tag{5.6}$$
$$A_z=(T_{11}^2-T_{22}^2)/(T_{11}^2+T_{22}^2)\ ,$$

where the above subscripts 1 and 2 denote a spin projection for the incident proton of $+\frac{1}{2}$ and $-\frac{1}{2}$ along the beam axis, and

$$T_{ab}^2=\sum_{m_2m_3m_4}(\langle m_3m_4|\mathbf{M}|am_2\rangle\cdot\langle m_3m_4|\mathbf{M}|bm_2\rangle^* -\langle m_3m_4|\hat{\mathbf{k}}\cdot\mathbf{M}|am_2\rangle\langle m_3m_4|\hat{\mathbf{k}}\cdot\mathbf{M}|bm_2\rangle^*) \tag{5.7}$$

By symmetry, $A_x=A_z=0$ in coplanar geometries. It should be noted that our convention that A_y is positive along the positive y axis, together with our choice $\phi_3=0$, which puts $\mathbf{p}_3$ in the first quadrant of the x-z plane, gives results which differ by a sign from the old calculation of McGuire and Pearce.[35]

VI. NUMERICAL RESULTS AND DISCUSSION

Before giving new results from our calculation, we outline some of the checks and comparisons with older calculations which have been made. First, we have checked our potential model calculation against a SPA calculation at laboratory energies of 25 and 280 MeV in the limit $k\to 0$. Potential models also satisfy the SPA, so differences in this limit should be primarily residual differences from different on-shell behavior only. Our potential model results for both cross section and analyzing power do converge to the soft photon results as $k\to 0$. As an additional check, we have calculated integrated ppγ cross sections for coplanar final protons with $\theta_3=\theta_4=30°$. These integrated cross sections agree, to about the 10%

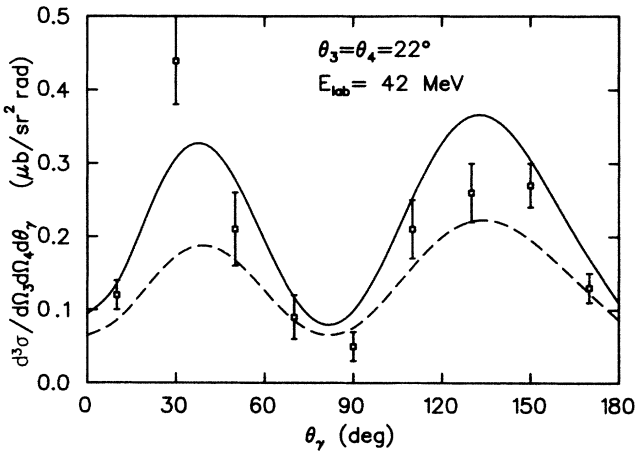

FIG. 2. Paris potential (solid line) and SPA (dashed line) calculations of $d\sigma/d\Omega_3 d\Omega_4 d\theta_\gamma$ as a function of θ_γ for an incident proton energy of 42 MeV. Final proton polar angles are $\theta_3=\theta_4=22°$. The experimental data are from Ref. 3.

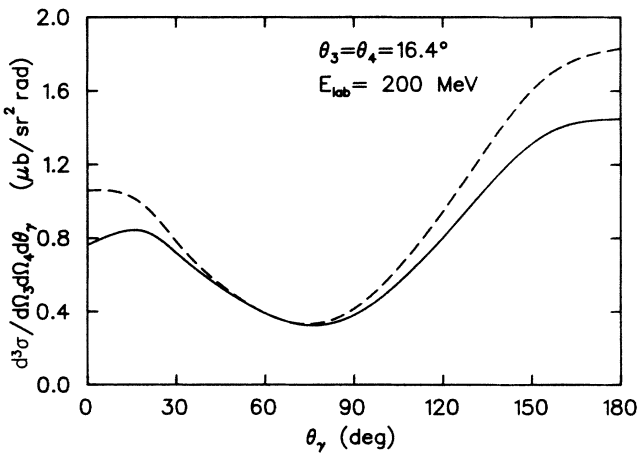

FIG. 4. Effect of dropping the relativistic spin corrections (dashed curve) to the cross section for the kinematic conditions of Fig. 3. The solid line corresponds to the full Paris potential calculation.

level, with those given by Drechsel and Maximon,[11] if we drop the relativistic spin corrections terms. Considering the differences in details of the two calculations, such agreement is quite satisfactory. Some comparisons with other previous calculations will be mentioned in the next section.

In Figs. 2 and 3 we compare our calculation to data from the 42 and 200 MeV ppγ experiments. For the purpose of comparison, SPA calculations have also been included. Figure 2 displays the cross section, $d\sigma/d\Omega_3 d\Omega_4 d\theta_\gamma$, as a function of the photon polar angle θ_γ for a laboratory energy of 42 MeV and coplanar final protons with $\theta_3=\theta_4=22°$. Here, our potential model fit to the data is not appreciably different from older Hamada-Johnston potential model calculations of Liou.[18] The apparent disagreement between theory and experiment is difficult to reconcile for this low energy, where one would expect the potential model to work best. In particular, the calculation of Brown[25] indicates that contributions from the double scattering term should be insignificant at this energy and likewise relativistic corrections should also be small.

In Fig. 3 we have again plotted cross sections versus the photon angle. The potential model and SPA calculations correspond to the 200 MeV TRIUMF experiment having coplanar protons with $\theta_3=\theta_4=16.4°$. Corrections due to the finite detector size have been incorporated into the final result. Clearly, the Paris potential curve lies only marginally closer to the data than earlier potential calculations. Both the Paris and extended Reid soft-core calculations are above the data over most of the angular range. The SPA curve appears to give a somewhat better fit. If the double scattering terms were to alter the cross section

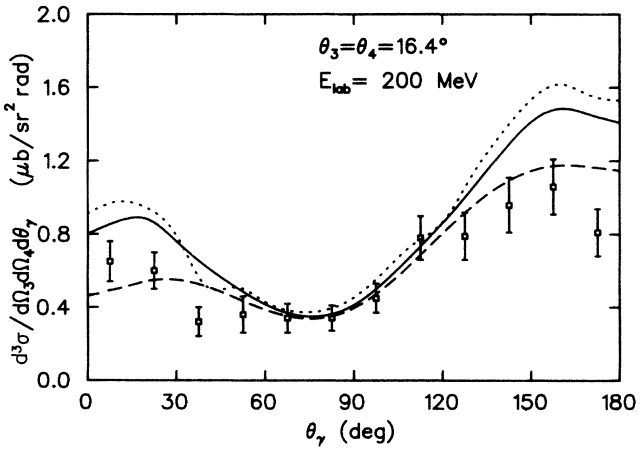

FIG. 3. Paris potential (solid line), SPA (dashed line), and extended Reid soft-core (dotted line) calculations of $d\sigma/d\Omega_3 d\Omega_4 d\theta_\gamma$ as a function of θ_γ for an incident proton energy of 200 MeV. Final proton polar angles are $\theta_3=\theta_4=16.4°$. The experimental data are from Ref. 2. In this figure the theoretical results have been multiplied by corrections for finite detector size.

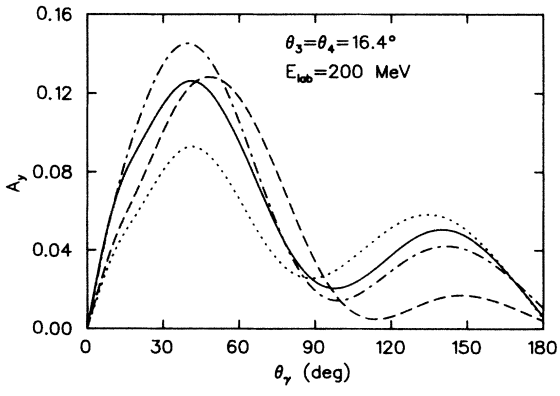

FIG. 5. Effects of dropping correction terms to the calculation of A_y, with the geometry and energy of Fig. 3. We compare the full Paris potential calculation (solid line) to the calculations neglecting individually the Coulomb corrections (dashed line), relativistic spin corrections (dotted line), and the one pion exchange for partial waves with $J \geq 6$ (dotted-dashed line).

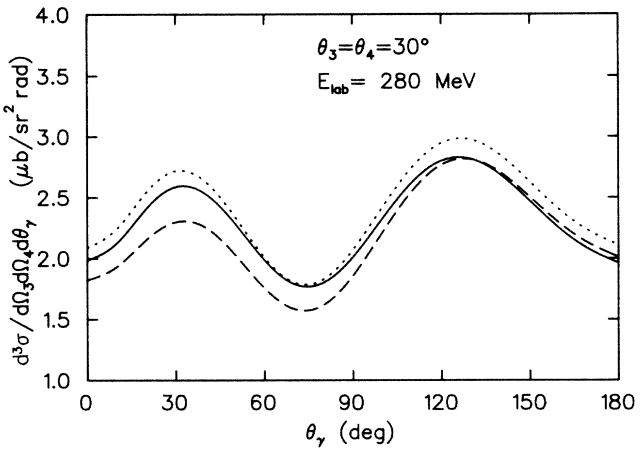

FIG. 6. Cross sections, labeled as in Fig. 3, for an incident proton energy of 280 MeV and final proton polar angles $\theta_3=\theta_4=30°$.

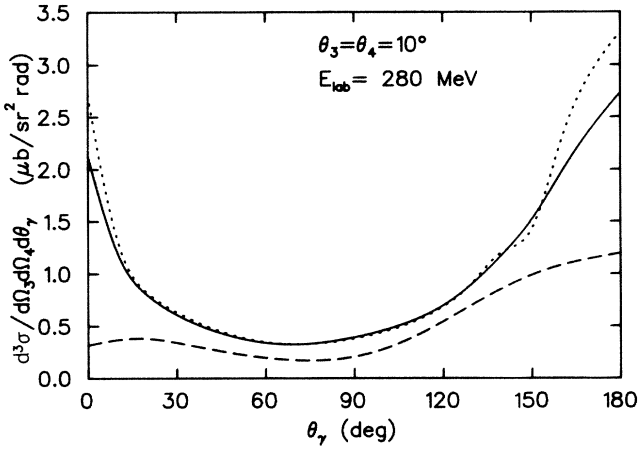

FIG. 8. Cross sections labeled as in Fig. 3, for an incident proton energy of 280 MeV and final proton polar angles $\theta_3=\theta_4=10°$.

by the order of 10%, as suggested in Brown's paper, and if they lowered the cross section, the fit to the data would be only marginally improved. It is not even clear that the addition of double scattering terms would lower the cross section. Brown's calculation for a laboratory energy of 158 MeV, with $\theta_3=\theta_4=30°$, shows that, in at least that geometry, the addition of double scattering terms increases the cross section.

In Figs. 4 and 5, at 200 MeV, we explore the effects of dropping individual contributions to our calculation. The most significant contribution is due to the relativistic spin correction terms. These terms lower the cross section by about 25% near $\theta_\gamma=0°$ and 180°. The Coulomb correction and the one pion exchange for high partial waves contribute negligibly to the cross section. The analyzing powers, however, are more sensitive to such corrections. The addition of one pion exchange amplitudes for high partial waves and both the Coulomb and relativistic spin corrections appear to be significant, at least at some angles, in the analyzing power calculation. However, the largest contribution is again due to the relativistic spin correction terms, the removal of which tends to flatten the analyzing power curve. This sensitivity provides an additional motivation for the measurement of analyzing powers in the 280 MeV TRIUMF experiment.

As a check on our calculation of analyzing powers, we have compared our results to the older calculation of McGuire and Pearce.[35] We find, at 200 MeV, $\theta_3=20°$, and $\theta_4=40°$, fairly strongly disagreement with the results of McGuire and Pearce. Our results do follow the SPA results rather well, however, as they should in this geometry. The calculation of McGuire and Pearce differs from ours in that they have not included relativistic spin corrections and have not calculated the higher partial waves which we obtained from the one pion exchange. They have also chosen to extend their amplitudes off shell using the one pion exchange potential alone. While these ingredients may account for some of the difference, most must be due to a different on-shell behavior, since the calculation of McGuire and Pearce does not have the proper limit as $\mathbf{k}\rightarrow 0$. Bohannon has also calculated[36] analyzing

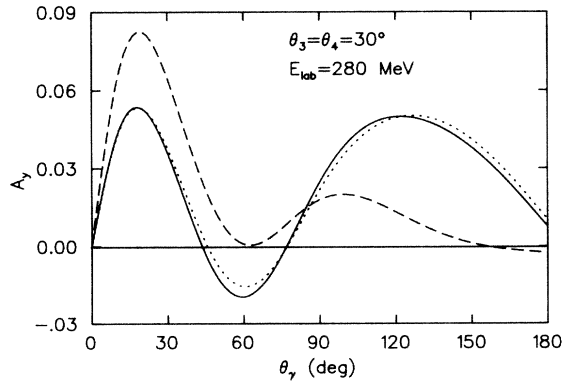

FIG. 7. Paris potential (solid line), SPA (dashed line), and extended Reid soft-core (dotted line) calculations of A_y as a function of θ_γ for an incident proton energy of 280 MeV and final proton polar angles $\theta_3=\theta_4=30°$.

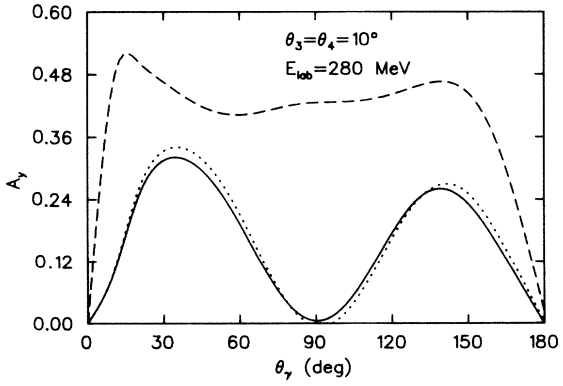

FIG. 9. Calculations of A_y labeled as in Fig. 7 for an incident proton energy of 280 MeV and final proton polar angles $\theta_3=\theta_4=10°$.

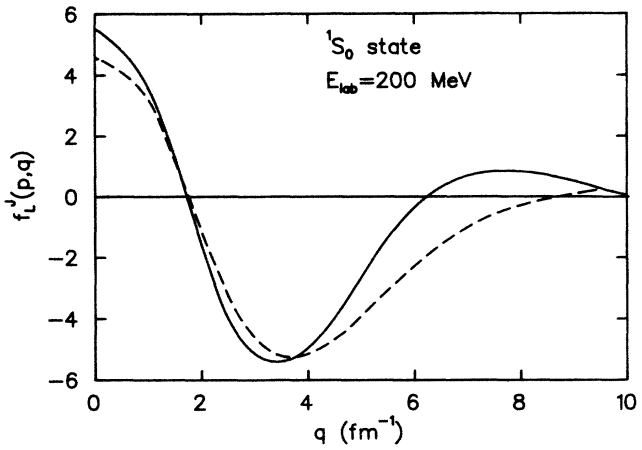

FIG. 10. Half-off-shell extension functions $f_l^J(p,q)$ for the $^1S_0$ state as a function of off-shell momentum q for an incident proton energy of 200 MeV. The solid line corresponds to the Paris potential and the dashed line corresponds to the extended Reid soft-core potential.

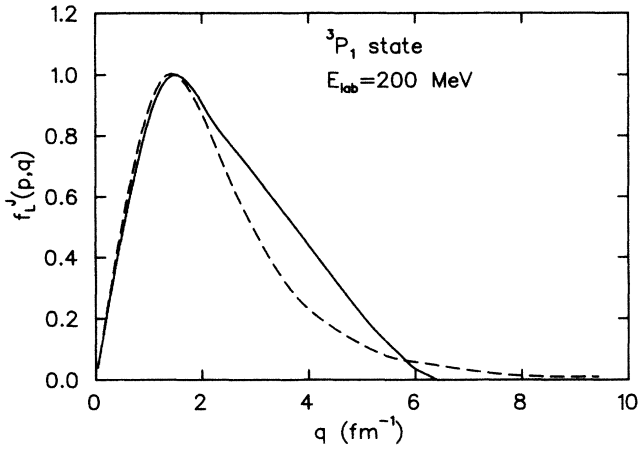

FIG. 11. Half-off-shell extension functions, as in Fig. 10, for the $^3P_1$ state.

powers within the potential model for a few specific cases.

Figures 6–9 show potential model and SPA predictions for cross section and analyzing powers at the energy of the new TRIUMF experiment. The qualitative behavior here is similar to that exhibited at 200 MeV. Note that at $\theta_3=\theta_4=10°$ there is a marked qualitative difference between the potential model and SPA analyzing powers which should be easily distinguishable in the new TRIUMF experiment. All of this should be contrasted with the results for $\theta_3=\theta_4=30°$. The photon now carries away much less momentum, so that the potential model and SPA calculations begin to merge. Thus, the greatest differences between the SPA and potential model calculations are visible for small final proton angles.

One of the initially rather surprising results evident from Figs. 7–9 was the very similar cross sections and analyzing powers obtained with the potentials we have used, keeping all other parts of the calculation fixed. In the $\mathbf{k}=0$ limit results should be the same since then they depend only on on-shell amplitudes which, for the cases we considered, fit the elastic data equally well and so are essentially the same. It turns out, however, that the potentials we considered have similar off-shell behavior as well, which in retrospect could possibly have been guessed since they both are parametrized in terms of Yukawa potentials.

Figures 10 and 11 show this similarity by comparing the half-off-shell extension functions for the extended Reid soft-core and Paris potentials in the $^1S_0$ and $^3P_1$ states, respectively. The two potentials have qualitatively similar extension functions. These extension functions are constrained to pass through unity at the on-shell point and differ from each other by small amounts over the range of off-shell momenta to which the calculation is sensitive. Furthermore, for a fixed off-shell momentum, the extension function obtained from the Paris potential is higher than that obtained from the Reid potential for some partial waves and lower for others. Hence there is some cancellation of the differences in the final result. With such similar off-shell behaviors for the various potentials, it is not surprising that the ppγ results are also similar for these N-N potentials. This does not mean that ppγ is insensitive to off-shell effects, just that the potentials we considered are similar both on and off shell.

We are currently extending this calculation,[37] in collaboration with R. Machleidt, so as to use the Bonn potential. Somewhat different technical considerations are necessary. Preliminary indications are that the Bonn potential results are again very similar to those for the other potentials.

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada.

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