Elastic collisions between light nuclei and the phase variation of the nucleon-nucleon scattering amplitude

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Elastic collisions between light nuclei are investigated at medium energies using Glauber theory. It is shown that a phase variation of the nucleon-nucleon elastic scattering amplitude leads to large increases in the calculated differential cross sections, even away from the minima that would occur if there were no phase variation. The relatively small number of minima that occur with no phase variation is explained in terms of the relative lack of strong destructive interference among multiple scattering amplitudes due to their relatively slow decrease with increasing momentum transfer. The introduction of a phase variation leads to even fewer minima. Calculations with no phase variation are in marked disagreement with $\alpha\,^4$He, $\alpha\,^3$He, $\alpha\,^2$H and $\alpha\,^1$H data at 7 GeV/c and $\alpha\,^4$He data at 5.07 and 4.32 GeV/c. The presence of a phase variation leads to a substantial improvement.

I. INTRODUCTION

Glauber theory has been quite successful during the past 20 years in describing hadron-nucleus elastic scattering at energies of approximately 500 MeV and higher. The usefulness of the theory in describing nucleus-nucleus ("heavy-ion") elastic scattering at corresponding energies of approximately 500 MeV/nucleon and higher has not been demonstrated in detail so clearly. This is, in part, due to the relative paucity of such medium-energy and high-energy nucleus-nucleus elastic scattering measurements. In addition, the extension of the theory to collisions between composite systems is significantly more complex and the computations are sufficiently more difficult and lengthy so that much fewer of these types of calculations exist.[1]

Recently, a comprehensive set of measurements of elastic scattering of α particles by four very light nuclei (^{4}He, ^{3}He, ^{2}H, and ^{1}H) was made[2] at an incident α-particle momentum of 7 GeV/c over a range of squared momentum-transfer values $0.07 \lesssim |t| \lesssim 4$ (GeV/c)^2. The measured cross sections fell from the barn to the nanobarn level over this range of t. Such data, in which several different systems of nuclei are involved, with intensities varying by so many orders of magnitude and over such a large range of momentum transfers, are extremely useful because they put enormous constraints on any theory. It no longer suffices to show that a theory describes measurements of collisions between just one pair of nuclei. Now it must describe measurements of collisions between each of the four different pairs of nuclei, and it must do so consistently. Whatever nucleon-nucleon (NN) elastic-scattering amplitude is used for collisions between one pair of nuclei should be used for collisions between the other pairs as well. In addition, since these measurements have gone out to rather large momentum transfers, the calculated intensities will be much more sensitive to the dynamical content of any theory.

The measurements for elastic scattering of α particles by the four light nuclei were accompanied[2] by theoretical analyses for the α-^{2}H, α-^{3}He, and α-^{4}He cross sections. These analyses were both by means of the so-called "rigid-projectile approximation" and by means of the conventional Glauber theory, with Gaussian densities for the nuclear ground states. The rigid-projectile approximation failed even qualitatively, except at very small momentum transfers. In the Glauber-theory calculations presented, the broad qualitative trends of the data were to some extent roughly described. Quantitatively, however, the results were in strong disagreement with the data, often being as much as an order of magnitude too low.

In the present analysis we calculate elastic scattering of light nuclei by light nuclei in Glauber theory. We evaluate the full multiple-scattering series (through sixteenth-order multiple collisions for α-^{4}He scattering, for example). Traditionally, calculations of nuclear scattering by means of Glauber theory generally employ parametrizations of the basic complex two-body hadron-nucleon (hN) elastic scattering amplitudes that have a phase which has a weak energy dependence but which is independent of the scattering angle (or momentum transfer). The value of the phase is taken to be $\frac{1}{2}\pi-\tan^{-1}\rho \approx \frac{1}{2}\pi-\rho$, where ρ is the ratio of the real part to the imaginary part of the hN forward elastic scattering amplitude. The actual phase of the hN elastic scattering amplitude cannot generally be determined from hN scattering measurements, except very close to the forward direction where the Coulomb effects are appreciable and result in interference between the Coulomb and the strong-interaction hN amplitudes. Away from the forward direction, however, hN measurements determine the scattering amplitude (or amplitudes) at best only to within an overall phase factor.

As we shall see, the variation of the phase of the hN elastic scattering amplitude affects hadron-nucleus scattering, and the variation of the phase of the NN elastic scattering amplitude also effects nucleus-nucleus scattering. Thus it may be possible to learn something about the phase variation of basic hN amplitudes by studying hadron-nucleus or nucleus-nucleus scattering. As we will see, a phase variation in the NN scattering amplitude brings Glauber theory into good agreement with the nucleus-nucleus elastic scattering data previously mentioned,[2,3] as well as with earlier $\alpha\alpha$ data[4] at the somewhat lower incident momenta of 4.31 and 5.07 GeV/c.

In Sec. II we describe the Glauber multiple-scattering series for nucleus-nucleus collisions and present explicit analytical expressions for the amplitudes for the case of Gaussian NN amplitudes and nuclear ground state densities. In Sec. II we describe the features of the differential cross sections for hadron-nucleus and nucleus-nucleus collisions, both with and without a phase variation of the NN amplitude. In Sec. IV we present applications to elastic scattering of α particles by ^{4}He, ^{3}He, ^{2}H, and ^{1}H at medium energies. In the Appendix we present the orbits, lengths, and Δ matrices (described in Sec. II) for $A=4$, $B=3$ (α^3He scattering, for example), and $A=B=4$ ($\alpha\alpha$ scattering, for example).

II. GLAUBER MULTIPLE-SCATTERING SERIES FOR NUCLEUS-NUCLEUS COLLISIONS

The formal generalization of Glauber theory to nucleus-nucleus collisions has been known for some time.[5,6] Significant difficulties occur, however, in the application of the theory to specific nuclear systems. For medium-weight or heavy nuclear systems, say for projectile and target mass numbers greater than approximately 10, the full Glauber multiple scattering series is rather unwieldy and contains too many terms to be of practical use. For such cases the Glauber scattering amplitude can be approximated by means of an optical phase shift function approach. Fourth-order calculations of the optical phase shift function have led to accurate and useful results[7] for describing such heavy-ion collisions.

If both the projectile and target are very light nuclei, then it may be useful to employ the full Glauber multiple scattering series to describe the collision. The scattering amplitude operator for collisions between two nuclei (A and B) with mass numbers A and B can be written as[5]

$$F(\mathbf{q};\mathbf{s}_i,\mathbf{s}'_j)=\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}} \times\left\{1-\prod_{i=1}^{A}\prod_{j=1}^{B}[1-\Gamma_{ij}(\mathbf{b}+\mathbf{s}_i-\mathbf{s}'_j)]\right\}, \tag{2.1}$$

where k is the wave number of the incident nucleus, $\hbar\mathbf{q}$ is the momentum transfer, and $\{\mathbf{s}_i\}$ and $\{\mathbf{s}'_j\}$ are the projections of the bound nucleon coordinates of nuclei A and B onto the plane of the impact parameter vector $\mathbf{b}$. The profile functions $\Gamma_{ij}(\mathbf{b})$ are related to the NN elastic scattering amplitudes $f_{ij}(\mathbf{q})$ by

$$\Gamma_{ij}(\mathbf{b})=\frac{1}{2\pi i k_{\rm N}}\int d^2q\, e^{-i\mathbf{q}\cdot\mathbf{b}}f_{ij}(\mathbf{q}) , \tag{2.2}$$

where $k_{\rm N}$ is the wave number of the incident nucleons.

The nucleus-nucleus elastic scattering amplitude is given by the expectation value of the scattering amplitude operator in the ground states of the two nuclei, i.e.,

$$F_{AB}(\mathbf{q})=\langle\psi_A\psi_B\,|\,F(\mathbf{q};\{\mathbf{s}_i\},\{\mathbf{s}'_j\}\,|\,\psi_A\psi_B\rangle , \tag{2.3}$$

where ψ_A,ψ_B are the ground state wave functions of the two nuclei. It is clear from Eqs. (2.1)–(2.3) that the scattering amplitude will be a sum of $2^{AB}-1$ terms, each term containing between one and AB factors of Γ_{ij} corresponding to single scattering through AB-tuple scattering.

To secure estimates for the nucleus-nucleus scattering amplitude, we need to know both the NN scattering amplitudes and the nuclear ground state wave functions. We shall assume, for simplicity, that all NN amplitudes are equal. This is approximately true at high energies. (This, however, is a matter of convenience and not of necessity. The generalization of our results to the case $f_{\rm pp}\neq f_{\rm np}$ is straightforward when the higher order corrections due to noncommutativity are ignored, and leads to very small effects. The calculation of the higher order corrections due to noncommutativity is much more tedious but is unlikely to yield major differences in the results.) We shall, further, neglect spin effects and use the conventional high-energy parametrization of $f_{ij}(\mathbf{q})$,

$$f_{ij}(\mathbf{q})=\frac{k_{\rm N}\sigma}{4\pi}(i+\rho)e^{-aq^2/2} , \tag{2.4}$$

where σ is the NN total cross section, and ρ is the ratio of the real part to the imaginary part of the NN forward scattering amplitude.

Typically, a is taken to be purely real and therefore equal to the slope parameter of the NN elastic scattering differential cross section $d\sigma/dt$. However, we will allow a to be complex, writing

$$a=\beta+i\gamma , \tag{2.5}$$

thereby introducing a simple phase variation of the NN amplitude, linear in q^2 (or in $t=-\hbar^2q^2$). The parameters σ, ρ, and β can all be obtained from NN scattering measurements. The parameter γ leads simply to an overall phase factor $e^{-i\gamma q^2/2}$ which cannot be obtained directly from NN scattering measurements. It will be treated as a free NN parameter; it will be fixed at any given velocity of the incident nucleus and hence will be independent of the nuclei involved in the collision, provided that the kinetic energies per nucleon are the same in all cases. Thus the *same* value of γ will be used in describing all nucleus-nucleus measurements at a given kinetic energy per nucleon.

One of us (V.F.) used such a phase variation many years ago to describe hadron-deuteron scattering[8] and also suggested this phase variation for quark-quark amplitudes in early hadron-hadron multiple-scattering analyses using quark models.[9] This parametrization does not affect the NN differential cross section and β is still the measured NN slope parameter.

With the NN scattering amplitude given by Eq. (2.4), the NN profile function of Eq. (2.2) can be written as

$$\Gamma_{ij}(\mathbf{b})=ge^{-b^2/2a} , \tag{2.6}$$

with

$$g=\frac{\sigma}{4\pi a}(1-i\rho) . \tag{2.7}$$

A. Elastic scattering amplitudes for collisions of projectiles and targets with mass numbers greater than 2 ($A,B>2$)

For nuclei with mass numbers greater than 2, we shall, for simplicity, consider ground state wave functions satisfying

$$|\psi_A(\{\mathbf{s}_i\})|^2=N_A\delta\left[\frac{1}{A}\sum_{i=1}^{A}\mathbf{s}_i\right]\prod_{i=1}^{A}e^{-\alpha_A^2 s_i^2}, \qquad (2.8)$$

with a corresponding expression for $|\psi_B(\{\mathbf{s}_i'\})|^2$. Here, N_A is the normalization factor

$$N_A=\frac{1}{A}\left[\frac{\alpha_A^2}{\pi}\right]^{A-1}, \qquad (2.8a)$$

with a similar expression for N_B. Thus Eq. (2.3) for the elastic scattering amplitude $F_{AB}(\mathbf{q})$ becomes

$$F_{AB}(\mathbf{q})=N_AN_B\int\{d\mathbf{s}_i\}\{d\mathbf{s}_j'\}\delta\left[\frac{1}{A}\sum_i\mathbf{s}_i\right]\delta\left[\frac{1}{B}\sum_j\mathbf{s}_j'\right]\exp\left[-\alpha_A^2\sum_i s_i^2-\alpha_B^2\sum_j s_j'^2\right]F(\mathbf{q};\{\mathbf{s}_i\},\{\mathbf{s}_j'\}). \qquad (2.9)$$

The effect of the δ functions is to refer the coordinates $\{\mathbf{s}_i\}$ and $\{\mathbf{s}_j'\}$ to the nuclear centers of mass. Consequently, these coordinates are internal coordinates. (Note that the longitudinal, or z, coordinates have been integrated over and F is independent of these coordinates.)

To evaluate the multi-dimensional [$2(A+B)$-dimensional] integral in Eq. (2.9), it is useful to first evaluate the following auxiliary integral:

$$\mathscr{F}_{AB}(\mathbf{q})\equiv\mathscr{N}_{AB}\int\{d\boldsymbol{\mathscr{s}}_i\}\{d\boldsymbol{\mathscr{s}}_j'\}\exp\left[-\alpha_A^2\sum_i \mathscr{s}_i^2-\alpha_B^2\sum_j \mathscr{s}_j'^2\right]F(\mathbf{q};\{\boldsymbol{\mathscr{s}}_i\},\{\boldsymbol{\mathscr{s}}_j'\}), \qquad (2.10)$$

where

$$\mathscr{N}_{AB}=\left[\frac{\alpha_A^2}{\pi}\right]^A\left[\frac{\alpha_B^2}{\pi}\right]^B. \qquad (2.11)$$

The integral $\mathscr{F}_{AB}$ differs from F_{AB} only in the normalization constant and in the absence of the δ functions. But this absence is significant, for now the coordinates $\{\boldsymbol{\mathscr{s}}_i\},\{\boldsymbol{\mathscr{s}}_j'\}$ are no longer internal coordinates referred to the nuclear centers of mass. They are, in fact, independent. But Eq. (2.10) is easier to evaluate directly than Eq. (2.9). Furthermore, the two sets of coordinates $\{\mathbf{s}_i\},\{\mathbf{s}_j'\}$ and $\{\boldsymbol{\mathscr{s}}_i\},\{\boldsymbol{\mathscr{s}}_j'\}$ are related by

$$\mathbf{s}_i=\boldsymbol{\mathscr{s}}_i-\mathbf{R}_A,\quad \mathbf{s}_j'=\boldsymbol{\mathscr{s}}_j'-\mathbf{R}_B, \qquad (2.12)$$

where

$$\mathbf{R}_A=\frac{1}{A}\sum_i\boldsymbol{\mathscr{s}}_i,\quad \mathbf{R}_B=\frac{1}{B}\sum_j\boldsymbol{\mathscr{s}}_j'. \qquad (2.13)$$

It then follows that

$$F(\mathbf{q};\{\boldsymbol{\mathscr{s}}_i\},\{\boldsymbol{\mathscr{s}}_j'\})=e^{-i\mathbf{q}\cdot\mathbf{R}_A+i\mathbf{q}\cdot\mathbf{R}_B}F(\mathbf{q};\{\mathbf{s}_i\},\{\mathbf{s}_j'\}). \qquad (2.14)$$

With the transformation of variables given by Eq. (2.12), we obtain, for $\mathscr{F}_{AB}(\mathbf{q})$, the expression

$$\mathscr{F}_{AB}(\mathbf{q})=\mathscr{N}_{AB}\int d\mathbf{R}_Ad\mathbf{R}_B\exp(-A\alpha_A^2R_A^2-i\mathbf{q}\cdot\mathbf{R}_A-B\alpha_B^2R_B^2+i\mathbf{q}\cdot\mathbf{R}_B)$$
$$\times\int\{d\mathbf{s}_i\}\{d\mathbf{s}_j'\}\delta\left[\frac{1}{A}\sum_i\mathbf{s}_i\right]\delta\left[\frac{1}{B}\sum_j\mathbf{s}_j'\right]$$
$$\times F(\mathbf{q},\{\mathbf{s}_i\},\{\mathbf{s}_j'\})\exp\left[-\alpha_A^2\sum_i s_i^2-\alpha_B^2\sum_j s_j'^2\right] \qquad (2.15)$$

$$=N_AN_B\exp\left[-q^2\left[\frac{1}{4A\alpha_A^2}+\frac{1}{4B\alpha_B^2}\right]\right]\int\{d\mathbf{s}_i\}\{d\mathbf{s}_j'\}\delta\left[\frac{1}{A}\sum_i\mathbf{s}_i\right]\delta\left[\frac{1}{B}\sum_j\mathbf{s}_j'\right]$$
$$\times\exp\left[-\alpha_A^2\sum_i s_i^2-\alpha_B^2\sum_j s_j'^2\right]F(\mathbf{q};\{\mathbf{s}_i\},\{\mathbf{s}_j'\}) \qquad (2.16)$$

$$=\exp\left[-q^2\left[\frac{1}{4A\alpha_A^2}+\frac{1}{4B\alpha_B^2}\right]\right]F_{AB}(\mathbf{q}). \qquad (2.17)$$

Therefore, we obtain the result

$$F_{AB}(\mathbf{q})=\exp\left[q^2\left[\frac{1}{4A\alpha_A^2}+\frac{1}{4B\alpha_B^2}\right]\right]\mathcal{F}_{AB}(\mathbf{q})\ , \qquad (2.18)$$

which allows us to obtain the scattering amplitude F_{AB} by evaluating $\mathcal{F}_{AB}$ instead.

The complete expansion of F_{AB} (or $\mathcal{F}_{AB}$) is exceedingly lengthy in general, there being $2^{AB}-1$ terms. Of course, not all terms are different, so the classification of the various terms becomes important. These terms can be grouped into "orbits." An orbit is a set of terms in which each term gives an equal contribution to the scattering amplitude. It characterizes the particular corresponding multiple collision. The "length" of an orbit is the number of terms contained in that orbit. With the aid of permutation group theory these orbits and their lengths can be found and the scattering amplitude can be obtained analytically.[10]

Each orbit belongs to a particular order of multiple scattering, μ, which ranges from 1 to AB. Let λ_μ be an index used to number the orbits of a given order μ. Then the orbits will be denoted by the order μ and the index λ_μ, with the notation (μ,λ_μ). The length of an orbit (i.e., the number of expansion terms in the orbit) will be denoted by $T(\mu,\lambda_\mu)$. The elastic scattering amplitude F_{AB} is then given by[10]

$$F_{AB}(\mathbf{q})=-\tfrac{1}{2}ik\exp\left[q^2\left[\frac{1}{4A\alpha_A^2}+\frac{1}{4B\alpha_B^2}\right]\right]\mathcal{N}_{AB}\sum_{\mu=1}^{AB}\sum_{\lambda_\mu}(-g)^\mu T(\mu,\lambda_\mu)\frac{A[\Delta(\mu,\lambda_\mu)]}{S[\Delta(\mu,\lambda_\mu)]}\exp\{-q^2/4S[\Delta(\mu,\lambda_\mu)]\}\ , \qquad (2.19)$$

where A, S, and $\Delta(\mu,\lambda_\mu)$ are now to be described. Each orbit (μ,λ_μ) is represented by an $A\times B$ Δ matrix, $\Delta(\mu,\lambda_\mu)$, whose ijth element is either 0 or 1 depending on whether or not Γ_{ij} appears in the expansion term corresponding to the orbit (μ,λ_μ). [In fact, there are $T(\mu,\lambda_\mu)$ such Δ matrices for each orbit, all related by the permutation operators of $S_A\otimes S_B$. Their contributions to the scattering amplitudes are all equal.] The ith nucleon of the incident nucleus collides with the jth nucleon of the target nucleus if $\Delta_{ij}=1$; if $\Delta_{ij}=0$, there is no such collision. To obtain A and S, we make the following definitions:

$$T_i\equiv\left[2a\left[2a\alpha_A^2+\sum_{j=1}^{B}\Delta_{ij}\right]\right]^{-1},\quad c_j(B)\equiv 2\alpha_B^2 \qquad (2.20a)$$

$$a_{jj}(B)\equiv\alpha_B^2+\sum_{i=1}^{A}\Delta_{ij}/2a-\sum_{i=1}^{A}T_i\Delta_{ij}\ , \qquad (2.20b)$$

$$a_{jk}(B)\equiv\sum_{i=1}^{A}2T_i\Delta_{ij}\Delta_{ik}\quad (j\neq k)\ . \qquad (2.20c)$$

Then,

$$A[\Delta]=\prod_{i=1}^{A}(4\pi a^2T_i)\prod_{j=1}^{B}\left[\frac{\pi}{a_{jj}(j)}\right]\ , \qquad (2.21a)$$

$$S[\Delta]=B\alpha_B^2-\sum_{l=1}^{B}\frac{c_l^2(l)}{4a_{ll}(l)}\ , \qquad (2.21b)$$

where the a_{jk}'s and c_j's are obtained by the recurrence relations

$$a_{jk}(l-1)=a_{jk}(l)+(\tfrac{1}{2}-\tfrac{3}{4}\delta_{jk})a_{jl}(l)a_{kl}(l)/a_{ll}(l)\ , \qquad (2.22a)$$

$$c_j(l-1)=c_j(l)+a_{jl}(l)c_l(l)/2a_{ll}(l)\ . \qquad (2.22b)$$

In the Appendix we present the orbits, lengths, and Δ matrices for the cases $(A=4, B=3)$ and $(A=4, B=4)$, corresponding, for example, to α^3He and $\alpha\alpha$ scattering.

B. Elastic scattering amplitude for nucleus-proton and nucleus-deuteron collisions

For the case $B=1$ the procedure described in Sec. II leads to the well-known[11] result

$$F_{A\mathrm{p}}(\mathbf{q})=-ike^{q^2/4A\alpha_A^2}\frac{1}{2W}\sum_{\mu=1}^{A}\frac{1}{\mu}\begin{pmatrix}A\\ \mu\end{pmatrix}(-G)^\mu e^{-q^2/4\mu W}\ , \qquad (2.23)$$

where

$$G=\frac{2\alpha_A^2a}{2\alpha_A^2a+1}g,\quad W=\frac{\alpha_A^2}{2\alpha_A^2a+1}\ . \qquad (2.24)$$

The amplitude contains A terms, each corresponding to a particular order of multiple collision.

For the case $B=2$ (the deuteron) we shall use a wave function that leads to a form factor given by a sum of Gaussians,

$$S_{\mathrm{d}}(q)=\sum_m\alpha_m e^{-\beta_m q^2}\ . \qquad (2.25)$$

The scattering amplitude for deuteron-nucleus elastic scattering can be written in a somewhat different way,[12]

$$F_{\mathrm{d}A}(\mathbf{q})=\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\langle\psi_{\mathrm{d}}\psi_A\,|\,\Gamma_{\mathrm{d}A}(\mathbf{b};\mathbf{s};\{\mathbf{s}_i\})\,|\,\psi_{\mathrm{d}}\psi_A\rangle\ , \qquad (2.26)$$

where

$$\Gamma_{\mathrm{d}A}(\mathbf{b};\mathbf{s};\{\mathbf{s}_i\})=1-\prod_{j=1}^{A}[1-\Gamma(\mathbf{b}+\tfrac{1}{2}\mathbf{s}-\mathbf{s}_j)-\Gamma(\mathbf{b}-\tfrac{1}{2}\mathbf{s}-\mathbf{s}_j)+\Gamma(\mathbf{b}+\tfrac{1}{2}\mathbf{s}-\mathbf{s}_j)\Gamma(\mathbf{b}-\tfrac{1}{2}\mathbf{s}-\mathbf{s}_j)]\ . \qquad (2.27)$$

After substituting Eq. (2.27) into Eq. (2.26) for $F_{\mathrm{d}A}$, we obtain

$$F_{dA}(\mathbf{q})=-e^{q^2/4A\alpha_A^2}\frac{ik}{2}\sum_{i=0}^{A-1}\sum_{j=0}^{A-i}\sum_{k=0}^{A-i-j}\frac{A!g_1^{j+k}g_2^{A-i-j-k}}{i!j!k!(A-i-j-k)!}\times\sum_m\frac{\alpha_m}{h_1+4\beta_m h_1h_2-\beta_m h_3^2}\exp\left[-q^2\frac{1+4\beta_m h_2}{4(h_1+4\beta_m h_1h_2-\beta_m h_3^2)}\right], \quad (2.28)$$

where

$$g_1=-\frac{\sigma(1-i\rho)}{2\pi(2a+R^2)},\quad g_2=\frac{[\sigma(1-i\rho)]^2}{16\pi^2a(a+R^2)},\quad R^2=\frac{1}{\alpha_A^2}, \quad (2.29a)$$

$$h_1(i,j,k)=(A-i)t+(A-i-j-k)(1-R^2t)/R_r, \quad (2.29b)$$

$$h_2(i,j,k)=\tfrac{1}{4}[(A-i)t+(A-i-j-k)(1+R^2t)/R_r], \quad (2.29c)$$

$$h_3(i,j,k)=-(A-i-2j)t+(A-i-j-k)(1-R^4t^2)/R_r, \quad (2.29d)$$

with

$$t=\frac{1}{2a+R^2},\quad R_r=2a+R^2-R^4t. \quad (2.30)$$

III. FEATURES OF DIFFERENTIAL CROSS SECTIONS

What are some features that can be expected for the differential cross section? The simplest case, of course, is $B=1$, corresponding to nucleus-hydrogen scattering or, equivalently, hadron-nucleus scattering. For real a ($\gamma=0$) the features are well known, although explanations given are not all quite correct. As can be seen from Eq. (2.23), there are A orders of multiple scattering, each with only one orbit (indicating there is only one type of μth-order scattering, the solitary nucleon colliding with μ nuclear nucleons). Typically, in the forward direction the magnitudes of the multiple-scattering terms decrease rapidly with increasing order (μ), but the rates of decrease of the magnitudes, with respect to q^2, decrease with increasing order. Since typically $|\rho|\ll 1$, successive amplitudes are almost 180° out of phase with each other and these phase differences between successive amplitudes are equal and constant (independent of q). These conditions are conducive to strong destructive interference and lead to a diffraction pattern exhibiting many minima, $A-1$ minima when $\rho\approx 0$. The standard lore has been that the appearance of the nth minimum is the results of the destructive interference between the $\mu=n$ and the $\mu=n+1$ orbits, i.e., between the multiple scattering amplitudes of orders n and $n+1$. This not quite the case when $A\gg 1$ and typical NN parameters are used. For example, for $A\approx 16$ and typical medium energy NN values for σ, a, and ρ, in general, near the minima, there are *a number* of important amplitudes. Near the first minimum, for example, four amplitudes are comparable, with the double- and triple-scattering amplitudes being the largest, and the single- and quadruple-scattering amplitudes being important as well. Only at extremely large momentum transfers [typically, $-t\gtrsim 30$ (GeV/c)^2] are the number of important amplitudes as few as 2, for in this region the highest-order (A) amplitude dominates and all amplitudes of order lower than $A-1$ are comparatively small.

What would be the effect of allowing a in Eq. (2.4) to have an imaginary part? The NN amplitude would then have a q-dependent phase. This leads to a *smaller modulus* for all orders of scattering amplitudes, except for the single-scattering amplitude whose modulus is unaffected. The reason for this is that the μth-order multiple scattering amplitude ($\mu\geq 2$) essentially arises from $\mu-1$ convolutions of NN amplitudes. If the phases of these amplitudes vary, the convolutions will be smaller. Furthermore, since the higher order multiple scattering amplitudes arise from more convolutions of NN amplitudes, the decrease of the moduli due to the phase variation will be larger the greater the order (μ) of multiple scattering. This is easily seen explicitly in the special case $B=1$ where, from Eq. (2.23), we observe that the μth order amplitude will now be multiplied by an additional factor whose modulus is

$$\{1+[\gamma/(\beta+\tfrac{1}{2}\alpha_A^{-2})]^2\}^{-(\mu-1)/2},$$

which is less than unity for $\mu\geq 2$ and decreases with increasing μ.

In addition, the resulting phase for the μth order scattering amplitude itself will now vary as $\exp(-i\gamma q^2/2\mu)$. Consequently, successive amplitudes will no longer be so nearly 180° out of phase with each other. Furthermore, the phase differences between successive amplitudes will no longer be equal or constant (independent of q). These conditions are not so conducive to the extremely strong destructive interference necessary to produce minima.

The two effects of the phase variation of the NN amplitudes, namely the smaller moduli of the individual multiple scattering amplitudes and changes in their phases, are essentially competing effects. In the region where interference is important, i.e., where several amplitudes are comparable, the latter effect is more significant. This is, by far, the most interesting and experimentally accessible region. In this region, the sharp minima which occur when there is no phase variation now become less sharp or vanish completely, often leaving, at most, only shoulders as vestiges of the original minima at the larger momentum transfers. In addition, since the strong destructive interference that appeared throughout this region when γ was zero is now greatly weakened, the cross section is larger throughout this region (even though the *individual* multiple scattering amplitudes are *smaller* in modulus). It is only at *extremely large* momentum transfers, past the

$(A-1)$th order minimum, where only the Ath order amplitude is important, that the decrease in the moduli of the multiple scattering amplitudes is more significant, since there the first $A-1$ amplitudes are too small to interfere with the Ath order amplitude. In that region the intensity is now *smaller* than for the case $\gamma=0$. From a practical point of view, however, in that region the momentum transfer may be unphysically large or the cross sections may be too small to be measured.

The situation for nucleus-nucleus $(A,B\geq 2)$ scattering is significantly more complex, even for light nuclei. There are very many orbits, in general, many more than the highest order of multiple collision, $\mu=AB$. (For $\alpha\alpha$ collisions, for example, $AB=16$, *but there are 191 orbits.*) There are new kinds of multiple collisions, not present in hadron-nucleus scattering. For example, a double collision can now involve two nucleons in nucleus A and two nucleons in nucleus B, each nucleon in A interacting with a single different nucleon in B. This is quite different from a double collision in which, for example, one nucleon in A interacts with two nucleons in B, as would occur in hadron-nucleus collisions (with $A=1$). These new kinds of multiple collisions were first described for deuteron-deuteron collisions[5,13] and for π-π scattering in the quark model.[14] In general, they and their generalizations when $A,B>2$ are much more important than the original type of multiple collisions, and they lead to differential cross sections with much different angular dependence, much greater intensities, and much slower rates of decrease with increasing q.

As a consequence of the resulting slower decrease of the individual multiple scattering amplitudes, very many of these multiple scattering amplitudes will be important at momentum transfers away from the forward direction, and for *each* multiple scattering amplitude several orbits can be important. If typical NN scattering amplitudes are used, such a plethora of significant orbits leads to nucleus-nucleus angular distributions with *only a few minima,* as it now becomes more difficult to achieve the extremely delicate cancellations among the complex amplitudes that are necessary for the appearance of the minima. (That is *not* to say that there is *not any* destructive interference. There is, but it is not complete enough to result in minima.) However, if the NN slope parameter (β) used in the calculation is larger (as it, in fact, is at very high energies), then the NN amplitude decreases more rapidly with q^2, and this leads to a more rapid decrease for each orbit. As a result, at any given momentum transfer there will not be as many important orbits (contributions) as when the slope parameter was smaller. Consequently, the possibility of complete destructive interference will be greatly enhanced and many more minima will appear. For example, in $\alpha\alpha$ scattering with $\sigma=44$ mb and $\rho=\gamma=0$, we find that with a slope parameter $\beta=1$ $(\mathrm{GeV}/c)^{-2}$ (which is much smaller than the slope in typical NN measurements), the calculated differential cross section $d\sigma/dt$ exhibits *only one* minimum [occurring between $t=0$ and -0.1 $(\mathrm{GeV}/c)^2$]. But for $2\lesssim\beta\lesssim 10$ $(\mathrm{GeV}/c)^{-2}$ it exhibits three mimima [occurring at $-t\lesssim 1.2$ $(\mathrm{GeV}/c)^2$], and for $\beta=15$ $(\mathrm{GeV}/c)^{-2}$ or 20 $(\mathrm{GeV}/c)^{-2}$ it exhibits nine minima [occurring at $0<-t\lesssim 22$ $(\mathrm{GeV}/c)^2$]. In fact, if β is made large enough [$\beta\gtrsim 70$ $(\mathrm{GeV}/c)^{-2}$], fifteen (i.e., $AB-1$) minima appear [occurring at $0<-t\lesssim 23$ $(\mathrm{GeV}/c)^2$]. Thus the number of minima in nucleus-nucleus elastic scattering depends significantly on the NN slope parameter, whereas the number of minima in hadron-nucleus elastic scattering does not.

We may ask why there are no more than $AB-1$ (e.g., 15 for $\alpha\alpha$ collisions) minima, since there are so many more (e.g., 191 for $\alpha\alpha$ collisions) orbits, each with its own modulus and phase. The main reason is that when β (the slope parameter) is relatively small, there are many important orbits and complete destructive interference is difficult to achieve, so that there are even fewer than $AB-1$ minima. But when β is large, the moduli of the various orbit terms decrease so rapidly that, away from the forward direction, for each order of multiple scattering only *one or two orbits* are significant and we have a situation akin to *hadron*-nucleus scattering.

Of course, at presently accessible energies, the NN amplitudes do not increase *so* rapidly. In addition, if they did the nucleus-nucleus cross sections would be too small to measure past the first few minima. And also, the momentum transfer for some of the higher order minima would be unphysically large if the incident energies were not very high.

Our analysis with $\gamma=0$ indicates that at medium energies [where $2\lesssim\beta\lesssim 10$ $(\mathrm{GeV}/c)^{-2}$], the differential cross section for $\alpha\alpha$ collisions will exhibit *no more than* three minima for $-t\lesssim 5$ $(\mathrm{GeV}/c)^2$, all occurring at $-t\lesssim 1$–2 $(\mathrm{GeV}/c)^2$.

What happens when we introduce a phase variation $(\gamma\neq 0)$. To some extent, the effects are similar to the case $B=1$ (hadron-nucleus collisions). The minima become shallower or barely visible or only a shoulder is present or there is only a very small or even no vestige of some of the higher-order minima. However, because of the greatly increased complexity of the phase relations among the various orbits, it is possible for a new kind of minimum at much larger momentum transfers, say $-t\gtrsim 5$ $(\mathrm{GeV}/c)^2$, to appear. In addition, the absence or weakening of the destructive interference when $\gamma\neq 0$ again leads to larger cross sections. These increases in the cross section *can be extremely large,* at times as much as an order of magnitude. Eventually, however, the decrease of the individual multiple scattering contributions due to $\gamma\neq 0$ becomes the dominant effect and at very large momentum transfers the cross section becomes smaller than for the case $\gamma=0$. This affect is *much greater* than for hadron-nucleus collisions $(B=1)$ and *can be very large,* capable of decreasing the cross section *by orders of magnitude* at extremely large momentum transfers. Again, however, from a practical point of view, in that region the momentum transfers may be unphysically large or the cross sections may be too small to be measured.

IV. APPLICATIONS

In this section we present our results for α^4He, α^3He, α^2H, and α^1H elastic scattering for an incident α particle momentum of 7 GeV/c and for α^4He elastic scattering at

TABLE I. Compilation of nucleon-nucleon parameters used in this analysis.

P_{lab} (GeV/c)	σ (mb)	Refs.	β (GeV/c)$^{-2}$	Refs.	ρ	Refs.
1.75	44.0	15	5.6	16, 17	-0.23	18, 19
1.27	39.9	15, 20–24	2.92	25–28	-0.20	18, 19
1.08	32.3	15, 20, 23, 24, 29	1.86	25, 27, 28	-0.02	19, 30, 31

incident momenta of 4.32 and 5.07 GeV/c. The NN input data are taken to be the mean values of the corresponding pp and pn data and are given in Table I along with the references[15–31] from which they were obtained. For the deuteron ground state we have used a three-term form factor given by Eq. (2.25) with the parameters given by[32] $(\alpha_1,\alpha_2,\alpha_3)=(0.34,0.58,0.08)$ and $(\beta_1,\beta_2,\beta_3)=(141.5,26.1,15.5)$ (GeV/c)$^{-2}$, with q in GeV/c. For the ground states of ^{3}He and ^{4}He we use Eq. (2.8) with $\alpha_3^2=0.3261$ fm^{-2} and $\alpha_4^2=0.5541$ fm^{-2}, which correspond to ^{3}He and ^{4}He rms radii of[33] 1.960 and 1.675 fm after corrections for center-of-mass recoil and finite-proton-size effects (we have used[34] $\langle r_p^2\rangle=0.7754$ fm^2).

In Fig. 1 we compare our results for α^4He, α^3He, α^2H, and α^1H elastic scattering at 7 GeV/c with the data of Satta *et al.*[2] The results with no phase variation ($\gamma=0$, dashed curves) exhibit three (two for α^1H scattering) rather well-defined minima. The data, however, do not seem to indicate that many minima. Furthermore, the data points generally are much higher than the theoretical results, often by an order of magnitude or more. When a phase variation is introduced ($\gamma\neq0$, solid and dotted curves), the minima became shallower or completely disappear and the cross sections are substantially increased. If ρ were identically zero, replacing γ by $-\gamma$ would not alter the cross section. For small nonzero values of ρ, two values of γ, one greater than ρ and one less than ρ, lead to somewhat similar angular distributions. We see that the values $\gamma=10$ and -15 (GeV/c)$^{-2}$ lead to cross sections which are very much improved compared with the case, $\gamma=0$, of no phase variation.[35] We see the two fold effect of the phase variation, namely making the minima shallow or eliminating them and significantly increasing the cross section, sometimes by more than an order of magnitude, even far from any minima.

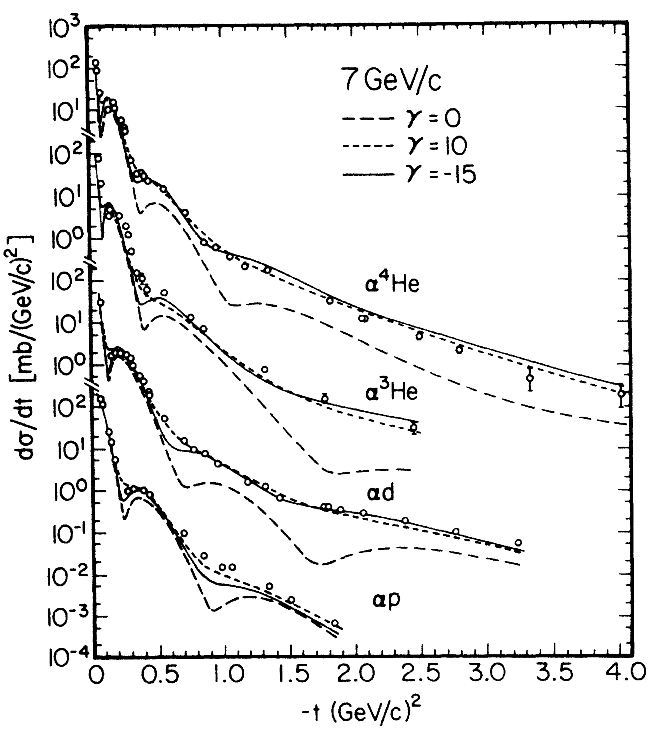

FIG. 1. Differential cross sections for elastic scattering of α particles by ^{4}He, ^{3}He, ^{2}H, and ^{1}H, at 7 GeV/c [data of Satta *et al.* (Ref. 2)]. The dashed curve is the constant phase result ($\gamma=0$). The dotted and solid curves are calculated with phase variations in the NN amplitude [$\gamma=10$ and -15 (GeV/c)$^{-2}$, respectively, in Eq. (2.5)].

In discussing the minima in αd scattering one may ask whether the deuteron D state is important, as it is in the case of pd scattering.[36] In pd scattering, the minimum (or shoulder) occurs near $-t\approx0.32$ (GeV/c)^2, where single and double scattering amplitudes are comparable. The single scattering amplitude depends on the form factors at $-t/4\approx0.08$ (GeV/c)^2. There the quadrupole form factor S_2 is comparable to the isotropic component S_0 (to within about a factor of 2), and so the D state is important in the region of the minimum. In the case of αd scattering, the first minimum (or shoulder) occurs near $-t\approx0.13$ (GeV/c)^2, so $-t/4\approx0.03$ (GeV/c)^2. At such a small value of $-t$ the quadrupole form factor S_2 is much smaller than S_0 and hence has no effect on the minimum. At the other minima (or shoulders) in αd collisions, the single scattering amplitude is negligibly small and hence the deuteron D state is of no significance.

In Fig. 2 we compare our results for $\alpha\alpha$ elastic scattering at 4.32 and 5.07 GeV/c with the data of Berger *et al.*[4] Again, the phase variation leads to very much improved results. At 5.07 GeV/c the values of γ are 7.5 and -13.0 (GeV/c)$^{-2}$ and at 4.32 GeV/c the values are 11.5 and -12.0 (GeV/c)$^{-2}$. The negative values are not very different from what would be obtained from phase shift analysis. For example, the spin- and isospin-averaged NN amplitude at 1.11 GeV/c (corresponding to an α-particle momentum of 4.44 GeV/c) yields[37] for γ a value of -13.2 (GeV/c)$^{-2}$.

We have not, in any quantitative manner, addressed the question of what effect intermediate Δ isobar states have on the differential cross sections, particularly near the minima or shoulders. Calculations exist for nucleon-nucleus collisions. Unfortunately, different parameterizations for the input needed in such calculations lead to rather different results.[38] At very high energies the effect is sometimes significant[38] and sometimes very small.[39] At

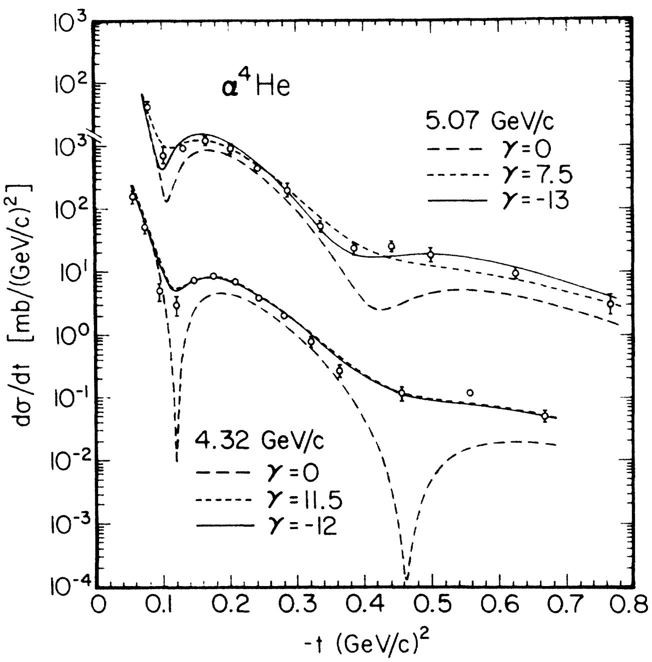

FIG. 2. Differential cross sections for elastic scattering of α particles by ^{4}He, at 5.07, and 4.32 GeV/c [data of Berger *et al.* (Ref. 4)]. The dashed curve is the constant phase result ($\gamma=0$). The dotted anbd solid curves are calculated with phase variations in the NN amplitude [$\gamma=7.5$ and -13 (GeV/c)$^{-2}$ at 5.07 GeV/c and $\gamma=11.5$ and -12 (GeV/c)$^{-2}$ at 4.32 GeV/c].

lower energies, such as for 0.8–1.0 GeV incident nucleons, the effect on the differential cross section appears to be quite small,[40] except near deep minima, which become somewhat less deep. Such an effect would improve our αp results for negative γ (see our Fig. 1), which are presently too low at the first minimum and at the next shoulder.

The question of intermediate Δ isobar states in αd, α^3He, and α^4He (and, in general, AB) collisions is more complex. The new kinds of multiple collisions possible in nucleus-nucleus collisions, described earlier, *do not allow contributions from inelastic intermediate states* since no nucleon is involved more than once in such collisions. Since these kinds of multiple collisions are often more important than the original type of multiple collisions, it is perhaps not unlikely that the effects of Δ isobar states are not so significant in nucleus-nucleus elastic scattering as they appear to be in nucleon-nucleus elastic scattering. (As an example of the relative importance of the new kind of multiple collisions, in $\alpha\alpha$ scattering at 7 GeV/c, near the first minimum [$-t\approx 0.1$ (GeV/c)^2], the modulus of the amplitude for the new kind of double collision is 50% larger than that for the original kind of double collision.) In addition, the first minimum in nucleus-nucleus collisions will occur at a smaller momentum transfer than in nucleon-nucleus collisions. This will tend to suppress the excitation to inelastic intermediate states. [For example, the first minimum in αp collisions occurs near $-t\approx 0.24$ (GeV/c)^2, whereas in α^3He and α^4He collisions they occur near $-t\approx 0.10$ (GeV/c)^2 and in αd collisions it occurs near $-t\approx 0.13$ (GeV/c)^2.] Nevertheless, the effect of inelastic intermediate states in nucleus-nucleus collisions is an open question and a reliable study of this question is needed.

Although the dependence of the phase of the NN amplitude on q^2 may not, in fact, be as simple as that assumed (i.e., linear) in Eqs. (2.4) and (2.5), the marked improvement in these light-ion results strongly indicates the presence of some significant phase variation. Furthermore, while it is also true that the NN phase we obtain varies by a few cycles between the forward direction and $-t\approx 4$ (GeV/c)^2, the modulus of the NN amplitude given by Eq. (2.4) varies by as much as 5 orders of magnitude over the same range of momentum transfer. With such a large modulus variation it is not unreasonable to expect a large phase variation. In addition, at values of t where nth-order multiple scattering dominates, the intensity often depends on the NN amplitude at t/n^2. For example, in αp collisions near $-t=1.8$ (GeV/c)^2 the dominant amplitude is the quadruple-scattering amplitude which depends mainly on the NN amplitudes at the much smaller value $-t\approx 0.1$ (GeV/c)^2 [where the phase has changed by less than $\frac{1}{2}\pi$ for $\gamma=-15$ (GeV/c)$^{-2}$].

We have seen that by allowing the NN elastic scattering amplitude to have a phase which varies with momentum transfer, the resulting calculated differential cross sections for elastic scattering of α particles by light nuclei at medium energies are markedly improved. Furthermore, even if there is *no* phase variation the large number ($A-1$) of minima possible in *hadron*-nucleus elastic scattering is not generalized to $AB-1$ in *nucleus*-nucleus elastic scattering at medium energies, even though the nucleus-nucleus amplitude contains AB multiple scattering amplitudes. For example, with typical NN values for σ and β we find only three (not 15) minima for $\alpha\alpha$ scattering (and even fewer when $\gamma\neq 0$; however, when β becomes very large, the number of minima does increase to $AB-1$).

The effect of the phase variation is to eliminate minima or to make them shallower and to generally increase cross sections significantly even at momentum transfers where no minima originally occurred. At extremely large momentum transfers, physically realizable only at very high energies, the phase variation can cause the cross sections to decrease much below their values with no phase variation.

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APPENDIX

In Table II we present the orbits, lengths, and Δ matrices for $A=4$, $B=3$. In the first two columns we give μ (the order of multiple scattering) and λ_μ (the index used to number the orbit of order μ). The third column is $T(\mu,\lambda_\mu)$, the length of the orbit. The 12-digit binary number in the fourth column gives the Δ matrix, the first

TABLE II. Orbits, lengths, and Δ matrices for $A=4$, $B=3$ scattering.

μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$	μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$
1	1	12	100000000000	5	6	12	011110001000
2	1	18	110000000000	5	7	36	111010001000
2	2	12	100010000000	5	8	144	110010100100
2	3	36	100001000000	5	9	36	110011000010
3	1	12	111000000000	5	10	36	110011001000
3	2	72	110000100000	5	11	72	110010101000
3	3	72	110010000000	5	12	72	110010100001
3	4	4	100010001000	5	13	72	110010000011
3	5	24	100001000010	6	1	36	111111000000
3	6	36	100010000100	6	2	36	111011010000
4	1	3	111100000000	6	3	12	111011100000
4	2	24	111000010000	6	4	72	111011000010
4	3	72	111010000000	6	5	144	111010010100
4	4	18	110000110000	6	6	72	111011000001
4	5	18	110011000000	6	7	72	111010010001
4	6	72	110010100000	6	8	144	111011001000
4	7	36	110010000100	6	9	24	110010100110
4	8	36	110010001000	6	10	72	110011001010
4	9	36	110000100001	6	11	72	110010100101
4	10	144	110010000010	6	12	18	110011000011
4	11	36	011010001000	6	13	6	110011001100
5	1	24	111110000000	6	14	24	110010101001
5	2	72	111010010000	6	15	12	111110001000
5	3	72	111011000000	6	16	36	111110000100
5	4	72	111010000100	6	17	72	111010011000
5	5	72	111010000001				

TABLE III. Orbits, lengths, and Δ matrices for $A=B=4$ scattering.

μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$	μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$	μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$
1	1	16	1000000000000000	6	8	576	1110110000010000	7	22	576	1110100110000100
2	1	48	1100000000000000	6	9	1152	1110100101000000	7	23	144	0111110010001000
2	2	72	1000001000000000	6	10	576	1110100110000000	7	24	144	1100110000110010
3	1	32	1110000000000000	6	11	576	1110100100010000	7	25	576	1100101001010010
3	2	144	1100100000000000	6	12	144	1110100010000001	7	26	96	1100101001100001
3	3	288	1100001000000000	6	13	192	1110100001000010	8	1	12	1111111100000000
3	4	96	1000010000100000	6	14	576	1110100001000001	8	2	576	1111111010000000
4	1	8	1111000000000000	6	15	288	1110100000010001	8	3	192	1111111000010000
4	2	288	1110100000000000	6	16	16	0111100010001000	8	4	144	1111110011000000
4	3	36	1100110000000000	6	17	144	1100110000100010	8	5	144	1111110000110000
4	4	96	1110000100000000	6	18	72	1100110000100001	8	6	576	1111110010100000
4	5	288	1100101000000000	6	19	576	1100101001000010	8	7	288	1111110010000100
4	6	72	1100001100000000	6	20	576	1100101001000001	8	8	144	1111110010001000
4	7	576	1100100000100000	6	21	288	1100101000010001	8	9	1152	1111110010000010
4	8	144	1100001000100000	6	22	288	1100100000110010	8	10	288	1111110000100001
4	9	288	1100001000010000	6	23	96	1100101001100000	8	11	288	1111110000100010
4	10	24	1000010000100001	7	1	96	1111111000000000	8	12	144	1110111011000000
5	1	96	1111100000000000	7	2	576	1111110000100000	8	13	288	1110110111000000
5	2	288	1110110000000000	7	3	576	1111110010000000	8	14	1152	1110110110010000
5	3	288	1110100100000000	7	4	192	1111100001000010	8	15	288	1110110100110000
5	4	144	1110100010000000	7	5	16	1111100010001000	8	16	288	1110111010010000
5	5	576	1110001000010000	7	6	288	1111100010000100	8	17	288	1110111010000001
5	6	576	1110100001000000	7	7	288	1110111010000000	8	18	48	1110111000010001
5	7	96	1110000100010000	7	8	576	1110110110000000	8	19	144	1110110110000100
5	8	144	1100110000100000	7	9	576	1110110100100000	8	20	288	1000111001110001
5	9	576	1100011000100000	7	10	576	1110100010010001	8	21	1152	1110110110000010
5	10	576	1100011000010000	7	11	96	1110111000010000	8	22	288	1110110100010001
5	11	576	1100001100100000	7	12	288	1110101011000000	8	23	576	1110110000110001

TABLE III. (*Continued*).

μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$	μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$	μ	λ_μ	T	$\Delta(\mu,\lambda_\mu)$
5	12	144	1100001000100001	7	13	1152	1110110010010000	8	24	288	1110110010100001
5	13	288	1100100000100001	7	14	576	1110110000110000	8	25	1152	1110110010010001
6	1	144	1111110000000000	7	15	576	1110100101010000	8	26	576	1110110010010010
6	2	96	1111100010000000	7	16	288	1110100110010000	8	27	576	1110100110000110
6	3	288	1111100001000000	7	17	576	1110110000100001	8	28	288	0111110010101000
6	4	48	1110111000000000	7	18	288	1110110000010001	8	29	576	1110100010010011
6	5	144	1110110100000000	7	19	576	1110100101000010	8	30	576	0111110010100001
6	6	576	1110110010000000	7	20	1152	0111110010000010	8	31	72	1100101001010011
6	7	576	1110110000100000	7	21	576	1110110010000001	8	32	18	1100110000110011

four digits being the elements Δ_{1j}, $j=1,2,3,4$, the next four being Δ_{2j}, and the last four being Δ_{3j}. The sequence μ, λ_μ, T, and $\Delta(\mu,\lambda_\mu)$ is repeated in the next four columns.

The Δ matrix $\Delta(AB,1)$ is $\Delta_{ij}=1$. The orbits, lengths, and Δ matrices for $\mu > AB/2$ can be obtained from those of lower orders by symmetry. To obtain the results for der $\mu > AB/2$, we use the results for order $\mu'=AB-\mu$ and interchange the 0's and 1's of $\Delta(\mu',\lambda_{\mu'})$. [The indices λ_μ and $\lambda_{\mu'}$ are the same, and the lengths $T(\mu,\lambda_\mu)$ and $T(\mu',\lambda_{\mu'})$ are equal.]

In Table III we present the orbits, lengths, and Δ matrices for $A=B=4$.

Although these results can be obtained "by hand," i.e., by enumerating and investigating all the possible combinations of collisions, such a procedure is extremely tedious and the possibility of making an error in the counting is not insignificant. A quicker and more reliable procedure is to program a computer to do the counting, which is the procedure we have used. As can be seen, there are 86 different orbits for the case $A=4, B=3$, and there are 191 different orbits for the case $A=4, B=4$.

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