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Boson mapping of the shell model algebra obtained from a seniority-dictated similarity transformation

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The qualitative ideas put forward by Geyer and Lee are given quantitative content by constructing a similarity transformation which reexpresses the Dyson boson images of the single-j shell fermion operators in terms of seniority bosons. It is shown that the results of Otsuka, Arima, and Iachello, or generalizations thereof which include g bosons or even bosons with $J>4$, can be obtained in an economic and transparent way without resorting to any comparison of matrix elements.

Any program which aims at the construction of a microscopically founded interacting boson model of nuclei has to address the two questions of transcribing the fermion problem to a boson space and of identifying the physically relevant bosons which facilitate a truncated description.

Of course, the ensuing boson model need not be restricted to the structure of any presently existing version of the successful phenomenological interacting boson model (IBM).[1-3] In fact, from microscopic considerations one hopes to identify the situations for which the existing IBM is applicable as well as those for which modifications should and could be made.

As far as the mapping between fermion and boson spaces is concerned, the Dyson boson mapping (DBM), characterized by finite boson images, has been demonstrated[4-10] to be a very efficient tool for formal investigations as well as ideally suited for the construction of boson models directly linked to fermion microscopy. Because of the nonunitarity of the DBM, the equivalent boson picture obtained from a direct application of this mapping is characterized by a Hamiltonian which is non-Hermitian when represented in an ideal boson basis. This, however, is no reason for concern[4,5] since one is dealing with an exact boson realization of some Lie algebra. In fact, the very nature of this pair boson Dyson Hamiltonian has served as the basis for economic and elegant further developments. Li (Ref. 9) exploited its structure to rederive the results of Otsuka, Arima, and Iachello (OAI)[11] without having to establish any correspondence between fermion and boson states; Kim and Vincent[12] used it to construct a Hermitian one-plus two-body boson Hamiltonian for Ginocchio's SO(8) model; and Geyer[13] utilized the non-Hermiticity of the Hamiltonian to construct seniority boson realizations for the Sp(4) algebra. In this paper it is shown that the ideas of Ref. 13 can be generalized to the case of a single-j shell model resulting in a description in terms of seniority bosons.

Direct application of the Dyson mapping leads to a description in terms of pair bosons, as of course does any boson mapping based on the preservation of the complete bifermion operator algebra. In general, however, this situation presents an unfavorable point of departure when considering a truncated description in terms of the physically relevant bosons, such as s and d bosons in the IBM. This point was emphasized by Geyer and Lee[8] and illustrated by numerical examples in a single-j shell model.

For this model Otsuka, Arima, and Iachello[11] had previously shown that, at least in the SU(5) vibrational limit of IBM, the physical s and d bosons are to be associated with a seniority description. In the OAI scheme, boson operators are accordingly constructed by requiring equality between boson matrix elements and the corresponding fermion matrix elements calculated in a seniority basis. In Ref. 8 it was indicated that the boson operators could also be constructed by requiring equality between pair boson matrix elements, obtained directly from the Dyson mapping, and seniority boson matrix elements. It was shown by Li (Ref. 9) that this procedure is indeed not only viable, but that it can elegantly exploit the nonunitarity of the Dyson mapping to achieve the OAI results while avoiding all complications due to the nonorthogonality of fermion states.

Perhaps a conceptually more satisfactory picture, which was also formulated in Ref. 8, envisages the OAI results, and generalizations thereof, as resulting from a similarity transformation on the mapped pair boson images of bifermion operators. Only a qualitative description of this transformation could be achieved at the time, its main characteristic being that it restores the physical requirement of a seniority mapping, namely, that twice the number of non-s bosons should be associated with seniority.

I now show that this requirement indeed produces the required transformation and that seniority boson images of bifermion operators can be obtained without having to resort, at any level, to the cumbersome process of equating matrix elements.

The generalized DBM is defined by the mapping

$$b^{\alpha\beta} \rightarrow R^{\alpha\beta} \equiv B^{\alpha\beta} - B^{\alpha\theta}B^{\beta\lambda}B_{\theta\lambda} \ , \tag{1}$$

$$b_{\alpha\beta} \rightarrow R_{\alpha\beta} \equiv B_{\alpha\beta} \ , \tag{2}$$

$$b^{\alpha}_{\beta} \rightarrow R^{\alpha}_{\beta} \equiv B^{\alpha\theta}B_{\beta\theta} \ , \tag{3}$$

between bifermion operators b and ideal boson operators B. (See, e.g., Ref. 10 for the notation used.) For a single-j shell it is convenient to introduce spherical boson operators

$$B^{JM} = \frac{1}{\sqrt{2}} \sum_{mm'} \langle jmjm' | JM \rangle B^{jmjm'}, \quad B_{JM} = (B^{JM})^{\dagger} \ . \tag{4}$$

From the basic mapping [(1)–(3)] and standard angular momentum coupling it is now straightforward to obtain the Dyson images of the spherical fermion operators

$$A^{JM}\equiv\frac{1}{\sqrt{2}}[c^j\times c^j]^J_M,\ A_{JM}=(A^{JM})^\dagger\ , \tag{5}$$

$$U^J_M=[c^j\times\tilde{c}_j]^J_M\ , \tag{6}$$

in terms of the spherical boson pair operators (4), namely,

$$(A^{JM})_D=B^{JM}-2\sum\hat{J}_1\hat{J}_2\hat{J}_3\hat{L}\begin{Bmatrix} j & J_1 & j\\ j & J_2 & j\\ J_3 & L & J\end{Bmatrix}[[B^{J_1}\times B^{J_2}]^L\times\tilde{B}_{J_3}]^J_M, \tag{7}$$

$$(A_{JM})_D=B_{JM}\ , \tag{8}$$

$$(U^J_M)_D=-2\sum\hat{J}_1\hat{J}_2\begin{Bmatrix} J_1 & J_2 & J\\ j & j & j\end{Bmatrix}[B^{J_1}\times\tilde{B}_{J_2}]^J_M\ . \tag{9}$$

Note that the pair boson image $(A^{00})_D$ contains terms such as $d^\dagger\cdot d^\dagger s$, which change the number of non-s bosons, in contrast to the fermion operator A^{00} which does not change the seniority. (Here $s\equiv B_{00}$, $d^\mu\equiv d^\dagger_\mu\equiv B^{2\mu}$, etc.)

Of particular importance for further developments is the structure of the Dyson image for the pairing Hamiltonian $H=-G\Omega A^{00}A_{00}$, where $2\Omega=2j+1$. From expressions (7)–(8) one finds

$$H_D=-G\left[\Omega n_s-n_s(n_s-1)-2n_s\sum_{J=1}^{\Omega-1}n_{2J}\right]+G\sum{}'B^J\cdot B^J ss$$
$$-G\sqrt{2\Omega}\sum{}'\hat{J}_1\hat{J}_2\hat{J}_3\begin{Bmatrix} J_1 & J_2 & J_3\\ j & j & j\end{Bmatrix}[[B^{J_1}\times B^{J_2}]^{J_3}\times\tilde{B}_{J_3}]^0_0 s. \tag{10}$$

The boson number operators n_s, $n_{2J=2}\equiv n_d$, etc. appear above, while $\sum'$ denotes a summation over J values other than zero. [Note the corrections to Eqs. (3.6) and (3.10) of Ref. 8 introduced above.]

The strategy for obtaining seniority boson images, as opposed to the pair boson images [(7)–(9)] is similar to that of Ref. 13 and again based on the special structure of the Dyson image (10) of the pairing interaction. We note that H_D, when represented in an ideal boson basis, such as the one defined by the SU(5)⊃O(5)⊃SO(3) chain, is of triangular form. (In Ref. 10 it has been proved that the use of an ideal boson basis in conjunction with a mapped operator, such as H_D, is always permissible.) This means that $H_D=H_0+W$ has the same spectrum as its diagonal part,

$$H_0=-G\left[\Omega n_s-n_s(n_s-1)-2n_s\sum_{J=1}^{\Omega-1}n_{2J}\right]\ , \tag{11}$$

while eigenstates $|\psi\rangle$ and $|\phi\rangle$ of H_D and H_0, respectively, are related by

$$|\psi\rangle=\sum_{k=0}^{\infty}\left[\frac{1}{E-H_0}W\right]^k|\phi\rangle\equiv Z^{-1}|\phi\rangle \tag{12}$$

for a specific energy E. (The range of k in the sum above can be extended to infinity since an automatic cutoff is associated with the annihilation part of W.)

If we now operate on the equation

$$(H_0+W)Z^{-1}Z|\psi\rangle=EZ^{-1}Z|\psi\rangle \tag{13}$$

from the left by Z, the resulting expression can be written as

$$Z(H_0+W)Z^{-1}|\phi\rangle=E|\phi\rangle=H_0|\phi\rangle\ , \tag{14}$$

which explicitly shows that the similarity transformation Z transforms away the term W in H_D. If Z also commutes with $(A_{00})_D=s$, it is clear that the same Z also transforms away those terms in $(A^{00})_D$ which change the number of non-s bosons. One then gets the structure envisaged previously, namely,

$$(A^{00})_{\rm sen}\equiv Z(A^{00})_D Z^{-1}=s^\dagger\left[1-\frac{n_s}{\Omega}-\frac{2}{\Omega}\sum{}'n_{2J}\right]\ , \tag{15}$$

and it becomes possible to directly associate seniority and boson number through $v=2\sum'n_{2J}$.

Restricting ourselves for the moment to the case $j=\frac{3}{2}$, we note that this choice allows only s and d bosons in the expression for $(A^{00})_D$. In particular, the term W in H_D consists of only one contribution which contains the combination $d^\dagger\cdot d^\dagger ss$. Because of this simplification a closed form for the transformation Z^{-1} can be obtained. This is accomplished by using the idea of Kim and Vincent[12] who define a positional operator through

$$Z^{-1}\equiv\sum_{k=0}^{\infty}\left[\frac{1}{E-H_0}W\right]^k\equiv\sum_{k=0}^{\infty}\left[\frac{1}{\hat{H}_0-H_0}W\right]^k\wedge\ . \tag{16}$$

The position of the wedge (or inverted wedge, below) indicates where in an expression a number dependent caret (or inverted caret) operator is to be evaluated. In this notation Z^{-1} and Z can be rewritten as

$$Z^{-1}=\vee e^{-1/2d^\dagger\cdot d^\dagger ss}\frac{(\Omega-1-n_d-\check{n}_d)!!}{(\Omega-1-2n_d)!!}\ , \tag{17a}$$

$$Z=\frac{(\Omega+1-2n_d)!!}{(\Omega+1-n_d-\hat{n}_d)!!}e^{1/2d^\dagger\cdot d^\dagger ss}\wedge\ . \tag{17b}$$

Construction of seniority boson images of the spherical fermion operators [(5),(6)] can now be carried out. In general one has, for any θ from expressions (7)–(9),

$$\theta_{\rm sen}=Z\theta_D Z^{-1}\ . \tag{18}$$

For $\theta=A^{00}$ one confirms that $(A^{00})_{\rm sen}$ is given by the expression (15). Furthermore $(A_{00})_{\rm sen}=s$, since Z indeed commutes with s. The other seniority images are more complicated and one finds, e.g.,

$$\tfrac{1}{2}(U^2_\mu)_{\rm sen}=s^\dagger\tilde{d}_\mu+\frac{(\Omega+1-N-n_d)}{(\Omega+1-2n_d)}\times\left[d^\mu s+\frac{1}{(\Omega+3-2n_d)}d^\dagger\cdot d^\dagger\tilde{d}_\mu s\right]\ , \tag{19}$$

where $N=n_d+n_s$ is the total boson number. In fact, the seniority images we obtain are given as Eq. (15) of Ref. 14, except for $(A_{2\mu})_{\rm sen}$ which is incorrectly given there. In Ref. 14, however, these images were obtained in OAI fashion from a comparison of matrix elements.[15]

Returning to the single-j shell model for general values of $j > \frac{3}{2}$, we note that in the nondiagonal part W of H_D in Eq. (13), there now appear two classes of terms with the structure $B^{J}\cdot B^{J}ss$ and $[B^{J_1}B^{J_2}\tilde{B}_{J_3}]^0_0 s$. While a closed form for the corresponding transformation Z now seems unattainable, we can still find approximate seniority boson images by truncating the sum over k in Eq. (12). It is clear from the structure of Z^{-1} that the inclusion of higher values of k will only introduce higher order multiboson contributions (characterized by having more and more s-boson annihilation operators) to these seniority images, without changing the structure of lower order contributions. To be more concrete we adopt the lowest order approximation to Z^{-1} which retains only the $k=0$ and $k=1$ terms, namely,

$$Z^{-1} \approx 1 + [(n_s - \hat{n}_s)(\Omega + 1 - 2N + \hat{n}_s + n_s)]^{-1}[F^{\dagger}ss + K^{\dagger}s]\wedge \ , \tag{20}$$

where

$$F^{\dagger} \equiv \sum_J{}' B^{J}\cdot B^{J} \ ,$$

$$K^{\dagger} \equiv -\sqrt{2\Omega}\sum_{L_i\mu_i}{}' \hat{L}_1\hat{L}_2\hat{L}_3 \begin{pmatrix} L_1 & L_2 & L_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{Bmatrix} L_1 & L_2 & L_3 \\ j & j & j \end{Bmatrix} B^{L_1\mu_1}B^{L_2\mu_2}\tilde{B}_{L_3\mu_3} \ . \tag{21}$$

On simplification, expression (20) becomes

$$Z^{-1} \approx 1 - \frac{1}{2}\frac{1}{\Omega + 3 - 2N + 2n_s}F^{\dagger}ss - \frac{1}{\Omega + 2 - 2N + 2n_s}K^{\dagger}s \ . \tag{22}$$

(Z differs from this expression by having positive signs for the last two terms.)

In this approximation the seniority image of, e.g., the fermion quadrupole operator can now be calculated as

$$(U^2_\mu)_{\rm sen} \approx \sqrt{2/\Omega}\left[s^{\dagger}\tilde{d}_\mu + d^{\mu}s\left(\frac{\Omega - N - n_d}{\Omega - 1 - 2n_d}\right)\right] - 10\begin{Bmatrix} 2 & 2 & 2 \\ j & j & j \end{Bmatrix}\left(\frac{\Omega - 2N}{\Omega - 2n_d}\right)[d^{\dagger}\times\tilde{d}]^2_\mu \ , \tag{23}$$

where only one-body operators with possible boson number dependent coefficients, and just those involving only s and d bosons, have been retained.

Expression (23) is not the OAI result yet, since $s^{\dagger}d_\mu$ and $d^{\mu}s$ have different coefficients. One has to bear in mind, however, that for a given fermion operator θ the Dyson scheme prescribes the association[5]

$$\langle\psi|\theta|\psi'\rangle = [{}_L(\psi|\theta_D|\psi')_R\, {}_L(\psi'|(\theta^{\dagger})_D|\psi)^*_R]^{1/2} \tag{24}$$

between the fermion matrix element and its boson equivalent, a direct result of the nonunitarity of the Dyson mapping. This relationship can be rewritten as

$$\langle\psi|\theta|\psi'\rangle = [(\psi|\theta_{\rm sen}|\psi')(\psi'|(\theta^{\dagger})_{\rm sen}|\psi)^*]^{1/2} \ , \tag{25}$$

where $(\psi| = {}_L(\psi|Z^{-1}$ and $|\psi') = Z|\psi')_R$. No distinction between left and right eigenstates is required anymore, since the Hamiltonian H_D in Eq. (10) is transformed to a Hermitian form H_0 by the transformation Z.

For any seniority boson image, such as $(U^2_\mu)_{\rm sen}$ in Eq. (23), of which the different boson terms connect different pairs of boson states of good seniority (good n_d) one can write a Hermitian equivalent boson operator $\bar{\theta}$ for which the boson matrix element is calculated in the usual way, namely,

$$\langle\psi|\theta|\psi'\rangle = (\psi|\bar{\theta}|\psi') \ . \tag{26}$$

The Hermitian equivalent $\bar{U}^2_\mu$ of $(U^2_\mu)_{\rm sen}$ is given by

$$\bar{U}^2_\mu = \left(\frac{2}{\Omega}\right)^{1/2}\left[d^{\mu}s\left(\frac{\Omega - N - n_d}{\Omega - 1 - 2n_d}\right)^{1/2} + \left(\frac{\Omega - N - n_d}{\Omega - 1 - 2n_d}\right)^{1/2} s^{\dagger}\tilde{d}_\mu\right] - 10\begin{Bmatrix} 2 & 2 & 2 \\ j & j & j \end{Bmatrix}\frac{\Omega - 2N}{\Omega - 2n_d}[d^{\dagger}\times\tilde{d}]^2_\mu \ , \tag{27}$$

and is written by noticing that, since $(U^2_\mu)_{\rm sen} = (U^{2\dagger}_\mu)_{\rm sen}$, one has the situation where if $d^{\mu}s$ connects $(\psi|$ and $|\psi')$ in Eq. (25), then $s^{\dagger}\tilde{d}_\mu$ will connect $(\psi'|$ and $|\psi)$. The result (27) is the same as that originally given in Ref. 16 and differs from the OAI result,[11] rederived by Li,[9] which is only valid when $n_d = 1$.

Similarly one can find the Hermitian boson equivalent for other operators in the approximation where only s and d bosons are allowed, and only the lowest order term which could connect boson states of good n_d is kept. For $A^{2\mu}$, for example, this means that only the boson terms d^{μ}, $s^{\dagger}s^{\dagger}\tilde{d}_\mu$ and $s^{\dagger}[d^{\dagger}\times\tilde{d}]^2_\mu$ are considered. A term $[d^{\dagger}d^{\dagger}\tilde{d}]^2_\mu$ can, of course, always be accommodated if we stick to expression (25) for calculating boson matrix elements and do not insist on finding a Hermitian equivalent. In the present example this latter option is ruled out, since both d^{μ} and $[d^{\dagger}d^{\dagger}\tilde{d}]^2_\mu$ can connect the same boson states. Such a situation prohibits a prescription that would reduce expression (25) to one of the form (26).

The lowest order Hermitian seniority boson image we obtain for $A^{2\mu}$ is the same as Eq. (2.30) obtained by Bonatsos *et al.*[17] who have to fix a number of unknown functions in order to satisfy the shell model algebra up to a chosen order. Again, the corresponding OAI result[9,11] is only valid for $n_d = 1$.

As in Ref. 17 we can now go beyond a seniority mapping in terms of s and d bosons by including g bosons or even bosons with $J > 4$ in the seniority boson images, as well as retaining higher order many-body boson operators. For $(U^2_\mu)_{\rm sen}$ this means that in expression (23) we also retain terms of the type $[d^\dagger\times\tilde{g}]^2_\mu$, $[g^\dagger\times\tilde{d}]^2_\mu$, $[d^\dagger d^\dagger\tilde{d}]^2_\mu s$, and $s^\dagger[d^\dagger\tilde{d}\tilde{d}]^2_\mu$, etc. As already stated, the inclusion of such terms does not change the structure of the terms $s^\dagger\tilde{d}_\mu$, $d^\mu s$, and $[d^\dagger\times\tilde{d}]^2_\mu$ already calculated. In contrast to the commutator method of Ref. 17, where the inclusion of such higher order terms requires increased effort to handle all the unknown functions introduced, the calculation of boson number dependent coefficients of terms like $[d^\dagger\times\tilde{g}]^2_\mu$ in expression (23) requires no more effort than that of, e.g., $d^\mu s$.

For the terms $[d^\dagger\times\tilde{g}]^2_\mu$ and $[g^\dagger\times\tilde{d}]^2_\mu$ we again find the same results as in Ref. 17. On the other hand, we find no term of the type $s^\dagger[d^\dagger\tilde{d}\tilde{d}]^2_\mu$. This only means that we cannot write down a Hermitian equivalent $\tilde{U}^2_\mu$ which contains the conjugate pair $s^\dagger[d^\dagger\tilde{d}\tilde{d}]^2_\mu$ and $[d^\dagger d^\dagger\tilde{d}]^2_\mu s$. However, as long as we observe the proper prescription (25) to calculate seniority boson matrix elements, these matrix elements will contain all information that can be carried by a seniority boson image of a chosen order.

In terms of the OAI prescription we have succeeded through the similarity transformation to obtain the seniority boson images of operators *valid for states with an arbitrary number of d bosons, g bosons,* or higher J bosons. This clearly goes beyond what is normally considered in the OAI framework; namely, to find matrix element equivalence only for some low seniority states containing one or two d or g bosons. In fact, the OAI method cannot be used to determine seniority images for general numbers of $J\geq 2$ bosons as soon as these images contain more than one term which connect the same set of basis states, since one would then attempt to find the (number dependent) coefficients of all these terms from one equation which equates the fermion and boson matrix element.

Finally I would like to comment on the claim in Ref. 17 that "inclusion of the g boson can be viewed as a theoretical necessity." In my formalism there is nothing special about the inclusion of g bosons, e.g., in Eq. (23). From this point of view the above claim seems to be an artificial consequence of the method of Ref. 17.

I believe that the treatment above provides the most economic way of obtaining seniority boson images. Of particular importance is the fact that higher order contributions are easily treated. This is in contrast to methods which either require matrix elements to be equated or a commutator algebra to be satisfied.

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