PHYSICAL REVIEW C
NUCLEAR PHYSICS

THIRD SERIES, VOLUME 34, NUMBER 6 DECEMBER 1986

Gauge-invariant nuclear Compton amplitude manifesting low-energy theorems

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(Received 18 July 1986)

The electromagnetic field representation of the nuclear electromagnetic interaction is developed, by extending previous work on the nuclear current (one-photon amplitude) to the nuclear Compton amplitude. The resulting form of the Compton amplitude is gauge invariant and, in transverse gauge, manifests the low-energy theorems for the $0^+ \to 0^+$ case, which is discussed in detail. Deuteron forward photodisintegration is also studied numerically, and it is shown that the generalized Siegert form of the retarded electric dipole interaction efficiently incorporates pion-exchange currents implicitly, while large explicit forms of these currents are needed in the standard representation of the interaction. Gauge invariance and Reiss's strong-field momentum-translation approximation are also discussed.

I. INTRODUCTION

The history of electromagnetic nuclear physics is largely a story of the use of perturbation theory, if one excludes those cases where the Schrödinger, Dirac, or Klein-Gordon equations are solved with a static Coulomb potential included. One obvious exception to this general rule is the field of radiative corrections[1] to electron-nucleus scattering or nuclear β decays. Not only is perturbation theory the rule, but the preponderance of applications employ first-order perturbation theory in e, the (positive) fundamental charge. To a much lesser extent, applications involving second-order perturbation theory (two photons) are considered.

The two-photon (nuclear Compton) amplitude[2] is a basic building block for a number of nuclear physics applications. Virtual excitation of the nucleus during electron scattering (dispersion corrections)[3,4] involves a minimum of two electromagnetic interactions, as do the analogous polarization corrections in atoms.[5] The electromagnetic energy shift in a nucleus[6] is calculated in lowest order using the Compton amplitude, and certain sum rules[7] are best discussed by considering the properties of that amplitude.[8]

Recently, nuclear Compton (elastic photon) scattering has become a more viable experimental tool; high intensity cw beams of electrons can produce experimentally useful beams of "tagged" photons, eliminating the necessity of using broad-spectrum bremsstrahlung beams. Although few nuclear Compton scattering experiments have been performed, there is a real hope that in the near future such experiments will be much more common. Recent experiments[9] have provided exciting new data that have resisted theoretical interpretation. In addition, a recent $0^+ \to 0^+$ two-photon decay experiment[10] has found evidence of strong $M1$-$M1$ competition with the dominant $E1$-$E1$ decay mode in ^{90}Zr and ^{40}Ca, an entirely unexpected result. The latter experiment is theoretically indistinguishable from nuclear Raman scattering (inelastic photon scattering).[2] In the more distant future the development of a gamma-ray laser[11] or a very intense XUV free-electron laser[12] could lead to multiphoton absorption and excitation of low-lying nuclear states, which would be a new experimental technique for electromagnetic nuclear physics.

Until this decade, interest in multiphoton ($n_\gamma > 2$) nuclear processes was virtually nonexistent. The recent controversial proposal by Reiss[13] that very-high-intensity long-wavelength electromagnetic waves could alter β-decay rates has modified the situation. Calculations of these processes must allow for the possibility of emitting or absorbing many orders of magnitude of (very low energy) photons. Clearly, formalisms which are nonperturbative must be used. The interaction of the outgoing electron with the wave is treated by using Volkov's wave function,[14] an exact solution of the Schrödinger, Dirac, or Klein-Gordon equations in the field of a classical traveling wave. Considerations of gauge invariance require a commensurate treatment of those nuclear processes involving multiphoton interactions. This has been done by Reiss;[15] the nuclear wave function which results is called the momentum-translation approximation (to the exact solution) or, simply, the MTA solution. Its genesis will be considered in Sec. VI. In view of the recent experimental and theoretical work which involves multiphoton processes and prospects of much more in the future, it is timely to consider electromagnetic nuclear processes from a unified perspective and this motivates the current work. Because many approaches are possible and because the history of these approaches is tangled and virtually unknown to nuclear physics, we outline the latter below in order to clarify our approach herein. Moreover, we treat a number of diverse topics, superficially unconnected, which are nonetheless intimately related, as we show below.

II. MOTIVATION

Although atomic and nuclear physics share a common origin, and to a large extent a common methodology for treating theoretical problems, there are qualitative differences, particularly for electromagnetic interactions. Atomic binding is achieved via the exchange of neutral quanta (i.e., photons), while roughly half of the nuclear binding is mediated by the exchange of *charged* constituents (e.g., mesons), which must be reflected in a sizable component of the electromagnetic current, $\mathbf{J}(\mathbf{x})$. These nonrelativistic meson-exchange, or potential-dependent, currents $\mathbf{J}_{\text{ex}}(\mathbf{x})$, modify the ordinary nonrelativistic convection and spin-magnetization currents common to atomic and nuclear physics: $\mathbf{J}_0(\mathbf{x})$.

The electromagnetic current operator, $\mathbf{J}(\mathbf{x})$, and charge operator, $\rho(\mathbf{x})$, must satisfy differential current conservation, in order that charge be locally conserved:

$$\nabla\cdot\mathbf{J}(\mathbf{x})=-i[H,\rho(\mathbf{x})]=-\dot{\rho}(\mathbf{x})\ , \tag{1}$$

where H is the strong interaction Hamiltonian. Writing $\mathbf{J}=\mathbf{J}_0+\mathbf{J}_{\text{ex}}$, and $H=T+V$, one finds that $\mathbf{J}_0$ on the left-hand side of the equation is conserved if one matches it with the kinetic energy, T, on the right-hand side, while the existence of $\mathbf{J}_{\text{ex}}$ corresponds to the lack of commutativity with ρ by those parts of the potential, V, corresponding to charged-meson exchange.

These considerations[4] have led to considerable theoretical concern about a consistent methodology for treating photonuclear reactions, in the absence of a detailed physical understanding of the origins of the nuclear potential, V. Such a methodology was initiated by Siegert,[16] who demonstrated that the long-wavelength form of the electric dipole current operator is completely determined by the electric dipole moment, calculated from the charge operator. Defining [with $\int d^3x\,\rho(\mathbf{x})=Z$]

$$\mathbf{J}(\mathbf{q})=\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\mathbf{J}(\mathbf{x}) \tag{2a}$$

and

$$\rho(\mathbf{q})=\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\rho(\mathbf{x})\ , \tag{2b}$$

we have

$$\mathbf{J}(\mathbf{q}=0)=\int d^3x\,\mathbf{J}(\mathbf{x})=-\int d^3x\,\mathbf{x}\,\nabla\cdot\mathbf{J}(\mathbf{x})$$
$$=i[H,\mathbf{D}]\rightarrow i\omega_{fi}\mathbf{D}_{fi}\ , \tag{3}$$

where $\mathbf{D}=\int d^3x\,\mathbf{x}\rho(\mathbf{x})$, and the last form results from taking matrix elements. Siegert also demonstrated that the nonrelativistic form of the charge operator should be dominant, and therefore $\mathbf{D}$ is given by the classical expression, $\sum_{i=1}^{A}e_i\mathbf{x}'_i$, where e_i is the charge of the ith nucleon in units of e, and $\mathbf{x}'_i$ is the coordinate of that nucleon relative to the nuclear center of mass. These two elements comprise what is known as Siegert's theorem, which forms the backbone of photonuclear physics because it affords a simple interpretation of electric (normal parity) transitions and obviates the need for explicit forms of the exchange currents, whose short-range behavior is not known.

Atomic physics has the advantage over nuclear physics in knowing the form and origin of the binding potential. Consequently, there is less need to work with general principles, such as current conservation, because the models which are used should satisfy those general principles automatically. Consequently, more diverse forms of the electromagnetic interaction are used[17] (or at least defined). In the long-wavelength regime, these forms are (1) dipole velocity, (2) dipole length, (3) dipole acceleration, and (4) dipole rate of change of acceleration. The ordinary convection current, $\mathbf{J}_c=e\mathbf{p}/mc=e\mathbf{v}/c$, forms the basis of the (standard) dipole-velocity form. The dipole-length form is just the Siegert current, Eq. (3). The third form is generated by formally writing

$$\omega_{fi}\mathbf{J}_c/\omega_{fi}=\frac{1}{\omega_{fi}}[H,\mathbf{J}_c]\ ,$$

and evaluating the commutator; repeating the process generates the fourth form. Because repeated commutators involve derivatives of the potential, the latter two forms are probably not very useful in nuclear physics. These forms will, however, play a role in our subsequent discussion. Although there is no need, in principle, to rely on current conservation to ensure that atomic electric dipole matrix elements are correctly calculated, in practice it can be useful in two cases: (1) approximate wave functions in many-body systems (either atomic[18] or nuclear[19]) will not satisfy Eq. (3), and (2) including relativistic corrections to order $(v/c)^2$ introduces a plethora of messy components to the atomic Hamiltonian and electromagnetic current operators,[20] while the charge operator remains quite simple.[21]

In nuclear physics, the restriction of Siegert's theorem to long wavelengths is acceptable provided that photon energies are not too high. Writing the photon wavelength as

$$\lambda(\text{fm})=2\pi(\hbar c)/E_\gamma=1240/E_\gamma(\text{MeV})\ ,$$

we see that energies approaching 100 MeV are needed before there is substantial retardation. (Recognizing that $\hbar c=1973$ eV Å illustrates why retardation effects are much less important in most atomic applications, where sizes are typically 1 Å.) Nevertheless, at a very early date Sachs and Austern[22] provided a generalization of Siegert's work by requiring that gauge invariance hold for the nuclear Hamiltonian. The resulting expressions for the electromagnetic interaction were written in a form which was inconvenient to use and difficult to interpret. Consequently, this highly novel work has been mainly used as a "justification" of Siegert's theorem.

Our own work on this problem began with a demonstration that Siegert's theorem, suitably modified to account for relativistic corrections, was a simple way to treat the transition strengths of intercombination lines in He-like atoms.[21] These lines correspond to $^3P_1 \rightarrow {}^1S_0$ spin-flip (i.e., forbidden) electric dipole transitions. Because the ordinary nonrelativistic electric dipole operator cannot flip spin, these lines are weak, and relativistic corrections to operators compete with spin-dependent retardation corrections (to the long-wavelength limit). It was found that the leading-order retardation corrections could be grouped into two categories by using a symmetry criterion; after expanding the electric dipole transition operator as a power series in the photon's momentum operator, $\mathbf{q}$, the terms were rearranged according to permutation symmetry in the (vector) indices on the various $\mathbf{q}$'s. All of the purely symmetric terms were found to be proportional to $\nabla\cdot\mathbf{J}$, which could be replaced by $\dot\rho$ according to Eq. (1), while the remaining terms were shown to be proportional to

$$\boldsymbol{\mu}(\mathbf{x})=\tfrac{1}{2}\mathbf{x}\times\mathbf{J}(\mathbf{x})\ , \tag{4}$$

the magnetization density operator. Thus, the residue of the current which is not determined by current continuity is given by the magnetization density. This approach was later extended[23] to all orders in the retardation for each of the electric multipoles, and provided a generalization of Siegert's theorem, or the dipole-length formula.

Subsequently,[24] we provided a detailed exposition of these results with several new features. A "trick" was found which allowed a decomposition of the current into the $\dot\rho(\mathbf{x})$ and $\boldsymbol{\mu}(\mathbf{x})$ parts without making a multipole decomposition, requiring only that there be no singularity in $\mathbf{J}(\mathbf{q})$ as a function of $\mathbf{q}$. The uniqueness aspect of the decomposition is obscured, however, in the simplicity of that rearrangement. If was also found that the effective (one-photon) Hamiltonian could be changed from

$$H_{\rm em}=-e\int d^3x\ \mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})+e\int d^3x\ \rho(\mathbf{x})\phi(\mathbf{x}) \tag{5a}$$

to

$$H'_{\rm em}=-e\int d^3x\ \mathbf{E}(\mathbf{x})\cdot\mathbf{d}(\mathbf{x})-e\int d^3x\ \mathbf{B}(\mathbf{x})\cdot\mathbf{m}(\mathbf{x})\ , \tag{5b}$$

where

$$\mathbf{d}(\mathbf{x})=\mathbf{x}\int_0^1 d\lambda\,\rho(\mathbf{x}/\lambda)/\lambda^4 \tag{5c}$$

and

$$\mathbf{m}(\mathbf{x})=2\int_0^1 d\lambda\,\boldsymbol{\mu}(\mathbf{x}/\lambda)/\lambda^2\ , \tag{5d}$$

and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic *fields*. The electromagnetic *potentials* $\mathbf{A}$ and ϕ do not explicitly appear in Eq. (5b).

Additionally, the asymptotic forms of the operators $\mathbf{d}$ and $\mathbf{m}$ (large $|\mathbf{q}|$, or small $|\mathbf{x}|$) were found to be different from the corresponding ρ and $\mathbf{J}$, and this pointed out a possible serious problem at short (photon) wavelengths. A numerical calculation by Haxton[19] with one of us demonstrated that, for nuclear models which are not inherently current conserving, removing defective small-$\mathbf{q}$ behavior according to the generalized Siegert prescription introduces a serious high-$\mathbf{q}$ problem. Our new electric multipole fields should *not* be used for the large momentum transfers associated with high-energy electron scattering for *such* models; we will show by means of an explicit calculation in Sec. IV that these fields work exceptionally well at modest momentum transfers corresponding to high-energy photon absorption, where retardation is important, but does not completely dominate.

While completing the work discussed above, we discovered that Sach's and Austern's expressions for the Fourier transform of the current could be manipulated into our forms. In retrospect this is perhaps not terribly surprising in view of Eq. (5b), which is manifestly gauge invariant, although we have already enumerated a number of alternative forms for the electromagnetic interaction, and the form (5b) above is only one of them. We were subsequently informed[25] that Foldy[26] had also completed the Sachs and Austern derivation using a (different) trick of his own. All of the one-photon work discussed heretofore treated the Fourier transform of the internal parts of the current; the convection current of the entire (composite) nucleus was separated and ignored. The latter vanishes in transverse gauge for all $\mathbf{q}$-congruent frames, where the average nuclear velocity, $\mathbf{V}$, is parallel to $\mathbf{q}$ or null:

$$\mathbf{V}=(\mathbf{P}_f+\mathbf{P}_i)/2m_t\ , \tag{6}$$

expressed in terms of the initial ($\mathbf{P}_i$) and final ($\mathbf{P}_f$) nucleus momenta and the total nuclear mass, $m_t=Am$.

In view of the overlap of different approaches separated in time by decades, it is not surprising that similar work has also been performed in other fields of physics. Competely unaware of the nuclear physics approach (and *vice versa*), Power and Zienau[27] (PZ) developed Eqs. (5) as a power series by means of a canonical transformation, keeping only the first few terms. This atomic physics work was motivated by a question which arose in the original Lamb shift experiment: In calculations which are only perturbative, which form of the electric dipole interaction, velocity or length, works better? Their conclusion that the dipole-length form (i.e., the Siegert form of the current) is better in most cases is supplemented by many delightful examples in Power's book.[28] Independently, Fiutak[29] developed the electric field part of Eq. (5) in closed form (i.e., a summed series) using a canonical transformation. Later, in the context of molecular physics, closed form versions[30,31] of PZ were developed, and the long-wavelength dipole-acceleration variant of the electric dipole operator was generalized[32] to the "space-translation" form of the electromagnetic interaction. The latter explicitly involves derivatives of the binding potential, and for this reason is probably not very useful in nuclear physics.

Amplitudes for processes such as Compton scattering, which involve more than one photon, necessarily contain energy denominators if a perturbative expansion is performed. Moreover, recoil (center-of-mass) motion (a consequence of translation invariance) is traditionally an important consideration in nuclear physics and is ignored in atomic physics. In developing the low-energy (long-wavelength) theorems for Compton scattering which are a consequence of gauge invariance, the standard methodology[2] is to find those combinations of terms which contain factors that exactly cancel energy denominators. If these denominators contain recoil energies, the nuclear convection current, which played an insignificant role in one-photon processes, must be consistently treated also. In Sec. III we develop a form of this current, consistent with Eqs. (5b) and (6), which is necessary for a treatment of the Compton amplitude. In Sec. IV we demonstrate the efficacy of our one-photon formalism in the context of deuteron photodisintegration.[33] In Sec. V the two-photon amplitudes are developed which are gauge invariant and manifest the low-energy theorems. In Sec. VI we present a discussion of gauge invariance[34–38] which clarifies the origin of our expressions for the electromagnetic interaction, and allows a consistent treatment of other processes, such as β decay in the presence of an external electromagnetic field.[13] In Sec. VII we apply our formalism to $0^+ \to 0^+$ two-photon emission. In Sec. VIII the momentum-translation approximation is derived and its role as a complex gauge, or seagull, term is explicated. Finally, we present our conclusions in Sec. IX.

III. ONE-PHOTON OPERATORS

A detailed derivation of the internal (i.e., no c.m. part) one-photon operators has been given elsewhere.[24] The primary result is that the internal electromagnetic current can be rearranged into the form

$$\mathbf{J}_{\text{int}}(\mathbf{q}) = i[H_0, \mathbf{d}(\mathbf{q})] - i\mathbf{q}\times\mathbf{m}_0(\mathbf{q}) , \tag{7a}$$

or

$$\mathbf{J}_{\text{int}}(\mathbf{x}) = i[H_0, \mathbf{d}(\mathbf{x})] + \nabla_x \times \mathbf{m}_0(\mathbf{x}) , \tag{7b}$$

where Eq. (7a) is the Fourier transform of Eq. (7b) and the subscript "0" refers to the internal part only. Moreover,

$$H = H_0 + H_R , \tag{8a}$$

$$H_0 \,|\, n\rangle = E_n^0 \,|\, n\rangle , \tag{8b}$$

where H_R is that part of H corresponding to center-of-mass motion,

$$H_R = \mathbf{P}^2/2m_t , \tag{9a}$$

$$\mathbf{P}\,|\,n\mathbf{P}_0\rangle = \mathbf{P}_0\,|\,n\mathbf{P}_0\rangle , \tag{9b}$$

$$|\,n\mathbf{P}_0\rangle = e^{i\mathbf{P}_0\cdot\mathbf{R}}\,|\,n\rangle , \tag{9c}$$

$$H_R\,|\,n\mathbf{P}_0\rangle = E_R\,|\,n\mathbf{P}_0\rangle . \tag{9d}$$

with $E_R = \mathbf{P}_0^2/2m_t$ and $m_t = Am$ for a nucleus composed of Z protons and N neutrons with a common nucleon mass, m. We have assumed translation invariance, labeling states with the momentum of the composite nucleus, which couples to the nuclear center-of-mass coordinate, $\mathbf{R}$. Clearly, the total nuclear energy corresponding to H is $E_n^0 + E_R$. We previously defined the average velocity, $\mathbf{V}$, corresponding to the matrix element of any operator, $\mathbf{O}$:

$$\left\langle m\mathbf{P}_f \left| \frac{\{\mathbf{P},\cdot\mathbf{O}\}}{2m_t} \right| n\mathbf{P}_i \right\rangle = \mathbf{V}\cdot\langle m\mathbf{P}_f\,|\,\mathbf{O}\,|\,n\mathbf{P}_i\rangle . \tag{10}$$

In addition to $\mathbf{V}$, one defines the momentum transfer: $\mathbf{q} = \mathbf{P}_f - \mathbf{P}_i$.

We note that Eqs. (7) correspond to a conserved current, which follows from Eq. (1) or

$$\mathbf{q}\cdot\mathbf{J}_{\text{int}}(\mathbf{q}) = i[H_0, \mathbf{q}\cdot\mathbf{d}(\mathbf{q})] = [H_0, \rho(\mathbf{q})] , \tag{11}$$

because one can show that[24]

$$\nabla\cdot\mathbf{d}(\mathbf{x}) = -\rho(\mathbf{x}) + Z\delta^3(\mathbf{x}) , \tag{12a}$$

or

$$\mathbf{q}\cdot\mathbf{d}(\mathbf{q}) = -i(\rho(\mathbf{q}) - Z) . \tag{12b}$$

The c-number terms proportional to Z commute with H_0, and were not included in Ref. 24. Thus, Eqs. (7) are a formal rearrangement of $\mathbf{J}(\mathbf{q})$, which isolates those terms determined by $\nabla\cdot\mathbf{J}(\mathbf{x})$, and then *uses* current continuity to produce a final current which satisfies current continuity, which is equivalent to gauge invariance for one-photon amplitudes. The rearrangement process dealt only with internal nuclear coordinates, the nuclear recoil or convection current

$$\mathbf{J}_{\text{conv}}(\mathbf{x}) = \left\{ \frac{\mathbf{P}}{2m_t}, \rho(\mathbf{x}) \right\} \tag{13}$$

having been ignored.

In order to include the recoil current, we must first examine the consequences of translation invariance in the matrix elements of any operator, $O(\mathbf{x})$:

$$\langle f\mathbf{P}_f\,|\,O(\mathbf{x})\,|\,i\mathbf{P}_i\rangle = e^{-i(\mathbf{P}_f-\mathbf{P}_i)\cdot\mathbf{x}}\langle f\mathbf{P}_f\,|\,O(0)\,|\,i\mathbf{P}_i\rangle , \tag{14a}$$

and for its Fourier transform

$$\int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}\langle f\mathbf{P}_f\,|\,O(\mathbf{x})\,|\,i\mathbf{P}_i\rangle = (2\pi)^3\delta^3(\mathbf{P}_f - \mathbf{P}_i - \mathbf{q})\langle f\mathbf{P}_f\,|\,O(0)\,|\,i\mathbf{P}_i\rangle . \tag{14b}$$

Equations (14) apply to each matrix element in perturbation theory, and lead to overall momentum conservation; they work without difficulty for nuclear operators such as the nonrelativistic charge and current operators which contain locality-preserving δ functions:

$$\rho(\mathbf{x}) = \sum_i e_i\delta^3(\mathbf{x} - \mathbf{x}_i) . \tag{15}$$

Separating the coordinate of the ith nucleon, $\mathbf{x}_i$, into the center-of-mass coordinate, $\mathbf{R}$, and a part, $\mathbf{x}_i'$, relative to the center of mass, and using Eq. (9c), immediately produces the results of Eqs. (14).

If one tries the same procedures for the operator $\mathbf{d}(\mathbf{x})$ defined in Eqs. (5), one finds the unsatisfactory result

$$\langle f\mathbf{P}_f\,|\,\mathbf{d}(\mathbf{x})\,|\,i\mathbf{P}_i\rangle=\mathbf{x}\int_0^1\frac{d\lambda}{\lambda^4}e^{-i(\mathbf{P}_f-\mathbf{P}_i)\cdot\mathbf{x}/\lambda}\times\langle f\mathbf{P}_f\,|\,\rho(0)\,|\,i\mathbf{P}_i\rangle\ , \tag{16}$$

whose origin lies in the superposition of δ functions implicit in Eqs. (5c) and (15). In fact, $\mathbf{d}(0)$ is infinite; as shown explicitly in Ref. 19, the singular part is

$$\mathbf{d}_s(\mathbf{x})\sim\frac{\mathbf{x}}{x^3}\int_0^\infty dz\, z^2\rho(\mathbf{z})\ . \tag{17}$$

This implies that

$$\nabla\cdot\mathbf{d}_s(\mathbf{x})\sim Z\delta^3(\mathbf{x})\ , \tag{18}$$

which is the singular part of Eq. (12a) and is the origin of the unphysical behavior at large momentum transfers (i.e., small $\mathbf{x}$) discussed in Sec. II. An identical singular term is also present in $\mathbf{m}_0(\mathbf{x})$ and cancellation occurs if and only if current continuity holds. In electrically neutral systems the singularity is not present. Thus, the rearrangement separates the new form of the current into two terms which separately violate translation invariance, while together they satisfy that condition, provided that current continuity holds for the original (i.e., dipole velocity) form of the current. For our purposes this is completely unsatisfactory, because our motivation was to produce a conserved current from one which is not necessarily conserved.

Precisely the same problem has arisen before in a different guise. Occasionally one encounters in the literature the "translation noninvariant" magnetic moment operator. This spurious term arises from Eq. (4) for the magnetization density operator. Using

$$\boldsymbol{\mu}=\tfrac{1}{2}\int d^3x\,\mathbf{x}\times\mathbf{J}(\mathbf{x})\ ,$$

one finds that $\boldsymbol{\mu}$ has a component proportional to $\mathbf{R}$ in addition to the usual magnetic moment operator which depends only on relative coordinates. Matrix elements of $\boldsymbol{\mu}(\mathbf{x})$ do not satisfy translation invariance, Eqs. (14), whereas the original current did. Indeed, the $\mathbf{R}$ term can be shown to arise from expanding the plane wave factors in Eq. (14a).

The obvious solution to the problem is to take matrix elements of the current and extract all momentum conserving δ functions *before* rearranging from the "standard" representation ($H_{\rm em}$) to the "electromagnetic field" (EMF) representation ($H'_{\rm em}$). This procedure leads to a proper magnetic moment operator which depends only on internal coordinates, and to the results of Ref. 24. Matrix elements of $\mathbf{J}_{\rm conv}$ can also be calculated in the same way.

$$\mathbf{J}_{\rm conv}(\mathbf{q},\mathbf{V})=\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\langle\mathbf{P}_f\,|\,\mathbf{J}_{\rm conv}(\mathbf{x})\,|\,\mathbf{P}_i\rangle\rightarrow\mathbf{V}\rho(\mathbf{q})\ , \tag{19}$$

where

$$\mathbf{q}\cdot\mathbf{J}_{\rm conv}=\mathbf{q}\cdot\mathbf{V}\rho(\mathbf{q})=\omega_R\rho(\mathbf{q})\ , \tag{20}$$

and $\omega_R=(\mathbf{P}_f^2-\mathbf{P}_i^2)/2m_t$ is the recoil energy. If we use the same technique to rearrange $\mathbf{J}_{\rm conv}$ that we employed in Sec. II of Ref. 24, and treat $\mathbf{V}$ as independent of $\mathbf{q}$, we obtain

$$\mathbf{J}_{\rm conv}(\mathbf{q},\mathbf{V})=Z\mathbf{V}-i\mathbf{q}\times[\mathbf{d}(\mathbf{q})\times\mathbf{V}]+i\omega_R\mathbf{d}(\mathbf{q})\ , \tag{21a}$$

or

$$\mathbf{J}_{\rm conv}(\mathbf{x},\mathbf{V})=Z\mathbf{V}\delta^3(\mathbf{x})+\nabla_x\times[\mathbf{d}(\mathbf{x})\times\mathbf{V}]-\mathbf{V}\cdot\nabla_x\mathbf{d}(\mathbf{x})\ , \tag{21b}$$

where it is understood that all center-of-mass coordinates have been removed from the problem. For example, we would replace $\mathbf{x}_i$ by $\mathbf{x}_i'$ in Eq. (15). A simple rearrangement of Eq. (21a) demonstrates that it is equivalent to Eq. (19). Moreover, it has precisely the form we exhibited in Eqs. (7), if we replace H_0 by H and define

$$\mathbf{J}(\mathbf{q},\mathbf{V})=\mathbf{J}_{\rm int}(\mathbf{q})+\mathbf{J}_{\rm conv}(\mathbf{q},\mathbf{V})=Z\mathbf{V}+i[H,\mathbf{d}(\mathbf{q})]-i\mathbf{q}\times\mathbf{m}(\mathbf{q},\mathbf{V})\ , \tag{22a}$$

where

$$\mathbf{q}\cdot\mathbf{J}(\mathbf{q},\mathbf{V})=[H,\rho(\mathbf{q})] \tag{22b}$$

and

$$\mathbf{m}(\mathbf{q},\mathbf{V})=\mathbf{m}_0(\mathbf{q})+\mathbf{d}(\mathbf{q})\times\mathbf{V}\ . \tag{22c}$$

Rearranging Eq. (12b) leads to

$$\rho(\mathbf{q})=Z+i\mathbf{q}\cdot\mathbf{d}(\mathbf{q})\ , \tag{22d}$$

and, therefore, by combining Eqs. (22) the four-vector current can be written as

$$J^\mu(\mathbf{q},\mathbf{V})=(\rho(\mathbf{q}),\mathbf{J}(\mathbf{q},\mathbf{V}))=J^\mu_{\rm pt}(\mathbf{q},\mathbf{V})+\Delta J^\mu+J'^\mu\ , \tag{23a}$$

$$J^\mu_{\rm pt}(\mathbf{q},\mathbf{V})=(Z,Z\mathbf{V})\ , \tag{23b}$$

$$\Delta J^\mu=i(0,[H,\mathbf{d}(\mathbf{q})]-q_0\mathbf{d}(\mathbf{q}))\ , \tag{23c}$$

$$J'^\mu=i(\mathbf{q}\cdot\mathbf{d}(\mathbf{q}),q_0\mathbf{d}(\mathbf{q})-\mathbf{q}\times\mathbf{m}(\mathbf{q},\mathbf{V}))\ , \tag{23d}$$

where we have added and subtracted a term, $iq_0\mathbf{d}$, proportional to the frequency, q_0, of the electromagnetic wave. Note that $q_\mu J'^\mu=0$, *without* enforcing energy conservation, and holds for both real and virtual photons.

It is possible to obtain other forms of $\mathbf{J}_{\rm conv}$, but none that mirror Eqs. (7). We have also paid a price for the rearrangement; what was originally a single term has become three, one of which is a point-particle current, $J^\mu_{\rm pt}$, which cannot excite the nucleus. We reiterate that in a q-congruent frame and transverse gauge $\mathbf{J}_{\rm conv}$ vanishes, as does $J^\mu_{\rm pt}$. In Eq. (22a) this is accommodated by the mutual cancellation of the last two terms.

IV. APPLICATION TO DEUTERON PHOTODISINTEGRATION

Deuteron forward photodisintegration offers a perfect testing ground for exploring the efficacy of our technique. This process is very sensitive to spin-dependent operators, and the dominant part of the meson-exchange currents (pion exchange) are spin dependent. We stated earlier that it is pointless to treat this process using any method which does not incorporate Siegert's theorem in the long wavelength limit. A traditional form of the electric multipoles (e.g., as summarized in Refs. 23 and 24) does this, but differs from our form. Keeping only (transverse) electric dipole terms up to order $\mathbf{q}^2$, these forms are

$$\mathbf{J}_{\text{old}} \simeq i[H_0,(\mathbf{D}-q^2\mathbf{O}/5)]+\frac{q^2}{2}\mathbf{N}_{\text{old}}\ , \tag{24a}$$

and from Eqs. (7),

$$\mathbf{J}_{\text{new}} \simeq i[H_0,(\mathbf{D}-q^2\mathbf{O}/30)]+\frac{q^2}{3}\mathbf{N}_{\text{new}}\ , \tag{24b}$$

where

$$\mathbf{O}=\sum_i e_i \mathbf{x}_i' x_i'^2\ ,$$

$$\mathbf{N}_{\text{new}}=\int d^3x\, \mathbf{x}\times\boldsymbol{\mu}(\mathbf{x})\ ,$$

and

$$\mathbf{N}_{\text{old}}=\int d^3x\, \mathbf{x}\,\mathbf{x}\cdot\mathbf{J}(\mathbf{x})\ .$$

In addition, the $\mathbf{N}$ terms can be decomposed into orbital ($\mathbf{N}_o$), pion-exchange ($\mathbf{N}_\pi$), and spin magnetization ($\mathbf{N}_S$) components. Because the latter is solenoidal, it is the same in both representations.[24] Explicit forms of all components of $\mathbf{N}_{\text{new}}$ were presented in Ref. 33. We have supplemented that work by also calculating the pion-exchange parts of $\mathbf{N}_{\text{old}}$, so that a direct comparison of all components of the retarded electric dipole operator could be made.

The various retardation terms are plotted in Fig. 1 versus photon energy, which depicts the percentage change in the forward photodisintegration cross section resulting from adding the retardation terms ($\sim q^2$) to the unretarded (i.e., $q^2=0$) electric dipole operator in Eqs. (24). The Reid soft core two-body potential was used. The largest correction arises from the spin-magnetization term, $\mathbf{N}_S$, while the $\mathbf{O}$ term is 6 times larger in the old representation than in the new. The two versions of the orbital part of $\mathbf{N}$ are comparable, though generally smaller in the new form. The pion-exchange currents contribute much less in the new representation. If the entire meson-exchange contribution was included in such a way as to guarantee a conserved current, the sum of all terms would be the same. This will not be exactly true for the pion-exchange currents, which are only the longest-range part of the meson-exchange currents, but is nonetheless approximately true.

Table I lists the percentage corrections to the unretarded (i.e., $q^2=0$) $E1$ cross section for the impulse approximation and for the pion-exchange current contributions (included in the operators $\mathbf{N}$) at a photon energy of 120 MeV. The latter are further decomposed into the isobar current (which is solenoidal), N_Δ, the seagull current, N_{SG}, and the true-exchange current, N_{ex}, and are plotted in Fig. 2. The solenoidal terms, N_S and N_Δ, must be the same in both representations. The remaining contributions are highly formalism dependent. Moreover, the latter terms are much larger in the "old" form than the "new," and strongly tend to cancel. This is particularly true for the sum of impulse and exchange currents (labeled "total" in Table I). They are not precisely the sum of the entries (they enter nonlinearly), but differ by less than 10%. This small difference in the total illustrates the primacy of the pion-exchange currents for the deuteron case. A related observation was made several years ago by Arenhövel.[39] The smallness and lack of cancellation among the nonsolenoidal terms demonstrates the superiority of the "new" forms.

We stated earlier that the new forms can be inferior to the old ones for very large momentum transfers, and we have demonstrated here that they are superior for small (or modest) momentum transfers. Where is the break-

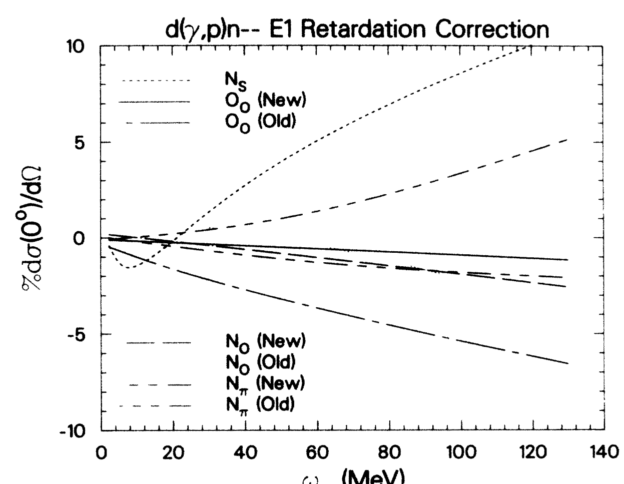

FIG. 1. Various retarded $E1$ contributions to the deuteron forward photodisintegration cross section, as a percentage of the unretarded contribution, plotted vs photon energy. The designations old and new refer to Eqs. (24a) and (24b), respectively, while S labels the spin-magnetization contribution to N from the impulse approximation, O labels the convection current contribution, and π labels the pion exchange-current contributions.

TABLE I. Percentage contributions of the retarded to unretarded $E1$ deuteron forward photodisintegration cross sections at $\omega_\gamma=120$ MeV using the "old" and "new" forms given by Eqs. (24a) and (24b), respectively. Each of these forms is broken down into impulse approximation and meson-exchange-current contributions, and their total. The MEC contributions are subdivided into isobar, seagull, and true-exchange contributions, labeled Δ, SG, and ex, respectively. The designations o and S refer to the convection and spin-magnetization parts of the impulse approximation current. Columns do not add precisely to the totals, because cross sections are quadratic in the amplitudes.

		New		Old	
	N_S	10.0		10.0	
	O_o	-1.0		-6.2	
	N_o	-2.4		-3.2	
Imp			6.6		0.6
	N_Δ	-1.7		-1.7	
	N_{SG}	-0.2		12.6	
	N_{ex}	-0.1		-5.9	
MEC			-1.9		4.6
Tot			4.6		4.8

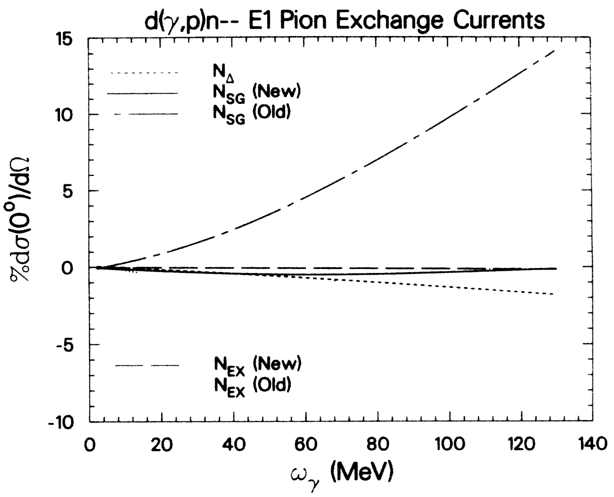

FIG. 2. Various retarded $E1$ pion-exchange-current contributions to N_π of Fig. 1, as a percentage of the unretarded contribution, plotted vs photon energy. The labels Δ, SG, and ex refer to the isobar, seagull, and true-exchange parts of the pion-exchange current, while old and new refer to Eqs. (24a) and (24b), respectively.

even point? While we have no precise answer to this question, we note that one important scale is provided by the analytic structure inherent in many problems: the location of branch points in the complex parameter plane, which determine the convergence of power series.[40] For electromagnetic interactions the appropriate parameter is q. For the elastic form factor the "edge" of the nucleus introduces a branch point at $q=\pm i\kappa$, where $\kappa=\sqrt{2ME_B}$, and E_B is the binding energy of the least bound wave function component. For photonuclear reactions, the analytic structure is more complicated because there is an extra variable, p, the momentum of the outgoing nucleon (either one in the deuteron case). For fixed p the branching ratios are located a distance κ above and below the real axis, which again sets the scale. Typically, this value is 100 MeV/c. A reasonable *guess* is that the efficacy of the "new" form of the electric multipoles extends to twice that value ($q=1$ fm^{-1}).

V. TWO-PHOTON AMPLITUDE

The standard representation for the nuclear electromagnetic interaction[3] has the form

$$H_{\rm em}=e\int d^3x[\rho(\mathbf{x})\phi(\mathbf{x})-\mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})] +\frac{e^2}{2}\int d^3x\int d^3y\, B^{mn}_{\rm SG}(\mathbf{x},\mathbf{y})A^m(\mathbf{x})A^n(\mathbf{y})+\cdots =H^{(1)}_{\rm em}+H^{(2)}_{\rm em}+\cdots \quad (25)$$

where we have included the seagull (A^2) terms explicitly and indicate with the ellipsis that there will, in general, be higher-order (in **A**) terms arising from meson currents. To the best of our knowledge none of these higher-order terms have ever been calculated, although the pionic contributions to $B^{mn}_{\rm SG}$ were calculated long ago.[41] The seagull vertex function satisfies the symmetry condition $B^{mn}_{\rm SG}(\mathbf{x},\mathbf{y})=B^{nm}_{\rm SG}(\mathbf{y},\mathbf{x})$. As we stated earlier, gauge invariance for $H^{(1)}_{\rm em}$ implies that the charge and current operators satisfy current conservation. If $H^{(2)}_{\rm em}$ is included, the additional requirement[2] is

$$[J^m(\mathbf{x}),\rho(\mathbf{y})]=i\nabla^k_y B^{mk}_{\rm SG}(\mathbf{x},\mathbf{y})\,, \quad (26a)$$

or, in momentum space,

$$[J^m_{\rm int}(\mathbf{q}_1),\rho(\mathbf{q}_2)]=-\frac{q_2^m}{m_t}\rho(\mathbf{q}_1)\rho(\mathbf{q}_2) +q_2^k B^{km}_{\rm SG}(\mathbf{q}_1,\mathbf{q}_2)\,. \quad (26b)$$

An equivalent statement can be obtained from crossing ($m\leftrightarrow n$, $\mathbf{x}\leftrightarrow\mathbf{y}$). Note[2] the recoil term ($\sim 1/m_t$) in Eq. (26b).

The two-photon amplitudes have the graphical form depicted in Figs. 3(a)—3(c), with all photons taken as incoming. The substitution rule[2] gives the corresponding two-photon emission, two-photon absorption, and photon scattering amplitudes. For the latter process, $i=f$ defines the (elastic) nuclear Compton case, while $i=$ground state$\neq f$ is the nuclear (Stokes) Raman (i.e., excitation) case and $i\neq f=$ground state is the (anti-Stokes) Raman (i.e., deexcitation) case. The intermediate state n is *any* of the states (8b). The three processes depicted in Figs. 3(a)—3(c) are the "direct," "crossed," and "seagull" processes. The corresponding S-matrix element is

$$S_{fi}=-ie^2N_1N_2(2\pi)^4\delta^4(P_f-P_i-q_i-q_2) \times[T^{\mu\nu}(q_1,q_2)\epsilon^\mu(\mathbf{q}_2)\epsilon^\nu(\mathbf{q}_1)]\,, \quad (27)$$

where $N_i=(\langle n\rangle_i/2\omega_i)^{1/2}$ is the normalization factor for photon i (with occupation number $\langle n\rangle_i$, usually taken to be 1) and $\epsilon^\mu(\mathbf{q})$ is the polarization four vector of a photon with momentum $\mathbf{q}$. Conservation of energy requires that

$$q_1^0+q_2^0=\omega_0+\omega_R\,, \quad (28a)$$

where

$$\omega_0=E_f^0-E_i^0\,. \quad (28b)$$

The amplitude $T^{\mu\nu}$ has the form

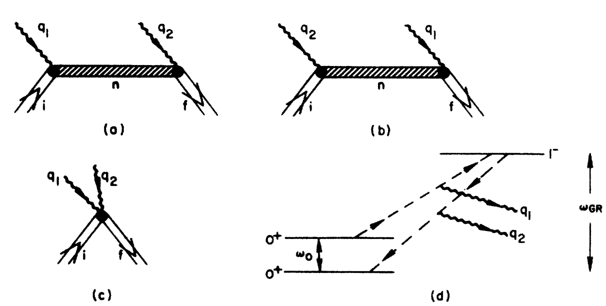

FIG. 3. The direct, crossed, and seagull components of the nuclear Compton amplitude are depicted in (a), (b), and (c), while (d) illustrates schematically the 2γ decay of a 0^+ excited state to a 0^+ ground state via an intermediate (virtual) giant dipole state.

$$T^{\mu\nu}(q_1,q_2)=\sum_n \frac{\langle f\,|\,J^{\mu}(\mathbf{q}_2,\mathbf{V}_L^a)\,|\,n\,\rangle\langle\, n\,|\,J^{\nu}(\mathbf{q}_1,\mathbf{V}_R^a)\,|\,i\,\rangle}{E_i^0-E_n^0+q_1^0-\omega_a+i\epsilon}+\sum_n \frac{\langle f\,|\,J^{\nu}(\mathbf{q}_1,\mathbf{V}_L^b)\,|\,n\,\rangle\langle\, n\,|\,J^{\mu}(\mathbf{q}_2,\mathbf{V}_R^b)\,|\,i\,\rangle}{E_i^0-E_n^0+q_2^0-\omega_b+i\epsilon}$$

$$+\langle f\,|\,B_{\rm SG}^{\mu\nu}(\mathbf{q}_1,\mathbf{q}_2)\,|\,i\,\rangle\equiv T_D^{\mu\nu}(J,J)+\langle f\,|\,B_{\rm SG}^{\mu\nu}\,|\,i\,\rangle\ , \tag{29}$$

where

$$\mathbf{V}_L^a=(2\mathbf{P}_f-\mathbf{q}_2)/2m_t\ ,$$

$$\mathbf{V}_R^a=(2\mathbf{P}_i+\mathbf{q}_1)/2m_t\ ,$$

$$\mathbf{V}_L^b=(2\mathbf{P}_f-\mathbf{q}_1)/2m_t\ ,$$

$$\mathbf{V}_R^b=(2\mathbf{P}_i+\mathbf{q}_2)/2m_t\ ,$$

$$\omega_a=\mathbf{q}_1\cdot\mathbf{V}_R^a\ ,$$

and

$$\omega_b=\mathbf{q}_2\cdot\mathbf{V}_R^b\ .$$

The three terms in (29) correspond to the first three diagrams of Fig. 3, respectively, and together satisfy the gauge invariance[2,3] conditions, $q_2^{\mu}T_{\mu\nu}=q_1^{\nu}T_{\mu\nu}\equiv 0$, which lead to Eqs. (26). We have explicitly separated the recoil energies (ω) from binding energies in the energy denominators of the dispersive terms (labeled "D") and have indicated the dependence on the nuclear recoil velocities, $\mathbf{V}$.

Our goal is to rewrite Eq. (29) (expressed in terms of $\mathbf{J}$ and ρ) in terms of $\mathbf{d}(\mathbf{q})$ and $\mathbf{m}(\mathbf{q})$. This is accomplished by using Eqs. (23). We note that the matrix elements of ΔJ^{μ} are proportional to the energy denominators, which are thereby cancelled. The resulting amplitude components can be grouped with the seagull amplitudes from $B_{\rm SG}^{\mu\nu}$. The point-nucleus current, $J_{\rm pt}^{\mu}$, cannot lead to nuclear transitions, while J'^{μ} is conserved; it leads to Compton amplitude components which are *manifestly* gauge invariant. We first define

$$\alpha(z)=\int_0^1 d\lambda\, e^{iz\lambda} \tag{30a}$$

and

$$\beta(z)=2\int_0^1 d\lambda\, \lambda e^{iz\lambda}\ , \tag{30b}$$

so that our previously defined quantities can be expressed as

$$\mathbf{d}(\mathbf{q})=\int d^3x\, \rho(\mathbf{x})\alpha(\mathbf{q}\cdot\mathbf{x})\mathbf{x}\ , \tag{31a}$$

and

$$\mathbf{m}_0(\mathbf{q})=\int d^3x\, \boldsymbol{\mu}(\mathbf{x})\beta(\mathbf{q}\cdot\mathbf{x})\ , \tag{31b}$$

and new operators are defined:

$$a(\mathbf{q})=\int d^3x\, \rho(\mathbf{x})\alpha(\mathbf{q}\cdot\mathbf{x})\ , \tag{32a}$$

$$\mathbf{b}(\mathbf{q})=\int d^3x\, \rho(\mathbf{x})\beta(\mathbf{q}\cdot\mathbf{x})\mathbf{x}\ . \tag{32b}$$

This allows us to write

$$\epsilon^{\mu}\epsilon'^{\nu}T^{\mu\nu}=\epsilon^{\mu}\epsilon'^{\nu}[T_D^{\mu\nu}(J'+J_{\rm pt},J'+J_{\rm pt})+\langle f\,|\,B_{\rm SG}^{\mu\nu}\,|\,i\,\rangle]+\langle f\,|\,\Gamma\,|\,i\,\rangle\ , \tag{33a}$$

where

$$\Gamma=-\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{m_t}\boldsymbol{\epsilon}'\cdot\mathbf{d}(\mathbf{q}_1)\boldsymbol{\epsilon}\cdot\mathbf{d}(\mathbf{q}_2)-\frac{i}{m_t}[\boldsymbol{\epsilon}\cdot\mathbf{q}_1\rho(\mathbf{q}_1)\boldsymbol{\epsilon}'\cdot\mathbf{d}(\mathbf{q}_1)+\boldsymbol{\epsilon}'\cdot\mathbf{q}_2\rho(\mathbf{q}_2)\boldsymbol{\epsilon}\cdot\mathbf{d}(\mathbf{q}_2)]$$

$$-i[\boldsymbol{\epsilon}\cdot\mathbf{J}_{\rm int}(\mathbf{q}_2),\,\boldsymbol{\epsilon}'\cdot\mathbf{d}(\mathbf{q}_1)]-i[\boldsymbol{\epsilon}'\cdot\mathbf{J}_{\rm int}(\mathbf{q}_1),\,\boldsymbol{\epsilon}\cdot\mathbf{d}(\mathbf{q}_2)]-[[H_0,\,\boldsymbol{\epsilon}\cdot\mathbf{d}(\mathbf{q}_2)],\,\boldsymbol{\epsilon}'\cdot\mathbf{d}(\mathbf{q}_1)]\ , \tag{33b}$$

and $\epsilon'^{\nu}\equiv\epsilon^{\nu}(\mathbf{q}_1)$ and $\epsilon^{\mu}\equiv\epsilon^{\mu}(\mathbf{q}_2)$. Using Eqs. (26) one can obtain the following variants of our basic gauge invariance condition:

$$i[J_{\rm int}^m(\mathbf{q}_2),d^n(\mathbf{q}_1)]=-\frac{\nabla_{q_1}^n q_1^m}{m_t}[a(\mathbf{q}_1)\rho(\mathbf{q}_2)]+\nabla_{q_1}^n q_1^k\int_0^1 d\lambda\, B_{\rm SG}^{mk}(\lambda\mathbf{q}_1,\mathbf{q}_2) \tag{34a}$$

and

$$[[H_0,d^m(\mathbf{q}_2)],\,d^n(\mathbf{q}_1)]=\nabla_{q_2}^m\nabla_{q_1}^n\left[\frac{q_2^k q_1^k}{m_t}a(\mathbf{q}_1)a(\mathbf{q}_2)-q_2^l q_1^k\int_0^1 d\lambda\int_0^1 d\lambda' B_{\rm SG}^{lk}(\lambda\mathbf{q}_1,\lambda'\mathbf{q}_2)\right]\ . \tag{34b}$$

With some foresight and the structure of Eqs. (33) in mind, we define

$$\widetilde{B}_{\rm SG}^{mn}=B_{\rm SG}^{mn}+\nabla_{q_2}^m\nabla_{q_1}^n q_2^l q_1^k\int_0^1 d\lambda\int_0^1 d\lambda' B_{\rm SG}^{lk}(\lambda\mathbf{q}_1,\lambda'\mathbf{q}_2)-\nabla_{q_1}^n q_1^k\int_0^1 d\lambda\, B_{\rm SG}^{mk}(\lambda\mathbf{q}_1,\mathbf{q}_2)-\nabla_{q_2}^m q_2^k\int_0^1 d\lambda' B_{\rm SG}^{kn}(\mathbf{q}_1,\lambda'\mathbf{q}_2)\ , \tag{35}$$

from which a tedious calculation leads to

$$\epsilon^m \epsilon'^n \widetilde{B}^{mn}_{\rm SG}(\mathbf{q}_1,\mathbf{q}_2) = -\tfrac{1}{4}(\boldsymbol{\epsilon}\times\mathbf{q}_2)^s(\boldsymbol{\epsilon}'\times\mathbf{q}_1)^r \epsilon^{rmn}\epsilon^{stv}\int d^3x\, \beta(\mathbf{q}_2\cdot\mathbf{x})x^t \int d^3y\, \beta(\mathbf{q}_1\cdot\mathbf{y})y^m B^{vn}_{\rm SG}(\mathbf{x},\mathbf{y})$$
$$\equiv (\boldsymbol{\epsilon}\times\mathbf{q}_2)^s(\boldsymbol{\epsilon}'\times q_1)^r \chi^{rs}_D(\mathbf{q}_1,\mathbf{q}_2)$$
$$= \left\{ [(\boldsymbol{\epsilon}\times\mathbf{q}_2)\times\nabla_{q_2}]^m [(\boldsymbol{\epsilon}'\times\mathbf{q}_1)\times\nabla_{q_1}]^n \int_0^1 d\lambda \int_0^1 d\lambda' B^{mn}_{\rm SG}(\lambda\mathbf{q}_1,\lambda'\mathbf{q}_2)\right\} . \tag{36}$$

Equations (33)–(36) can now be combined. The commutators in Eqs. (33) are eliminated in terms of $B^{mn}_{\rm SG}$ and recoil terms. The $B^{mn}_{\rm SG}$ terms all combine to form the single $\widetilde{B}^{mn}_{\rm SG}$ term. Using the Cartesian tensor identity (53) of Ref. 2 and the identity

$$a(\mathbf{q}) - \rho(\mathbf{q}) = -\frac{i}{2}\mathbf{q}\cdot\mathbf{b}(\mathbf{q}) \tag{37}$$

allows the recoil terms to be grouped conveniently. The final result for T is now labeled with a prime, T', to indicate the massive rearrangement:

$$T'^{\mu\nu}(q_1,q_2) = \{T^{\mu\nu}_D(J',J') + \langle f \,|\, [\widetilde{B}^{\mu\nu}_{\rm SG}(\mathbf{q}_1,\mathbf{q}_2) + B^{\mu\nu}_R(\mathbf{q}_1,\mathbf{q}_2) + B^{\mu\nu}_T(\mathbf{q}_1,\mathbf{q}_2)] \,|\, i\rangle\} + \delta_{fi}R^{\mu\nu}_P + R^{\mu\nu}_1 + R^{\mu\nu}_2 , \tag{38}$$

where $T^{\mu\nu}_D(J',J')$ are the two dispersive terms in Eq. (29) calculated with J' in Eq. (23d) replacing J, $\widetilde{B}_{\rm SG}$ is defined in Eq. (35), the recoil correction to the diamagnetic susceptibility is given by

$$\epsilon^m\epsilon'^n B^{mn}_R(\mathbf{q}_1,\mathbf{q}_2) = \frac{1}{m_t}(\boldsymbol{\epsilon}\times\mathbf{q}_2)^m(\boldsymbol{\epsilon}'\times\mathbf{q}_1)^n[d^m(\mathbf{q}_1)d^n(\mathbf{q}_2) - b^m(\mathbf{q}_1)b^n(\mathbf{q}_2)/4 - \delta^{mn}(\mathbf{d}(\mathbf{q}_1)\cdot\mathbf{d}(\mathbf{q}_2) - \mathbf{b}(\mathbf{q}_1)\cdot\mathbf{b}(\mathbf{q}_2)/4)] , \tag{39}$$

and the generalized Thomson amplitude is given by

$$\epsilon^m\epsilon'^n B^{mn}_T(\mathbf{q}_1,\mathbf{q}_2) = \frac{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}'}{m_t} Z(\rho(\mathbf{q}_1) + \rho(\mathbf{q}_2) - Z) . \tag{40}$$

In addition, we have

$$R^{\mu\nu}_P = \frac{J^\mu_{\rm pt}(\mathbf{q}_2,\mathbf{V}^a_L)J^\nu_{\rm pt}(\mathbf{q}_1,\mathbf{V}^a_R)}{q^0_1 - \omega_a} + \frac{J^\nu_{\rm pt}(\mathbf{q}_1,\mathbf{V}^b_L)J^\mu_{\rm pt}(\mathbf{q}_2,\mathbf{V}^b_R)}{q^0_2 - \omega_b} , \tag{41}$$

$$\epsilon^m\epsilon'^n R^{mn}_1 = -\frac{iZ}{m_t}[\boldsymbol{\epsilon}\cdot\mathbf{q}_1\boldsymbol{\epsilon}'\cdot\langle f \,|\, \mathbf{d}(\mathbf{q}_1) \,|\, i\rangle + \boldsymbol{\epsilon}'\cdot\mathbf{q}_2\boldsymbol{\epsilon}\cdot\langle f \,|\, \mathbf{d}(\mathbf{q}_2) \,|\, i\rangle] , \tag{42}$$

$$R^{\mu\nu}_2 = \frac{J^\mu_{\rm pt}(\mathbf{q}_2,\mathbf{V}^a_L)\langle f \,|\, J'^\nu(\mathbf{q}_1,\mathbf{V}^a_R) \,|\, i\rangle}{q^0_1 - \omega_0 - \omega_a} + \frac{J^\nu_{\rm pt}(\mathbf{q}_1,\mathbf{V}^b_L)\langle f \,|\, J'^\mu(\mathbf{q}_2,\mathbf{V}^b_R) \,|\, i\rangle}{q^0_2 - \omega_0 - \omega_b}$$
$$+ \frac{\langle f \,|\, J'^\mu(\mathbf{q}_2,\mathbf{V}^a_L) \,|\, i\rangle J^\nu_{\rm pt}(\mathbf{q}_1,\mathbf{V}^a_R)}{q^0_1 - \omega_a} + \frac{\langle f \,|\, J'^\nu(\mathbf{q}_1,\mathbf{V}^b_L) \,|\, i\rangle J^\mu_{\rm pt}(\mathbf{q}_2,\mathbf{V}^b_R)}{q^0_2 - \omega_b} . \tag{43}$$

The terms in Eqs. (41) and (43) involve the intermediate states $n = i$ or $n = f$.

These results look exceptionally complex, but, in fact, are not. In transverse gauge, ($\boldsymbol{\epsilon}\cdot\mathbf{q}_2 = \boldsymbol{\epsilon}'\cdot\mathbf{q}_1 = 0$), R_P vanishes in the lab frame as do the last two terms in R_2. For $0^+ \to 0^+$ transitions the R_1 term and the remaining R_2 terms cancel, leaving only the curly-bracketed terms in Eq. (38). The complete amplitude $T'^{\mu\nu}$ is gauge invariant, as are $T^{\mu\nu}_D(J',J')$, $\widetilde{B}^{\mu\nu}_{\rm SG}$, $B^{\mu\nu}_R$, and $R^{\mu\nu}_2$. The non-gauge-invariant terms $R^{\mu\nu}_P$ and $R^{\mu\nu}_1$ together with the constant and ρ terms of $B^{\mu\nu}_T$ are, respectively, gauge invariant. Each of the curly-bracketed terms in Eq. (38) leads to one or more terms in the usual low-energy theorem expansion for long photon wavelengths, which we describe in Sec. VI. Parts of these results have been obtained before.[17,42]

VI. $0^+ \to 0^+$ TRANSITIONS

The $0^+ \to 0^+$ transitions form the single most important case for the Compton amplitude, and we treat this case exclusively here. We will also restrict ourselves to transverse gauge in the lab frame: $\boldsymbol{\epsilon}\cdot\mathbf{q}_2 = \boldsymbol{\epsilon}'\cdot\mathbf{q}_1 \equiv 0$. These restrictions eliminate all of the ugly terms, which exhibit elastic (i.e., purely recoil) denominators for $f = i$. We expand the remaining terms to order q^2. We also define

$$\langle 0^+ \,|\, \rho(\mathbf{q}) \,|\, 0^+\rangle \equiv F(q^2) \cong Z\left[\delta_{fi} - \frac{q^2\langle r^2\rangle}{6} + \cdots\right] , \tag{44}$$

which leads trivially to

$$\epsilon^m\epsilon'^n B^{mn}_T \cong \frac{\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}' Z^2}{m_t}\left[\delta_{fi} - (\mathbf{q}_1^2 + \mathbf{q}_2^2)\frac{\langle r^2\rangle}{6} + \cdots\right] . \tag{45}$$

In the long-wavelength limit, $\mathbf{d} \to \mathbf{D}$ and $\mathbf{b} \to \mathbf{D}$, and we obtain

$$\epsilon^m\epsilon'^n B^{mn}_R \cong -\frac{(\boldsymbol{\epsilon}\times\mathbf{q}_2)\cdot(\boldsymbol{\epsilon}'\times\mathbf{q}_1)}{2m_t}\langle f \,|\, \mathbf{D}^2 \,|\, i\rangle . \tag{46}$$

The seagull term $\widetilde{B}_{\rm SG}$ contains contributions[2] from both the impulse approximation and nonrelativistic meson-exchange currents,[41] unlike the terms above which are determined solely by $\rho(\mathbf{q})$ and hence by the impulse approximation in the nonrelativistic limit. The general long-wavelength limit of $\widetilde{B}_{\rm SG}$ is given trivially by Eq. (36) with $\beta(0)=1$:

$$\epsilon^m \epsilon'^n \widetilde{B}\,^{mn}_{\rm SG}=(\boldsymbol{\epsilon}\times\mathbf{q}_2)^s(\boldsymbol{\epsilon}'\times\mathbf{q}_1)^r \chi_D^{sr} \ , \tag{47a}$$

where χ_D^{mn} is the unretarded diamagnetic susceptibility tensor,

$$\chi_D^{sr}=-\frac{\epsilon^{rmn}\epsilon^{stu}}{4}\int d^3x \int d^3y\, x^t y^m B^{\rm un}_{\rm SG}(\mathbf{x},\mathbf{y}) \rightarrow \chi_D \delta^{sr} \ , \tag{47b}$$

and the last form applies to $0^+\rightarrow 0^+$:

$$\chi_D=-\tfrac{1}{12}\int d^3x \int d^3y [\mathbf{x}\cdot\mathbf{y} B^{nn}_{\rm SG}(\mathbf{x},\mathbf{y})-x^n y^m B^{mn}_{\rm SG}(\mathbf{x},\mathbf{y})] \ . \tag{47c}$$

The impulse approximation to $B^{mn}_{\rm SG}$ has the form

$$B^{mn}_{\rm imp}(\mathbf{x},\mathbf{y})=\delta^{mn}\delta^3(\mathbf{x}-\mathbf{y})\frac{\rho(\mathbf{x})}{m} \ , \tag{48a}$$

from which we obtain, with the aid of Eq. (39), the standard result

$$\chi_D^{\rm imp}=-\frac{Z}{6m}\langle r^2\rangle \ . \tag{48b}$$

Practical forms of $\widetilde{B}_{\rm imp}$ are given in the Appendix.

The remaining term is $T_D(J',J')$. In the long-wavelength limit $\mathbf{m}_0\rightarrow\boldsymbol{\mu}$, $\mathbf{d}\rightarrow\mathbf{D}$, and the recoil part of $\mathbf{m}(\mathbf{q},\mathbf{V})$ can be neglected because it is of order q^2 and generates terms of higher order than q^2 in the low-energy expansion. Assuming the nuclear states have good parity, one finds, for $0^+\rightarrow 0^+$ transitions,

$$\begin{aligned} T_D \rightarrow & -\mathbf{E}_1\cdot\mathbf{E}_2\left[2\sum_{n\neq 0}\frac{\langle f\,|\,D_z\,|\,n\rangle\langle n\,|\,D_z\,|\,i\rangle}{\Delta E_n}\right] \\ & -\mathbf{B}_1\cdot\mathbf{B}_2\left[2\sum_{n\neq 0}\frac{\langle f\,|\,\mu_z\,|\,n\rangle\langle n\,|\,\mu_z\,|\,i\rangle}{\Delta E_n}\right] \\ & =-\mathbf{E}_1\cdot\mathbf{E}_2\alpha_E-\mathbf{B}_1\cdot\mathbf{B}_2\chi_P \ , \end{aligned} \tag{49}$$

where

$$\mathbf{E}_i=i(q_i^0\boldsymbol{\epsilon}_i-\mathbf{q}_i\epsilon_i^0) \ , \tag{50a}$$

$$\mathbf{B}_i=i\mathbf{q}_i\times\boldsymbol{\epsilon}_i \ , \tag{50b}$$

and

$$-2\Delta E_n^{-1}=(E_i^0-E_n^0+q_1^0)^{-1}+(E_i^0-E_n^0+q_2^0)^{-1} \ . \tag{50c}$$

Obviously, T_D is gauge invariant, and depends only on the (frequency-dependent) electric polarizability, α_E, and paramagnetic susceptibility, χ_P. Each term in our rearranged Compton amplitude generates an important term in the low-energy theorem.[2]

Most of the terms in the low-energy theorem are relatively easy to calculate, even in a many-body system. The primary problem lies with T_D, which involves a complicated energy denominator, ΔE_n. For elastic scattering one can simply ignore the q_i^0 factors in the denominators. For $0^+\rightarrow 0^+$ two-photon decays, shown in Fig. 3(d), one can approximate q_i^0 by $\omega_0/2$ and then expand about this point. Defining $\overline{E}_0=(E_f^0+E_i^0)/2$ and ignoring recoil, one finds

$$\begin{aligned} \Delta E_n^{-1} &= \frac{-(\overline{E}_0-E_n)}{[\overline{E}_0-E_n+(q_1^0-\omega_0/2)][\overline{E}_0-E_n+(q_2^0-\omega_0/2)]} \\ &\cong (E_n-\overline{E}_0)^{-1}(1+\delta_n) \ , \end{aligned} \tag{51a}$$

where

$$\delta_n=(q_1^0-\omega_0/2)^2/(E_n-\overline{E}_0)^2 \ . \tag{51b}$$

The maximum value of $(q_1^0-\omega_0/2)^2$ is $\omega_0^2/4$. Consequently, a good estimate of the maximum value of the correction, $\overline{\delta}_n$, is obtained by replacing (E_n-E_0) with the giant dipole resonance energy, $\omega_{\rm GR}$. Six $0^+\rightarrow 0^+$ ($E1$-$E1$) transitions are listed in Table II; the percentage corrections are very simply estimated and range from small to tiny.

Performing the sum over intermediate states for two-step processes such as Compton scattering or double β decay is usually quite difficult unless one is certain that only a few states saturate the sum. A better way to proceed is to use Podolsky's method,[43–45] which relies on the formal properties of perturbation theory. One evaluates

$$\alpha_E=2\sum_{n\neq f,i}\frac{\langle f\,|\,D_z\,|\,n\rangle\langle n\,|\,D_z\,|\,i\rangle}{E_n-\overline{E}_0} \tag{52a}$$

by writing

$$\alpha_E=-2\langle f\,|\,D_z\,|\,\Delta\Psi\rangle \ , \tag{52b}$$

where

$$\begin{aligned}(\overline{E}_0-H_0)\,|\,\Delta\Psi\rangle &= D_z\,|\,i\rangle-|\,i\rangle\langle i\,|\,D_z\,|\,i\rangle \\ &\quad -|\,f\rangle\langle f\,|\,D_z\,|\,i\rangle \end{aligned} \tag{52c}$$

is subject to the conditions $\langle f\,|\,\Delta\Psi\rangle=\langle i\,|\,\Delta\Psi\rangle=0$, which are trivially satisfied for $0^+\rightarrow 1^-\rightarrow 0^+$ two-step processes because of parity conservation. In such cases very different approximation techniques may be used in the many-body problem for the intermediate states n, and for

TABLE II. First-excited (0^+) state (ω_0) and giant dipole resonance ($\omega_{\rm GR}$) energies for six nuclei which manifest $0^+\rightarrow 0^+$ 2γ decays. The maximum-estimated percentage corrections for the static ($E1$-$E1$) denominator approximation is given by $\overline{\delta}_n$.

	ω_0 (Mev)	$\omega_{\rm GR}$ (MeV)	$\overline{\delta}_n$ (%)
^{16}O	6.05	23.0	1.7
^{40}Ca	3.35	20.0	0.7
^{72}Ge	0.69	16.4	0.04
^{90}Zr	1.76	16.8	0.27
^{96}Zr	1.59	16.0	0.24
$^{98}M_0$	0.74	15.8	0.05

the states i and f. If $f=i$, only a single term is subtracted on the right-hand side of (52c). We note that the same technique can be used for each of the complex energy denominators in Eq. (50c) (or for T_D itself), but this requires solving Eq. (52c) with $\tilde{E}_0 \to E_i^0 + q_i^0$ or $E_i^0+q_2^0$ for *each* value of q_i^0.

VII. GAUGE INVARIANCE

The structure of the current in Eq. (23) suggests that we can rearrange the standard form of the electromagnetic Hamiltonian, $H_{\rm em}^{(1)}$, using the identities (22a) and (12) for $\rho(\mathbf{x})$ and for $\mathbf{J}(\mathbf{x})$:

$$e\int d^3x[\rho(\mathbf{x})\phi(\mathbf{x})-\mathbf{J}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})]=eZ[\phi(0)-\mathbf{V}\cdot\mathbf{A}(0)] -e\int d^3x[\mathbf{E}(\mathbf{x})\cdot\mathbf{d}(\mathbf{x})+\mathbf{m}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})]-e\int d^3x\{\mathbf{d}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x},t)+i[H,\mathbf{d}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})]\}\ . \tag{53}$$

Explicit time dependence of the fields is ignored, except where required for clarity. Matrix elements of the last term clearly vanish if we enforce energy conservation (i.e., an on-shell one-photon process), but not otherwise. The last term in Eq. (53) looks like a first-order (in e) unitary transformation on the Schrödinger equation, using a Hermitian operator Λ:

$$\Lambda=e\int d^3x\, \mathbf{d}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x},t)\ , \tag{54}$$

$$H_t=H+H_{\rm em}^{(1)}+H_{\rm em}^{(2)}\ , \tag{55a}$$

$$H_{\rm em}^{(1)}=H_{\rm em}^{(1)\prime}-\dot{\Lambda}-i[H,\Lambda]\ . \tag{55b}$$

Performing the indicated unitary transformation of H_t, we find

$$H_t'=\left[e^{-i\Lambda}\left[H_t-i\frac{\partial}{\partial t}\right]e^{i\Lambda}\right]=H+H_{\rm em}^{(1)}+i[H,\Lambda]+\dot{\Lambda} +H_{\rm em}^{(2)}-i[\Lambda,\dot{\Lambda}]/2-i[\Lambda,H_{\rm em}^{(1)}]-\tfrac{1}{2}[\Lambda,[\Lambda,H]]+\cdots=H+H_{\rm em}' \tag{56a}$$

and

$$\Psi'=e^{-i\Lambda}\Psi\ . \tag{56b}$$

This defines $H_{\rm em}'$ as

$$H_{\rm em}'=H_{\rm em}'^{(1)}+H_{\rm em}'^{(2)}+\cdots=H_{\rm em}'^{(1)}+(H_{\rm em}^{(2)}-i[\Lambda,\dot{\Lambda}]/2-i[\Lambda,H_{\rm em}^{(1)}]-\tfrac{1}{2}[\Lambda,[\Lambda,H]])+\cdots\ . \tag{56c}$$

We also note that, in the *impulse approximation*, $H_{\rm em}^{(2)}$ is a functional of ρ, as is Λ (in general); because

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0 \tag{57}$$

for nonrelativistic charge operators, and because repeated commutators of $\mathbf{J}$ or H with ρ also will vanish, the series in Eqs. (56) will terminate at order e^2. Moreover, it is easy to show that the definition of $H_{\rm em}'^{(2)}$ corresponds to Eq. (36):

$$H_{\rm em}'^{(2)}=-\tfrac{1}{2}\int d^3x\int d^3y\,\chi_D^{mn}(\mathbf{x},\mathbf{y})B^m(\mathbf{x})B^n(\mathbf{y})\ , \tag{58}$$

which is manifestly gauge invariant, as is $H_{\rm em}'^{(1)}$. The fact that $H_{\rm em}'$ has this property is a significant result.

The unitary transformation function Λ also plays an important role in gauge transformations. The function Λ can be reexpressed in terms of nuclear coordinates by using Eq. (15):

$$\Lambda=e\int_0^1 d\lambda\int d^3y\,\mathbf{y}\cdot\mathbf{A}(\lambda\mathbf{y})\rho(\mathbf{y}) =e\sum_{i=1}^{A}e_i\mathbf{x}_i\cdot\int_0^1 d\lambda\,\mathbf{A}(\lambda\mathbf{x}_i) \tag{59a}$$

$$=e\sum_{i=1}^{A}e_i\int_{c_i}^{\mathbf{x}_i}d\mathbf{s}\cdot\mathbf{A}(\mathbf{s})\ . \tag{59b}$$

The last form expresses the ith term in the sum as a line integral over the path c_i, which is a *straight-line* path from $\mathbf{s}=0$ to $\mathbf{s}=\mathbf{x}_i$. Such paths produce the minimal coupling scheme[35,36] in the familiar $[(\mathbf{p}-e\mathbf{A}/c)^2/2m+V]$ nonrelativistic Schrödinger Hamiltonian. Indeed, a formal solution, Ψ, for that Hamiltonian can sometimes be developed[37,38] from the solution, Ψ_0, of the field-free case, Ψ_0 by writing

$$\Psi=\exp\left[ie\int_c^{\mathbf{r}}d\mathbf{s}\cdot\mathbf{A}(\mathbf{s})+\cdots\right]\Psi_0=e^{i\Phi}\Psi_0\ , \tag{60}$$

where the ellipsis indicates further contributions which depend only on the electromagnetic *fields,* rather than the *potentials.* Under a gauge transformation, $\mathbf{A}(\mathbf{s})\to\mathbf{A}(\mathbf{s})+\nabla\Lambda(\mathbf{s})$, the exponential of Φ develops the necessary additional phase

$$e\int_c^{\mathbf{r}}d\mathbf{s}\cdot\nabla\Lambda(\mathbf{s})=e[\Lambda(\mathbf{r})-\Lambda(0)]\ , \tag{61}$$

which is independent of the path c, by Stokes's theorem. Thus, we obtain the familiar

$$\Psi\to e^{i(\Lambda(\mathbf{r})-\Lambda(0))}\Psi\ , \tag{62}$$

as shown by Foldy.[26] The result (60) is largely useless, except for a few special cases,[38] because the integral is path dependent, as is its gradient with respect to $\mathbf{r}$. The path dependence can be expressed solely in terms of the magnetic field, $\mathbf{B}(\mathbf{r})$, using Stokes's theorem. Nevertheless, it is easy to show that

$$e^{-i\Phi}\frac{(\mathbf{p}-e\mathbf{A})^2}{2m}e^{i\Phi}\sim\frac{\mathbf{p}^2}{2m}+\mathbf{B}\text{ terms}\ , \tag{63}$$

or, more succinctly, the transformation $e^{i\Phi}$ "undoes" the minimal coupling, at the cost of introducing additional magnetic-field-dependent terms. This is the essence of Power's transformation and why it "miraculously" produces a manifestly gauge-invariant Hamiltonian.

We have previously shown that the electromagnetic field representation (dipole-length form) of the electromagnetic interaction is very effective for treating deuteron photodisintegration, but can work poorly for electron scattering at large momentum transfers. Thus, the enforcement of gauge invariance does not resolve all problems. If one begins with a standard representation model Hamiltonian which is gauge invariant, is there a gauge which is best? If one believes that all observables in a consistent calculation using a gauge-invariant Hamiltonian must be independent of gauge, then the question is, in principle, moot. In practice, this is not the case, because the choice of gauge can make a calculation easy or complicated. A good example is the microwave-induced Lamb shift amplitude discussed so nicely in the appendix of Power and Zienau.[27] That calculation is relatively simple using the dipole-length form, but very cumbersome in dipole-velocity form, although both should give the same answer if performed to *all* orders in e.

VIII. THE MTA APPROXIMATION

The momentum-translation approximation (MTA) of Reiss[15] treats a situation not often encountered in nuclear physics: nuclei interacting with very intense classical electromagnetic waves that have a very long wavelength. This application illustrates nicely the power and utility of the techniques developed previously. That is, the wavelength is so large compared to the nuclear radius that any matrix element of $e\mathbf{E}\cdot\mathbf{D}$ is negligible. In this context what role does the electromagnetic potential play?

We neglect center-of-mass motion and suppose, according to the argument above, that $\langle H'_{\rm em}\rangle$ is very small. The processes in Figs. 4(a)—4(c), which depict the effect on the wave function Ψ of the electromagnetic interaction, are tiny when calculated with $H'_{\rm em}$. We can, however, calculate with $H_{\rm em}$ instead. These processes are *not* negligible. Using schematic forms which label wave function components according to the powers of e they contain, we obtain

$$\Psi^{(1)}=\frac{1}{E-H_0+q_0}H^{(1)}_{\rm em}\Psi^{(0)}\ , \tag{64a}$$

and

$$\Psi^{(2)}=\frac{1}{E-H_0+2q_0}H^{(1)}_{\rm em}\frac{1}{E-H_0+q_0}H^{(1)}_{\rm em}\Psi^{(0)}+\frac{1}{E-H_0+2q_0}H^{(2)}_{\rm em}\Psi^{(0)}\ , \tag{64b}$$

where $\Psi^{(0)}$ is the nuclear ground state in the absence of electromagnetic interactions. Equation (23) can be used to rewrite J in terms of J' and $\Delta J'$. The analogous calculation of Ψ using J' we label with a prime: Ψ'. A tedious rearrangement of Eqs. (64) then leads to

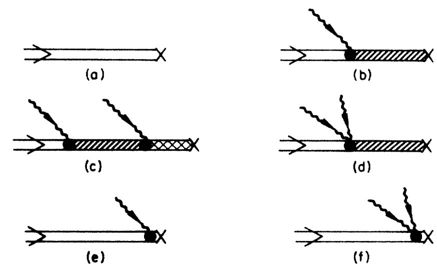

FIG. 4. Graphic representation of the nuclear wave function in the presence of an external electromagnetic interaction. The unperturbed state is shown in (a), while (b), (c), and (d) represent the one-photon, sequential two-photon, and two-photon seagull components. The one-photon and two-photon gauge terms of the MTA are illustrated in (e) and (f).

$$\begin{aligned}\Psi&\simeq\Psi^{(0)}+\Psi^{(1)}+\Psi^{(2)}+\cdots\\&=\Psi^{(0)}+id\Psi^{(0)}+\Psi'^{(1)}+id\Psi'^{(1)}\\&\quad-\frac{d^2}{2}\Psi^{(0)}+\Psi'^{(2)}+\cdots\simeq e^{id}\Psi'\ ,\end{aligned} \tag{65a}$$

where

$$d=e\mathbf{D}\cdot\mathbf{A}(0)\ . \tag{65b}$$

If we exploit the excellent approximation $\Psi'\simeq\Psi_0$ in the long-wavelength limit, we find

$$\Psi\equiv\Psi_{\rm MTA}\simeq e^{id}\Psi_0\ , \tag{65c}$$

which is Reiss's momentum-translation approximation to the *complete* wave function. Representative d terms in Eq. (65c) are represented graphically by Fig. 4(f).

Because $H'_{\rm em}$ is gauge invariant and the operator d is not, we see that this rearrangement has isolated the gauge or seagull terms which are necessary when the nuclear wave function is used to calculate a decay, for example. The form of $\Psi_{\rm MTA}$ is probability conserving and in most nuclear charge-conserving reactions d would commute with the transition operator, $\hat{T}$, and therefore not contribute at all. In contradistinction, the β-decay process changes Z, and $\mathbf{D}$ will fail to commute with the transition operator. Nonrelativistic radiative Gamow-Teller transitions require no gauge terms when calculated using the standard representation of the electromagnetic interaction. When one transforms to the electromagnetic field representation, one finds that gauge terms are needed, because of Eqs. (65):

$$\hat{T}'=e^{-id}\hat{T}e^{id}\ . \tag{66}$$

We note that Eq. (65a) is nothing more than the long-wavelength limit of Eq. (56b), which follows immediately from the canonical transformation.

The MTA has been criticized[46] as unreliable in a variety of contexts, and this is hardly surprising; no approximation works well in all problems. It has also been dismissed as simply a gauge transformation. In view of the intimate connection in gauge theories between gauge and canonical transformations, this is superficially true, but greatly understates the MTA's utility in the nuclear context where charge-changing reactions are common, and where gauge or seagull terms play a vital role. Our derivation has made *no* choice of gauge other than the usual Lorentz condition. In addition, for didactic reasons, the wave function Ψ has been defined[47] as "noninteracting," and the MTA dismissed as trivial. This combination is clearly an oxymoron in the nuclear physics context, because the MTA provides the gauge terms for electromagnetically modified β-decay processes. Such terms are not understood to be "noninteracting" in the nuclear physics context.

Thus we see that the MTA is an approximation, valid for long wavelengths, which gives the form of the electromagnetically "dressed" nuclear wave function. It sums (approximately) a series of virtual nuclear transitions calculated with the standard $(J\cdot A)$ form of the electromagnetic interaction, and is also obtainable most easily via a (time-dependent) unitary transformation. No specific choice of gauge is needed.

IX. CONCLUSIONS

In Sec. II we reviewed alternative formulations of the electromagnetic interaction, and the necessity in the nuclear physics context for those formulations which manifest the constraints of currents conservation in the long-wavelength limit (i.e., Siegert's theorem).[16] In Sec. III we extended the work of Refs. 23 and 24 to include the nuclear recoil current. The "standard" and "electromagnetic field" representations of the electromagnetic interaction were contrasted and compared in Sec. IV in the context of deuteron forward photodisintegration.[33] For that process, the model current was the sum of impulse approximation and pion-exchange-current operators. The effects of this current are nearly the same in both forms, indicating that for low-energy photonuclear reactions in the deuteron the pion-range exchange currents dominate the shorter-range meson currents. In the standard representation large mutually canceling terms lead to a small result; in the EMF representation all contributions are small.

In Sec. V the Compton amplitude, including recoil, was calculated in the EMF representation. This amplitude is gauge invariant, and in transverse gauge the individual terms in the amplitude correspond to terms in the low-energy theorem expansion. Important amplitude components are manifestly gauge invariant. In Sec. VI, the low energy expansion is carried out for $0^+\to 0^+$ transitions and the low-energy theorem verified. The efficacy of Podolsky's method[43–45] for calculating the "dispersive" terms (i.e., the generalized polarizability and susceptibility amplitudes) was also explored for six nuclear isotopes whose ground and first excited states have $J^\pi=0^+$.

Gauge invariance was extensively discussed in Sec. VII, in the context of the PZ canonical transformation[27] and the "minimal" substitution. Reiss's momentum-translation approximation[15] was derived in Sec. VIII by the canonical transformation method and, alternatively, by summing a series of Feynman diagrams. Its importance in the context of high-field nuclear reactions was emphasized.

Finally, this work was motivated by the expectation that multiphoton $(n_\gamma > 1)$ processes will become increasingly important in nuclear physics. Tagged photon facilities at cw electron machines should encourage Compton and Raman scattering experiments.[9] Should lasers of sufficient power and energy become available,[11,12] other multiphoton reactions will become a part of electromagnetic nuclear physics. The nuclear Compton amplitude is also important theoretically, where it is a basic building block for such processes as dispersion corrections[3,4] in electron scattering, which may have been seen in a recent experiment,[48] and in the analogous polarization corrections in atoms.[5,6] We expect that interest in all these processes will continue to grow.

ACKNOWLEDGMENTS

The work of one of us (S.F.) was supported in part by the U. S. Department of Energy, while that of J.L.F. was performed under the auspices of the U.S. Department of Energy. We would like to thank Dr. E. V. Fuller for his assistance in assembling Table II.

APPENDIX

The impulse approximation to $B_{\rm SG}$ is given by Eq. (48a), and can be combined with Eq. (36) to generate the impulse approximation χ_D,

$$\begin{aligned}\chi_D^{rs}(\mathbf{q}_1,\mathbf{q}_2)&=-\frac{1}{4m}\epsilon^{rmn}\epsilon^{stn}\int d^3x\, x^t x^m \rho(\mathbf{x})\beta(\mathbf{q}_1\cdot\mathbf{x})\beta(\mathbf{q}_2\cdot\mathbf{x})\\&=\frac{1}{m}\epsilon^{rmn}\epsilon^{nst}\nabla^t_{q_1}\nabla^m_{q_2}\\&\quad\times\int_0^1 d\lambda\int_0^1 d\lambda'\rho(\lambda\mathbf{q}_1+\lambda'\mathbf{q}_2)\,,\end{aligned}\tag{A1}$$

using

$$\begin{aligned}\mathbf{z}\beta(\mathbf{q}\cdot\mathbf{z})&=2\int_0^1 d\lambda\,\lambda e^{i\mathbf{q}\cdot\mathbf{z}\lambda}\mathbf{z}\\&=-2i\nabla_q\int_0^1 d\lambda\, e^{i\mathbf{q}\cdot\mathbf{z}\lambda}\\&=-2i\nabla_q\alpha(\mathbf{q}\cdot\mathbf{z})\,.\end{aligned}\tag{A2}$$

Moreover, we can explicitly calculate $\alpha(z)$:

$$\alpha(z)=\frac{e^{iz}-1}{iz}\,.\tag{A3}$$

For the important $0^+\to 0^+$ case, the double integral in (A1) becomes

$$\xi(\mathbf{q}_1,\mathbf{q}_2)=\int_0^1 d\lambda\int_0^1 d\lambda'\rho(\,|\lambda\mathbf{q}_1+\lambda'\mathbf{q}_2|\,)\,.\tag{A4}$$

If we perform a partial wave expansion of $\alpha(\mathbf{q}\cdot\mathbf{z})$, we obtain

$$\alpha(\mathbf{q}\cdot\mathbf{z})=\sum_l(2l+1)i^lP_l(\hat{\mathbf{q}}\cdot\hat{\mathbf{z}})H_l(qz)\,,\tag{A5}$$

where the auxiliary function H_l is defined by

$$H_l(x)=\int_0^1 d\lambda\, j_l(\lambda x)\,.\tag{A6}$$

These functions can be calculated recursively from

$$lH_{l-1}(z)-(l+1)H_{l+1}(z)=\frac{2l+1}{z}[j_l(z)-\delta_{l,0}] , \quad \text{(A7)}$$

and

$$H_0(z)=\mathrm{Si}(z)/z , \quad \text{(A8)}$$

$$H_1(z)=-\frac{1}{z}[j_0(z)-1] , \quad \text{(A9)}$$

where Si(z) is the sine integral. For the $0^+\to 0^+$ case, this leads to

$$\xi(\mathbf{q}_1,\mathbf{q}_2)=\int d^3x\,\rho(\mathbf{x})\sum_{l=0}^{\infty}(-1)^l(2l+1)\times P_l(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{q}}_2)H_l(q_1x)H_l(q_2x)$$
$$=\sum_{i=1}^{A}e_i\sum_{l=0}^{\infty}(-1)^l(2l+1)P_l(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{q}}_2)\times H_l(q_1x_i)H_l(q_2x_i) , \quad \text{(A10)}$$

which is a tractable expression for values of q_ix not too large.

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