"Exact" relativistic theory of two-body bound-state wave functions

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An exact three-dimensional reduction of the Bethe-Salpeter equation is presented for the case of two spin-$\frac{1}{2}$ fermions exchanging scalar and pseudoscalar bosons. The resulting three-dimensional integral equation involves only a truncated basis, and once it is solved to get an auxiliary scattering amplitude $\tilde{T}$, the fully off-shell Bethe-Salpeter amplitude T is given essentially as an integral over $\tilde{T}$, thereby considerably simplifying its calculation. Similar remarks apply to bound-state equations as well. Partial-wave decomposition of the latter equations is carried out leading to Schrödinger-type equations for two-body relativistic wave functions. By means of a simple order-of-magnitude estimate we show that in the deuteron wave functions the admixture of virtual states with both nucleons in negative-energy states is more likely than that of the states with one nucleon in a positive-energy and the other in a negative-energy state, provided the coupling is pseudoscalar-pseudoscalar. This effect has not received sufficient attention in the literature. Another interesting result is that if the effective interaction in our equations is calculated to the lowest order in the coupling constant, only six out of the eight components of the deuteron wave functions are nonzero, the other two appear only in the higher order.

I. INTRODUCTION

The relativistic two-body problem is of fundamental importance with applications in atomic (e.g., positronium, muonium), nuclear (e.g., deuteron, NN scattering), and particle (e.g., $q\bar{q}$ bound states) physics. The knowledge of a two-body scattering equation is of central importance also in a variety of microscopic many-body theories. Recent progress in meson and baryon spectroscopy and in relativistic nuclear many-body theories has revived the interest in this problem. If the approximation of nonrelativistic energies is valid, one may use the Lippmann-Schwinger equation. However, it has long been recognized that in a two-hadron system relativistic effects have to be taken into account even at relatively low energies, particularly in the case of spin and large momentum transfer observables.[1] In the relativistic domain the Bethe-Salpeter equation[2,3] (BSE) or, more commonly, one of the several approximate equations derived from it, is often the choice (see below for a discussion of other approaches). The need for an approximate equation arises because in the realistic case of two spin-$\frac{1}{2}$ particles (with nonzero total angular momentum J), the BSE actually represents eight coupled singular four-dimensional integral equations, the solution of which involves considerable numerical difficulties. For this reason the BSE is rarely solved (see, however, Ref. 4). More commonly, it is reduced to a three-dimensional equation by replacing the exact two-body propagator in it by one involving an energy-delta function which allows the energy integration to be performed trivially. Many such three-dimensional equations are available in the literature. Some of them have been compared with each other and with the BSE, for the simple case of two scalar particles with equal mass in Refs. 5 and 6 and two scalar particles with unequal mass in Refs. 7 and 8. We shall reconsider some of these equations for the case of two spin-$\frac{1}{2}$ particles in Sec. III. We mention in passing that the Blanckenbecler-Sugar equation[9] assumes the total energy to be distributed equally between the two particles. This puts them equally off mass shell (in the center-of-momentum frame) and makes the interaction instantaneous, thereby neglecting all retardation effects. The Gross equation,[10] on the other hand, puts one of the fermions on mass shell throughout the equation and keeps only the contribution of positive-energy poles of the fermion propagator.

We present below a few examples of the kind of two-body equations that have been used in some of the most recent theoretical calculations. In Ref. 11 Machleidt and Brockmann describe their Dirac-Brueckner calculations of nuclear matter. These are based on an approximate three-dimensional two-body equation which assumes the basic two-body interaction to be instantaneous; that is, the retardation effects are neglected. Secondly, this two-body equation requires that both the intermediate fermions be in positive-energy states, thereby neglecting those intermediate states where one or both the fermions have negative energies.

The second example deals with calculations of meson ($q\bar{q}$), baryon (qqq), and glueball spectra. Here again the BSE in the ladder approximation is a common starting point.[12] Many of these calculations further assume instantaneous two-body interactions and neglect intermediate states with one or both particles in negative-energy states.

The final example deals with a recent paper by Matheltsch and Garcilazo.[13] They have constructed a set of separable potentials for the NN, πN, and $\pi\pi$ subsystems in the framework of the relativistic two-body Kadyshevsky equation.[14] They plan to use these separable interactions in relativistic Faddeev calculations of the three-body sys-

tems NNN, NNπ, N$\pi\pi$, and $\pi\pi\pi$. Their formalism satisfies relativistic three-body unitarity, is Lorentz-invariant, and unlike earlier formalisms is applicable in the scattering as well as in the bound-state domains. We note that the Kadyshevsky equation restricts one of the two particles to its mass shell and neglects the contribution of the negative-energy poles of the two-body propagator.

The two most common approximations, namely the neglect or a crude treatment of the retardation effects in the two-body interaction and the neglect of intermediate states where one or both the fermions are in negative-energy states ($+-$, $-+$, and $--$ states), are highly questionable. The importance of the retardation effects was stressed in Ref. 15. It was shown there that a careful treatment of these effects greatly improves the agreement between two-body phase shifts calculated with a three-dimensional equation and those calculated with the BSE. However, Ref. 15 dealt only with scalar "nucleons" exchanging scalar bosons, and negative-energy poles of the two-body propagator were neglected. In this paper we reconsider this long-standing problem and present an exact three-dimensional reduction of the BSE for the case of two spin-$\frac{1}{2}$ fermions exchanging scalar as well as pseudoscalar bosons (Sec. II). The resulting three-dimensional equation is far easier to handle numerically, while it retains the desirable properties of the BSE in that the retardation effects are present, positive-energy as well as negative-energy intermediate states are allowed and the ladder approximation is not made. Two-body scattering as well as bound-state equations are discussed. The bound-state equations are applied to the deuteron and some interesting conclusions are drawn regarding the relative sizes of the various components of the deuteron relativistic wave functions. Finally, in Sec. III, we compare our work with some of the earlier work on this problem.

Before we proceed further, we describe briefly some of the other approaches to the problem of a relativistic two-body system.

(a) One approach, following the original suggestion by Dirac, is based on the consideration of the ten generators of the Poincaré group. A recent work based on this approach is described in Ref. 16. Starting from a field theory that gives the ten generators of the Poincaré group in the full space of particles, antiparticles, and bosons, they project on the pure n-particle subspace and by a unitary transformation separate the subspace from the rest of the space. They show in the framework of the perturbation theory that this procedure does not invalidate commutation relations satisfied by the ten generators. They thus get a relativistic theory expressed in the particle variables only, to a certain order in the coupling constant.

(b) Another approach is based on the use of the light-cone variables ($\tau=t+z$, $Z=t-z$); for a review see Ref. 17. This approach offers several advantages over other approaches. In the τ-ordered perturbation theory fewer diagrams result from a given Feynman diagram than in the t-ordered perturbation theory, and the diagram rules are as simple as the old-fashioned perturbation theory rules, in the infinite-momentum frame. As an example of a recent application of this approach see Ref. 18, where Brodsky *et al.* have studied bound states of two scalar particles. They also discuss the relation of this approach to the standard Bethe-Salpeter approach.

(c) There exists yet another approach to this problem, that through the relativistic quantum mechanics of two interacting particles (see Refs. 19 and 20, and references therein). In Ref. 19 Crater and Alstine have used Dirac's constraint mechanics and supersymmetry to obtain a consistent description of two interacting particles, either or both of which may have spin $\frac{1}{2}$. In Ref. 20 Rizov *et al.* have presented a Hilbert-space formulation of the relativistic quantum mechanical two-body problem for a range of (quasi)potentials which include the electromagnetic interaction of two spinless particles. The relation of this approach with the Bethe-Salpeter approach has been discussed recently by Sazdjian.[21]

(d) Bilal and Schuck[22] have applied the memory-function approach of Mori[23] to the relativistic two-body problem. They have obtained integral equations similar to the BSE, taking all retardation effects fully into account and eliminating the relative time variable. They found retardation effects to be extremely important in the calculation of binding energy of two scalar particles (Cutkosky model). Their treatment is relativistic but noncovariant.

(e),(f) Relativistic bound states, particularly the hadrons, are also being studied in the frameworks of the lattice quantum chromodynamics[24] (QCD) and the QCD sum rules.[25]

II. THEORY

A. Exact three-dimensional reduction of the Bethe-Salpeter equation

Consider the Bethe-Salpeter equation (BSE) satisfied by the T matrix for scattering of two distinguishable spin-$\frac{1}{2}$ particles of equal mass m (see Ref. 26),

$$T(P,Q,Q')=V(Q,Q')+i\int\frac{V(Q,Q'')(\not{Q}''+m)_1(\not{P}-\not{Q}''+m)_2T(P,Q'',Q')d^4Q''}{(Q''^2-m^2+i\delta)[(P-Q'')^2-m^2+i\delta](2\pi)^4}\,. \tag{1}$$

Here, Q, Q'', and Q' are the four-momenta of the fermion labeled 1 in the initial, intermediate, and final states, respectively, $P=(P_0,\mathbf{0})$ is the total four-momentum in the center-of-momentum (c.m.) frame, and $V(Q,Q')$ is the interaction which in the one-boson–exchange approximation (ladder approximation) is given by

$$V(Q,Q')=\frac{g^2O^{(1)}O^{(2)}}{(Q'-Q)^2-\mu^2+i\delta}\,. \tag{2}$$

In Eq. (2), g is the coupling constant, μ is the exchanged-boson mass, and $O^{(1)}$ and $O^{(2)}$ are operators defined as follows:

$$O^{(1)}O^{(2)}=1^{(1)}1^{(2)}\,, \tag{3a}$$

for a scalar boson and scalar coupling (SS), and

$$O^{(1)}O^{(2)}=-\gamma_5^{(1)}\gamma_5^{(2)}\,, \tag{3b}$$

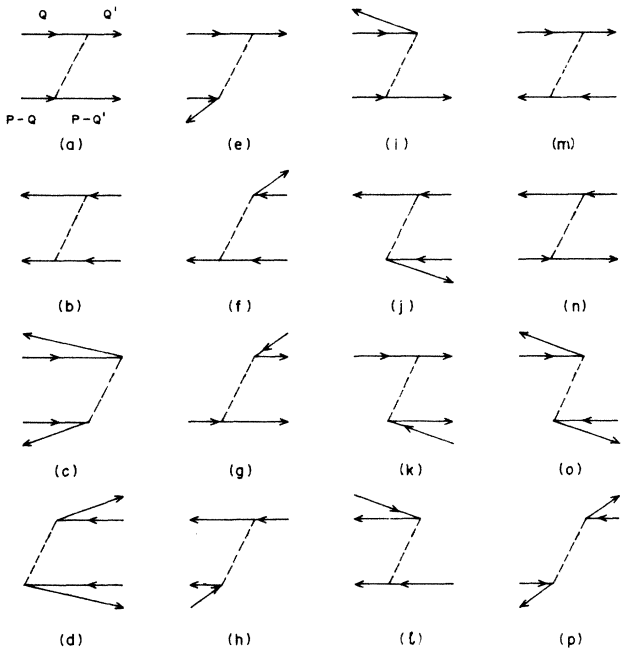

FIG. 1. Representation of the 16 matrix elements of $V_+(Q,Q')$ in Eq. (4). Here and in all the figures solid lines represent fermions and dashed lines bosons. The solid lines pointing to the right (left) refer to positive (negative) -energy spinors.

for a pseudoscalar boson and pseudoscalar coupling (PP). These operators are based on interaction Lagrangian densities,

$$\mathscr{L}=g\bar{\psi}\psi\phi$$

and

$$\mathscr{L}=ig\bar{\psi}\gamma_5\psi\phi\ ,$$

respectively. We shall drop the approximation made in Eq. (2) and discuss a more general interaction at a later stage. We shall work in the c.m. frame.

Our first objective is to perform an exact three-dimensional reduction of Eq. (1) by carrying out the Q_0'' integration. We adopt the following approach.[15] We expand the right-hand side (rhs) of Eq. (1) in an infinite Born series, with the hope that if the energy integrations are then performed in each term separately, we may be able to write the resulting series or at least a part of it in a closed form.

Our procedure thus consists of three steps.

Step 1: Equation (1) is iterated to get the infinite Born series for T:

$$T=V+VG_0V+VG_0VG_0V+\cdots\ ,$$

where G_0 is the two-fermion propagator.

Step 2: We then carry out the energy integration(s) in each term of the above series exactly using the standard techniques of contour integration. (This is possible because the only singularities that V and G_0 have are simple poles.) The contribution of each term (say nth order) is arranged in such a way that it corresponds to a set of distinct time-ordered diagrams of that order. This way T in Eq. (1) appears as a series of an infinite number of time-ordered diagrams.

Step 3: Next we try to rewrite this series in a closed form. We first define an auxiliary amplitude $\tilde{T}$ as the sum of all the above time-ordered diagrams, provided $Q_0=\pm(\mathbf{Q}^2+m^2)^{1/2}$ or $P_0\mp(\mathbf{Q}^2+m^2)^{1/2}$, $Q_0'=\pm(\mathbf{Q}'^2+m^2)^{1/2}$ or $P_0\mp(\mathbf{Q}'^2+m^2)^{1/2}$, and the external lines are of the particle-particle or hole-hole type. We also define an effective interaction $\tilde{V}_{\rm eff}$ as the sum of all noniterative time-ordered diagrams with the same conditions as above. ($\tilde{V}_{\rm eff}$ is an infinite series which cannot be written in a closed form.) The remaining (iterative) diagrams can be summed, giving us a Lippmann-Schwinger–type three-dimensional integral equation for $\tilde{T}$ with $\tilde{V}_{\rm eff}$ as the kernel. We finally write T in terms of $\tilde{T}$.

We have

$$\begin{aligned}V(Q,Q')&=\frac{g^2O^{(1)}O^{(2)}}{(Q_0'-Q_0-\omega+i\delta)(Q_0'-Q_0+\omega-i\delta)}\\&=\frac{g^2O^{(1)}O^{(2)}}{2\omega}\left[\frac{1}{Q_0'-Q_0-\omega+i\delta}+\frac{1}{Q_0-Q_0'-\omega+i\delta}\right]\\&\equiv V_+(Q,Q')+V_-(Q,Q')\ ,\end{aligned}\tag{4}$$

where $\omega\equiv\omega_{Q'-Q}$, $\omega_{\mathbf{x}}\equiv(\mathbf{x}^2+\mu^2)^{1/2}$. Equation (4) is an operator identity. We can take matrix elements of $V_\pm(Q,Q')$ between any two states from the following two-particle basis: $u_1(\mathbf{Q})u_2(-\mathbf{Q})$, $v_1(-\mathbf{Q})v_2(\mathbf{Q})$, $u_1(\mathbf{Q})v_2(\mathbf{Q})$, and $v_1(-\mathbf{Q})u_2(-\mathbf{Q})$. Here, u_i and v_i are the positive- and negative-energy solutions of the free Dirac equation, respectively.[26] The resulting 16 matrix elements of $V_+(Q,Q')$ are represented by the 16 time-ordered diagrams in Fig. 1. (In all the figures time flows from left to right.) In Fig. 1 the solid lines pointing to the right (left) refer to the positive (negative) -energy spinors. The matrix elements of $V_-(Q,Q')$ can be represented by a similar set of diagrams.

For any four-vector p we have the identity

$$\begin{aligned}\frac{\not{p}+m}{p^2-m^2+i\delta}&=\frac{m}{E_{\mathbf{p}}}\sum_s\left[\frac{u(\mathbf{p},s)\bar{u}(\mathbf{p},s)}{p_0-E_{\mathbf{p}}+i\delta}+\frac{v(-\mathbf{p},s)\bar{v}(-\mathbf{p},s)}{p_0+E_{\mathbf{p}}-i\delta}\right]\\&\equiv\frac{m}{E_{\mathbf{p}}}\left[\frac{\Lambda_+(\mathbf{p})}{p_0-E_{\mathbf{p}}+i\delta}-\frac{\Lambda_-(-\mathbf{p})}{p_0+E_{\mathbf{p}}-i\delta}\right]\ ,\end{aligned}\tag{5}$$

where $E_{\mathbf{x}} \equiv (\mathbf{x}^2+m^2)^{1/2}$ and $\Lambda_\pm$ are the positive- and negative-energy projection operators.[26] Equation (5) simply states that the propagation of an off-shell fermion with four-momentum p can be looked upon as a mixture of on-shell propagation with positive energy with momentum $\mathbf{p}$ and on-shell propagation with negative energy with the opposite momentum $-\mathbf{p}$.

Substitution of Eqs. (4) and (5) in the term of order g^4 in Eq. (1) gives

$$i\int [V_+(Q,Q'')+V_-(Q,Q'')]\frac{m^2}{E_{Q''}^2}\left[\frac{\Lambda_+^{(1)}(\mathbf{Q}'')}{Q_0''-E_{Q''}+i\delta}-\frac{\Lambda_-^{(1)}(-\mathbf{Q}'')}{Q_0''+E_{Q''}-i\delta}\right]$$
$$\times\left[\frac{\Lambda_+^{(2)}(-\mathbf{Q}'')}{P_0-Q_0''-E_{Q''}+i\delta}-\frac{\Lambda_-^{(2)}(\mathbf{Q}'')}{P_0-Q_0''+E_{Q''}-i\delta}\right][V_+(Q'',Q')+V_-(Q'',Q')]\frac{d^4Q''}{(2\pi)^4}\ . \tag{6}$$

In each of the 16 terms resulting from this expression, the Q_0'' integration can be performed exactly by using the techniques of contour integration in the complex-Q_0'' plane. Consider, for example, the terms proportional to $\Lambda_+^{(1)}(\mathbf{Q}'')\Lambda_+^{(2)}(-\mathbf{Q}'')$. After Q_0'' integration and some rearrangement these terms can be written as

$$\int\frac{d^3Q''}{(2\pi)^3}\frac{g^4}{4\omega_1\omega_2}\frac{m^2}{E_{Q''}^2}O^{(1)}O^{(2)}\Lambda_+^{(1)}(\mathbf{Q}'')\Lambda_+^{(2)}(-\mathbf{Q}'')O^{(1)}O^{(2)}$$
$$\times[(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}(Q_0'-Q_0-\omega_1-\omega_2+i\delta)^{-1}(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}$$
$$+(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}(P_0-2E_{Q''}+i\delta)^{-1}(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}$$
$$+(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}(P_0-2E_{Q''}+i\delta)^{-1}(P_0-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}$$
$$+(Q_0-E_{Q''}-\omega_1+i\delta)^{-1}(P_0-2E_{Q''}+i\delta)^{-1}(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}$$
$$+(Q_0-E_{Q''}-\omega_1+i\delta)^{-1}(P_0-2E_{Q''}+i\delta)^{-1}(P_0-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}$$
$$+(Q_0-E_{Q''}-\omega_1+i\delta)^{-1}(Q_0-Q_0'-\omega_1-\omega_2+i\delta)^{-1}(P_0-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}]\ , \tag{7}$$

where $\omega_1\equiv\omega_{Q''-Q}$ and $\omega_2\equiv\omega_{Q'-Q''}$. All the terms in (7) are operators, as in Eq. (4). The first two terms in (7) correspond to the combination V_+V_+ in (6), the third term to the combination V_+V_-, the fourth term to V_-V_+, and the last two terms to V_-V_-. All this is consistent with the experience one has of the time-ordered perturbation theory and thus we associate these six terms with diagrams (a)—(f) in Fig. 2, respectively. It is easy to check that these are the only diagrams one can draw in this case. Note that we have not drawn lines corresponding to the external fermions because (7) is an operator expression and not a matrix element. As in the case of Eq. (4) and Fig. 1, each term in (7) gives rise to 16 matrix elements; some of them are represented in Fig. 3. Except for the spinors corresponding to the external lines, each diagram in Fig. 3 is given by the first term in (7). Expressions for these diagrams are consistent with the rules of the time-ordered perturbation theory given in the Appendix.

Consider once again the expression (6). The Q_0'' integration in the remaining terms can be done in a similar manner. As before, this gives rise to six terms proportional to $\Lambda_-^{(1)}(-\mathbf{Q}'')\Lambda_-^{(2)}(\mathbf{Q}'')$, six proportional to $\Lambda_+^{(1)}(\mathbf{Q}'')\Lambda_-^{(2)}(\mathbf{Q}'')$, and six proportional to $\Lambda_-^{(1)}(-\mathbf{Q}'')\Lambda_+^{(2)}(-\mathbf{Q}'')$. These are represented by diagrams (g)—(l), (m)—(r), and (s)—(x) in Fig. 2. For later use we write below the six terms proportional to $\Lambda_-^{(1)}(-\mathbf{Q}'')\Lambda_-^{(2)}(\mathbf{Q}'')$:

$$\int\frac{d^3Q''}{(2\pi)^3}\frac{g^4}{4\omega_1\omega_2}\frac{m^2}{E_{Q''}^2}O^{(1)}O^{(2)}\Lambda_-^{(1)}(-\mathbf{Q}'')\Lambda_-^{(2)}(\mathbf{Q}'')O^{(1)}O^{(2)}$$
$$\times[(Q_0'-\omega_2-P_0-E_{Q''}+i\delta)^{-1}(Q_0'-Q_0-\omega_1-\omega_2+i\delta)^{-1}(-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}$$
$$+(Q_0'-\omega_2-P_0-E_{Q''}+i\delta)^{-1}(-P_0-2E_{Q''}+i\delta)^{-1}(-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}$$
$$+(-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}(-P_0-2E_{Q''}+i\delta)^{-1}(-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}$$
$$+(Q_0'-\omega_2-P_0-E_{Q''}+i\delta)^{-1}(-P_0-2E_{Q''}+i\delta)^{-1}(Q_0-P_0-E_{Q''}-\omega_1+i\delta)^{-1}$$
$$+(-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}(-P_0-2E_{Q''}+i\delta)^{-1}(Q_0-P_0-E_{Q''}-\omega_1+i\delta)^{-1}$$
$$+(-E_{Q''}-Q_0'-\omega_2+i\delta)^{-1}(Q_0-Q_0'-\omega_1-\omega_2+i\delta)^{-1}(Q_0-P_0-E_{Q''}-\omega_1+i\delta)^{-1}]\ . \tag{8}$$

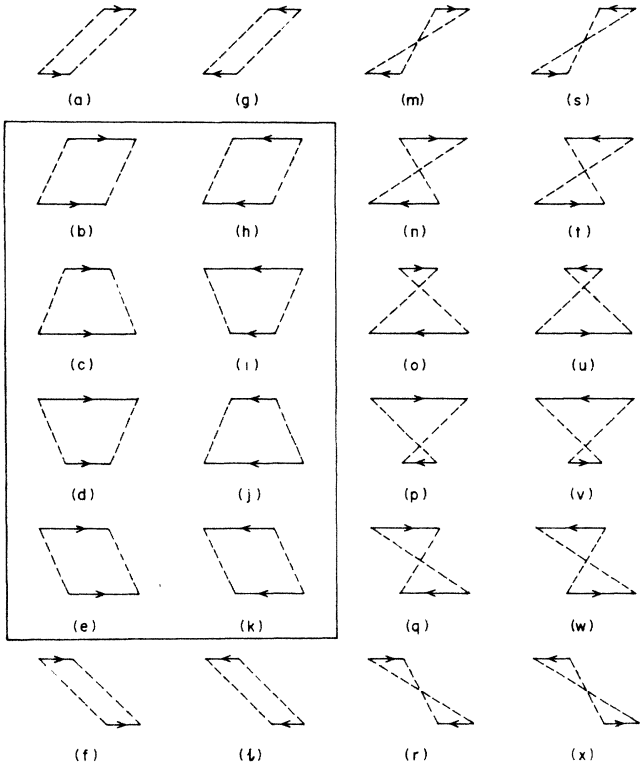

FIG. 2. Diagrams (a)—(f), six terms in (7); (g)—(l), six terms in (8); (m)—(r), six terms proportional to $\Lambda_+^{(1)}\Lambda_-^{(2)}$; and (s)—(x), six terms proportional to $\Lambda_-^{(1)}\Lambda_+^{(2)}$.

These six terms as well as those proportional to $\Lambda_+^{(1)}\Lambda_-^{(2)}$ and $\Lambda_-^{(1)}\Lambda_+^{(2)}$ can also be obtained by using the rules of the time-ordered perturbation theory, given in the Appendix.

So far we have considered time-ordered diagrams resulting from the term VG_0V. We next consider the term VG_0VG_0V. Here again we can substitute for V and G_0 from Eqs. (4) and (5), respectively, and perform the energy integrations exactly. This will give rise to additional time-ordered diagrams. In principle, we can repeat this procedure in the case of all higher-order terms in the expansion of Eq. (1). Having generated an infinite number of time-ordered diagrams this way, we now try to rewrite their sum in a closed form.

Consider the eight diagrams in the box in Fig. 2 and the corresponding terms in (7) and (8). The matrix element of the second term in (7) between $\bar{u}_1(\mathbf{Q}')\bar{u}_2(-\mathbf{Q}')$ and $u_1(\mathbf{Q})u_2(-\mathbf{Q})$ is represented by Fig. 4(a). (We are now

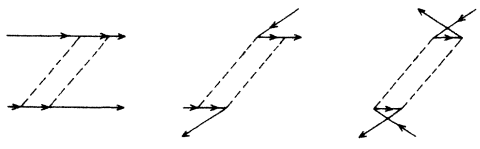

FIG. 3. Representation of some of the matrix elements of the first term in (7).

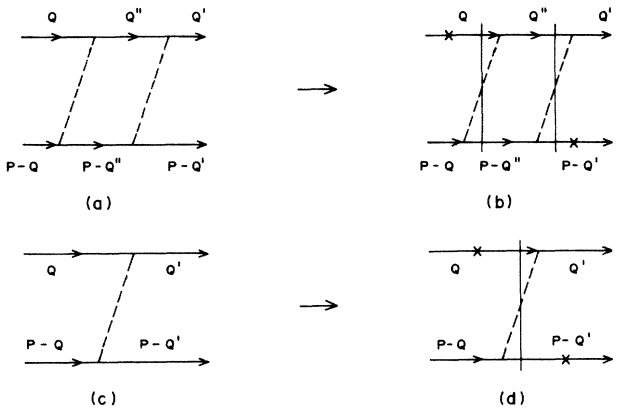

FIG. 4. (a) A diagram resulting from Fig. 2(b). (a) is not iterative, but (b) is iterative because fermions intersected by vertical lines (indicated by $\times$) are on mass shell and the interaction V_+ in (c) has been redefined in a similar way in (d).

considering off-shell scattering.) Note that the middle factor in the denominator of this term, $P_0-2E_{Q''}+i\delta$, is independent of ω_1 and ω_2 and depends only on $\mathbf{Q}''$. Correspondingly, this diagram can be "cut" in two by a vertical line without intersecting a boson line. Therefore we expect it to be iterative; that is, of the general form $f(\mathbf{Q},\mathbf{Q}'')g(\mathbf{Q}'')f(\mathbf{Q}'',\mathbf{Q}')$, where f and g are any arbitrary functions. However, the other factors in the denominator do not bear this out. That is, the factors $(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}$ and $(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}$, which correspond to the two interactions in Fig. 4(a), are formally different from each other and from the basic interaction

$$V_+(Q,Q')\sim(Q_0'-Q_0-\omega_{Q'-Q}+i\delta)^{-1}$$

represented by Fig. 4(c). The reason is that in deriving (7) contour integration was performed in the complex Q_0'' plane and some of the terms were rearranged so that there was one-to-one correspondence between these terms and Figs. 2(a)–(f). As a result, in the expression $(Q_0'-Q_0''-\omega_2+i\delta)^{-1}$, Q_0'' was replaced by $E_{Q''}$, giving us $(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}$. Thus the obvious remedy is to redefine the interaction V_+ with the implied choice of Q_0 and Q_0',

$$V_+(Q,Q')\rightarrow V_+(Q,Q')_{Q_0=E_Q,\ Q_0'=P_0-E_{Q'}},$$

which amounts to replacing Fig. 4(c) by 4(d), and make the replacements

$$(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}\rightarrow(P_0-Q_0-\omega_1-E_{Q''}+i\delta)^{-1}_{Q_0=E_Q}$$

and

$$(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}\rightarrow(Q_0'-E_{Q''}-\omega_2+i\delta)^{-1}_{Q_0'=P_0-E_{Q'}}$$

in the expression for Fig. 4(a), which amounts to replacing Fig. 4(a) by Fig. 4(b). With these replacements, the second term in (7) acquires the general form $f(\mathbf{Q},\mathbf{Q}'')g(\mathbf{Q}'')f(\mathbf{Q}'',\mathbf{Q}')$ of an iterative term. In sum-

mary, Fig. 4(c), when iterated, does not give Fig. 4(a), but Fig. 4(d), when iterated, does give Fig. 4(b).

The above argument can be easily extended to the other diagrams in the box in Fig. 2. It can be checked that in each case the external fermions intersected by vertical lines are required to be taken on mass shell to cast the corresponding term in Eqs. (7) and (8) in the iterative form (cf. Fig. 4). This amounts to replacing Q_0 by E_Q or P_0-E_Q, and Q_0' by $E_{Q'}$ or $P_0-E_{Q'}$. So far we have considered external lines of the particle-particle type only. However, the argument is applicable also when they are of the hole-hole type. In this case Q_0 is replaced by $-E_Q$ or P_0+E_Q, and Q_0' by $-E_{Q'}$ or $P_0+E_{Q'}$ (see rule 2 in the Appendix). It is convenient to express these conclusions diagrammatically. We show in Figs. 5(a)–5(c) some examples of iterative diagrams, and in Figs. 5(d)–5(f) the corresponding (redefined) basic interactions which appear in these diagrams.

The rest of the diagrams in Fig. 2 (that is, those outside the box) are noniterative because there is no boson-free intermediate state in them; they cannot be cut in two by a vertical line without intersecting a boson line. This fact is also reflected in their expressions [see, for example, the corresponding terms in (7) and (8)], which cannot be cast in the form $f(\mathbf{Q},\mathbf{Q}'')g(\mathbf{Q}'')f(\mathbf{Q}'',\mathbf{Q}')$.

Thus we have separated the diagrams in Fig. 2 into two classes. One class gives rise to iterative diagrams with a special choice of the external variables Q_0 and Q_0' [see, for example, Figs. 4(b) and 5(a)–5(c)], while the other class gives rise to noniterative diagrams. This classification can be easily extended to diagrams of all orders. It is important to note, however, that in the above discussion the various diagrams did not play any essential role; they only serve to make the arguments more transparent and easier to follow.

We are now in a position to write a three-dimensional integral equation for an auxiliary amplitude which we denote by $\tilde{T}(P_0,\mathbf{Q},\mathbf{Q}')$. $\tilde{T}(P_0,\mathbf{Q},\mathbf{Q}')$ is the sum of all the time-ordered diagrams obtained from the expansion of Eq. (1) provided the lines corresponding to $Q, P-Q$, Q', and $P-Q'$ are of the particle-particle or hole-hole type, and Q_0 and Q_0' have been fixed as described in the preceding paragraphs. Note that $T(P_0,Q,Q')$ in Eq. (1) is also the sum of all the diagrams, but without any restrictions on the values of Q_0 and Q_0' or on the directions of lines labeled Q, $P-Q$, Q', and $P-Q'$. Later on we will express $T(P_0,Q,Q')$ in terms of $\tilde{T}(P_0,\mathbf{Q},\mathbf{Q}')$.

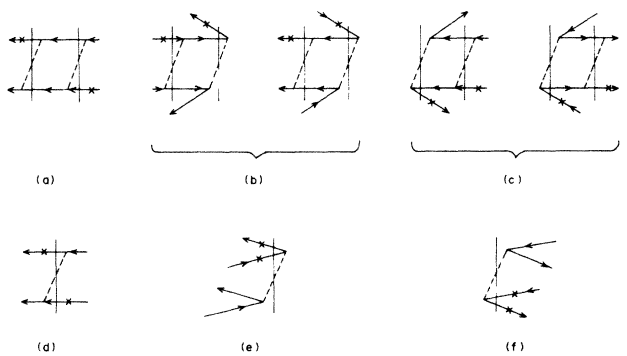

FIG. 5. (a)–(c) Some iterative diagrams and (d)–(f) the corresponding basic interactions which enter those diagrams. Fermions marked by $\times$ are on mass shell.

In order to write equations for $\tilde{T}(P_0,\mathbf{Q},\mathbf{Q}')$ and $T(P_0,Q,Q')$, we define the following basis,

$$\begin{aligned}
&\Psi^{++}(\mathbf{p}\lambda\lambda')\equiv u_1(\mathbf{p}\lambda)u_2(-\mathbf{p}\lambda')\,,\\
&\Psi^{--}(\mathbf{p}\lambda\lambda')\equiv v_1(-\mathbf{p}\lambda)v_2(\mathbf{p}\lambda')\,,\\
&\Psi^{e}(\mathbf{p}\lambda\lambda')\equiv\frac{1}{\sqrt{N_1}}[u_1(\mathbf{p}\lambda)v_2(\mathbf{p}\lambda')+v_1(-\mathbf{p}\lambda)u_2(-\mathbf{p}\lambda')]\,,\\
&\Psi^{o}(\mathbf{p}\lambda\lambda')\equiv\frac{1}{\sqrt{N_2}}[u_1(\mathbf{p}\lambda)v_2(\mathbf{p}\lambda')-v_1(-\mathbf{p}\lambda)u_2(-\mathbf{p}\lambda')]\,,
\end{aligned}\tag{9}$$

where N_1 and N_2 are the normalization constants. Ψ^{++}, Ψ^{--}, and Ψ^{e} are even (e), and Ψ^{o} is odd (o) in the ρ-spin space.[27] We have the relation

$$\bar{\Psi}^{i}(\mathbf{p}\lambda\lambda')\Psi^{i}(\mathbf{p}\mu\mu')=\delta_{\lambda\mu}\delta_{\lambda'\mu'}\tag{10}$$

for $i=++$, $--$, e, or o. We also define the operators

$$\Lambda^{i}(\mathbf{p})\equiv\sum_{\lambda\lambda'}\Psi^{i}(\mathbf{p}\lambda\lambda')\bar{\Psi}^{i}(\mathbf{p}\lambda\lambda')\,,\tag{11}$$

which satisfy

$$\begin{aligned}
&[\Lambda^{i}(\mathbf{p})]^2=\Lambda^{i}(\mathbf{p})\,,\\
&\Lambda^{i}(\mathbf{p})\Psi^{i}(\mathbf{p}\lambda\lambda')=\Psi^{i}(\mathbf{p}\lambda\lambda')\,,\\
&\bar{\Psi}^{i}(\mathbf{p}\lambda\lambda')\Lambda^{i}(\mathbf{p})=\bar{\Psi}^{i}(\mathbf{p}\lambda\lambda')\,.
\end{aligned}\tag{12}$$

We introduce the notation

$$A^{i,j}(P_0,\mathbf{p},\mathbf{p}')\equiv\Lambda^{i}(\mathbf{p})A(P_0,\mathbf{p},\mathbf{p}')\Lambda^{j}(\mathbf{p}')$$

and similarly for $A^{i,j}(P_0,\mathbf{p},p')$, $A^{i,j}(P_0,p,\mathbf{p}')$, and $A^{i,j}(P_0,p,p')$.

We have

$$\begin{aligned}
\tilde{T}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')=&\tilde{V}_{\rm eff}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')\\
&+\sum_{k''}\int\tilde{V}_{\rm eff}^{k,k''}(P_0,\mathbf{Q},\mathbf{Q}'')G_0^{k''}(P_0,\mathbf{Q}'')\\
&\times\tilde{T}^{k'',k'}(P_0,\mathbf{Q}'',\mathbf{Q}')d^3Q''/(2\pi)^3\,,
\end{aligned}\tag{13}$$

where the superscripts k, k', and k'' can be $++$ and $--$ only. In Eq. (13),

$$G_0^{k}(P_0,\mathbf{Q})\equiv\frac{m^2}{E_Q^2}\frac{1}{H_k(|\mathbf{Q}|)+i\delta}\,,\tag{14}$$

with

$$H_{\pm\pm}(|\mathbf{Q}|)\equiv\pm P_0-2E_Q\,,$$

are the two propagators or Green functions, and $\tilde{V}_{\rm eff}^{k,k'}$ are the effective interactions which are defined in Fig. 6. Note that an effective interaction is the sum of all noniterative diagrams of the given type. To order g^2 these ef-

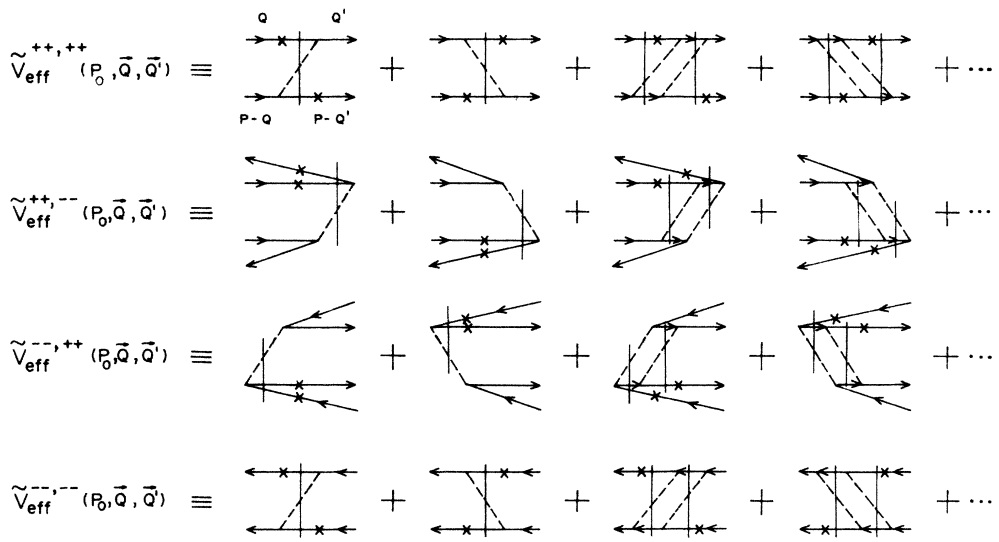

FIG. 6. Matrix elements of the effective interactions appearing in Eq. (13).

fective interactions are given by the following expressions:

$$\tilde{V}^{k,k'}_{\text{eff},1}(P_0,\mathbf{Q},\mathbf{Q}')\equiv A^{k,k'}(\mathbf{Q},\mathbf{Q}')\mathscr{V}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')\ , \tag{15}$$

where

$$A^{k,k'}(\mathbf{Q},\mathbf{Q}')\equiv\Lambda^k(\mathbf{Q})O^{(1)}O^{(2)}\Lambda^{k'}(\mathbf{Q}')\ ,$$

$$\mathscr{V}^{\pm\pm,\pm\pm}(P_0,\mathbf{Q},\mathbf{Q}')\equiv\frac{g^2}{2\omega}\frac{2}{\pm P_0-E_{\mathbf{Q}}-E_{\mathbf{Q}'}-\omega+i\delta}\ ,$$

$$\mathscr{V}^{\pm\pm,\mp\mp}(P_0,\mathbf{Q},\mathbf{Q}')\equiv\frac{g^2}{2\omega}\frac{2}{-E_{\mathbf{Q}}-E_{\mathbf{Q}'}-\omega+i\delta}\ ,$$

and

$$\omega=[(\mathbf{Q}'-\mathbf{Q})^2+\mu^2]^{1/2}\ .$$

Expressions for the higher-order diagrams in Fig. 6 can be written in a straightforward way using the rules of the time-ordered perturbation theory (see Appendix). Equation (13) represents two (not four) coupled integral equations in two unknowns, namely $\tilde{T}^{++,k'}(P_0,\mathbf{Q},\mathbf{Q}')$ and $\tilde{T}^{--,k'}(P_0,\mathbf{Q},\mathbf{Q}')$.

The fully off-shell Bethe-Salpeter amplitude $T(P_0,Q,Q')$ of Eq. (1) is related to $\tilde{T}(P_0,\mathbf{Q},\mathbf{Q}')$ by the following relation:

$$\begin{aligned}T^{i,j}(P_0,Q,Q')=&V^{i,j}_{\text{eff}}(P_0,Q,Q')+\sum_k\int V^{i,k}_{\text{eff}}(P_0,Q,\mathbf{Q}'')G^k_0(P_0,\mathbf{Q}'')V^{k,j}_{\text{eff}}(P_0,\mathbf{Q}'',Q')d^3Q''/(2\pi)^3\\&+\sum_{kk'}\int\int V^{i,k}_{\text{eff}}(P_0,Q,\mathbf{Q}'')G^k_0(P_0,\mathbf{Q}'')\tilde{T}^{k,k'}(P_0,\mathbf{Q}'',\mathbf{Q}''')G^{k'}_0(P_0,\mathbf{Q}''')\\&\times V^{k',j}_{\text{eff}}(P_0,\mathbf{Q}''',Q')d^3Q''d^3Q'''/(2\pi)^6\ .\end{aligned} \tag{16}$$

Here, as well as in the rest of this paper, i and j can be $++$, $--$, e, or o, while k and k' can be only $++$ or $--$. The effective interactions appearing in Eq. (16) are defined below, to order g^2. Higher-order terms can be written easily using the rules of the time-ordered perturbation theory:

$$V^{i,j}_{\text{eff},1}(P_0,Q,Q')\equiv A^{i,j}(\mathbf{Q},\mathbf{Q}')\mathscr{V}^{i,j}(P_0,Q,Q')\ ,$$

$$V^{i,k}_{\text{eff},1}(P_0,Q,\mathbf{Q}')\equiv A^{i,k}(\mathbf{Q},\mathbf{Q}')\mathscr{V}^{i,k}(P_0,Q,\mathbf{Q}')\ ,$$

$$V^{k,i}_{\text{eff},1}(P_0,\mathbf{Q},Q')\equiv A^{k,i}(\mathbf{Q},\mathbf{Q}')\mathscr{V}^{k,i}(P_0,\mathbf{Q},Q')\ ,$$

where

$$A^{i,j}(\mathbf{Q},\mathbf{Q}')\equiv\Lambda^i(\mathbf{Q})O^{(1)}O^{(2)}\Lambda^j(\mathbf{Q}')\ ,$$

$$\mathscr{V}^{i,j}(P_0,Q,Q')\equiv\frac{g^2}{2\omega}\left[\frac{1}{Q'_0-Q_0-\omega+i\delta}+\frac{1}{Q_0-Q'_0-\omega+i\delta}\right]\ ,$$

$$\mathscr{V}^{i,++}(P_0,Q,\mathbf{Q}')\equiv\frac{g^2}{2\omega}\left[\frac{1}{P_0-E_{\mathbf{Q}'}-Q_0-\omega+i\delta}+\frac{1}{Q_0-E_{\mathbf{Q}'}-\omega+i\delta}\right]\ , \tag{17}$$

$$\mathcal{V}^{i,--}(P_0,Q,Q')\equiv\frac{g^2}{2\omega}\left[\frac{1}{-E_{Q'}-Q_0-\omega+i\delta}+\frac{1}{Q_0-P_0-E_{Q'}-\omega+i\delta}\right],$$

$$\mathcal{V}^{++,i}(P_0,\mathbf{Q},Q')\equiv\frac{g^2}{2\omega}\left[\frac{1}{Q_0'-E_Q-\omega+i\delta}+\frac{1}{P_0-E_Q-Q_0'-\omega+i\delta}\right],$$

and

$$\mathcal{V}^{--,i}(P_0,\mathbf{Q},Q')\equiv\frac{g^2}{2\omega}\left[\frac{1}{Q_0'-P_0-E_Q-\omega+i\delta}+\frac{1}{-E_Q-Q_0'-\omega+i\delta}\right].$$

Equation (16) can be justified as follows. Consider once again the infinite set of time-ordered diagrams obtained from Eq. (1) (see step 2 above). All those diagrams which cannot be cut in two by a vertical line without intersecting a boson line or particle-hole lines together constitute $V^{i,j}_{\text{eff}}$, which is the first term on the rhs of Eq. (16). Among the remaining diagrams there will be some which can be cut only once. They have to be of the form $i\rightarrow k\rightarrow j$. Their sum can be written in a compact form as the second term in Eq. (16). The rest of the diagrams can be summed to get the third term in Eq. (16).

Equation (16) is exactly equivalent to the BSE [Eq. (1)] with the interaction in Eq. (2). $\widetilde{T}$ appearing on the rhs is obtained by solving the three-dimensional equation (13) which involves only a truncated basis. All quantities on the rhs of Eq. (16) are known and it expresses the fully off-shell amplitude T essentially as an integral over $\widetilde{T}$. In comparison, Eq. (1) is a four-dimensional integral equation defined over the full basis. If one is interested in on-shell elastic scattering of, say, two nucleons (NN→NN), then it is sufficient to consider Eq. (13) only, because

$$\widetilde{T}^{++,++}_{\text{on-shell}}(P_0,\mathbf{Q},\mathbf{Q}')=T^{++,++}_{\text{on-shell}}(P_0,Q,Q')\ .$$

The price that has been paid is that the effective interaction is an infinite series. However, experience[15] shows that this may be a rapidly converging series and, in practice, one is not likely to be required to go beyond the terms of order g^4 or g^6. An approximate method which in the low-energy limit greatly simplifies the calculation of these higher-order terms was presented in Ref. 15. Like the BSE, Eq. (16) enables one to treat the relativistic, off-shell, and boson-exchange effects in a two-fermion (each with spin $\frac{1}{2}$) system in a consistent way. Equation (16) can be applied, for example, to the problem of NN scattering below pion-production threshold. It can also serve as a starting point to derive two-body bound-state equations which can be applied, for example, to the deuteron. Before we consider the bound-state equations, we would like to overcome the only approximation that we made so far, namely the ladder approximation [Eq. (2)],

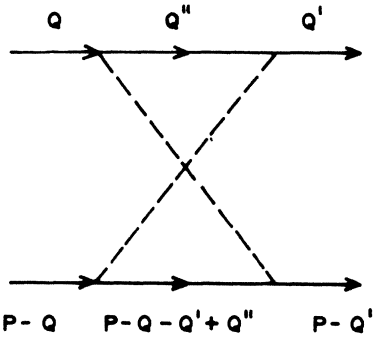

FIG. 7. The "crossed-box" (Feynman) diagram.

and prove that Eqs. (13) and (16) are valid even in the general case.

1. *Beyond the ladder approximation*

The driving term in the BSE [Eq. (1)], $V(Q,Q')$, is an infinite series representing all connected two-particle irreducible diagrams, the lowest-order term in the series being given by Eq. (2). The next higher term is the so-called "crossed-box diagram" (Fig. 7), not to be confused with the apparently similar time-ordered diagrams in Fig. 2. Its expression involves

$$i\int V(Q,Q'')\frac{(\not{Q}''+m)_1}{Q''^2-m^2+i\delta}\frac{(\not{D}+m)_2}{D^2-m^2+i\delta}V(Q'',Q')\frac{d^4Q''}{(2\pi)^4}$$

$$(D\equiv P-Q-Q'+Q'')\ , \quad (18)$$

where the interactions V are given by Eq. (2). We substitute Eqs. (4) and (5) in Eq. (18) and perform the Q_0'' integration exactly. The resulting 24 terms (operators) are represented schematically in Fig. 8. Their expressions are consistent with the rules in the Appendix. These expressions show that none of the diagrams in Fig. 8, not even those in the box, can give rise to an iterative diagram when external fermion lines are drawn. In other words, these expressions cannot be cast in the form $f(\mathbf{Q},\mathbf{Q}'')g(\mathbf{Q}'')f(\mathbf{Q}'',\mathbf{Q}')$, where f and g are any arbitrary functions. This is essentially because, unlike the g^4-order terms in Eq. (6), one of the internal fermions in (18) has a complicated four-momentum $D\equiv P-Q-Q'+Q''$ that depends on Q and Q'. The result that Fig. 7 does not give rise to iterative time-ordered diagrams is to be expected because the crossed-box diagram itself is an irreducible diagram. This result will hold true also in the case of all higher-order terms in $V(Q,Q')$ because they all correspond to irreducible diagrams. It should be obvious now that even in the general case the structure of Eqs. (13) and (16) remains unchanged; one is only required to include additional noniterative time-ordered diagrams of order g^4 and higher in $\widetilde{V}_{\text{eff}}$ and V_{eff}.

B. Relativistic two-body bound-state equations

In order to obtain two-body bound-state equations, we use

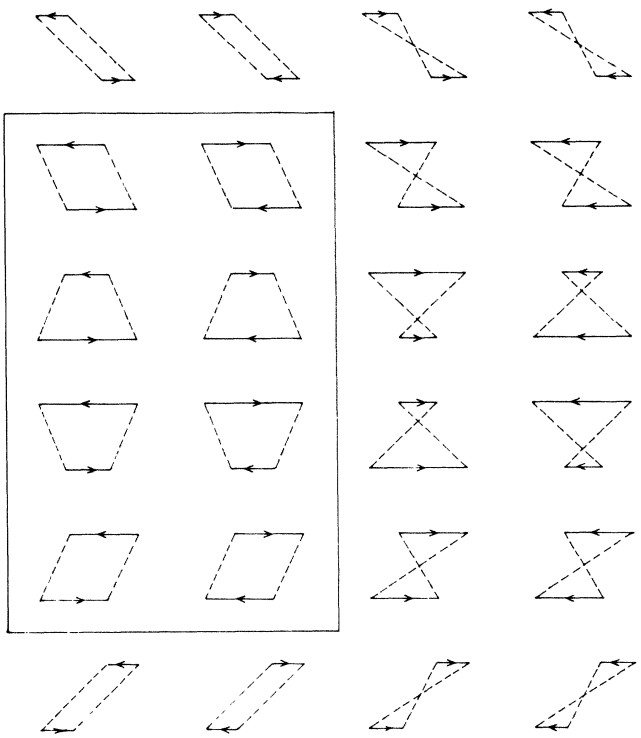

FIG. 8. Schematic representation of terms resulting from Eq. (18).

$$\widetilde{T}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')=\frac{\widetilde{\Gamma}^k(P_0,\mathbf{Q})\widetilde{\Gamma}^{k'\dagger}(P_0,\mathbf{Q}')}{P_0^2-m_B^2}+R \tag{19}$$

and

$$T^{i,j}(P_0,Q,Q')=\frac{\Gamma^i(P_0,Q)\Gamma^{j\dagger}(P_0,Q')}{P_0^2-m_B^2}+R' . \tag{20}$$

Here, $\widetilde{\Gamma}$ and Γ are the vertex functions, R and R' are (regular) functions with no singularity at $P_0^2=m_B^2$, and m_B is the bound-state mass. Upon substitution of Eqs. (19) and (20) in Eqs. (13) and (16), respectively, and equating the residues at $P_0^2=m_B^2$, we get

$$\widetilde{\Gamma}^k(P_0,\mathbf{Q})=\sum_{k'}\int \widetilde{V}_{\text{eff}}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}') \times G_0^{k'}(P_0,\mathbf{Q}')\widetilde{\Gamma}^{k'}(P_0,\mathbf{Q}')\frac{d^3Q'}{(2\pi)^3} \tag{21}$$

and

$$\Gamma^i(P_0,Q)=\sum_{k'}\int V_{\text{eff}}^{i,k'}(P_0,Q,\mathbf{Q}') \times G_0^{k'}(P_0,\mathbf{Q}')\widetilde{\Gamma}^{k'}(P_0,\mathbf{Q}')\frac{d^3Q'}{(2\pi)^3} , \tag{22}$$

where $P_0=m_B$. Equation (21) represents two coupled integral equations in two unknowns, namely $\widetilde{\Gamma}^{++}(P_0,\mathbf{Q})$ and $\widetilde{\Gamma}^{--}(P_0,\mathbf{Q})$, Fig. 9. Once these two vertex functions are obtained, they can be substituted in Eq. (22) to obtain $\Gamma^i(P_0,Q)$, where $i=++$, $--$, e (even), or o (odd). Note that Eq. (22) is not an integral equation for $\Gamma^i(P_0,Q)$ since all quantities on the rhs are known. Moreover, it involves only a three-dimensional integration.

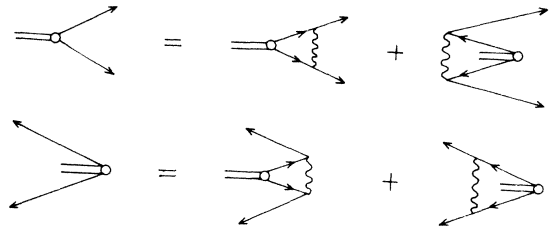

FIG. 9. Diagrammatic representation of Eq. (21). The wavy line represents $\widetilde{V}_{\text{eff}}$.

Now we would like to write partial-wave expansions of Eqs. (21) and (22) for arbitrary values of J. Let us concentrate on Eq. (21) first. We define

$$|\varphi^k(P_0,\mathbf{Q})\rangle\equiv\frac{m}{E_Q}\frac{\widetilde{\Gamma}^k(P_0,\mathbf{Q})}{H_k(|\mathbf{Q}|)} \quad (k=\pm\pm) . \tag{23}$$

With this definition, Eq. (21) becomes

$$H_k(|\mathbf{Q}|)|\varphi^k(P_0,\mathbf{Q})\rangle = \sum_{k'}\int\frac{m^2}{E_Q E_{Q'}}\widetilde{V}_{\text{eff}}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')|\varphi^{k'}(P_0,\mathbf{Q}')\rangle\frac{d^3Q'}{(2\pi)^3} . \tag{24}$$

We expand $|\varphi^k(P_0,\mathbf{Q})\rangle$ as

$$|\varphi^k(P_0,\mathbf{Q})\rangle=\sum_{l'S'J'M'}\varphi^k_{l'S'J'M'}(P_0,|\mathbf{Q}|)|\psi^k(\mathbf{Q}l'S'J'M')\rangle , \tag{25}$$

where

$$\varphi^k_{l'S'J'M'}(P_0,|\mathbf{Q}|)\equiv\langle\bar{\psi}^k(\mathbf{Q}l'S'J'M')|\varphi^k(P_0,\mathbf{Q})\rangle . \tag{26}$$

In Eq. (26) and in the following discussion angular integration is implicit in a scalar product. The basis is defined as follows:

$$|\psi^k(\mathbf{Q}lSJM)\rangle = \sum_{m_l M_S}\langle lSm_lM_S|JM\rangle Y_{lm_l}(\hat{Q})|\mathbf{Q},\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle_k ,$$

$$|\mathbf{Q},\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle_{++}=\sum_{\lambda\lambda'}\langle\tfrac{1}{2}\tfrac{1}{2}\lambda\lambda'|SM_S\rangle u_1(\mathbf{Q}\lambda)u_2(-\mathbf{Q}\lambda') ,$$

$$|\mathbf{Q},\tfrac{1}{2}\tfrac{1}{2}SM_S\rangle_{--}=\sum_{\lambda\lambda'}\langle\tfrac{1}{2}\tfrac{1}{2}\lambda\lambda'|SM_S\rangle v_1(-\mathbf{Q}\lambda)v_2(\mathbf{Q}\lambda') .$$

We have the following orthogonality and closure relations:

$$\langle\bar{\psi}^k(\mathbf{Q}lSJM)|\psi^k(\mathbf{Q}l'S'J'M')\rangle=\delta_{ll'}\delta_{SS'}\delta_{JJ'}\delta_{MM'} ,$$

$$\sum_{lSJM}|\psi^k(|\mathbf{Q}|,\hat{Q},lSJM)\rangle\langle\bar{\psi}^k(|\mathbf{Q}|,\hat{Q}',lSJM)|=\delta(\Omega_Q-\Omega_{Q'})\Lambda^k(\mathbf{Q}) .$$

Substituting Eq. (25) in Eq. (24) and premultiplying by $\langle\bar{\psi}^k(\mathbf{Q}lSJM)|$, we get two coupled equations for $\varphi^{\pm\pm}_{lSJM}$:

$$H_k(|\mathbf{Q}|)\varphi^k_{lSJM}(P_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\frac{m^2}{E_Q E_{Q'}}\times\sum_{l'S'J'M'}\sum_{k'}\langle\bar{\psi}^k(\mathbf{Q}lSJM)|\,\tilde{V}^{k,k'}_{\rm eff}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{k'}(\mathbf{Q}'l'S'J'M')\rangle\varphi^{k'}_{l'S'J'M'}(P_0,|\mathbf{Q}'|)\,. \tag{27}$$

Recall that $\int d\Omega_Q d\Omega_{Q'}$ is implicit in the matrix element in Eq. (27).

For SS coupling [Eq. (3a)] using Eq. (15), we get

$$\begin{aligned}\langle\bar{\psi}^{++}(\mathbf{Q}lSJM)|\,\tilde{V}^{++,++}_{\rm eff,1}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{++}(\mathbf{Q}'l'S'J'M')\rangle&\\ =\langle\bar{\psi}^{++}(\mathbf{Q}lSJM)|\,\mathscr{V}^{++,++}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{++}(\mathbf{Q}'l'S'J'M')\rangle&\\ =\int\sum\langle\tfrac{1}{2}\tfrac{1}{2}\mu\mu'|S'M_{S'}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}\lambda\lambda'|SM_S\rangle\langle l'S'm_{l'}M_{S'}|J'M'\rangle\langle lSm_lM_S|JM\rangle&\\ \times Y^*_{lm_l}(\hat{Q})Y_{l'm_{l'}}(\hat{Q}')\mathscr{V}^{++,++}(P_0,\mathbf{Q},\mathbf{Q}')&\\ \times\bar{u}_1(\mathbf{Q}\lambda)\bar{u}_2(-\mathbf{Q}\lambda')u_1(\mathbf{Q}'\mu)u_2(-\mathbf{Q}'\mu')d\Omega_Q d\Omega_{Q'}\,.&\end{aligned} \tag{28}$$

Using

$$\bar{u}_1(\mathbf{Q}\lambda)\bar{u}_2(-\mathbf{Q}\lambda')u_1(\mathbf{Q}'\mu)u_2(-\mathbf{Q}'\mu')=\frac{\epsilon_Q\epsilon_{Q'}}{(2m)^2}\chi^\dagger_\lambda(1)\chi^\dagger_{\lambda'}(2)\left[1-\frac{\boldsymbol{\sigma}\cdot\mathbf{Q}\,\boldsymbol{\sigma}\cdot\mathbf{Q}'}{\epsilon_Q\epsilon_{Q'}}\right]_1\Big[\ \Big]_2\chi_\mu(1)\chi_{\mu'}(2)\ \ (\epsilon_Q\equiv E_Q+m)\,,$$

and defining

$$\psi_{lSJM}(\mathbf{Q})\equiv\sum\langle\tfrac{1}{2}\tfrac{1}{2}\mu\mu'|SM_S\rangle\langle lSm_lM_S|JM\rangle Y_{lm_l}(\hat{Q})\chi_\mu(1)\chi_{\mu'}(2)\,,$$

the above matrix element becomes

$$\langle\bar{\psi}^{++}(\mathbf{Q}lSJM)|\,\tilde{V}^{++,++}_{\rm eff,1}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{++}(\mathbf{Q}'l'S'J'M')\rangle=\frac{\epsilon_Q\epsilon_{Q'}}{(2m)^2}\int\mathscr{V}^{++,++}(P_0,\mathbf{Q},\mathbf{Q}')\psi^\dagger_{lSJM}(\mathbf{Q})f_-(\sigma_1,\mathbf{Q},\mathbf{Q}')f_-(\sigma_2,\mathbf{Q},\mathbf{Q}')\psi_{l'S'J'M'}(\mathbf{Q}')d\Omega_Q d\Omega_{Q'}\,, \tag{29}$$

where we have introduced the following notation:

$$f_\pm(\sigma_i,\mathbf{Q},\mathbf{Q}')\equiv 1\pm\frac{\sigma_i\cdot\mathbf{Q}\,\sigma_i\cdot\mathbf{Q}'}{\epsilon_Q\epsilon_{Q'}}\,,\quad g_\pm(\sigma_i,\mathbf{Q},\mathbf{Q}')\equiv\sigma_i\cdot\left[\frac{\mathbf{Q}}{\epsilon_Q}\pm\frac{\mathbf{Q}'}{\epsilon_{Q'}}\right].$$

Similarly, it can be shown that

$$\langle\bar{\psi}^{\pm\pm}(\mathbf{Q}lSJM)|\,\tilde{V}^{\pm\pm,\mp\mp}_{\rm eff,1}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{\mp\mp}(\mathbf{Q}'l'S'J'M')\rangle=-\frac{\epsilon_Q\epsilon_{Q'}}{(2m)^2}\int\mathscr{V}^{\pm\pm,\mp\mp}(P_0,\mathbf{Q},\mathbf{Q}')\psi^\dagger_{lSJM}(\mathbf{Q})g_+(\sigma_1,\mathbf{Q},\mathbf{Q}')g_+(\sigma_2,\mathbf{Q},\mathbf{Q}')\psi_{l'S'J'M'}(\mathbf{Q}')d\Omega_Q d\Omega_{Q'} \tag{30}$$

and

$$\langle\bar{\psi}^{--}(\mathbf{Q}lSJM)|\,\tilde{V}^{--,--}_{\rm eff,1}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi^{--}(\mathbf{Q}'l'S'J'M')\rangle=\frac{\epsilon_Q\epsilon_{Q'}}{(2m)^2}\int\mathscr{V}^{--,--}(P_0,\mathbf{Q},\mathbf{Q}')\psi^\dagger_{lSJM}(\mathbf{Q})f_-(\sigma_1,\mathbf{Q},\mathbf{Q}')f_-(\sigma_2,\mathbf{Q},\mathbf{Q}')\psi_{l'S'J'M'}(\mathbf{Q}')d\Omega_Q d\Omega_{Q'}\,. \tag{31}$$

Matrix elements of higher-order terms in $\tilde{V}_{\rm eff}$, appearing in Eq. (27), can be calculated in a similar fashion. For the sake of simplicity and for later use, we choose to present below equations that are based on effective interactions calculated to order g^2. These will be useful when we discuss relative magnitudes of various components of the relativistic deuteron wave functions. Substitution of Eqs. (29)–(31) in Eq. (27) then gives

$$H_k(|\mathbf{Q}|)\varphi^k_{lSJM}(P_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\sum_{l'S'J'M'}\sum_{k'}\langle\psi^\dagger_{lSJM}(\mathbf{Q})|\,W^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')\,|\psi_{l'S'J'M'}(\mathbf{Q}')\rangle\varphi^{k'}_{l'S'J'M'}(P_0,|\mathbf{Q}'|)\,. \tag{32}$$

For SS coupling [(Eq. (3a)]

$$W^{\pm\pm,\pm\pm}(P_0,\mathbf{Q},\mathbf{Q}')=\frac{\epsilon_Q\epsilon_{Q'}}{4E_QE_{Q'}}\mathscr{V}^{\pm\pm,\pm\pm}(P_0,\mathbf{Q},\mathbf{Q}')f_-(\sigma_1,\mathbf{Q},\mathbf{Q}')f_-(\sigma_2,\mathbf{Q},\mathbf{Q}')$$

and

$$W^{\pm\pm,\mp\mp}(P_0,\mathbf{Q},\mathbf{Q}')=-\frac{\epsilon_Q\epsilon_{Q'}}{4E_QE_{Q'}}\mathscr{V}^{\pm\pm,\mp\mp}(P_0,\mathbf{Q},\mathbf{Q}')g_+(\sigma_1,\mathbf{Q},\mathbf{Q}')g_+(\sigma_2,\mathbf{Q},\mathbf{Q}')\ . \tag{33}$$

$W^{k,k'}$ for PP coupling [Eq. (3b)] are obtained from the corresponding W in Eq. (33) by interchanging f and g.

Partial-wave projection of Eq. (22) is only slightly more involved. We only quote the final results. Instead of Eq. (27), we now get

$$\Gamma^i_{lSJM}(P_0,Q_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\frac{m^2}{E_QE_{Q'}}\sum_{l'S'J'M'}\sum_{k'}\langle\bar{\psi}^i(\mathbf{Q}lSJM)\,|\,V^{i,k'}_{\text{eff}}(P_0,Q,Q')\,|\,\psi^{k'}(\mathbf{Q}'l'S'J'M')\rangle\varphi^{k'}_{l'S'J'M'}(P_0,|\mathbf{Q}'|)\ , \tag{34}$$

where

$$\Gamma^i_{lSJM}(P_0,Q_0,|\mathbf{Q}|)\equiv\frac{m}{E_Q}\int\langle\bar{\psi}^i(\mathbf{Q}lSJM)\,|\,\Gamma^i(P_0,Q)d\Omega_Q\ ,$$

$\varphi^{k'}_{l'S'J'M'}$ on the rhs is to be obtained by solving Eq. (27), and ψ^i is defined in the same manner as ψ^k.

Matrix elements in Eq. (34) can be calculated as before. Below we present equations that are based on effective interactions calculated to order g^2. We have, analogous to Eq. (32),

$$\Gamma^i_{lSJM}(P_0,Q_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\sum_{l'S'J'M'}\sum_{k'}\langle\psi^\dagger_{lSJM}(\mathbf{Q})\,|\,W^{i,k'}(P_0,Q,Q')\,|\,\psi_{l'S'J'M'}(\mathbf{Q}')\rangle\varphi^{k'}_{l'S'J'M'}(P_0,|\mathbf{Q}'|)\ . \tag{35}$$

The $W^{k,k'}(P_0,Q,Q')$ appearing in Eq. (35) are given by expressions similar to those in Eq. (33), except that $\mathscr{V}^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')$ is replaced by $\mathscr{V}^{k,k'}(P_0,Q,Q')$. $W^{e,\pm\pm}$ and $W^{o,\pm\pm}$ are defined below. For SS coupling [Eq. (3a)] we have

$$W^{e,\pm\pm}(P_0,Q,Q')=\frac{\pm1}{\sqrt{N_1}}\frac{\epsilon_Q\epsilon_{Q'}}{4E_QEsubb\mathbf{Q}'}\mathscr{V}^{e,\pm\pm}(P_0,Q,Q')[f_-(\sigma_1,\mathbf{Q},\mathbf{Q}')g_+(\sigma_2,\mathbf{Q},\mathbf{Q}')-f_-(\sigma_2,\mathbf{Q},\mathbf{Q}')g_+(\sigma_1,\mathbf{Q},\mathbf{Q}')]$$

and

$$W^{o,\pm\pm}(P_0,Q,Q')=\frac{1}{\sqrt{N_2}}\frac{\epsilon_Q\epsilon_{Q'}}{4E_QE_{Q'}}\mathscr{V}^{o,\pm\pm}(P_0,Q,Q')[f_-(\sigma_1,\mathbf{Q},\mathbf{Q}')g_+(\sigma_2,\mathbf{Q},\mathbf{Q}')+f_-(\sigma_2,\mathbf{Q},\mathbf{Q}')g_+(\sigma_1,\mathbf{Q},\mathbf{Q}')]\ . \tag{36}$$

Finally, for PP coupling [Eq. (3b)], $W^{e,\pm\pm}$ and $W^{o,\pm\pm}$ are obtained from the corresponding W in Eq. (36) by interchanging f and g and changing the overall sign.

We stress that Eqs. (32) and (35) are valid in general; only the expressions for W given above are valid to order g^2.

Equations (32) and (35) can be simplified by using the symmetry properties of the interaction, W. The following observations can be made by studying the expressions for the W's:

(i) All W's are rotationally invariant so that the interaction matrix elements are diagonal in J and M, and are independent of M.

(ii) $W^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')$, $W^{k,k'}(P_0,Q,\mathbf{Q}')$, and $W^{o,k'}(P_0,Q,\mathbf{Q}')$ are symmetric under the operation $\sigma_1\leftrightarrow\sigma_2$, so that they cannot connect states of different spin symmetry ($S'\neq S$). On the other hand, $W^{e,k'}(P_0,Q,\mathbf{Q}')$ is antisymmetric under this operation and so it cannot connect states of the same spin symmetry ($S'=S$).

(iii) $W^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')$ and $W^{k,k'}(P_0,Q,\mathbf{Q}')$ are symmetric under the operation $\mathbf{Q}\rightarrow-\mathbf{Q}$, $\mathbf{Q}'\rightarrow-\mathbf{Q}'$ so that they cannot connect states with different spatial parity ($l-l'=$ an odd integer). On the other hand, $W^{e,k'}(P_0,Q,\mathbf{Q}')$ and $W^{o,k'}(P_0,Q,\mathbf{Q}')$ are antisymmetric under this operation and so they cannot connect states with the same spatial parity ($l-l'=$ an even integer). This, however, does not mean that the parity is not conserved, because the intrinsic parity of $\psi^e(\mathbf{Q}lSJM)$ and $\psi^o(\mathbf{Q}lSJM)$ is opposite to that of $\psi^k(\mathbf{Q}lSJM)$.

As a result, Eqs. (32) and (35) simplify considerably:

$$H_k(|\mathbf{Q}|)\varphi^k_{lSJM}(P_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\sum_{l'-l=\text{even}}\sum_{k'}\langle\psi^\dagger_{lSJM}(\mathbf{Q})\,|\,W^{k,k'}(P_0,\mathbf{Q},\mathbf{Q}')\,|\,\psi_{l'SJM}(\mathbf{Q}')\rangle\varphi^{k'}_{l'SJM}(P_0,|\mathbf{Q}'|) \tag{37}$$

and

$$\Gamma^i_{lSJM}(P_0,Q_0,|\mathbf{Q}|)=\int\frac{Q'^2dQ'}{(2\pi)^3}\sum_{l'S'}\sum_{k'}\langle\psi^\dagger_{lSJM}(\mathbf{Q})\,|\,W^{i,k'}(P_0,Q,\mathbf{Q}')\,|\,\psi_{l'S'JM}(\mathbf{Q}')\rangle\varphi^{k'}_{l'S'JM}(P_0,|\mathbf{Q}'|)\ . \tag{38}$$

In Eq. (38), if $i=\pm\pm$, then $l'-l=$even and $S'=S$; if $i=e$, then $l'-l=$odd and $S'\neq S$; and if $i=o$, then $l'-l=$odd and $S'=S$.

Equations (37) are the Schrödinger-type equations satisfied by the relativistic two-fermion (each with spin $\frac{1}{2}$) bound-state wave functions if the effective interaction is calculated to order g^2. Remarks similar to those made in connection with Eqs. (13) and (16) can be made in connection with Eqs. (37) and (38) as well.

1. *Application to the deuteron*

If we use the experimental fact that the deuteron is a $J^P=1^+$ object, then the only values of orbital angular momentum allowed in [Eqs. (27), (32), and] Eq. (37) are 0 and 2, giving us coupled equations for the ${}^3S_1^{\pm\pm}$ and ${}^3D_1^{\pm\pm}$ wave functions. Solutions of Eq. (37) when substituted on the rhs of Eq. (38) give us Q_0-dependent vertex functions for the ${}^3S_1^{\pm\pm}$, ${}^3D_1^{\pm\pm}$, ${}^1P_1^e$, and ${}^3P_1^o$ states. Interestingly, one does not get the remaining two states, namely ${}^3P_1^e$ and ${}^1P_1^o$. This is essentially because the effective interaction was calculated to order g^2. In order to get the ${}^3P_1^e$ state from ${}^3S_1^{\pm\pm}$ and ${}^3D_1^{\pm\pm}$ states [see Eq. (38)], one needs a $W^{e,k}$ that is symmetric under the operation $\sigma_1\leftrightarrow\sigma_2$ and antisymmetric under the operation $\mathbf{Q}\rightarrow-\mathbf{Q}$, $\mathbf{Q}'\rightarrow-\mathbf{Q}'$. Similarly, ${}^1P_1^o$ requires a $W^{o,k}$ that is antisymmetric under both of these operations. Such W's do not exist to order g^2. However, if the effective interaction is calculated to order g^4, such W's do appear, giving rise to the ${}^3P_1^e$ and ${}^1P_1^o$ states.

It follows from Eq. (17) and the expressions for $W^{i,k}$ that $\mathscr{V}^{i,k}(P_0,Q,Q')$, and hence $W^{i,k}(P_0,Q,Q')$ are symmetric under the interchange of energies of the two fermions, $Q_0\leftrightarrow P_0-Q_0$. As a result [see Eq. (38)],

$$\Gamma^i_{lSJM}(P_0,Q_0,|\mathbf{Q}|)=\Gamma^i_{lSJM}(P_0,P_0-Q_0,|\mathbf{Q}|)\ .$$

In other words, the Q_0-dependent vertex functions for the ${}^3S_1^{\pm\pm}$, ${}^3D_1^{\pm\pm}$, ${}^1P_1^e$, and ${}^3P_1^o$ states are symmetric under the interchange $Q_0\leftrightarrow P_0-Q_0$. Vertex functions for the ${}^3P_1^e$ and ${}^1P_1^o$ states, on the other hand, are antisymmetric under this interchange. Of course, all the vertex functions are antisymmetric, as they should be, under the combined operation of space, spin, isospin, ρ-spin,[27] and energy exchange.

In Table I we present the relative-order-of-magnitude estimates of the quantities

$$W^{i,k}(P_0,Q,Q')/\mathscr{V}^{i,k}(P_0,Q,Q')\ ,$$

TABLE I. Relative-order-of-magnitude estimates of the quantities $W^{i,k}(P_0,Q,Q')/\mathscr{V}^{i,k}(P_0,Q,Q')$ in the static limit, for the SS and PP couplings [Eqs. (3)].

i,k	SS	PP
$\pm\pm,\pm\pm$	$1+O(v^2/c^2)$	$O(v^2/c^2)$
$\mp\mp,\pm\pm$	$O(v^2/c^2)$	$1+O(v^2/c^2)$
$e,\pm\pm$	$O(v/c)$	$O(v/c)$
$o,\pm\pm$	$O(v/c)$	$O(v/c)$

in the static limit, for the SS and PP couplings [Eqs. (3)]. Though the static limits of $\mathscr{V}^{i,++}(P_0,Q,Q')$ and $\mathscr{V}^{i,--}(P_0,Q,Q')$ are different, this is not expected to alter the following conclusions.

The following is clear from Table I and Eq. (38).

(i) For the PP coupling the off-diagonal matrix elements $\mp\mp,\pm\pm$ are larger than the diagonal and other off-diagonal matrix elements. [This is easy to understand because $\gamma_5 u(\mathbf{p},s)=v(\mathbf{p},s)$ and $\gamma_5 v(\mathbf{p},s)=u(\mathbf{p},s)$.] Thus, in the deuteron the admixture of virtual states with both nucleons in negative-energy states is more likely than that of the states with one nucleon in a positive-energy and the other in a negative-energy state, provided the coupling is pseudoscalar-pseudoscalar. This feature of the deuteron vertex functions, despite its simple origin, has not received sufficient attention in the literature (see, however, Ref. 4).

(ii) The P-wave components of the deuteron are smaller than the S-wave and D-wave components by roughly a factor v/c. In the nonrelativistic limit only S- and D-wave components survive.

It is desirable to have the above estimates supported by detailed numerical calculations. The deuteron wave functions displayed in Ref. 4 are consistent with these estimates.

III. DISCUSSION

We have presented in Sec. II an exact three-dimensional reduction of the BSE [Eq. (1)] for the case of two spin-$\frac{1}{2}$ fermions exchanging scalar and pseudoscalar bosons. Scalar and pseudoscalar couplings were considered. We derived the three-dimensional equation (13) for an auxiliary amplitude $\tilde{T}$ and Eq. (16) for the fully off-shell Bethe-Salpeter amplitude T of Eq. (1). Equations (13) and (16) are valid in the general case and not merely when the ladder approximation is made. Note that Eq. (13) involves only a truncated basis and once it is solved to get $\tilde{T}$, T is obtained simply by substituting $\tilde{T}$ on the rhs of Eq. (16). The effective interactions appearing in these equations are infinite series, but that is not expected to cause any problem in practical applications because they are expected to converge rapidly (see Sec. II). From Eqs. (13) and (16) we obtained Eqs. (21) and (22), respectively, for the two-body bound-state vertex functions. General partial-wave decomposition was performed, leading to Eqs. (32) and (35), where, for simplicity, we retained terms of order g^2 only in the effective interactions. Finally, the symmetry properties of the interaction were used to rewrite Eqs. (32) and (35) in a simplified form of Eqs. (37) and (38). These are our equations for the relativistic two-fermion (each with spin $\frac{1}{2}$) bound state. Finally, we used these equations to estimate the relative orders of magnitude of the various components of the deuteron relativistic wave functions. These results were presented at the end of Sec. II. We now compare our work with some of the earlier work reported in the literature.

Gross equation:[10] In the integrand in Eq. (1), if the singularities of $V(Q,Q'')$ and $T(P,Q'',Q')$ in the complex-Q_0'' plane and the contribution of the negative-energy poles of the fermion propagator are neglected, and the Q_0'' integration is performed by closing the contour in the lower half plane, one gets, in the c.m. frame,

$$T(P_0,Q,Q')=V(Q,Q')+\int \frac{V(Q,Q''_{\rm on})(\not{Q}''_{\rm on}+m)_1(\not{P}-\not{Q}''_{\rm on}+m)_2+T(P_0,Q''_{\rm on},Q')d^3Q''}{2E_{Q''}P_0(P_0-2E_{Q''}+i\delta)(2\pi)^3}, \tag{39}$$

where, in general, $p_{\rm on}\equiv(E_{\rm p},\mathbf{p})$. Equivalently, one can obtain Eq. (39) from Eq. (1) by the replacement

$$\frac{(\not{Q}''+m)_1(\not{P}-\not{Q}''+m)_2}{(Q''^2-m^2+i\delta)[(P-Q'')^2-m^2+i\delta]}\rightarrow(\not{Q}''_{\rm on}+m)_1(\not{P}-\not{Q}''_{\rm on}+m)_2\frac{-2\pi i\delta(Q''_0-E_{Q''})}{2E_{Q''}P_0(P_0-2E_{Q''}+i\delta)}. \tag{40}$$

If one further replaces Q and Q' in Eq. (39) by $Q_{\rm on}$ and $Q'_{\rm on}$, respectively, and makes the ladder approximation, one gets the Gross equation[28]

$$T(P_0,Q_{\rm on},Q'_{\rm on})=V(Q_{\rm on},Q'_{\rm on})+\int \frac{V(Q_{\rm on},Q''_{\rm on})(\not{Q}''_{\rm on}+m)_1(\not{P}-\not{Q}''_{\rm on}+m)_2T(P_0,Q''_{\rm on},Q'_{\rm on})d^3Q''}{2E_{Q''}P_0(P_0-2E_{Q''}+i\delta)(2\pi)^3}, \tag{41a}$$

with

$$V(Q_{\rm on},Q'_{\rm on})=\frac{g^2O^{(1)}O^{(2)}}{(E_{Q'}-E_Q)^2-\omega^2_{Q'-Q}+i\delta}. \tag{41b}$$

Note that in Eq. (41a) the particle labeled 1 is on mass shell throughout the scattering process, while the particle labeled 2 is generally off mass shell. The bound-state equation corresponding to Eq. (41) is

$$\Gamma(P_0,Q_{\rm on})=\int \frac{V(Q_{\rm on},Q''_{\rm on})(\not{Q}''_{\rm on}+m)_1(\not{P}-\not{Q}''_{\rm on}+m)_2\Gamma(P_0,Q''_{\rm on})d^3Q''}{2E_{Q''}P_0(P_0-2E_{Q''}+i\delta)(2\pi)^3}. \tag{42}$$

This equation has been used[29] in detailed numerical work, resulting in relativistic deuteron wave functions which are perhaps the most sophisticated deuteron wave functions available in the literature today.

It is interesting to compare Eqs. (41) and (42) with our Eqs. (13) and (21), respectively. In Eqs. (41) and (42),

$$(\not{Q}''_{\rm on}+m)_1=2m\Lambda^{(1)}_+(\mathbf{Q}''),$$

and

$$(\not{P}-\not{Q}''_{\rm on}+m)_2=\frac{m}{E_{Q''}}[P_0\Lambda^{(2)}_+(-\mathbf{Q}'')-(P_0-2E_{Q''})\Lambda^{(2)}_-(\mathbf{Q}'')].$$

Thus they allow only those processes which proceed through intermediate states of the following two types. (i) Both intermediate particles are in positive-energy states ($++$), and (ii) the intermediate particle labeled 1 is in a positive-energy state and that labeled 2 in a negative-energy state ($+-$). Thus there is no possibility of admixture of states of the types $--$ and $-+$. In contrast to Eqs. (41) and (42), four types of intermediate states are inherent in Eqs. (13) and (21); they are $++$, $--$, $+-$, and $-+$. Secondly, we have shown that the diagrams such as those in Figs. 2(m)–2(r) represent noniterative processes when external lines are drawn. The Gross equation treats such processes as iterative. Finally, unlike Eq. (41b), our effective interaction is an infinite series. Consider, for example, the series for $\tilde{V}^{++,++}_{\rm eff}(P_0,\mathbf{Q},\mathbf{Q}')$. The first term in it involves $\mathscr{V}^{++,++}(P_0,\mathbf{Q},\mathbf{Q}')$, which may be compared with the interaction in Eq. (41b). These two quantities, though different, are identical in the static limit. The higher-order terms in the series for $\tilde{V}^{++,++}_{\rm eff}$, however, do not vanish in the static limit.

Schaden and Baier equation:[30] Recently, these authors have presented a new three-dimensional reduction of the BSE and have applied it to the case of NN scattering. In the c.m. frame Eq. (1) can be rewritten as

$$T(P_0,Q,Q')=V(Q,Q')+i\int V(Q,Q'')\frac{m^2}{E^2_{Q''}}\left[\frac{\Lambda^{(1)}_+(\mathbf{Q}'')}{Q''_0-E_{Q''}+i\delta}-\frac{\Lambda^{(1)}_-(-\mathbf{Q}'')}{Q''_0+E_{Q''}-i\delta}\right]\times\left[\frac{\Lambda^{(2)}_+(-\mathbf{Q}'')}{P_0-Q''_0-E_{Q''}+i\delta}-\frac{\Lambda^{(2)}_-(\mathbf{Q}'')}{P_0-Q''_0+E_{Q''}-i\delta}\right]T(P_0,Q'',Q')\frac{d^4Q''}{(2\pi)^4}. \tag{43}$$

Equation (39) would follow if the term containing $\Lambda^{(1)}_-$ is dropped, $\Lambda^{(2)}_-(\mathbf{Q}'')/(P_0-Q''_0+E_{Q''}-i\delta)$ is replaced by $\Lambda^{(2)}_-(\mathbf{Q}'')/P_0$, singularities of $V(Q,Q'')$ and $T(P_0,Q'',Q')$ in the complex-Q''_0 plane are neglected, and the Q''_0 integration is performed closing the contour in the lower half plane. The procedure of Schaden and Baier is a slight improvement over this in that they retain the terms containing $\Lambda^{(1)}_-$ and $\Lambda^{(2)}_-$ as they are. They too, however, neglect the singularities of $V(Q,Q'')$ and $T(P_0,Q'',Q')$ in performing the Q''_0 integration. [Strictly speaking, this approximation is valid only when $V(Q,Q'')$ and $T(P_0,Q'',Q')$ are independent of Q''_0.] Thus they need to consider only

$$T(P_0,Q,Q')=V(Q,Q')+i\int V(Q,Q'')\frac{m^2}{E^2_{Q''}}\left[\frac{\Lambda^{(1)}_+(\mathbf{Q}'')\Lambda^{(2)}_+(-\mathbf{Q}'')}{(Q''_0-E_{Q''}+i\delta)(P_0-Q''_0-E_{Q''}+i\delta)}+\frac{\Lambda^{(1)}_-(-\mathbf{Q}'')\Lambda^{(2)}_-(\mathbf{Q}'')}{(Q''_0+E_{Q''}-i\delta)(P_0-Q''_0+E_{Q''}-i\delta)}\right]T(P_0,Q'',Q')\frac{d^4Q''}{(2\pi)^4}, \tag{44}$$

because the cross terms do not contribute. Note that intermediate states of the types $+-$ and $-+$ do not occur in this equation. Now there are four ways of performing the Q_0'' integration in the last equation, because the integration contour can be closed in the upper or lower half planes in the first and the second terms; the results, in general, are different.

Blankenbecler-Sugar (BbS) equation:[9] In Ref. 9 they proposed an approximate three-dimensional reduction of the BSE for *scalar* particles only. Their equation can be obtained from Eq. (1) by dropping the factors $(Q''+m)_1$ and $(P-Q''+m)_2$, replacing Q_0'' in $V(Q,Q'')T(P_0,Q'',Q')$ as well as Q_0 and Q_0' everywhere in the equation by $\frac{1}{2}P_0$ and making the ladder approximation. One thus gets

$$T(P,\mathbf{Q},\mathbf{Q}')=V(\mathbf{Q},\mathbf{Q}')+i\int\frac{V(\mathbf{Q},\mathbf{Q}'')T(P_0,\mathbf{Q}'',\mathbf{Q}')d^4Q''}{(Q''^2-m^2+i\delta)[(P-Q'')^2-m^2+i\delta](2\pi)^4}$$

$$=V(\mathbf{Q},\mathbf{Q}')+\int\frac{V(\mathbf{Q},\mathbf{Q}'')T(P_0,\mathbf{Q}'',\mathbf{Q}')d^3Q''}{4E_{Q''}(P_0^2/4-E_{Q''}^2+i\delta)(2\pi)^3}\ , \tag{45}$$

where

$$V(\mathbf{Q},\mathbf{Q}')=\frac{-g^2O^{(1)}O^{(2)}}{\omega_{\mathbf{Q}-\mathbf{Q}'}^2}$$

is an instantaneous interaction. Note that the above procedure (i) neglects the singularities of $V(Q,Q'')$ and $T(P_0,Q'',Q')$ in the complex-Q_0'' plane, (ii) neglects all retardation effects, and (iii) assumes that the two interacting particles are always equally off mass shell. In order to generalize the BbS equation (45) to the case of two spin-$\frac{1}{2}$ particles, one may start with Eq. (43), which is identical to Eq. (1). If the Q_0'' integration is performed, as before, after making the replacements

$$V(Q,Q'')\rightarrow V(Q,Q'')_{Q_0''=P_0/2}\ ,$$

$$T(P_0,Q'',Q')\rightarrow T(P_0,Q'',Q')_{Q_0''=P_0/2}\ ,$$

only two terms—namely those with $\Lambda_+^{(1)}\Lambda_+^{(2)}$ and $\Lambda_-^{(1)}\Lambda_-^{(2)}$—survive [cf. Eq. (44)]. In other words, intermediate states of the types $+-$ and $-+$ would not occur; without them the P-wave components of the deuteron wave function are absent.

Zmora and Gersten[31] have also presented a set of three-dimensional equations which are exactly equivalent to the four-dimensional BSE. Their work, however, is restricted to the BSE in the ladder approximation for spinless particles. Their method is quite different from ours and appears to be far more complicated even for scalar particles. It is based on the so-called energy analytic representation of the scattering matrix and makes use of the generalized Wick rotation.

In summary, we have presented an exact three-dimensional reduction of the BSE for the case of two spin-$\frac{1}{2}$ fermions; the resulting equations have several attractive features. Application of these equations to the deuteron has provided valuable insight into the various components of the relativistic deuteron wave function and their couplings to each other.

Note added: Though we have considered only scalar and pseudoscalar bosons in this paper, the method is also applicable to the case of massless vector bosons with vector coupling. In that case we have $\mathcal{L}=g\bar{\psi}\gamma^u\psi\phi_\mu$ and $O^{(1)}O^{(2)}=-\gamma^{(1)}\gamma^{(2)}$.

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APPENDIX

In this appendix we give the rules to write expressions for the time-ordered diagrams representing scattering of two spin-$\frac{1}{2}$ fermions. We use the convention that time flows from left to right.

(1) Given a diagram, specify four-momenta of all the lines: Let Q and Q' be the initial and final four-momenta of a fermion labeled 1 and let $Q'',Q''',\ldots$ be its four-momenta in the intermediate states. Four-momenta of all other lines are determined using the energy-momentum conservation. In particular, $P-Q$ and $P-Q'$ will be the initial and final four-momenta of the other fermion if P is the total four-momentum.

(2) If the external line with four-momentum $Q(Q_0,\mathbf{Q})$ is going forward (backward) in time, we associate energy $Q_0(-Q_0)$ with it. Similar statements apply to the external lines with four-momenta Q', $P-Q$, and $P-Q'$. Energies of all relevant internal lines—bosons and fermions—are determined assuming they are on mass shell.

(3) With the energies of the lines, Q, Q', $P-Q$, and $P-Q'$ determined this way, calculate the total energy (E) in the initial or final state ($E=E_i=E_f$). Note that E may or may not be equal to P_0.

(4) The expression for the diagram is an integral over the internal three-momenta $\mathbf{Q}'',\mathbf{Q}''',\ldots$, the integrand containing the following factors.

(i) For each distinct intermediate state a factor $(E-\sum_n E_n+i\delta)^{-1}$, where E_n are the energies associated with the lines in that intermediate state. Recall the relevant internal lines are on shell while the external lines are on or off shell.

(ii) $\Lambda_+(\mathbf{Q}'')$ or $\Lambda_-(-\mathbf{Q}'')$ for each internal fermion with four-momentum $Q''(Q_0'',\mathbf{Q}'')$ going forward or back-

ward in time, respectively [see Eq. (5)].

(iii) A factor of the type $g^2 O^{(1)} O^{(2)}/(2\omega_{\mathbf{q}})$ for each internal boson with three-momentum $\mathbf{q}$ [see Eq. (4)], and a factor of the type $m/E_{\mathbf{Q}''}$ for each internal fermion with three-momentum $\mathbf{Q}''$ [see Eq. (5)].

(iv) Finally,

$$\frac{d^3Q''}{(2\pi)^3}\frac{d^3Q'''}{(2\pi)^3}\cdots .$$

(The Λ's and O's, being operators, should appear in the order in which the corresponding elements appear in the diagram.)

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