Relativistic scattering operators for Dirac particles: Structure, symmetries, and reconstruction

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A systematic construction of the general forms of relativistic scattering operators for spin-$\frac{1}{2}$ particles is presented. Lorentz-invariant two-body scattering operators for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$-spin-0 systems are explicitly represented in a manner which is the relativistic generalization of the Wolfenstein representation of Galilean-invariant scattering operators. The simpler spin-$\frac{1}{2}$-spin-0 case serves as a prototype for the much more complicated spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case. For each case a complete set of independent Lorentz invariants is obtained directly in terms of the available four-momenta and the Dirac γ matrices. The most general forms of the scattering operators are derived in terms of these. The implications of parity and time-reversal symmetries are obtained and the resultant constraints are imposed upon the scattering operator. The resulting analysis is in terms of independent true scalar amplitudes that are invariant to all the symmetries imposed. In this way a separation of the purely off-mass-shell components of the scattering operators is obtained and the characteristic structure of such components is explicated. Partially and fully on-mass-shell limits are subsequently obtained. The full implications of the Pauli principle are developed for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ identical particle case and a generalization of the Fierz transformation is developed. Isospin symmetry is also incorporated into the analysis. The connection to the optical potential for the scattering of a spin-$\frac{1}{2}$ particle from a composite spin-0 object (such as a nucleus) is described, completing a technical basis for the construction of such optical models from spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ scattering operators. The systematic development also provides a natural basis for extensions to investigate symmetry violating contributions in a relativistic context for both the identical and nonidentical particle spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ cases. A general framework for the reconstruction of invariant scattering operators from (known) matrix elements in the barycentric frame is developed. This framework reveals the relationship between the relativistic invariants and the Pauli spin structure of their matrix elements in each of the ρ-spin sectors of the full space. Constraints upon, ambiguities of, and alternatives in the reconstruction procedure are made apparent. A specific reconstruction scheme is developed which directly maps Pauli-space rotational invariant operators to Dirac-space Lorentz invariant operators via a covariant extension technique. This scheme for direct reconstruction on the full Dirac space precludes the inadvertent introduction of kinematic singularities, as well as instabilities in fitting and approximations; it also provides a vehicle for the analysis of alternative schemes. The formalism developed here is compared to methods employed in recent work on the reconstruction of the invariant nucleon-nucleon scattering operator from matrix elements of the solution of a relativistic scattering equation. A prospectus for further work is given.

I. INTRODUCTION

The Wolfenstein representation[1] of nonrelativistic Galilean-invariant scattering operators in terms of invariants compatible with symmetry principles has been enormously useful in treatments of the nuclear many-body problem. For the physical nonrelativistic spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ identical particle scattering operator in Pauli spin space, there are five independent terms[1] compatible with invariance under space inversion, rotations, Galilean frame transformations, reciprocity (time reversal), and the Pauli principle. The explicit structure in spin and spatial degrees of freedom (and in isospin degrees of freedom when this symmetry is employed) of the Wolfenstein form of the nucleon-nucleon (NN) scattering operator has served as a focus for systematically correlating the rich variety of nuclear excitations that are possible with nucleon-nucleus inelastic scattering.[2] On the theoretical side, knowledge of the invariant structure of a NN t matrix is invaluable as a control upon approximate treatments of nuclear matrix elements of this operator that arise in multiple scattering approaches.[3] In general, the NN t matrix is required off shell for this purpose, and the symmetry constraints specify the one additional (nonrelativistic) invariant which is needed in addition to the on-shell representation. The additional freedoms that arise in the relativistic circumstance have not been systematically explored. Given the recent advances which have resulted from simple relativistic approaches to the nuclear many-body bound state and scattering problems, it seems clear that further development of relativistic approaches is called for. It is also evident that the replacement of the Wolfenstein representation of scattering and interaction operators by a Lorentz covariant representation in terms of a complete set of invariants is a natural first step.

In this paper we develop the Lorentz invariant counterpart of the Wolfenstein representation of both the on-mass-shell and fully off mass-shell structure of the scattering (and general interaction) operators for the spin-$\frac{1}{2}$-spin-0 and spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ systems. These representations are manifestly covariant and are minimal in the sense that they consist of a sum of terms, each of which is, at most, linear in the Dirac γ matrices of a given fermion. For each system a set of invariants that is complete on the full Dirac space is formed explicitly in terms of the available four momenta and γ matrices such that the discrete symmetries of parity and time reversal are obeyed. These covariant representations are expected to serve as a technical basis for extensions of recent work that incorporates selected relativistic features into nuclear scattering and bound state processes. An example is the construction of spin-$\frac{1}{2}$-spin-0 optical potentials from interactions with the spin-$\frac{1}{2}$ constituent particles of the spin-0 "quasiparticle." The constraints we find on the functional forms of the purely off-mass-shell terms (how they go to zero as the mass shell is approached) may be expected to prove important in the necessary development of consistent approximation methods.

A general covariant form for the spin-$\frac{1}{2}$-spin-0 optical potential has recently been presented.[4] Off mass-shell features of relativistic scattering operators and invariants have received some attention recently,[5] but a complete set of explicit invariants in the full Dirac space for a spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ scattering operator does not appear to have been obtained previously. We provide a derivation of the spin-$\frac{1}{2}$-spin-0 case in parallel to our treatment of the more complex spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case. The latter case we treat both with and without identical particle constraints. We obtain a complete treatment of the twin implications of the Pauli principle in the identical particle circumstance and, as a by-product of this, a generalization of the Fierz transformation. The systematic development of the implications of the discrete symmetries of parity and time reversal which we employ provides a natural basis for investigations of various symmetry violating effects within a relativistic context. Tjon and Wallace[6] have recently developed a representation of the NN scattering operator in terms of Lorentz invariants which must be used in conjunction with ρ-spin projection operators. These invariants for the various ρ-spin sectors are not required to have definite time reversal or Pauli exchange symmetries. In contrast, the present work deals with Lorentz invariants on the full Dirac space, and because we insist that each be a (true) scalar with respect to all the symmetries imposed, the number of amplitudes involved here is much larger in the off-mass-shell case. An advantage of this approach is that the purely off-mass-shell components are isolated and have their characteristic structure revealed.

We derive a completely general framework by which Lorentz invariant scattering operators, both on and off the mass shell, can be reconstructed from knowledge of a full set of matrix elements given in a particular frame. Constraints, ambiguities, and alternatives in the reconstruction process are clarified by means of a combined Pauli-spin and ρ-spin decomposition. A specific scheme based upon the unique covariant extension of Pauli-space rotational invariant operators to Dirac-space Lorentz invariant operators is developed for reconstruction on the full Dirac space. This scheme eliminates the inadvertent introduction of problems associated with kinematic singularities and instabilities in fitting and approximation; it also provides a basis for the analysis of such questions for alternative procedures. The relationship of our general formalism to the method employed recently by Tjon and Wallace[6] for reconstruction of the invariant NN scattering operator is identified.

The role of relativistic dynamics in nuclear physics at the level of meson-nucleon degrees of freedom is a subject to which considerable effort is being devoted at present. A relativistic version of mean field theory has been developed over a number of years[7] and provides descriptions of nuclear matter saturation and the shell structure of nuclei which depend explicitly upon relativistic dynamical mechanisms. Considerable success for the description of spin observables in nucleon-nucleus scattering at intermediate energy has followed from recent work on a relativistic treatment of the first-order optical potential.[8,9] The scattering developments are particularly interesting because the nucleon-nucleus dynamical input is inferred from the NN scattering operator. The inferred strengths of the dominant Lorentz components of the optical potential have been found[8] to be consistent with the phenomenological Lorentz strengths obtained by fitting the parameters of relativistic mean field models to the bulk properties of nuclear matter. These initial successes must be treated with caution. In neither the bound state nor the scattering sector do we have an underlying relativistic theory which is unambiguous in relating the NN system and the many nucleon system. A relativistic version of Brueckner theory has been postulated[10] and employed recently[10,11] to incorporate the important NN correlation contributions to nuclear matter properties. Knowledge of the invariant forms allowed for the NN scattering operator are necessary here, as well as for first-order multiple scattering constructions of an elastic optical potential or effective interaction for inelastic scattering.

The initial work[8] on a Dirac-based first-order optical potential adopted an *ad hoc* form[12] for the NN scattering operator in the product space of Dirac spinors for each nucleon. The five Lorentz invariants for a physical fermion-fermion interaction introduced by Fermi to describe β decay of the nucleon were employed for the NN operator. These are sufficient for β decay since the nuclear four-momentum transfer is negligible and the weak coupling effectively suppresses mechanisms beyond the lowest order where only positive energy Dirac spinors are required for the matrix element. For the strong coupling process of NN scattering under the influence of nuclear binding and distortion effects, mechanisms beyond lowest order are important, the full spectrum of spinors is required in intermediate states, and the momentum transfer is significant. Use of just the Fermi invariants can guarantee only that the positive energy spinor matrix elements are correct, and even then, only for on-mass-shell kinematics. The large number of matrix elements that involve negative energy spinors, or off-mass-shell kinematics, are then implicitly determined by the kinematic properties of the invariants without the benefit of dynamical input or control. As we will see, only one or the other of a pair of the seven independent ρ-spin sectors can be spanned by the Fermi invariants, and then only on mass shell. This lack of control over the six independent sectors that connect with negative energy spinors is unfortunate since the dynamical content embedded in the Dirac equation over and above the Schrödinger dynamics *is* the coupling to negative energy states.[9] Control over all the independent ρ-spin sectors can be maintained by reconstruction of the scattering operator from a complete set of positive and negative energy spinor matrix elements obtained, for example, from the solution of a relativistic wave equation of the Bethe-Salpeter[13] type. Consequently, in developing such a formalism, we employ an orthogonal basis composed of products of single particle states in the full space that are solutions of the free Dirac equation. The natural projection operators for this basis are not manifestly covariant. New projection operators having covariant form are introduced and employed. A second method, introduced many years ago by Stapp[14] in work on a relativistic density matrix description of polarization phenomena, is adapted here to provide a more direct reconstruction formalism. This method employs a covariant extension of the special Lorentz boost operator to find Lorentz invariant forms for barycentric frame quantities.

In Sec. II we outline the notation and variables which we employ and define the scattering operators that we shall deal with. The discrete symmetries of space inversion and time reversal and the resultant constraints which they yield are discussed in operator form in Sec. III. The general forms of spin-$\frac{1}{2}$ scattering operators, constrained by these symmetries, are developed in Sec. IV. The results obtained for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system are applicable to the case of nonidentical particles. In Sec. V the simplifications which arise as a result of on mass-shell kinematic constraints are displayed and the relation between the transition operators we employ, on-mass-shell constraints, physical and virtual amplitudes, and cross-channel processes is described. In Sec. VI a comprehensive treatment of the operator constraints due to the Pauli principle is given for the relativistic spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system. The general form of the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ identical particle scattering operator is obtained and an extension of the Fierz[15] transformation is developed to cover the complete set of invariants that we deal with in this work. In Sec. VII we develop the general framework for the reconstruction of the invariant operators from matrix elements in the barycentric frame. The Pauli spin structure of matrix elements in each ρ-spin sector is explicated. In Sec. VIII a specific scheme for reconstructing the invariant operator in the full Dirac space by means of a covariant extension of Pauli matrix elements from each ρ-spin sector is presented. In Sec. IX a summary and outlook for future work is presented. In the Appendix the operator techniques of this work are extended in a compatible way to the symmetries of charge conjugation and PCT to facilitate related applications that require explicit imposition of those symmetries.

II. NOTATION, VARIABLES, AND SCATTERING OPERATORS

One of the objectives of this paper is to provide a basis for relating an elementary relativistic spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ scattering amplitude to the description of relativistic spin-$\frac{1}{2}$-spin-0 scattering in the circumstance that the spin-0 "particle" is an atomic nucleus. The relationship between the general form of a spin-$\frac{1}{2}$-spin-0 optical potential and a spin-$\frac{1}{2}$-spin-0 scattering amplitude will be clarified in Sec. III. However, if one presumes something like an impulse or "$t\rho$" source[3] for the optical potential, then it is clearly necessary to consider the general forms to be expected of both the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ scattering operator (t) and spin-$\frac{1}{2}$-spin-0 scattering operator, as well as the restrictions imposed upon them by general symmetry principles. In this section we define a notation, a choice of scattering variables, and an integral representation of the scattering operators which is convenient for dealing with both cases. In this section we are mainly concerned with preparations for subsequent formal considerations. The connection of the scattering operators and amplitudes which we employ to virtual and physical processes, to the Feynman propagator formalism, and to the particle-antiparticle field theoretic formalism is described in Sec. V, where special mass-shell results are considered.

A schematic representation of the scattering process is depicted in Fig. 1. The particles are labeled as particles (1) and (2). Particle (1) is always a Dirac (spin-$\frac{1}{2}$) particle and its initial and final four momenta are denoted by k and k', respectively. Particle (2) is either a spin-$\frac{1}{2}$ or a spin-0 particle and its initial and final four momenta are p and p', respectively. The invariant scattering operator is denoted by $\widetilde{T}$, and it is defined such that it produces an invariant amplitude when matrix elements are taken with

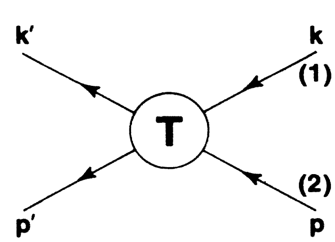

FIG. 1. Schematic representation of the scattering process showing the four momenta employed. Particle (1) is a spin-$\frac{1}{2}$ particle, while particle (2) may have either spin $\frac{1}{2}$ or spin 0.

(appropriately normalized) plane wave states describing the (virtual or physical) momenta and spins of the particles. Dirac adjoint matrix elements are assumed for the Dirac particles.

To be more specific, denote plane wave Dirac states of four momentum k and rest-frame spin s by

$$|k(\pm),s\rangle = |k,\pm\rangle\chi_s = |k\rangle u(\mathbf{k},\pm)\chi_s\ , \tag{2.1}$$

where the abstract (momentum-space) kets satisfy

$$\langle k'|k\rangle = \delta^4(k'-k)\ , \tag{2.2}$$

the positive and negative energy Dirac spinors $u(\mathbf{k},\pm)$ are given by

$$u(\mathbf{k},+) = \left[\frac{E_k+m}{2E_k}\right]^{1/2} \begin{bmatrix} 1 \\ \dfrac{\sigma\cdot\mathbf{k}}{E_k+m} \end{bmatrix} \tag{2.3}$$

and

$$u(\mathbf{k},-) = \left[\frac{E_k+m}{2E_k}\right]^{1/2} \begin{bmatrix} \dfrac{-\sigma\cdot\mathbf{k}}{E_k+m} \\ 1 \end{bmatrix} \tag{2.4}$$

and χ_s is a Pauli spinor. In Eqs. (2.3) and (2.4), $E_k^2 = |\mathbf{k}|^2 + m^2$, m is the mass, σ is the usual 2×2 Pauli spin matrix, and 1 is the 2×2 unit matrix. If $k^2 = m^2$ and $k^0 = \pm E_k$ in Eq. (2.1), then the states are on the mass shell and are positive and negative energy solutions, respectively, of the free Dirac equation. The orthonormality relations for the basis states of Eq. (2.1) are $(i,j=\pm)$

$$\langle k'(j),s'|k(i),s\rangle = \delta^4(k'-k)\delta_{i,j}\delta_{s,s'}\ . \tag{2.5}$$

The completeness relation is

$$\sum_s \int d^4k[\,|k(+),s\rangle\langle k(+),s| + |k(-),s\rangle\langle k(-),s|\,] = 1\ , \tag{2.6}$$

and the Dirac adjoint spinor is

$$\langle \overline{k'(\pm),s'}| = \langle k'(\pm),s'|\gamma^0\ . \tag{2.7}$$

Our gamma matrix conventions follow those of Bjorken and Drell.[16] The basis states for the spin-0 case are simply the kets $|k\rangle$, with the property of Eq. (2.2).

With these definitions we can now write the Lorentz invariant scattering amplitude in terms of the scattering operator $\widetilde{T}$. For spin-$\frac{1}{2}$-spin-0 scattering we have, e.g.,

$$\mathcal{F} = \left[\frac{E_{k'}E_k}{m^2}\right]^{1/2} \langle \overline{k'(j),s'};p'|\widetilde{T}|k(i),s;p\rangle\ , \tag{2.8a}$$

while for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case we have, e.g.,

$$\mathcal{F} = \left[\frac{E_{k'}E_{p'}E_kE_p}{m^4}\right]^{1/2} \times \langle \overline{k'(j),s'};\overline{p'(m),\sigma'}|\widetilde{T}|k(i),s;p(1),\sigma\rangle\ . \tag{2.8b}$$

The square root factors are needed in Eqs. (2.8) in order that $\mathcal{F}$ transform as a Lorentz scalar. It is often useful to define new basis states such that the square root factors are absorbed into the states and a covariant normalization condition may be employed.[16] This is especially useful in a field theoretic context employing particles and antiparticles. It is also convenient in certain other circumstances, such as in manipulating spinor matrix elements once momentum-space or position-space integrations have been performed. For our purposes the completeness relation on the full space, Eq. (2.6), assumes a central role. Thus it is convenient to use the basis states defined above. The reasons for this and the relationships between the different procedures are described in more detail in Sec. VI.

It is also convenient to introduce an integral representation of the operator $\widetilde{T}$ on the full space. For either the spin-$\frac{1}{2}$-spin-0 or the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ circumstance we write

$$\widetilde{T} = \int d^4\{k'p'kp\}\,|k';p'\rangle \times \hat{T}(k',k,\{\Gamma(1)\};p',p,\{\Gamma(2)\})\langle k;p|\ , \tag{2.9}$$

where $d^4\{k'p'kp\}$ denotes the product of the separate differentials, the two-particle momentum-space ket is $|k';p'\rangle = |k'\rangle|p'\rangle$, and $\{\Gamma(i)\}$ denotes the complete set of Dirac gamma matrices for particle (i), $(1,\gamma^5,\gamma^\mu,\gamma^5\gamma^\mu,\sigma^{\mu\nu})$, if it is a spin-$\frac{1}{2}$ particle. It is evident that the quantity

$$\hat{T}(k',k,\{\Gamma(1)\};p',p,\{\Gamma(2)\})$$

is an operator in the spin space(s) of the Dirac particle(s). It is also evident that the complete spin-space operator $\hat{T}$ must be constructed from its arguments, the three independent momenta and the Γ matrices, and the characteristic tensors of the space: the metric tensor $g_{\mu\nu}$ and the Levi-Cività tensor $\epsilon^{\mu\nu\rho\sigma}$ (after total four-momentum conservation is applied there remain three independent momenta). No other quantities are available.

The operator

$$\hat{T}(k',k\{\Gamma(1)\};p',p,\{\Gamma(2)\})$$

must transform as a Lorentz scalar. The set of Γ matrices contains Lorentz vectors as well as second-rank tensors. Overall scalars can be formed by contraction of the vectors with appropriate momentum vectors and also by contraction of the tensors with second-rank tensors constructed from the available momentum vectors. It is thus necessary to construct a complete set of Lorentz four vectors from the available momenta. This set can then be used to build a complete set of second-rank tensors as well. The complete sets of vectors and second-rank tensors are then available for use in obtaining the most general form of $\widetilde{T}$ in Secs. IV–VI.

Although Pauli symmetries in the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case are not introduced until Sec. VI, it is advantageous to begin from three linearly independent momenta with simple Pauli exchange properties:

$$q = k'-k = p-p'\ , \tag{2.10a}$$

$$K=k'-p=k-p'=\frac{(k'+k)-(p'+p)}{2}\,, \tag{2.10b}$$

and

$$\omega=k'+p'=k+p=\frac{(k'+k)+(p'+p)}{2}\,. \tag{2.10c}$$

The particle-label exchange symmetries of these variables are summarized in Table I. The relationship of the above variables to the (scalar) Mandelstam[17] variables s, t, and u is $s=\omega^2$, $t=q^2$, and $u=K^2$. A fourth linearly independent momentum vector is required to span the four dimensional Lorentz space. In view of the simple Pauli symmetries of the variables of Eqs. (2.10), it is convenient to construct this variable from the previous three. We choose

$$\xi^\mu=\epsilon^{\mu\nu\rho\sigma}q_\nu K_\rho \omega_\sigma\,, \tag{2.11}$$

where the completely antisymmetric Levi-Civita tensor satisfies $\epsilon_{0123}=+1$, the convention of Ref. 16. The antisymmetric nature of the Levi-Civita tensor immediately yields $\xi\cdot q=\xi\cdot K=\xi\cdot\omega=0$, thus establishing the linear independence of ξ.

In the following, we employ the complete set of vector variables, q, K, ω, and ξ as our basis set. We therefore rewrite Eq. (2.9) as

$$\widetilde{T}=\int d^4\{k'p'kp\}\,|k';p'\rangle \hat{T}(q,K,\omega,\xi,\{\Gamma(i)\})\langle k;p\,|\,, \tag{2.12}$$

where $i=1$ or 1,2 depending upon the context. This integral representation serves as the basis of our further developments.

III. SYMMETRIES: PARITY AND TIME REVERSAL CONSTRAINTS

Before proceeding to the construction of the general forms of the scattering operators in the next section, it is convenient to develop first the operator statements of, and the consequent constraints imposed by, the discrete symmetries of parity and time reversal.

The parity operator for either two-particle system is given as the product of parity operators for each particle:

$$\Pi=\Pi(1)\Pi(2)\,. \tag{3.1}$$

Here, $\Pi(1)$ is the parity operator for a Dirac particle, and is given by

$$\Pi(1)=\gamma^0(1)P(1)\,, \tag{3.2}$$

where the three-space inversion (parity) operator satisfies

$$P\,|p\rangle=|\widetilde{p}\rangle\,. \tag{3.3}$$

Throughout this paper we adopt the notation that $p^\mu=(p^0,\mathbf{p})$ implies $\widetilde{p}^{\,\mu}=p_\mu=(p^0,-\mathbf{p})$. If particle (2) is a Dirac particle, then its parity operator is given by Eq. (3.2) with the replacement (1)→(2), while if particle (2) is spin 0, then we have simply

$$\Pi(2)=P(2)\,. \tag{3.4}$$

In either case, $\Pi^{-1}=\Pi$ and the constraint of parity conservation is that the S matrix, or, equivalently, the scattering operator $\widetilde{T}$, is invariant under the parity transformation; that is,

$$\widetilde{T}=\Pi\widetilde{T}\Pi^{-1}\,. \tag{3.5}$$

In a Hamiltonian formalism Eq. (3.5) follows directly from the assumption that both the full and free Hamiltonians are invariant under Π,[18] while in a propagator formalism it results from the invariance of the free propagator and the interaction. Use of Eq. (3.5) together with Eq. (2.12) for the integral representation of the scattering operator yields

$$\widetilde{T}=\int d^4\{k'p'kp\}\,|\widetilde{k}';\widetilde{p}'\rangle \times\hat{T}(q,K,\omega,\xi,\{\gamma^0(i)\Gamma(i)\gamma^0(i)\})\langle \widetilde{k};\widetilde{p}\,|\,, \tag{3.6}$$

or

$$\widetilde{T}=\int d^4\{\widetilde{k}'\widetilde{p}'\widetilde{k}\widetilde{p}\}\,|k';p'\rangle \times\hat{T}(\widetilde{q},\widetilde{K},\widetilde{\omega},-\widetilde{\xi},\{\widetilde{\Gamma}(i)\})\langle k;p\,| \tag{3.7}$$

$$=\int d^4\{k'p'kp\}\,|k';p'\rangle \times\hat{T}(\widetilde{q},\widetilde{K},\widetilde{\omega},-\widetilde{\xi},\{\widetilde{\Gamma}(i)\})\langle k;p\,|\,, \tag{3.8}$$

where we have introduced the notation

$$\widetilde{\Gamma}=\gamma^0\Gamma\gamma^0\,, \tag{3.9}$$

for the parity transform of γ matrices. The particular transforms are summarized in Table I. The parity constraint upon the form of $\hat{T}$ is thus

$$\hat{T}(q,K,\omega,\xi,\{\Gamma(i)\})=\hat{T}(\widetilde{q},\widetilde{K},\widetilde{\omega},-\widetilde{\xi},\{\widetilde{\Gamma}(i)\})\,, \tag{3.10}$$

for either spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ or spin-$\frac{1}{2}$-spin-0 scattering.

Next we consider the implications of time-reversal symmetry or reciprocity.[16,18] The time-reversal operator is necessarily antiunitary and must thus be treated somewhat differently than the parity operator. The time-reversal

TABLE I. Parity, reciprocity transform, and particle-label exchange (E_{12}^k) properties of the momentum variables and the Dirac gamma matrices.

	Transform		
Variable	Parity	Reciprocity	(E_{12}^k)
Momentum			
q^μ	q_μ	$-q_\mu$	$-q^\mu$
K^μ	K_μ	K_μ	$-K^\mu$
ω^μ	ω_μ	ω_μ	ω^μ
ξ^μ	$-\xi_\mu$	ξ_μ	ξ^μ
Γ matrix	($\widetilde{\Gamma}$)	($\overline{\Gamma}$)	
γ^5	$-\gamma^5$	$-\gamma^5$	
γ^μ	γ_μ	γ_μ	
$\gamma^5\gamma^\mu$	$-\gamma^5\gamma_\mu$	$\gamma^5\gamma_\mu$	
$\sigma^{\mu\nu}$	$\sigma_{\mu\nu}$	$-\sigma_{\mu\nu}$	

operator for a Dirac particle is

$$\mathscr{T}=U\mathscr{K}\ , \tag{3.11}$$

where U is the unitary (and Hermitian) operator

$$U=i\gamma^1\gamma^3 \tag{3.12}$$

and $\mathscr{K}$ is an antilinear operator which, in a position-time representation, is given by KP_0. Here, P_0 is the operator which reverses the time coordinate and K denotes the usual complex conjugation operation. The complex conjugation operation is representation dependent. The operator which produces the complex conjugate in the position-time representation becomes, in the momentum representation, P_4K, where P_4 is the four-momentum inversion operator. Since we work in the momentum representation, the operator $\mathscr{K}$ is given by

$$\mathscr{K}=PK\ , \tag{3.13}$$

where P $(=P_4P_0)$ inverts three momenta. Thus the time-reversal operator, in a form suited to the momentum representation, is

$$\mathscr{T}=i\gamma^1\gamma^3PK\ , \tag{3.14}$$

with the inverse given by

$$\mathscr{T}^{-1}=\mathscr{K}U^\dagger=PK(i\gamma^1\gamma^3)\ .$$

For a spin-0 particle we can simply take $U=1$, and thus $\mathscr{T}=\mathscr{K}=PK$. Because of the presence of the global complex conjugation operator, the two-particle time-reversal operator is not simply the product of the one-particle operators. Rather, we have, for the spin-$\frac{1}{2}$-spin-0 case,

$$\mathscr{T}=U(1)P(1)P(2)K\ . \tag{3.15}$$

Similarly, for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case the time-reversal operator is

$$\mathscr{T}=U(1)U(2)P(1)P(2)K\ . \tag{3.16}$$

For either the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ or spin-$\frac{1}{2}$-spin-0 case, the pertinent time-reversal symmetry, or more precisely the reciprocity relation, is that the scattering operator $\tilde{T}$ satisfies[16,18]

$$\mathscr{T}\tilde{T}\mathscr{T}^{-1}=\tilde{T}^\ddagger\ , \tag{3.17}$$

where $\ddagger$ denotes the Dirac adjoint *operator*

$$A^\ddagger=\Gamma^0A^\dagger\Gamma^0\ . \tag{3.18}$$

In Eq. (3.18), Γ^0 denotes γ^0 in the spin-$\frac{1}{2}$-spin-0 case and $\gamma^0(1)\gamma^0(2)$ in the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case. The origin of the Γ^0 factor is the use in Eqs. (2.8) of a biorthogonal basis in terms of Dirac, rather than Hermitian, adjoint spinors.[16]

For a relativistic one-particle system the reciprocity relation given in Eq. (3.17) follows[16,18] directly from the assumptions that the full Hamiltonian satisfies the reciprocity relation

$$\mathscr{T}H\mathscr{T}^{-1}=H^\dagger\ , \tag{3.19}$$

and that the unperturbed Hamiltonian is time-reversal invariant, i.e., $\mathscr{T}H_0\mathscr{T}^{-1}=H_0$. The circumstance that $H^\dagger=H$ in addition, in which case H is time-reversal invariant, is a special case of Eq. (3.19). For a relativistic multiparticle system, a field theory is needed to define a covariant Hamiltonian formalism and then the above-mentioned properties of H and H_0 under time-reversal remain the fundamental assumptions that produce the time-reversal symmetry. Equivalently, a dynamical equation of the Bethe-Salpeter[13] type may be adopted. Then the reciprocity relation (3.17) follows if time-reversal invariance holds for the inverse of the system's free Feynman propagator (instead of H_0), and if Eq. (3.17) is satisfied by the interaction term of the full covariant propagator (instead of $\tilde{T}$). [In such a covariant propagator formalism the development of the time-reversal property of $\tilde{T}$ parallels that of charge conjugation. The development of the charge conjugation property of $\tilde{T}$ is sketched in the Appendix; the analogous treatment of time reversal is pointed out there.]

It is useful to employ the property $\mathscr{T}^{-1}=\pm\mathscr{T}$, so that Eq. (3.17) can be rewritten as

$$\tilde{T}=\mathscr{T}\tilde{T}^\ddagger\mathscr{T}^{-1}\ . \tag{3.20}$$

The right-hand side of Eq. (3.20) then defines a useful transformation with which to classify operators. We call this the reciprocity transform to distinguish it from the time-reversal transform which appears, for example, in Eq. (3.17). The advantage of dealing with the reciprocity transform is that certain important operators can be classified according to whether they are even or odd under this transformation. The time-reversal transformation does not lend itself to such a classification in the case of scattering operators. Since invariance or antisymmetry (evenness or oddness) under the reciprocity transformation is also a time-reversal symmetry, we sometimes denote reciprocity symmetries by $\mathscr{T}$, when the context eliminates the possibility of confusion. We also note that the time-reversal and reciprocity transforms are identical in the case of the momentum variables.

To obtain the reciprocity relation satisfied by the spinor-space scattering operator $\hat{T}$, we substitute Eq. (2.12) for the integral representation of $\tilde{T}$ into Eq. (3.20) and obtain

$$\tilde{T}=\int d^4\{k'p'kp\}\,|\tilde{k};\tilde{p}\rangle \times\hat{T}(q,K,\omega,\xi,\{\bar{\Gamma}(i)\})\langle\tilde{k}';\tilde{p}'|\ , \tag{3.21}$$

which, after a change of variables similar to that performed earlier in Eq. (3.7), can be written as

$$\tilde{T}=\int d^4\{k'p'kp\}\,|k';p'\rangle \times\hat{T}(-\tilde{q},\tilde{K},\tilde{\omega},\tilde{\xi},\{\bar{\Gamma}(i)\})\langle k;p|\ . \tag{3.22}$$

In the above we have introduced the notation

$$\bar{\Gamma}=U[\Gamma^\ddagger]^*U \tag{3.23}$$

for the reciprocity transform of the γ matrices for each Dirac particle, with U given by Eq. (3.12). These properties of the γ matrices are summarized in Table I.

Comparison of Eqs. (3.22) and (2.12) shows that the constraint upon the operator $\hat{T}$ due to the assumption of reciprocity is

$$\hat{T}(q,K,\omega,\xi,\{\Gamma(i)\})=\hat{T}(-\tilde{q},\tilde{K},\tilde{\omega},\tilde{\xi},\{\overline{\Gamma}(i)\}) . \qquad (3.24)$$

Henceforth we deal almost exclusively with the operator $\hat{T}$. We often omit its arguments and refer to it as the scattering operator. In fact, it is an operator in Dirac spinor space but a matrix element in momentum space, as is evident from Eq. (2.12). Properties and general forms derived for $\hat{T}$ determine those of the complete scattering operator $\tilde{T}$ via Eq. (2.12).

Equations (3.10) and (3.24) are the parity and time-reversal constraints on the form of $\hat{T}$, with $\tilde{\Gamma}(i)$ and $\overline{\Gamma}(i)$ given by Eqs. (3.9) and (3.23), respectively. Put another way, $\hat{T}$ must remain invariant as its arguments transform under parity and reciprocity according to Eqs. (3.10) and (3.24), respectively. The transformation properties of the arguments of $\hat{T}$, momentum vectors and the Dirac Γ matrices, are summarized in Table I. We have also included a summary of the symmetries of the momentum variables under particle-label exchange (E_{12}^{k}) in Table I for later use.

Before proceeding to the construction of the general form of $\hat{T}$ and the application of the parity and reciprocity restrictions, we digress to note the connection between these considerations and the properties of the optical potential for the scattering of a spin-$\frac{1}{2}$ particle from a composite spin-0 object such as a nucleus. The assumption of parity conservation for the inverse of the free (two-particle) propagator and for the optical potential implies the same property for the optical model scattering operator. Thus, the presumption of this property for the scattering operator indicates that the covariant optical potential $\hat{U}$ should satisfy

$$\hat{U}=\Pi\hat{U}\Pi^{-1} , \qquad (3.25)$$

and hence the parity constraint for spinor operators given in Eq. (3.10). Similarly, if the inverse of the free two-particle propagator is time-reversal invariant and if the covariant interaction satisfies the reciprocity relation (3.17), then so will the resulting scattering operator. Thus, the assumption that the scattering operator satisfies the reciprocity relation indicates that the optical potential should satisfy

$$\mathscr{T}\hat{U}\mathscr{T}^{-1}=\hat{U}^{\ddagger} , \qquad (3.26)$$

and hence also the reciprocity constraint for spinor operators given in Eq. (3.24). It is thus evident that the results we obtain for the general invariant form of $\hat{T}$ in the spin-$\frac{1}{2}$-spin-0 case suggest analogous results for the spin-$\frac{1}{2}$-spin-0 optical potential. In this regard, it is to be stressed that neither of the reciprocity relations, Eq. (3.19), or its covariant analog Eq. (3.26), require hermiticity. In particular, Eq. (3.19) is generally satisfied by non-Hermitian nucleon-nucleus optical potentials which contain substantial absorptive parts. Although the one-body Hamiltonian is not time-reversal invariant, it does satisfy the reciprocity relation (3.19) and this is sufficient to yield a scattering operator which also obeys a reciprocity relation.

Finally, in the construction of the general invariant forms of scattering operators in the following section, we do not explicitly consider the discrete symmetry of charge conjugation. The reason for this is that the Lorentz-invariant forms that we shall deal with automatically introduce invariance under the combined PCT transformation. This is a consequence of the so-called PCT theorem.[16,18] Thus, imposition of Lorentz, Π, and $\mathscr{T}$ symmetries automatically yields the desired charge conjugation symmetry. We denote the combined operator by $\Theta=\Pi C\mathscr{T}$, where C is the charge conjugation operator. In the Appendix we give, for the sake of clarity, the explicit forms of the operator C for the cases considered in this work and sketch the derivation of the associated symmetry of the scattering operators. Also pointed out there are that $\Theta=i\gamma^5 P_4$ for a spin-$\frac{1}{2}$ particle, $\Theta=P_4$ for a spin-0 particle, $\Theta=\Theta(1)\Theta(2)$ for a two-particle system, and that the PCT symmetry of the scattering operator is

$$\Theta\tilde{T}\Theta^{-1}=\tilde{T} . \qquad (3.27)$$

Use of the integral representation for $\tilde{T}$ given by Eq. (2.12) yields, for the PCT symmetry,

$$\hat{T}(q,K,\omega,\xi,\{\Gamma(i)\})=\hat{T}(-q,-K,-\omega,-\xi,\{\hat{\Gamma}(i)\}) , \qquad (3.28)$$

where $\hat{\Gamma}=\gamma^5\Gamma\gamma^5$ for each Dirac particle. The manner in which the requirement that $\hat{T}$ is to be a Lorentz scalar operator constructed from its momentum and γ-matrix arguments automatically produces PCT symmetry can now be seen. For any term of $\hat{T}$, the reversal of each four momentum produces a factor of -1 (corresponding to the implicit Lorentz four-vector index associated with the momentum). Then, left and right multiplication by γ^5 produces a factor of -1 for each implicit four-vector index associated with a γ matrix. The total number of such indices is even due to Lorentz invariance; thus PCT invariance, i.e., Eq. (3.28), is assured.

IV. SCATTERING OPERATORS: GENERAL FORMS AND APPLICATION OF CONSTRAINTS

The Lorentz scalar scattering operator $\hat{T}$ must be constructed from the momentum vectors q, K, ω, and ξ and the available Dirac γ matrices. The elementary scalar quantities that can be constructed are those made up solely from momenta and called momentum-space scalars (MSS's), and those formed from contractions of γ matrices with momenta and called mixed scalars. The available MSS's are q^2 $(=t)$, K^2 $(=u)$, and ω^2 $(=s)$, as well as $q\cdot K$, $q\cdot\omega$, and $\omega\cdot K$. The scalar ξ^2 is not independent of these, and we recall that $q\cdot\xi=K\cdot\xi=\omega\cdot\xi=0$. The MSS's are all true scalars under parity. Their time-reversal properties may be obtained from Table I; the results are summarized in Table II along with their particle-label symmetry for later use.

Because the matrices $\Gamma(i)$ span the Dirac spinor space of particle (i), $\hat{T}$ consists of terms which are, at most, linear in these matrices. Since $\hat{T}$ is Lorentz scalar, the 1

TABLE II. Classification of momentum-space scalar quantities according to whether they are odd or even under the parity (Π) and time-reversal ($\mathscr{T}$) (more precisely, reciprocity) transformations. The sign in parentheses to the right of each quantity indicates whether it is even (+) or odd (−) under E_{12}^{k} (interchange of the momenta of the two particles).

$\mathscr{T}$ even	$\mathscr{T}$ odd
q^2,ω^2,K^2 (+)	$q\cdot K$ (+)
$\omega\cdot K$ (−)	$q\cdot\omega$ (−)

and γ^5 matrices may appear multiplicatively, while γ^μ and $\gamma^5\gamma^\mu$ must be contracted with another vector and the antisymmetric second-rank tensor $\sigma^{\mu\nu}$ must be contracted with another antisymmetric second-rank tensor. Since the four linearly independent momenta span the Lorentz vector space, any tensor can be expanded in this basis. Thus, the vectors to be contracted with γ^μ and $\gamma^5\gamma^\mu$ can be restricted to the momenta q, K, ω, and ξ and the antisymmetric tensors to be contracted with $\sigma^{\mu\nu}$ can be restricted to those which can be constructed from these momenta. This is clearly true for both the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$-spin-0 cases. Thus the available mixed (Lorentz) scalars which can be constructed from the Dirac matrices $\Gamma(i)$ may be restricted to the set I, whose 16 elements are 1, γ^5, $\gamma\cdot\lambda_i$, $\gamma^5\gamma\cdot\lambda_i$, and $\sigma:[\lambda_i,\lambda_j]$, $i\neq j$. Here, the symbol : signifies the contraction (dot product) of two second-rank tensors; $\lambda_i,\lambda_j=q$, K, ω, or ξ; and the usual commutator notation has been employed. Because of the antisymmetric property of $\sigma^{\mu\nu}$, we have

$$\sigma:[a,b]=2\sigma:(ab)=2\sigma:ab\ .$$

Thus to simplify notation we may employ the $\sigma:ab$ as the mixed scalars which can be formed from $\sigma^{\mu\nu}$, with the understanding that only the six independent (*unordered*) momentum pairs are needed.

The parity and time-reversal properties of the members of the set I of mixed scalars may be obtained from those of their constituents (given in Table I). This set may then be divided into four sets of scalars depending upon their parity and time-reversal properties: a set A, with members both parity and reciprocity even; a set B, with members which are parity even but reciprocity odd; a set C, with members which are parity odd and reciprocity even; and a set D, with members both parity and reciprocity odd. The results of this decomposition are summarized in Table III. Again, particle-label symmetries *relative to the momenta* are also given in Table III for later use.

Equipped with these results, we can now write down the most general forms for the scattering operators $\hat{T}$ relevant to both the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$-spin-0 cases and impose the restrictions implied by the assumptions of parity conservation and reciprocity. Unlike the procedure to this point, it is now convenient to treat the two cases separately. Because of its simplicity we treat the spin-$\frac{1}{2}$-spin-0 case first.

A. Spin $\frac{1}{2}$-spin 0

In this case, $\hat{T}$ must be a Lorentz scalar function of the MSS's and the 16 mixed scalars $I_i(1)$, $i=1$–16, constructed from the $\Gamma(1)$. It may be, at most, linear in the $I_i(1)$. The most general form for $\hat{T}$ is thus

$$\hat{T}=\sum_{i=1}^{(16)} F_i'(s,t,u,q\cdot K,q\cdot\omega,K\cdot\omega)\,I_i(1)\ . \tag{4.1}$$

In Eq. (4.1) the F_i' are arbitrary functions of the MSS's, while the properties of the 16 Lorentz scalars I_i are summarized in Table III. However, not all of the terms in Eq. (4.1) are allowed by the Π and $\mathscr{T}$ symmetries and other terms are further constrained. Parity conservation precludes any contribution from the Π pseudoscalars of the set I_i; that is, no contribution from sets C or D is allowed. This is due to the fact that all of the MSS's are (parity) true scalars, so that the F_i must also be true scalars: hence there is no way to compensate for the odd parity of the elements of sets C and D and produce an overall parity-even operator. Thus Eq. (4.1) becomes

TABLE III. Classification of the one-body mixed scalar quantities according to whether they are odd or even under the parity (Π) and time-reversal ($\mathscr{T}$) (more precisely, reciprocity) transformations. The sign in parentheses to the right of each quantity indicates whether it is even (+) or odd (−) under E_{12}^{k} (interchange of the momenta of the two particles).

A_i Π even, $\mathscr{T}$ even	B_i Π even, $\mathscr{T}$ odd	C_i Π odd, $\mathscr{T}$ even	D_i Π odd, $\mathscr{T}$ odd
A_i^+:	B_i^-:	C_i^+:	D_i^+:
1 (+)	$\gamma\cdot q$ (−)	$\gamma\cdot\xi$ (+)	γ^5 (+)
$\gamma\cdot\omega$ (+)	$\sigma:\omega K$ (−)	$\gamma^5\gamma\cdot\omega$ (+)	$\sigma:\xi\omega$ (+)
$\gamma^5\gamma\cdot\xi$ (+)			
$\sigma:qK$ (+)		C_i^-:	D_i^-:
		$\gamma^5\gamma\cdot K$ (−)	$\gamma^5\gamma\cdot q$ (−)
A_i^-:		$\sigma:\xi q$ (−)	$\sigma:\xi K$ (−)
$\gamma\cdot K$ (−)			
$\sigma:q\omega$ (−)			

$$\hat{T}=\sum_{I_i\in A,B}^{(8)} F_i'(s,t,u,q\cdot K,q\cdot\omega,K\cdot\omega)\,I_i(1)\ . \quad (4.2)$$

On the other hand, two of the MSS's ($q\cdot K$ and $q\cdot\omega$; see Table II) are reciprocity odd. These scalars are available to compensate for the reciprocity-odd property of set B and thereby produce a net reciprocity-even operator. The most general form of $\hat{T}$ for spin-$\frac{1}{2}$-spin-0 scattering, consistent with the restrictions of parity and reciprocity, is therefore

$$\hat{T}=\sum_{I_i\in A}^{(6)} F_i(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)\,I_i(1)$$
$$+\sum_{I_i\in B}^{(2)}[q\cdot KG_i(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)+q\cdot\omega\overline{G}_i(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)]\,I_i(1)\ . \quad (4.3a)$$

The off-mass-shell operator $\hat{T}$ is thus determined by 10 independent functions (the F_i, G_i, and $\overline{G}_i$) that are scalars with respect to the Lorentz, parity, and time-reversal symmetries. That is, they are true scalars, as are the operators that multiply them. The seven independent arguments are also true scalars. Previously, Celenza and Shakin[4] obtained a set of eight Lorentz invariants for this case which are equivalent to the set $I_i(1)$ that appears above. Those authors obtained eight amplitude functions because they chose to treat the square-bracketed expression in Eq. (4.3a) as a single (reciprocity odd) function. It should be noted that the necessary linear dependence of the terms in the second summation of Eq. (4.3a) (and the lack of it in the first summation) upon particular momentum-space scalars ($q\cdot K, q\cdot\omega$) is an important consequence of reciprocity. Since $q\cdot K$ and $q\cdot\omega$ in Eq. (4.3a) are linearly independent quantities, so are the amplitudes G_i and $\overline{G}_i$, and one can envisage a variety of applications in which these amplitudes will be needed separately rather than in the combination shown. We therefore choose to explicitly decompose in this way the amplitudes corresponding to those spinor Lorentz invariants that are reciprocity odd. The resulting number of independent amplitudes that is invariant with respect to all the symmetries being considered, i.e., true scalars, will always be larger than the number of amplitudes that is required to satisfy only Lorentz invariance. The distinction between these two ways of defining and counting amplitudes will be much more evident in the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case. As we will see, in the fully on-mass-shell case $q\cdot K=q\cdot\omega=0$, so that here, as well as throughout the remainder of this work, explicit linear dependences upon momentum-space scalars serve to identify the purely off-mass-shell parts of scattering operators. This explicit separation is an advantageous feature of the representation that we develop, and is one of the main results of the paper. If one is prepared to employ amplitudes that have no particular reciprocity symmetry, then, as will become evident in Sec. VII, a total of six such amplitudes (combined with projection operators) can carry the same information as Eq. (4.3a)

Before leaving the spin-$\frac{1}{2}$-spin-0 case we display the explicit structure of $\hat{T}$. With the arguments of the amplitude functions suppressed, and with the explicit forms of the mixed scalars substituted from Table III, Eq. (4.3a) for the general off-mass-shell form of $\hat{T}$ is

$$\hat{T}=F_1+F_2\gamma\cdot\omega+F_3\gamma\cdot K+F_4\gamma^5\gamma\cdot\xi+F_5\sigma{:}q\omega+F_6\sigma{:}qK+(q\cdot K\,G_1+q\cdot\omega\,\overline{G}_1)\gamma\cdot q+(q\cdot K\,G_2+q\cdot\omega\,\overline{G}_2)\sigma{:}\omega K\ . \quad (4.3b)$$

We note that early work[8,9] on a relativistic optical potential for nucleon elastic scattering from spin-0 nuclei has included contributions from only three amplitude functions (Lorentz scalar, vector, and tensor) from among the ten which are found in Eqs. (4.3). The more recent work of Tjon and Wallace[6] has included contributions from the first six terms of Eq. (4.3b) since the last four terms vanish under the approximation of on-mass-shell momenta for both particles, as will be discussed in Sec. V.

B. Spin $\frac{1}{2}$-spin $\frac{1}{2}$

Although considerably more complicated, the analysis in this case is a straightforward extension of that employed in the spin-$\frac{1}{2}$-spin-0 case. The Lorentz scalar $\hat{T}$ must be constructed from the MSS's, the mixed scalars $I_i(1)$, $i=1-16$, and the mixed scalars $I_j(2)$, $j=1-16$. Again, $\hat{T}$ may be at most linear in the $I_i(1)$ and the $I_j(2)$. It follows that the most general form of $\hat{T}$ is

$$\hat{T}=\sum_{i=1}^{16}\sum_{j=1}^{16} F_{ij}'(s,t,u,q\cdot K,q\cdot\omega,K\cdot\omega)I_i(1)I_j(2)\ . \quad (4.4)$$

Evidently, there are $16^2=256$ independent invariants, as expected, to span the two-particle Dirac space. Again, all of the MSS's are true (parity) scalars, so that all of the F_{ij}' are also. Conservation of parity then requires that parity even (odd) $I_i(1)$ must be matched with parity even (odd) $I_j(2)$ in order to form an overall parity-even operator. Thus Eq. (4.4) immediately reduces to

$$\hat{T}=\sum_{I_i\in A,B}^{(8)}\sum_{I_j\in A,B}^{(8)}F'_{ij}(s,t,u,q\cdot K,q\cdot\omega,K\cdot\omega)I_i(1)I_j(2)+\sum_{I_i\in C,D}^{(8)}\sum_{I_j\in C,D}^{(8)}F'_{ij}(s,t,u,q\cdot K,q\cdot\omega,K\cdot\omega)I_i(1)I_j(2)\ , \tag{4.5}$$

for a total of 128 independent terms.

As in the spin-$\frac{1}{2}$-spin-0 case, time reversal does not further reduce the total number of terms due to the presence of the two time-reversal odd MSS's, $q\cdot K$ and $q\cdot\omega$. These two MSS's are available to compensate for products $I_i(1)I_j(2)$ which are reciprocity odd and thus produce an overall reciprocity-even operator. In order to write a simpler equation for $\hat{T}$ it is convenient to classify the (parity even) products $I_i(1)I_j(2)$ into two sets according to their reciprocity properties. One set, which we call E_e, consists of the reciprocity even operator products and the other set, which we denote by E_o, contains the reciprocity odd operator products. Clearly, then,

$$E_e\supset A(1)\otimes A(2),\ B(1)\otimes B(2),\ C(1)\otimes C(2),\ D(1)\otimes D(2) \tag{4.6a}$$

and

$$E_o\supset A(1)\otimes B(2),\ B(1)\otimes A(2),\ C(1)\otimes D(2),\ D(1)\otimes C(2)\ , \tag{4.6b}$$

where we have employed the usual notation $\supset$ to mean "contains." We have also employed the direct product notation whereby the set $A(1)\otimes B(2)$ consists of the products $I_i(1)I_j(2)$, where $I_i(1)$ is contained in the set $A(1)$ and $I_j(2)$ is contained in the set $B(2)$. The set E_e consists of $(6\times6)+(2\times2)+2(4\times4)=72$ elements, whereas the set E_o consists of $(6\times2)+(2\times6)+2(4\times4)=56$ elements.

With this notation we can now write the most general form of $\hat{T}$, which is consistent with parity and reciprocity symmetries, as

$$\hat{T}=\sum_{I_iI_j\in E_e}^{(72)}F_{ij}(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)I_i(1)I_j(2)$$
$$+\sum_{I_iI_j\in E_o}^{(56)}[q\cdot K\,G_{ij}(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)+q\cdot\omega\,\overline{G}_{ij}(s,t,u,K\cdot\omega,(q\cdot K)^2,(q\cdot\omega)^2,q\cdot Kq\cdot\omega)]I_i(1)I_j(2)\ . \tag{4.7}$$

In Eq. (4.7) the F_{ij}, G_{ij}, and $\overline{G}_{ij}$ are Lorentz scalar functions of their arguments. They are also true (parity and time reversal) scalars since their arguments are. Equation (4.7) determines the operator $\hat{T}$, consisting of 128 Lorentz invariant operators [the parity even $I_i(1)I_j(2)\in E_e,E_o$], and 184 true scalar amplitude functions which are otherwise arbitrary. Again, the lack of linearity in $q\cdot K$ or $q\cdot\omega$ of the terms in the first summation of Eq. (4.7) and the necessity of such linearities in the terms of the second summation in Eq. (4.7) are important consequences of reciprocity. These restrictions may be expected to play an important role in subsequent developments; for example, in the construction of approximations to a covariant spin-$\frac{1}{2}$-spin-0 optical model. If one is prepared to employ amplitudes that have no particular time-reversal symmetry, then, as will be seen in Sec. VII, a total of 80 such amplitudes (combined with projection operators) can carry the same information as Eq. (4.7).

V. MASS-SHELL CONSTRAINTS AND SIMPLIFICATIONS

The general forms of the scattering operators developed in the preceding section simplify considerably when one or both particles are subjected to on-mass-shell constraints. In this section we describe the constraints and obtain the simplified forms. Throughout the remainder of this paper we take an "on-mass-shell" constraint to mean *only* that the magnitude of a four momentum is set equal to the rest mass of the associated particle. Clearly, this restriction operates within the momentum-space sector of the full Hilbert space associated with the description of the particle and *does not* imply a restriction to a particular Dirac-spinor sector of the space. We return to this topic shortly, but we note that, in contrast, other authors[4,5] often take the terminology "on mass shell" to specify, in addition, a projection upon either positive or negative energy Dirac spinors, according to whether the fourth component of the momentum is positive or negative.

With reference to Fig. 1, the two constraints which arise when particle (1) is put on its mass shell, in both the initial and final state, are $k'^2=m_1^2=k^2$. In terms of the three independent momentum variables which we employ, these constraints are

$$q\cdot(\omega+K)=0\ , \tag{5.1}$$

$$q^2+(\omega+K)^2=4m_1^2\ . \tag{5.2}$$

If, instead, particle (2) is on mass shell, the constraints $p'^2=m_2^2=p^2$ can be expressed as

$$q\cdot(\omega-K)=0\ , \tag{5.3}$$

$$q^2+(\omega-K)^2=4m_2^2\ . \tag{5.4}$$

The important case in which both particles are on mass shell in both the initial and final states is summarized by the four constraints

$$q\cdot\omega=q\cdot K=0\ , \tag{5.5}$$

$$\omega\cdot K=m_1^2-m_2^2\ , \tag{5.6}$$

$$s+t+u\equiv q^2+\omega^2+K^2=2(m_1^2+m_2^2)\ . \tag{5.7}$$

In the special case of identical particles, we see from Eqs. (5.5)–(5.7), together with Eq. (2.13), that when all particles are on mass shell the momenta q, K, ω, and ξ constitute a set of four mutually orthogonal momenta.

Before applying these constraints to simplify the general forms of scattering operators, we digress to remark upon the relation between the transition operators, physical and virtual scattering amplitudes, the above on-mass-shell kinematic constraints, and cross-channel processes.[17] As noted above, the sign of the time component of a four momentum is not specified by the on-mass-shell kinematic constraint. Moreover, a constraint that a particle's momentum be on mass shell does not imply a choice of the positive or negative energy Dirac spinor-space sector even if we, in addition, specify the signs of the time components of its momentum vectors. In other words, the time component of the four momentum in the momentum-space ket [see Eqs. (2.1)–(2.6)] and the Dirac spinor space represent independent degrees of freedom in the full Hilbert space.

Given this, let us consider as an example the simpler spin-$\frac{1}{2}$-spin-0 case where particle (2) is a boson. Consider the transition matrix elements (we suppress the inessential Pauli spinors for convenience)

$$M_{ij}(q,K,\omega)=\bar{u}(\mathbf{k}',i)\hat{T}(k',p';k,p;\{\Gamma(1)\})u(\mathbf{k},j)\ , \tag{5.8}$$

where, we recall, the i,j labels indicate the sign of the energy *associated with the Dirac spinors.* Since we want to focus on the description of the Dirac particle, we suppress discussion concerning the character of the four momenta of the boson in the following. There are four different combinations of the pair of labels i,j. These determine the spinor character of the matrix elements so that there are four spinor degrees of freedom for each set of values of the four momenta. An analysis of general (off-mass-shell) spinor operators such as that developed in this paper clearly requires that the full range of degrees of freedom be considered. The four spinor degrees of freedom correspond to four distinct physical or virtual processes. Whether these processes are physical or virtual depends upon the particular kinematical circumstances alone. For k_0', and k_0 both positive and both on mass shell, one combination of (i,j) in Eq. (5.8) [namely $(i,j)=(+,+)$] yields a physical scattering amplitude. This scattering amplitude corresponds to that for the scattering of a spin-$\frac{1}{2}$ particle, in accord with either a Feynman propagator or field theoretic viewpoint. This physical process is often called the s-channel process and the amplitude M_{++} in Eq. (5.8) is called the s-channel physical amplitude since for the specified kinematic circumstances the Mandelstam s variable attains values which correspond to physical values of the invariant mass of the system of particles.

The other three combinations of i,j yield scattering amplitudes for other processes which are virtual (off shell) for this choice of kinematic conditions. That is to say, M_{--}, M_{-+}, and M_{+-} are virtual amplitudes in the s channel. For example, $(i,j)=(-,-)$ in Eq. (5.8) corresponds to virtual antiparticle scattering for the process $(-k'\rightarrow -k)$, in accord with the Feynman picture. When the time components k_0' and k_0 are both negative and satisfy the on-mass-shell condition, then the case $(i,j)=(-,-)$ in Eq. (5.8) corresponds to the physical antiparticle scattering amplitude for $(-k'\rightarrow -k)$. In this case (assuming the appropriate boson kinematics) it is the Mandelstam variable u which has the character of a physical invariant mass (squared). Consequently, this process is termed a u-channel process and M_{--} is called the u-channel physical amplitude.

This is perhaps made a bit clearer if we note that the variables s and u are, in fact, interchanged under the crossing transformation $(k,k')\rightarrow(-k',-k)$. Consider again the kinematic circumstance where k_0 and k_0' are positive and the s-channel process is physical. Under the crossing transformation, which then effectively reverses the signs of the time components of the momenta, Eq. (5.8) becomes

$$M_{ij}(q,-\omega,-K)=\bar{u}(-\mathbf{k},i)\hat{T}(-k,p';-k',p;\{\Gamma(1)\})\times u(-\mathbf{k}',j)\ . \tag{5.9}$$

Thus we explicitly see that under this transformation of variables the former s-channel virtual amplitude M_{--} now represents the u-channel physical amplitude, and the former s-channel physical amplitude M_{++} now represents a u-channel virtual amplitude. Similarly, the s- (and u-) channel virtual amplitudes for the cases $(i,j)=(+,-)$ and $(-,+)$ can correspond to physical particle-antiparticle creation and annihilation processes in the t channel [when the Mandelstam t variable has the characteristics of a physical invariant mass (squared)]. For example, the $(-,+)$ annihilation case in Eq. (5.8) represents a physical amplitude, according to the Feynman rule, when k_0 is positive and k_0' negative, and both are on mass shell. It is clear that in this case the variable t attains values corresponding to a physical invariant mass for the particles.

The spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case is similarly characterized. Here, there are $2^4=16$ possible combinations of Dirac spinor sectors (16 ρ-spin[19] sectors). Many of the corresponding amplitudes becomes physical in some Mandelstam channel, where it represents a physical particle and/or antiparticle process. The interpretation of scattering matrix elements involving negative energy spinors in terms of antiparticles follows the Feynman rule, as illustrated in the spin-$\frac{1}{2}$-spin-0 example above. This concludes our discussion of the physical interpretation of the scattering operators and their matrix elements.

A. Spin $\frac{1}{2}$-spin 0

From Eqs. (5.1)–(5.3) we see that when either particle (1) or particle (2) is on mass shell $q\cdot\omega$ can be eliminated in favor of $q\cdot K$, and $\omega\cdot K$ can be eliminated in favor of s, t, and u. The general form of the scattering operator $\hat{T}$, given previously in Eq. (4.3), thus simplifies to

$$\hat{T}=\sum_{I_i\in A}^{(6)} F_i'(s,t,u,(q\cdot K)^2)\,I_i(1)+\sum_{I_i\in B}^{(2)} q\cdot K\,G_i'(s,t,u,(q\cdot K)^2)\,I_i(1)\ , \tag{5.10}$$

when one of the particles is on mass shell. In this case only eight scalar amplitude functions F_i' and G_i' are required and the number of independent scalar variables is reduced to four. When both particles are on mass shell, Eqs. (5.5)—(5.7) show that $q\cdot K$ and one of s, t, and u are eliminated as independent variables as well. Thus Eq. (5.10) reduces to

$$\hat{T}=\sum_{I_i\in A}^{(6)} F_i''(s,t)I_i' , \tag{5.11}$$

when both particles are on mass shell. Now only six scalar amplitude functions of two variables are required to completely specify the scattering operator $\hat{T}$. Explicit forms showing the mixed scalars and suppressing the arguments of the amplitude functions are as follows. For one particle on mass shell,

$$\hat{T}=F_1'+F_2'\gamma\cdot\omega+F_3'\gamma\cdot K+F_4'\gamma^5\gamma\cdot\xi+F_5'\sigma{:}q\omega+F_6'\sigma{:}qK$$
$$+q\cdot K(G_i'\gamma\cdot q+G_2'\sigma{:}\omega K) , \tag{5.12}$$

and for both particles on mass shell,

$$\hat{T}=F_1''+F_2''\gamma\cdot\omega+F_3''\gamma\cdot K+F_4''\gamma^5\gamma\cdot\xi$$
$$+F_5''\sigma{:}q\omega+F_6''\sigma{:}qK . \tag{5.13}$$

B. Spin $\frac{1}{2}$-spin $\frac{1}{2}$

In a similar way, with one particle constrained to be on mass shell, the general form of $\hat{T}$ given by Eq. (4.7) reduces to

$$\hat{T}=\sum_{I_iI_j\in E_e}^{(72)} F_{ij}'(s,t,u,(q\cdot K)^2)I_i(1)I_j(2)$$
$$+\sum_{I_iI_j\in E_o}^{(56)} q\cdot K\, G_{ij}'(s,t,u,(q\cdot K)^2)I_i(1)I_j(2) , \tag{5.14}$$

so that 128 scalar amplitude functions F_{ij}' and G_{ij}' are needed instead of the 184 necessary in the off-mass-shell case. For both particles on mass shell, Eq. (5.14) reduces to

$$\hat{T}=\sum_{I_iI_j\in E_e}^{(72)} F_{ij}''(s,t)I_i(1)I_j(2) , \tag{5.15}$$

which involves only 72 scalar amplitude functions.

The formulae in this section are our final results for the general forms of, and the symmetry constraints upon, the scattering operators $\hat{T}$ for two nonidentical particles. Reconstruction of scattering operators from "known" amplitudes is treated in Secs. VII and VIII. Further results concerning the special case of the scattering of two identical fermions are developed in the next section.

VI. SPIN-$\frac{1}{2}$-SPIN-$\frac{1}{2}$ SCATTERING OPERATORS: PAULI SYMMETRIES

For the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system we are particularly interested in the identical particle case. This section consists of a detailed exposition of the implications of Pauli symmetries for the scattering operator $\tilde{T}$.

The precise constraints imposed upon $\tilde{T}$ by the Pauli principle depend upon the choice of the particle exchange symmetry presumed for the two-particle initial and final states used in forming physical matrix elements. A general statement of the Pauli prescription is that operators involving identical particles must be symmetric under particle interchange, while states are to be antisymmetric under particle interchange. Schematically, the scattering amplitude is then given by

$$\mathscr{F}=\langle\Psi_A'(1,2)\,|\,\tilde{T}'(1,2)\,|\,\Psi_A(1,2)\rangle , \tag{6.1}$$

where the two-particle operator $\tilde{T}'(1,2)$ and the two-particle states, e.g., $\Psi_A(1,2)$, satisfy the stated particle-exchange symmetries. However, it is generally convenient to employ a scattering operator $\tilde{T}(1,2)$ which contains the requirement of initial and final state antisymmetry, together with corresponding simple (unsymmetrized) product states, $\Psi(1,2)=\phi(1)\psi(2)$. This can be obtained by introducing an operator antisymmetrizer $\mathscr{A}$ defined by

$$\mathscr{A}=\frac{1-E_{12}}{\sqrt{2}} \tag{6.2}$$

such that

$$|\,\Psi_A(1,2)\rangle=\mathscr{A}\,|\,\Psi(1,2)\rangle , \tag{6.3}$$

where E_{12} is the particle-exchange operator. It then follows that the desired antisymmetrized scattering operator is

$$\tilde{T}=\mathscr{A}\tilde{T}'(1,2)\mathscr{A} , \tag{6.4}$$

in terms of which the scattering amplitude from Eq. (6.1) becomes

$$\mathscr{F}=\langle\Psi'(1,2)\,|\,\tilde{T}\,|\,\Psi(1,2)\rangle . \tag{6.5}$$

It is advantageous for us to work with a scattering operator $\tilde{T}$ consistent with Eqs. (6.5) and (6.4). In this way we clearly preserve the results of preceding sections, and scattering amplitudes remain as defined previously in Eqs. (2.8).

Since $\tilde{T}'(1,2)$ is symmetric under particle interchange, that is

$$\tilde{T}'(1,2)=E_{12}\tilde{T}'(1,2)E_{12} , \tag{6.6}$$

it follows immediately from Eq. (6.4), and the property $E_{12}^2=1$, that

$$\tilde{T}=(1-E_{12})\tilde{T}'(1,2)=\tilde{T}'(1,2)(1-E_{12}) . \tag{6.7}$$

It is also readily verified that

$$\tilde{T}=E_{12}\tilde{T}E_{12} , \tag{6.8}$$

so that the antisymmetrized operator $\tilde{T}$ is also required to be symmetric under particle-label exchange. The two terms in Eq. (6.7) are often referred to as the direct and exchange processes.

Equation (6.8) does not express the full implications of the Pauli principle, however, since the required antisymmetry in the initial and final states is not represented [Eq. (6.8) contains no more information than Eq. (6.6)]. Equation (6.8) holds equally well for bosons and fermions. The additional antisymmetry requirement for fermions is readily seen from Eq. (6.7) to be

$$\widetilde{T}=-E_{12}\widetilde{T}\ . \qquad (6.9)$$

The two independent constraints of Eqs. (6.8) and (6.9) embody the full implications of the Pauli principle. This is clear since, from Eq. (6.8),

$$\widetilde{T}=\tfrac{1}{2}(E_{12}\widetilde{T}E_{12}+E_{12}\widetilde{T}E_{12}) \qquad (6.10)$$

$$=\tfrac{1}{2}(\widetilde{T}+E_{12}\widetilde{T}E_{12})\ , \qquad (6.11)$$

so that use of Eqs. (6.8) and (6.9) gives

$$\widetilde{T}=(1-E_{12})(\tfrac{1}{2}\widetilde{T})\ , \qquad (6.12)$$

which shows that $\widetilde{T}$ has the required form, Eq. (6.7).

It is convenient at this point to recall the integral representation of $\widetilde{T}$ given in Eq. (2.12), and to write it here in the form

$$\widetilde{T}=\int d^4\{k'p'kp\}\,|\,k';p'\rangle \times\hat{T}(k',p';k,p;\Gamma(1),\Gamma(2))\langle k;p\,|\ , \qquad (6.13)$$

so that we may express the two requirements of the Pauli principle, Eqs. (6.8) and (6.9), in terms of simpler requirements upon the spinor operator $\hat{T}$. In the case of Eq. (6.9) this necessitates the construction of a particular representation for E_{12}. Furthermore, it turns out that the constraint of Eq. (6.8) restricts the Lorentz invariants that may appear in the general form of $\hat{T}$, while Eq. (6.9) only provides further constraints upon the properties of the invariant amplitude functions, such as the F_{ij} of Eq. (4.7). For these reasons it is convenient to treat the implications of Eqs. (6.8) and (6.9) separately and in that order. We defer consideration of Eq. (6.9), the exchange antisymmetry constraint, until subsection B.

The implications of Eq. (6.8), the particle-label symmetry constraint, are developed in the next subsection, and in preparation for this we note that the combination of Eq. (6.13) with Eq. (6.8) implies that

$$\widetilde{T}=\int d^4\{k'p'kp\}\,|\,k';p'\rangle \times\hat{T}(p',k';p,k;\Gamma(2),\Gamma(1))\langle k;p\,|\ . \qquad (6.14)$$

Comparison of Eqs. (6.13) and (6.14) yields, as the restatement of Eq. (6.8) for the constraint of particle-label symmetry,

$$\hat{T}(k',p';k,p;\Gamma(1),\Gamma(2))=\hat{T}(p',k';p,k;\Gamma(2),\Gamma(1))\ . \qquad (6.15)$$

Equation (6.15) simply states that the operator $\hat{T}$ must be symmetric under the simultaneous interchange of the momenta and the γ matrices of the two particles.

A. Particle-label symmetry

We subject the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ results for the form of $\hat{T}$ obtained in Secs. IV and V to the further constraint of particle label symmetry by applying Eq. (6.8) in the form of Eq. (6.15). In particular, we need to find the necessary modifications to the general result of Eqs. (4.6) and (4.7) and to the on-mass-shell limit, Eqs. (5.14) and (5.15). In order to do this it is convenient to subdivide the sets E_e and E_o of two-body spinor operators defined in Eqs. (4.6). In particular, we divide each set into its parts which are symmetric, E_e^+ and E_o^+, and antisymmetric, E_e^- and E_o^-, under particle-label exchange, making use of the classification of the one-body mixed scalars in Table III according to their symmetry under the operation E_{12}^k which interchanges the momenta of the two particles (for example, the A_i^+ are symmetric under E_{12}^k). The construction proceeds as follows. Introduction of the notation

$$[A_i^+A_j^+]^\pm=\tfrac{1}{2}\{A_i^+(1)A_j^+(2)\pm A_i^+(2)A_j^+(1)\} \qquad (6.16)$$

allows the $A(1)\otimes A(2)$ part of E_e, as given by Eq. (4.6a), to be subdivided into the set

$$E_e^+\supset[A_i^+A_j^+]^+,\ [A_i^-A_j^-]^+,\ [A_i^+A_j^-]^-\ , \qquad (6.17a)$$

which is symmetric, and the set

$$E_e^-\supset[A_i^+A_j^+]^-,\ [A_i^-A_j^-]^-,\ [A_i^+A_j^-]^+\ , \qquad (6.17b)$$

which is antisymmetric under particle label interchange. In Eqs. (6.17), i and j specify the particular members of the set A of mixed scalars and are unrestricted, except for $i\neq j$ in the case of the first two contributors to Eq. (6.17b). Thus the number of elements contributing to E_e^+ above is $10+3+8=21$, and the number contributing to E_e^- above is $6+1+8=15$, for a total of $6\times6=36$, as is necessary. A completely analogous procedure can be car-

TABLE IV. Contents of the set E_e of (two-particle) reciprocity-even mixed scalar operators, given in terms of the one-particle mixed scalars of Table III and separated according to their symmetric (E_e^+) or antisymmetric (E_e^-) nature under particle-label interchange using the notation of Eq. (6.16). The numbers in parentheses give the number of contributing elements of each type.

E_e^+ (E_{12} even)		E_e^- (E_{12} odd)	
$[A_i^+A_j^+]^+$	(10)	$[A_i^+A_j^+]^-$	(6)
$[A_i^-A_j^-]^+$	(3)	$[A_i^-A_j^-]^-$	(1)
$[A_i^+A_j^-]^-$	(8)	$[A_i^+A_j^-]^+$	(8)
	(21)		(15)
$[B_i^-B_j^-]^+$	(3)	$[B_i^-B_j^-]^-$	(1)
	(3)		(1)
$[C_i^+C_j^+]^+$	(3)	$[C_i^+C_j^+]^-$	(1)
$[C_i^-C_j^-]^+$	(3)	$[C_i^-C_j^-]^-$	(1)
$[C_i^+C_j^-]^-$	(4)	$[C_i^+C_j^-]^+$	(4)
	(10)		(6)
$[D_i^+D_j^+]^+$	(3)	$[D_i^+D_j^+]^-$	(1)
$[D_i^-D_j^-]^+$	(3)	$[D_i^-D_j^-]^-$	(1)
$[D_i^+D_j^-]^-$	(4)	$[D_i^+D_j^-]^+$	(4)
	(10)		(6)
Total	(44)	Total	(28)

ried out for the $B(1)\otimes B(2)$, $C(1)\otimes C(2)$, and $D(1)\otimes D(2)$ contributions to the reciprocity-even operator set E_e. The results are summarized in Table IV.

The reciprocity-odd operators E_o are easier to separate. The two sets $A(1)\otimes B(2)$ and $A(2)\otimes B(1)$ are combined into particle-label symmetric elements $[A_i^+B_j^-]^-$ and $[A_i^-B_j^-]^+$ and particle label antisymmetric elements $[A_i^+B_j^-]^+$ and $[A_i^-B_j^-]^-$. Thus, these contributions to E_o can be subdivided into the symmetric set

$$E_o^+ \supset [A_i^+B_j^-]^-,\ [A_i^-B_j^-]^+ \tag{6.18a}$$

and the antisymmetric set

$$E_o^- \supset [A_i^+B_j^-]^+,\ [A_i^-B_j^-]^- \ . \tag{6.18b}$$

The number of contributors to E_o^+ above is thus $(4\times2)+(2\times2)=12$ and the number of contributors to E_o^- above is also 12, for the necessary total of 24. The contributions of $C(1)\otimes D(2)$ and $C(2)\otimes D(1)$ are handled similarly. The results are summarized in Table V.

Each of the reciprocity-odd operators in E_o must be combined with one of the two time-reversal odd MSS's, $q\cdot K$ and $q\cdot\omega$, as in Eq. (4.7), in order to form an overall reciprocity-even operator, the full set of which we denote by $\hat{E}_o$. Since $q\cdot K$ is even and $q\cdot\omega$ is odd under particle interchange, it is useful to categorize the set of reciprocity-even operators formed in this fashion ($\hat{E}_o$) according to whether they are symmetric ($\hat{E}_o^+$) or antisymmetric ($\hat{E}_o^-$) under particle interchange. The results of this task are summarized in Table V.

Before writing down the general form of $\hat{T}$ consistent with identical particles, it remains only to treat the particle exchange properties of the seven arguments of Eq. (4.7): s, t, u, $K\cdot\omega$, $(q\cdot K)^2$, $(q\cdot\omega)^2$, and $q\cdot Kq\cdot\omega$. Two of the seven, $K\cdot\omega$ and $q\cdot Kq\cdot\omega$, are E_{12} odd. We refer to them as MSS2. The other five arguments are E_{12} even and form part of a set of E_{12}-even scalars which we refer to as MSS1. The other elements contained in the set MSS1 are $(K\cdot\omega)^2$ and the product $q\cdot Kq\cdot\omega K\cdot\omega$. Evidently, an arbitrary function of the initial seven variables above can be expressed as some function of the seven MSS1's together with possible linear dependences upon the elements of MSS2. The elements of MSS1 form a set of Lorentz, parity, and time-reversal scalars which are also E_{12} even.

An expression for the most general form of $\hat{T}$ consistent with invariance under Lorentz and parity transformations, reciprocity under time-reversal, and with particle indistinguishability can now be read from Tables II–V with the help of Eq. (4.7). Let an arbitrary element of the sets E_e and $\hat{E}_o$ of mixed scalar operators be denoted by $\hat{I}_{ij}$. Then,

$$\hat{T}=\sum_{\hat{I}_{ij}\in E_e^+}^{(44)} F_{ij}\hat{I}_{ij}+\sum_{\hat{I}_{ij}\in E_e^-}^{(28)} (K\cdot\omega\, G_{ij}+q\cdot Kq\cdot\omega\, G'_{ij})\hat{I}_{ij} + \sum_{\hat{I}_{ij}\in\hat{E}_o^+}^{(56)} H_{ij}\hat{I}_{ij}+\sum_{\hat{I}_{ij}\in\hat{E}_o^-}^{(56)} K\cdot\omega\, H'_{ij}\hat{I}_{ij}\ , \tag{6.19}$$

where the arguments of the 212 independent, true scalar functions F_{ij}, G_{ij}, G'_{ij}, H_{ij}, and H'_{ij} are the seven momentum scalars in MSS1. Since the MSS1's are Lorentz, parity, and time-reversal scalars and are symmetric under particle-label exchange, so is each of the 212 amplitude functions.

The argument which yields Eq. (6.19) is as follows. From Eq. (4.7), $\hat{T}$ must be constructed from the sets of reciprocity-even operators E_e of Table IV and $\hat{E}_o$ of Table V, together with the seven variables manifested in Eq. (4.7). This variable set may be replaced by the MSS1's together with possible linear dependences upon

TABLE V. Contents of the set E_o of (two-particle) reciprocity-odd mixed scalar operators, given in terms of the one-particle mixed scalars of Table III and separated according to their symmetric (E_o^+) or antisymmetric (E_o^-) nature under particle-label interchange using the notation of Eq. (6.16). The third column categorizes the set of reciprocity-even operators $\hat{E}_o$, constructed from the operators $\hat{E}_o$ and the reciprocity-odd MSS's ($q\cdot K$ and $q\cdot\omega$), according to their particle-label symmetry ($\hat{E}_o^+$) or antisymmetry ($\hat{E}_o^-$). The numbers in parentheses give the number of contributing elements in each case. The final column lists the set of available time-reversal even, particle-exchange odd, momentum-space scalars: MSS2.

E_o^+ (E_{12} even)		E_o^- (E_{12} odd)		$\hat{E}_o$ ($\mathscr{T}$ even)		MSS2
$[A_i^+B_j^-]^-$	(8)	$[A_i^+B_j^-]^+$	(8)	$\hat{E}_o^+$ (E_{12} even)		$K\cdot\omega$
$[A_i^-B_j^-]^+$	(4)	$[A_i^-B_j^-]^-$	(4)	$q\cdot K\, E_o^+$	(28)	$q\cdot Kq\cdot\omega$
	(12)		(12)	$q\cdot\omega\, E_o^-$	(28)	
					(56)	
$[C_i^+D_j^-]^-$	(4)	$[C_i^+D_j^-]^+$	(4)			
$[C_i^-D_j^-]^+$	(4)	$[C_i^-D_j^-]^-$	(4)	$\hat{E}_o^-$ (E_{12} odd)		
$[C_i^+D_j^+]^+$	(4)	$[C_i^+D_j^+]^-$	(4)	$q\cdot\omega\, E_o^+$	(28)	
$[C_i^-D_j^+]^-$	(4)	$[C_i^-D_j^+]^+$	(4)	$q\cdot K\, E_o^-$	(28)	
	(16)		(16)		(56)	
Total	(28)	Total	(28)	Total	(112)	

the MSS2's. Each of the 184 Lorentz scalar operators $\hat{I}_{ij} \in E_e, \hat{E}_o$ defines a possible term of $\hat{T}$. Each of these terms is multiplied by an arbitrary function of the MSS1's. For E_{12}-even $\hat{I}_{ij}$ this fixes the term [first and third summations of Eq. (6.19)] since no dependence upon the E_{12}-odd elements of MSS2 is permitted. For the E_{12}-odd $\hat{I}_{ij} \in E_e^-, \hat{E}_o^-$ a linearity in an element of MSS2 is required. For $\hat{I}_{ij} \in E_e^-$, either element is permitted, yielding the second summation of Eq. (6.19). For $\hat{I}_{ij} \in \hat{E}_o^-$, the element $K\cdot\omega$ of MSS2 is allowed [yielding the fourth summation of Eq. (6.19)]. The other element of MSS2, $q\cdot Kq\cdot\omega$, is also allowed, but, due to the fact that each element of $\hat{E}_o^-$ is already necessarily linear in either $q\cdot\omega$ or in $q\cdot K$ (see Table V), this possibility simply reproduces the result of the preceding one. Thus, Eq. (6.19) is obtained.

Equation (6.19) expresses the general, off-mass-shell result. Complete specification of $\hat{T}$ requires 212 true scalar functions (of the MSS1 variables), even after the imposition of the symmetry principles discussed above. The number of reciprocity-even, parity-even covariants required is 184. These are constructed from the basic set of 128 parity-even, but reciprocity even or odd, mixed scalars. If the only requirement upon amplitudes is Lorentz invariance and no particular symmetry under time reversal and particle label exchange is demanded, then, as will become clear in Sec. VII, a total of 56 such amplitudes (combined with projectors) can carry the same information as Eq. (6.19).

The restriction of this result to the case wherein both particles are on mass shell is readily obtained from Eq. (6.19) and the considerations of the preceding section for the on-mass-shell limits in the distinguishable particle case. With both particles on their mass-shell, $K\cdot\omega = q\cdot\omega = q\cdot K = 0$, so that the second and fourth summations in Eq. (6.19) are immediately eliminated. The third summation in Eq. (6.19) is also eliminated since all of the elements of the set $\hat{E}_o^+$ are linear in either $q\cdot\omega$ or $q\cdot K$. The expression for $\hat{T}$ in the limit of on-mass-shell particles is thus

$$\hat{T} = \sum_{\hat{I}_{ij} \in E_e^+}^{(44)} F'_{ij}(s,t)\hat{I}_{ij} , \tag{6.20}$$

which involves 44 invariant amplitude functions $F'_{ij}(s,t)$ of two of the Mandelstam variables.

B. Antisymmetry

In this subsection we develop the further restrictions upon the scattering operator which arise from the antisymmetry constraint given by Eq. (6.9); that is,

$$\tilde{T} = -E_{12}\tilde{T} . \tag{6.21}$$

We wish, first, to express this as a constraint upon the spinor operator $\hat{T}$, and to this end we introduce, for the exchange operator E_{12}, the representation

$$E_{12} = \int d^4\{kp\} \, |p;k\rangle\langle k;p| \, E_{12}^{\gamma} , \tag{6.22}$$

where the spinor-space operator E_{12}^{γ} is constructed so as to interchange the Dirac spinors of the two particles. It is convenient to express Eq. (6.22) as

$$E_{12} = E_{12}^{k} E_{12}^{\gamma} , \tag{6.23}$$

where $(E_{12}^{\gamma})^2 = 1 = (E_{12}^{k})^2$, and E_{12}^{k} interchanges the momenta of the two particles. The exchange operator E_{12}^{γ} is defined in the direct product space of Dirac spinors such that

$$E_{12}^{\gamma}\psi(1)\phi(2) = \psi(2)\phi(1) , \tag{6.24}$$

where ψ and ϕ are four-component spinors. From the algebra of Dirac matrices[20] the standard representation

$$E_{12}^{\gamma} = \tfrac{1}{4}(S + V + \tfrac{1}{2}T - A + P) \tag{6.25}$$

can be obtained (as a special case of crossing matrices[21]) in terms of the Fermi invariants denoted by $S=1$, $V=\gamma(1)\cdot\gamma(2)$, $T=\sigma(1):\sigma(2)$, $P=\gamma^5(1)\gamma^5(2)$, and $A=PV$. Other representations of E_{12}^{γ} are possible. For example, it can be useful in some extensions of the present work to expand E_{12}^{γ} in direct products from the set of 16 one-body invariants I_i introduced in Sec. IV. A particular representation is not needed for the present discussion.

Use of Eqs. (6.13) and (6.22) for the integral representations of $\tilde{T}$ and E_{12} converts Eq. (6.21) into the form

$$\tilde{T} = -\int d^4\{k'p'kp\} \, |k';p'\rangle E_{12}^{\gamma} \times \hat{T}(p',k';k,p;\Gamma(1),\Gamma(2))\langle k;p| . \tag{6.26}$$

Comparison of Eqs. (6.13) and (6.26) yields, as a restatement of the antisymmetry constraint for the spinor-space scattering operator, the equation

$$\hat{T}(k',p';k,p;\Gamma(1),\Gamma(2)) = -E_{12}^{\gamma}\hat{T}(p',k';k,p;\Gamma(1),\Gamma(2)) . \tag{6.27}$$

The general form for $\hat{T}$ derived in the preceding subsection and given in Eq. (6.19) can be summarized in the schematic form

$$\hat{T} = \sum_{\alpha} F_\alpha(k',p';k,p)\hat{I}_\alpha(k',p';k,p) , \tag{6.28}$$

where the F_α are the amplitudes and the $\hat{I}_\alpha$ are mixed scalars. Here we are concerned with the nature of any modifications to this form that are necessary to satisfy Eq. (6.27).

In fact, the effect of Eq. (6.27) is only to restrict the nature of the amplitudes F_α. The set of mixed scalars $\hat{I}_\alpha$ is not affected. The reason for this is the property (established below) that the particle exchange operator E_{12} is invariant under parity, time reversal, and Lorentz transformations. To see this, we note from Eqs. (3.1) and (3.16) that the parity Π and time-reversal $\mathscr{T}$ operators are symmetric in the particle labels. Thus E_{12} commutes with both operators, so that the equations

$$E_{12}\Pi E_{12} = \Pi , \tag{6.29a}$$

$$E_{12}\mathscr{T}E_{12} = \mathscr{T} , \tag{6.29b}$$

together with $E_{12}^2=1$, yield

$$\Pi E_{12}\Pi^{-1}=E_{12}=\mathcal{T}E_{12}\mathcal{T}^{-1} , \tag{6.30}$$

the invariance of E_{12} under parity and time reversal. From its definition, E_{12} is Lorentz invariant. The symmetry properties of $E_{12}\tilde{T}$ then follow directly from the symmetry properties of $\tilde{T}$. In particular, we have

$$\Pi(E_{12}\tilde{T})\Pi^{-1}=E_{12}\Pi\tilde{T}\Pi^{-1}=(E_{12}\tilde{T}) \tag{6.31a}$$

and

$$\mathcal{T}(E_{12}\tilde{T})\mathcal{T}^{-1}=E_{12}\mathcal{T}\tilde{T}\mathcal{T}^{-1}$$
$$=E_{12}\tilde{T}^{\ddagger}=\tilde{T}^{\ddagger}E_{12}=[E_{12}\tilde{T}]^{\ddagger} . \tag{6.31b}$$

Equations (6.31) establish the result that if $\tilde{T}$ has been constructed to satisfy parity, time-reversal, and Lorentz transformation symmetries, then $E_{12}\tilde{T}$ will automatically satisfy the same symmetries. Also, if $\tilde{T}$ is even under particle label exchange, then so is $E_{12}\tilde{T}$. Thus the general operator forms obtained for $\tilde{T}$ necessarily suffice for $E_{12}\tilde{T}$.

The set of mixed scalar operators $\hat{I}_\alpha$ found in the schematic form of $\tilde{T}$ given in Eq. (6.28) is a complete set for expanding an operator consistent with the presumed symmetry properties. Because of this completeness, E_{12} maps the set $\hat{I}_\alpha$ into itself. Put another way, the "exchange" covariants $E_{12}\hat{I}_\alpha$ are contained in the space spanned by the "direct" covariants, and can thus be expanded in terms of them. To be more specific, we combine Eqs. (6.27) and (6.28) to obtain the antisymmetry constraint in the form

$$\sum_\alpha F_\alpha(k',p';k,p)\hat{I}_\alpha(k',p';k,p)$$
$$=-\sum_\beta F_\beta(p',k';k,p)E_{12}^{\gamma}\hat{I}_\beta(p',k';k,p) . \tag{6.32}$$

Use of the above-mentioned properties of E_{12} and the completeness of the set of $\hat{I}_\beta$ allows the expansion

$$E_{12}^{\gamma}\hat{I}_\beta(p',k';k,p)=\sum_\alpha a_{\beta\alpha}\hat{I}_\alpha(k',p';k,p) , \tag{6.33}$$

where the matrix a of scalars has the property $a^2=1$. Substitution of Eq. (6.33) into Eq. (6.32) and use of the linear independence of the $\hat{I}_\alpha$ leads to

$$F_\alpha(k',p';k,p)=-\sum_\beta F_\beta(p',k';k,p)\,a_{\beta\alpha} . \tag{6.34}$$

Thus the antisymmetry constraint is upon the amplitudes F_α and not upon the set of mixed scalars. The general form for $\hat{T}$ obtained in the preceding subsection therefore fully subsumes the Pauli principle for identical fermions.

The matrix a of scalars introduced in Eq. (6.33) is a generalization of the Fierz[15] matrix which relates the "direct" and "exchange" versions of the five Fermi invariants. The generalized matrix that appears here is larger than 5×5 because the direct and exchange character of the complete operator is being described rather than just the physical (positive energy, on mass shell) matrix elements. A representation for E_{12}^{γ} in terms of the complete set $\hat{I}_\beta$ would, through Eq. (6.33) and the use of trace techniques, allow the generalized Fierz matrix a to be constructed.

For applications to the nucleon-nucleon scattering operator, the isospin structure of the problem needs to be incorporated into the analysis. This is easily accomplished for the usual case in which the system is to be invariant with respect to rotations in isospin space. The allowable independent isospin scalars are then 1 and $\tau_1\cdot\tau_2$, both of which are symmetric under particle-label interchange. The previous general forms can be multiplied by either of the isospin scalars, and thus the total number of amplitudes is doubled.

VII. RECONSTRUCTION OF SCATTERING OPERATORS: GENERAL FRAMEWORK

In this section we develop a general formalism for the reconstruction of the invariant scattering operators $\tilde{T}$, or $\hat{T}$, from knowledge of their matrix elements in a particular frame. We assume the "known" matrix elements to be supplied in the frame of zero total three momentum, the barycentric frame. The matrix elements may be thought of as being the result of a solution of, e.g., the Bethe-Salpeter equation with a given interaction model.[19,22,23] In the general case the matrix elements may be presumed to be known off, as well as on, the mass shell and the complete off-mass-shell representation of $\tilde{T}$, or $\hat{T}$, is to be constructed. Specific motivation for the considerations of this section lies in the realm of applications of two-body scattering operators to many-body contexts.

It turns out to be advantageous to relate the Lorentz invariant form of $\tilde{T}$ (or $\hat{T}$) to the Pauli-spin structure of its Dirac spinor matrix elements (in the barycentric frame) in each of the possible ρ-spin channels. A particular ρ-spin channel is defined[19] by the characteristic sign of the energy ($\pm E_k$) associated with each of the Dirac spinors employed in forming matrix elements (i.e., positive and/or negative energy spinors in the initial and final states). In the case of two identical particles a simple ρ-spin product space is sufficient because the full Pauli principle is incorporated into the operators; see Sec. VI. Casting the analysis in this form not only subdivides the elucidates the reconstruction problem, but it also provides a convenient (and easily obtained) intermediate representation of the known matrix elements. The results obtained here can easily be transcribed into the singlet and triplet ρ-spin representation (the ρ-spin algebra is identical to that of the Pauli or σ spin), which is often used in numerical work with the Bethe-Salpeter equation.[19,22]

In order to maintain close contact with standard conventions, we first note that our basis states, defined in Sec. II, are related to those of Ref. 16 by

$$u(\mathbf{k},+)=\left[\frac{m}{E_k}\right]^{1/2}u_B(\mathbf{k}) \tag{7.1a}$$

and

$$u(\mathbf{k},-)=\left[\frac{m}{E_k}\right]^{1/2} v_B(-\mathbf{k}) . \tag{7.1b}$$

The orthonormality relations satisfied by the spinors of Ref. 16 are

$$\bar{u}_B(\mathbf{k})u_B(\mathbf{k})=-\bar{v}_B(\mathbf{k})v_B(\mathbf{k})=1 \tag{7.2a}$$

and

$$\bar{u}_B(\mathbf{k})v_B(\mathbf{k})=\bar{v}_B(\mathbf{k})u_B(\mathbf{k})=0 . \tag{7.2b}$$

In the treatment of the full scattering operator in terms of its matrix elements in the different ρ-spin sectors, it is necessary to introduce projectors which divide the full and the Dirac spinor spaces according to their positive- and negative-energy spinor character. From the completeness relation (2.6), together with Eq. (2.1), we find that the full Hilbert space (for a single Dirac particle) may be decomposed into positive- and negative-energy spinor spaces according to

$$1=\mathscr{P}_+ + \mathscr{P}_- \tag{7.3}$$

where the positive and negative energy projectors $\mathscr{P}_i$, $i=\pm$, are given by

$$\mathscr{P}_i=\int d^4k \mid k,i\rangle\langle k,i \mid \tag{7.4a}$$

$$=\int d^4k \mid k\rangle P_i(\mathbf{k})\langle k \mid , \tag{7.4b}$$

where the

$$P_i(\mathbf{k})=u(\mathbf{k},i)u^\dagger(\mathbf{k},i) \tag{7.5}$$

operate in the spinor subspace; they are the purely spinor-space projectors onto positive and negative energy Dirac spinors of momentum $\mathbf{k}$. Both the full and spinor-space projectors are Hermitian and they satisfy

$$\mathscr{P}_i\mathscr{P}_j=\mathscr{P}_i\delta_{i,j} , \tag{7.6a}$$

$$P_i(\mathbf{k})P_j(\mathbf{k})=P_i(\mathbf{k})\delta_{i,j} , \tag{7.6b}$$

and

$$P_+(\mathbf{k})+P_-(\mathbf{k})=1 . \tag{7.6c}$$

The essential features of the projectors $\mathscr{P}_i$ and $P_i(\mathbf{k})$ are summarized by

$$\mathscr{P}_i \mid k,j\rangle=\delta_{i,j} \mid k,j\rangle \tag{7.7a}$$

and

$$P_i(\mathbf{k})u(\mathbf{k},j)=\delta_{i,j}u(\mathbf{k},j) , \tag{7.7b}$$

which follow from Eqs. (7.4) and (7.5), and Eq. (2.5).

Explicit representations of the spinor-space projectors are

$$P_\pm(\mathbf{k})=\pm\frac{m}{E_k}\Lambda_\pm(\pm\mathbf{k})\gamma^0 , \tag{7.8}$$

where the Λ_+ and Λ_- are the more familiar (invariant) spinor-space projectors of Ref. 16:

$$\Lambda_+(\mathbf{k})=u_B(\mathbf{k})\bar{u}_B(\mathbf{k}) , \tag{7.9a}$$

and

$$\Lambda_-(\mathbf{k})=-v_B(\mathbf{k})\bar{v}_B(\mathbf{k}) , \tag{7.9b}$$

or more simply,

$$\Lambda_\pm(\mathbf{k})=\frac{\pm\not{k}+m}{2m} . \tag{7.9c}$$

In Eq. (7.9c), $k=(E_k,\mathbf{k})$ and the usual slash notation $\not{k}=\gamma\cdot k$ has been used. Since the spinor space is also orthogonally decomposed by the $\Lambda_i(\mathbf{k})$,

$$\Lambda_+(\mathbf{k})+\Lambda_-(\mathbf{k})=1 , \tag{7.10a}$$

with

$$\Lambda_i(\mathbf{k})\Lambda_j(\mathbf{k})=\Lambda_i(\mathbf{k})\delta_{i,j} , \tag{7.10b}$$

the completeness relation for the spinors of Eqs. (7.9) is

$$1=\int d^4k \mid k\rangle[\Lambda_+(\mathbf{k})+\Lambda_-(\mathbf{k})]\langle k \mid \tag{7.11}$$

as opposed to Eq. (7.3), which is

$$1=\int d^4k \mid k\rangle[P_+(\mathbf{k})+P_-(\mathbf{k})]\langle k \mid . \tag{7.12}$$

The advantage of the Λ_i is their invariance property. The advantage of Eq. (7.12) is its association of the Dirac spinors of three momentum $\mathbf{k}$ with the momentum space kets $\mid k\rangle$ which have the same three momentum. Thus Eq. (7.12) is an expansion in the (positive and negative energy) eigenstates of the free Dirac Hamiltonian. The (spinor) negative-energy part of Eq. (7.11), on the other hand, associates kets of three momenta $\mathbf{k}$ with Dirac spinors of momentum $-\mathbf{k}$ [see Eqs. (7.1b) and (7.9b)]. The projectors $\Lambda_\pm(\mathbf{k})$, which do not yield a particularly useful representation of completeness in the *full* Hilbert space, find their main use in the computation of cross sections via closure and trace techniques in spinor space after momentum-space integrations have been performed. The case of interest here is the reconstruction of the scattering operator $\tilde{T}$ from its matrix elements, which are naturally given in terms of eigenstates of the free Dirac Hamiltonian. These are the matrix elements that arise, for example, in the solution of an equation of the Bethe-Salpeter type, since it is natural to employ a basis which diagonalizes the free Hamiltonian[19,22] and the free propagator.[19,22] Thus, in our considerations we employ the noncovariant projectors $\mathscr{P}_i$ and $P_i(\mathbf{k})$. As we shall see, a covariant form for the reconstructed scattering operator can still be achieved.

Because the invariant amplitude is given in terms of Dirac adjoint matrix elements [see Eqs. (2.8)], it is useful to introduce the auxiliary quantities $\bar{\mathscr{P}}_i$ and $\bar{P}_i(\mathbf{k})$ defined by

$$\bar{\mathscr{P}}_i=\gamma^0\mathscr{P}_i\gamma^0 \tag{7.13a}$$

and

$$\bar{P}_i(\mathbf{k})=\gamma^0P_i(\mathbf{k})\gamma^0=\gamma^0u(\mathbf{k},i)\bar{u}(\mathbf{k},i) , \tag{7.13b}$$

so that

$$\bar{\mathscr{P}}_i=\int d^4k \mid k\rangle\bar{P}_i(\mathbf{k})\langle k \mid \tag{7.13c}$$

with

$$\overline{\mathscr{P}}_{+}+\overline{\mathscr{P}}_{-}=1\ , \tag{7.14a}$$

$$\overline{\mathscr{P}}_{i}\overline{\mathscr{P}}_{j}=\overline{\mathscr{P}}_{i}\delta_{i,j}\ , \tag{7.14b}$$

and

$$\bar{P}_{+}(\mathbf{k})+\bar{P}_{-}(\mathbf{k})=1\ , \tag{7.14c}$$

$$\bar{P}_{i}(\mathbf{k})\bar{P}_{j}(\mathbf{k})=\bar{P}_{i}(\mathbf{k})\delta_{i,j}\ . \tag{7.14d}$$

Evidently, the operators $\overline{\mathscr{P}}_i$ and $\bar{P}_i(\mathbf{k})$ are also Hermitian projectors. The essential properties of the projectors $\overline{\mathscr{P}}$ and $\bar{P}(\mathbf{k})$ follow as a consequence of their definitions, Eqs. (7.13), and of Eqs. (7.7):

$$\langle \overline{k,j} \mid \overline{\mathscr{P}}_{i}=\langle \overline{k,j} \mid \delta_{i,j} \tag{7.15a}$$

and

$$\bar{u}(\mathbf{k},j)\bar{P}_{i}(\mathbf{k})=\bar{u}(\mathbf{k},j)\delta_{i,j}\ , \tag{7.15b}$$

so that these operators project onto positive and negative energy Dirac adjoint states.

Before proceeding to the development of a general framework for the reconstruction of the scattering operators for the spin-$\frac{1}{2}$-spin-0 (subsection A) and spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ (subsection B) cases, it is useful to first obtain the parity and time-reversal properties of the projectors $\mathscr{P}_i$, $\overline{\mathscr{P}}_i$, $P_i(\mathbf{k})$, and $\bar{P}_i(\mathbf{k})$. If we use A to denote either the parity operator Π given in Eq. (3.2) or the time-reversal operator $\mathscr{T}$ given in Eq. (3.14), then from Eqs. (7.4) and (7.8) we find the symmetry properties

$$A\mathscr{P}_{i}A^{-1}=\mathscr{P}_{i}\quad (A=\Pi,\mathscr{T}) \tag{7.16}$$

in the full space, and

$$P_{i}(\mathbf{k})\underset{(\Pi,\mathscr{T})}{\longrightarrow}P_{i}(-\mathbf{k})=\bar{P}_{i}(\mathbf{k}) \tag{7.17}$$

in just the spinor space. For the auxiliary projectors we find from Eqs. (7.16) and (7.13) that

$$A\overline{\mathscr{P}}_{i}A^{-1}=\overline{\mathscr{P}}_{i}\quad (A=\Pi,\mathscr{T}) \tag{7.18}$$

in the full space and, of course,

$$\bar{P}_{i}(\mathbf{k})\underset{(\Pi,\mathscr{T})}{\longrightarrow}\bar{P}_{i}(-\mathbf{k})=P_{i}(\mathbf{k}) \tag{7.19}$$

in just the spinor space.

A. Spin $\frac{1}{2}$-spin 0

It is convenient to begin from the integral representation, Eq. (2.12), of the full scattering operator $\tilde{T}$, which we write in the form

$$\tilde{T}=\int d^4\{k'p'kp\}\mid k';p'\rangle\hat{T}(k',p';k,p;\{\Gamma\})\langle k;p\mid\ . \tag{7.20}$$

The ρ-spin decomposition of $\tilde{T}$ is then introduced by means of the identity

$$\tilde{T}=\sum_{i,j}\overline{\mathscr{P}}_{j}\tilde{T}\mathscr{P}_{i}\ , \tag{7.21}$$

where $i,j=\pm$. Equation (7.21) expresses $\tilde{T}$ in terms of its projections in each of the four ρ-spin channels. Given the parity and reciprocity symmetries satisfied by $\tilde{T}$, Eqs. (3.5) and (3.20), respectively, it is immediate from the invariance of the $\mathscr{P}_i$ and $\overline{\mathscr{P}}_i$ under parity and time reversal, Eqs. (7.16) and (7.18), that the three operators $\overline{\mathscr{P}}_{+}\tilde{T}\mathscr{P}_{+}$, $\overline{\mathscr{P}}_{-}\tilde{T}\mathscr{P}_{-}$, and $\overline{\mathscr{P}}_{+}\tilde{T}\mathscr{P}_{-}+\overline{\mathscr{P}}_{-}\tilde{T}\mathscr{P}_{+}$ share these symmetries. Given covariant extensions of the projectors (which we obtain shortly), it is then evident that these three operators can be expressed in terms of expansions which are form identical to those already obtained for $\tilde{T}$. Such is not the case for the "difference" operator $\overline{\mathscr{P}}_{+}\tilde{T}\mathscr{P}_{-}-\overline{\mathscr{P}}_{-}\tilde{T}\mathscr{P}_{+}$, which is easily seen to be reciprocity odd, unlike the other operators. Thus, although the sum of the off-diagonal elements automatically has the same symmetry properties as $\tilde{T}$, this is not the case for the individual off-diagonal projections which appear in Eq. (7.21). (We find explicit examples of such behavior at the end of this subsection.) It is important to note that the fact that the off-diagonal elements do not have the same symmetry properties as $\tilde{T}$ does not pose an "in principle" impediment to the reconstruction process. This is because the "difference" operator does not contribute to Eq. (7.21) in any event and since, more generally, it is evident that the decomposition of Eq. (7.21) is such that [see Eqs. (2.8a), (7.7), and (7.15)] the relevant matrix elements satisfy

$$\langle \overline{k',j};p' \mid \tilde{T} \mid k,i;p\rangle=\langle \overline{k',j};p' \mid \overline{\mathscr{P}}_{j}\tilde{T}\mathscr{P}_{i} \mid k,i;p\rangle\ . \tag{7.22}$$

Equation (7.22) implies that a valid procedure for constructing the $\overline{\mathscr{P}}_j\tilde{T}\mathscr{P}_i$ would be to fit the appropriate matrix elements with an operator of the same form as $\tilde{T}$ and then to attach the appropriate projectors. As we will see, a better method can be obtained.

In terms of the spinor-space operator $\hat{T}$, the decomposition of Eq. (7.21) is easily seen to yield

$$\hat{T}(k',p';k,p;\{\Gamma\})=\sum_{i,j}\bar{P}_{j}(\mathbf{k}')\hat{T}(k',p';k,p;\{\Gamma\})P_{i}(\mathbf{k}) \tag{7.23a}$$

$$=\sum_{i,j}\hat{T}_{ji}(k',p';k,p;\{\Gamma\})\ , \tag{7.23b}$$

while if we define

$$M_{ji}(k',p';k,p)=\bar{u}(\mathbf{k}',j)\hat{T}(k',p';k,p;\{\Gamma\})u(\mathbf{k},i)\ , \tag{7.24}$$

then the analog of Eq. (7.22) is

$$M_{ji}(k',p';k,p)=\bar{u}(\mathbf{k}',j)\bar{P}_{j}(\mathbf{k}')\times\hat{T}(k',p';k,p;\{\Gamma\})P_{i}(\mathbf{k})u(\mathbf{k},i) \tag{7.25a}$$

$$=\bar{u}(\mathbf{k}',j)\hat{T}_{ji}(k',p';k,p;\{\Gamma\})u(\mathbf{k},i)\ . \tag{7.25b}$$

Note that M_{ji} is a 2×2 operator in Pauli spin space. Combining Eq. (7.25b) with Eqs. (7.5) and (7.13b) yields

$$\hat{T}_{ji}(k',p';k,p;\{\Gamma\})=\gamma^{0}u(\mathbf{k}',j)M_{ji}(k',p';k,p)u(\mathbf{k},i)^{\dagger}\ . \tag{7.26}$$

Although Eq. (7.26) essentially reconstructs each of the operators $\hat{T}_{ji}$ [and hence the operator $\hat{T}$ via Eq. (7.23b)] in the barycentric frame from knowledge of the matrix elements M_{ji}, the resultant operator will not be in Lorentz scalar form since its constituent elements are not covariant. Thus, although matrix elements of $\tilde{T}$ or $\hat{T}$ in other reference frames are obtainable from Eq. (7.26), the necessary procedure is complicated. Equation (7.26) does not provide the convenience and conciseness of a Lorentz scalar operator form.

One way of overcoming the inconvenience of Eq. (7.26) is to fit the matrix elements M_{ji}, as given by Eq. (7.25b), directly in terms of some "consistent" invariant representation for the $\hat{T}_{ji}$. Once this is done the full $\hat{T}$ may be constructed via Eq. (7.23a). However, just what is required to obtain "consistency" must be determined. Furthermore, in order to guarantee an invariant representation for the operator $\hat{T}$ constructed in this fashion, covariant forms for the $\bar{P}_i(\mathbf{k})$ and $P_i(\mathbf{k})$ must be developed and employed in Eq. (7.23a) instead of the noncovariant forms used so far.

Put differently, presumed invariant forms of the operators $\hat{T}_{ji}$ may be determined by fitting them to the relevant invariant amplitudes $\mathscr{F}$ [see Eq. (2.8a)]

$$\mathscr{F}_{ji}=\left[\frac{E_{k'}E_k}{m^2}\right]^{1/2} M_{ji}(k',p';k,p) \tag{7.27}$$

or, equivalently,

$$\mathscr{F}_{ji}=\left[\frac{E_{k'}E_k}{m^2}\right]^{1/2} \bar{u}(\mathbf{k}',j)\hat{T}_{ji}(k',p';k,p;\{\Gamma\})u(\mathbf{k},i) . \tag{7.28}$$

The full scattering operator $\hat{T}$ is then formed from Eq. (7.23a). Two well-defined issues in this reconstruction method require resolution. First, covariant forms of the projectors $\bar{P}_j$ and P_i must be found for use in Eq. (7.23a) in order to produce $\hat{T}$ in invariant form, given Lorentz scalar representations of the $\hat{T}_{ji}$. Secondly, the constraints which limit the choice of the invariant forms that can be adopted for each of the operators $\hat{T}_{ji}$ must be clarified and a convenient fitting procedure must be developed. In particular, must the general operator form of $\hat{T}$ [i.e., the ten terms in Eq. (4.3a)] be employed for the $\hat{T}_{ji}$ in each ρ-spin channel? Are there simple alternative fitting methods? What constraints arise from the parity and time-reversal symmetries, etc.?

To resolve the first issue, we recall from Eqs. (7.8) and (7.13) that the projectors are

$$P_\pm(\mathbf{k})=\Lambda_\pm(\pm\mathbf{k})\left[\frac{\pm m}{E_k}\gamma^0\right] ,$$
$$\bar{P}_\pm(\mathbf{k})=\left[\frac{\pm m}{E_k}\gamma^0\right]\Lambda_\pm(\pm\mathbf{k}) , \tag{7.29}$$

where the $\Lambda_\pm$ are already in covariant form. Thus we need only cast the factors $\gamma^0 m/E_k$ and $\gamma^0 m/E_{k'}$ in Lorentz scalar form. In the barycentric frame the kinematic variable ω [see Eq. (2.10c)] has only a time component so that a covariant extension of γ^0 is given by the simple expression $\omega\cdot\gamma/\sqrt{s}$. Similarly, it is easily verified that a covariant extension of the barycentric quantity m/E_k is given by the somewhat more complicated expression

$$R(|\mathbf{k}|)=\frac{2m}{[\{\omega\cdot(K-q)\}^2/s-(K-q)^2+4m^2]^{1/2}} , \tag{7.30}$$

while the covariant extension of the barycentric quantity $m/E_{k'}$ is obtained from Eq. (7.30) by the replacement $q\rightarrow -q$. However, despite the fact that all three of the momentum variables (q,K,ω) appear in Eq. (7.30), they are not needed. The covariant extension of the barycentric quantity m/E_k ($m/E_{k'}$) depends only upon the initial (final) momenta $\omega=k+p$ and $K-q=k-p$ ($\omega=k'+p'$ and $K+q=k'-p'$). Thus if we regard ω as a symbol for the sum of the four momenta of the two particles and $K-q$ as a symbol for the difference of the four momenta, then *both* results are expressed by Eq. (7.30). We adopt this interpretation of Eq. (7.30); with it no distinction between the descriptions of the initial and final states is needed and we avoid the introduction of an inessential asymmetry.

Thus if one introduces the covariant realizations of the barycentric projectors of Eqs. (7.29) as

$$\Delta_\pm(\mathbf{k})=\Lambda_\pm(\pm\mathbf{k})\left[\pm R(|\mathbf{k}|)\frac{\omega\cdot\gamma}{\sqrt{s}}\right] \tag{7.31a}$$

and

$$\bar{\Delta}_\pm(\mathbf{k})=\left[\pm R(|\mathbf{k}|)\frac{\omega\cdot\gamma}{\sqrt{s}}\right]\Lambda_\pm(\pm\mathbf{k}) , \tag{7.31b}$$

where the $\Lambda_\pm(\mathbf{k})$ satisfy Eq. (7.9c), then an invariant form of $\hat{T}$ is given by

$$\hat{T}(k',p';k,p;\{\Gamma\})=\sum_{i,j}\bar{\Delta}_j(\mathbf{k}')\hat{T}_{ji}(k',p';k,p;\{\Gamma\})\Delta_i(\mathbf{k}) , \tag{7.32}$$

with the manifest Lorentz scalar operators $\hat{T}_{ji}$ determined by a fitting procedure yet to be described.

The question of the forms to be presumed for the $\hat{T}_{ji}$ and of a convenient form of the fitting procedure remain. It is especially useful in this regard to develop the symmetry properties of the M_{ji} under the parity and time-reversal transformations. In the case of parity we make use of Eq. (3.10) to rewrite Eq. (7.25b) as

$$M_{ji}(k',p';k,p;\sigma)=\bar{u}(\mathbf{k}',j)\gamma^0\hat{T}(\tilde{k}',\tilde{p}';\tilde{k},\tilde{p};\{\Gamma\})\gamma^0 u(\mathbf{k},i) \tag{7.33a}$$

$$=ij\,\bar{u}(-\mathbf{k}',j)\hat{T}(\tilde{k}',\tilde{p}';\tilde{k},\tilde{p};\{\Gamma\})u(-\mathbf{k},i) , \tag{7.33b}$$

where $i,j=\pm$ and where we have made explicit the possible linear dependences of the M_{ji} upon the Pauli spin matrices. Equation (7.33b) states that the constraint of parity conservation upon the M_{ji} is

$$M_{ji}(k',p';k,p;\sigma)=ij\,M_{ji}(\tilde{k}',\tilde{p}';\tilde{k},\tilde{p};\sigma) . \quad (7.34)$$

Thus M_{ji} is even (odd) under the reversal of the three-momenta of both particles (the Pauli-space parity operation π) if $i=j$ ($i\neq j$).

To obtain the time-reversal symmetry of the M_{ji}, we make use of Eq. (3.24) to rewrite Eq. (7.25b) as

$$M_{ji}(k',p';k,p;\sigma)=\bar{u}(\mathbf{k}',j)(\gamma^0 i\gamma^1\gamma^3)[\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma\})^\dagger]^*(i\gamma^1\gamma^3\gamma^0)u(\mathbf{k},i) \quad (7.35a)$$

$$=-\bar{u}(\mathbf{k}',j)(\gamma^5\gamma^2)[\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma\})^\dagger]^*(\gamma^2\gamma^5)u(\mathbf{k},i) . \quad (7.35b)$$

Upon noting that $\gamma^2\gamma^5=\Sigma^2\gamma^0=\sigma^2\gamma^0$ (where $\sigma^2=\boldsymbol{\sigma}\cdot\hat{2}$), and making use of the complex conjugation operator K, Eq. (7.35b) becomes

$$M_{ji}(k',p';k,p;\sigma)=K[\bar{u}(\mathbf{k}',j)^*\Sigma^2\gamma^0\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma\})^\dagger\gamma^0\Sigma^2u(\mathbf{k},i)^*]K \quad (7.36)$$

$$=K\sigma^2[\bar{u}(-\mathbf{k}',j)\gamma^0\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma\})^\dagger\gamma^0u(-\mathbf{k},i)]\sigma^2K . \quad (7.37)$$

In terms of the usual Pauli-space time-reversal operator given by

$$\tau=\sigma^2K , \quad (7.38)$$

we have the result

$$M_{ji}(k',p';k,p;\sigma)=\tau[\bar{u}(-\mathbf{k},i)\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma\})u(-\mathbf{k}',j)]^\dagger\tau^{-1} \quad (7.39a)$$

$$=\tau M_{ij}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\sigma)^\dagger\tau^{-1} . \quad (7.39b)$$

After implementation of the τ operation, Eq. (7.39b) yields a constraint of the form familiar from nonrelativistic scattering theory:

$$M_{ji}(k',p';k,p;\sigma)=M_{ij}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';-\sigma) . \quad (7.40)$$

Equation (7.40) is the time-reversal (reciprocity) constraint upon the form of the M_{ji}. It constrains the form of the M_{ii}, while for $i\neq j$ it determines M_{ij} from M_{ji} such that the sum $M_{ij}+M_{ji}$ satisfies the same time-reversal constraint as the M_{ii}. The parity and time-reversal constraints upon the (invariant) $\mathscr{F}_{ji}$ of Eq. (7.27) are, of course, identical to those upon the M_{ji}.

A convenient and informative representation of the M_{ji} can now be obtained as follows. Due to rotational invariance, each of the M_{ji} (or $\mathscr{F}_{ji}$) can be expanded in components which are linear in either the unit matrix or in the rotational invariants which can be formed from the Pauli spin vector $\boldsymbol{\sigma}$: $\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{K})$, $\boldsymbol{\sigma}\cdot\mathbf{q}$, and $\boldsymbol{\sigma}\cdot\mathbf{K}$. We refer to this set of four operators as the set E_σ. The parity (π) and time-reversal (τ) symmetries of these operators are summarized in Table VI. In forming a particular one of the M_{ji}, each of these operators is multiplied by an arbitrary scalar function G_i of the seven available parity and time-reversal even momentum-space (Lorentz) scalars of Eq. (4.3a): s, t, u, $K\cdot\omega$, $(q\cdot K)^2$, $(q\cdot\omega)^2$, and $q\cdot Kq\cdot\omega$. In addition, linear dependence upon the time-reversal odd (but parity even) Lorentz scalars $q\cdot K$ and $q\cdot\omega$ is permitted, just as in Eq. (4.3a). Before the imposition of parity and time-reversal constraints, each of the M_{ji} may therefore by expanded as

$$M_{ji}=\sum_{I_m\in E_\sigma}^{(4)}(G_m+\tilde{G}_m q\cdot K+G'_m q\cdot\omega)I_m . \quad (7.41)$$

Imposition of the symmetry constraints of Eqs. (7.34) and (7.40) yields, for the form of each of the two diagonal elements M_{ii},

$$M_{ii}=G_1+G_2\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{K}) , \quad (7.42a)$$

i.e., the familiar form for a Pauli spin-$\frac{1}{2}$-spin-0 scattering operator.[1] For each of the nondiagonal ($i\neq j$) M_{ji}, the structure, as determined by the odd parity constraint of Eq. (7.34) and the lack of a time-reversal constraint, is

$$M_{ji}=(G_3+\tilde{G}_3q\cdot K+G'_3q\cdot\omega)\,\boldsymbol{\sigma}\cdot\mathbf{q}+(G_4+\tilde{G}_4q\cdot K+G'_4q\cdot\omega)\,\boldsymbol{\sigma}\cdot\mathbf{K} . \quad (7.42b)$$

Since M_{ij} ($i\neq j$) is fully determined by M_{ji}, Eqs. (7.42) require the determination of 10 true scalar functions, precisely the number known to be required for the specification of $\hat{T}$ via Eq. (4.3a). Evidently, when one (both) particles are on mass shell, this number is reduced to eight (six), as described in Sec. V A. We note that the quantities in parentheses in Eq. (7.42b) could be treated as single amplitudes that do not have a particular time-reversal symmetry. A total of six such amplitudes (two for each of the

TABLE VI. Classification of the available Pauli one-body mixed scalars E_σ in the barycentric frame according to their parity (π) and time reversal (τ) symmetry. Their symmetry character under the interchange of momenta (E^k_{12}) is included for later use.

(Pauli) E_σ	π	τ	(E^k_{12})
1	+	+	+
$\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{K})$	+	+	+
$\boldsymbol{\sigma}\cdot\mathbf{q}$	−	−	−
$\boldsymbol{\sigma}\cdot\mathbf{K}$	−	+	−

three independent ρ-spin sectors) are needed to reconstruct the full operator $\hat{T}$.

Equations (7.42) reveal alternatives and simplifications in the structures needed in the fitting procedure used to obtain Lorentz scalar realizations of the $\hat{T}_{ij}$. The full set of 10 independent invariants [See Eq. (4.3a)] is not needed in any of the ρ-spin channels. Only two terms are required for $\hat{T}_{++}$ and $\hat{T}_{--}$. Six independent terms are necessary for either $\hat{T}_{+-}$ or $\hat{T}_{-+}$. Evidently, the set of four invariants linear in the time-reversal odd MSS ($q\cdot K$, $q\cdot\omega$) should be included as a subset of these six, and the remaining two invariants of the ρ-spin channels should evidently be chosen from among the set of six invariants which do not vanish when both particles are on their mass shells. This choice seems otherwise constrained only in that the chosen pair must be nonvanishing and linearly independent on the particular projected space where they are used. Clearly, considerable flexibility in the choice of the particular invariants to be used in each of the ρ-spin channels remains.

In short, the Pauli spin representations of Eqs. (7.42) provide a convenient intermediate representation of the known matrix elements. Fitting these expressions to the matrix elements could be the first step of the fitting process (a variant of this approach is considered in detail in Sec. VIII). For the off-diagonal terms, fitting the sum $M_{ij}+M_{ji}$ and the difference $M_{ij}-M_{ji}$ is advantageous since this further subdivides the problem [to see that this is the case, use Eq. (7.42b) for M_{ij}, use Eq. (7.40) to obtain M_{ji}, and then form the sum and the difference]. With this accomplished, essentially any two of the six (Lorentz) scalar operators of Eq. (4.3a) which have no linear dependence upon either $q\cdot K$ or $q\cdot\omega$ may then be used in fitting to the Pauli representations in each ρ-spin channel. In fitting one of the off-diagonal terms, the four operators linear in one or the other of $q\cdot K$ or $q\cdot\omega$ should also be used. The choice of invariants employed in the fitting process seems to be otherwise unconstrained, except, of course, that the chosen invariants must be nonvanishing and linearly independent on the projected spaces where they are used. With Lorentz scalar representations determined for each of the $\hat{T}_{ij}$, the full operator is obtained by means of Eqs. (7.32) and (7.20).

Although these results provide for significant simplification, considerable flexibility in the general reconstruction method also remains. Such flexibility might be used to advantage. However, methods which guarantee the requisite linear independence in each of the ρ-spin sectors and methods for choosing a set of invariants with desirable interference and stability properties remain to be obtained. A particular reconstruction procedure which both avoids these problems and provides a means for assessing the properties of other procedures is obtained in Sec. VIII.

B. Spin $\frac{1}{2}$-spin $\frac{1}{2}$

Although technically more complicated than the spin-$\frac{1}{2}$-spin-0 case due to the presence of two Dirac spinors in both the initial and final states, the considerations of this subsection are a straightforward extension of those of subsection A. For this reason we adopt a treatment which parallels that of the preceding subsection. It is again convenient to begin from the integral representation, Eq. (2.12), of the full scattering operator $\tilde{T}$, which we write in the form

$$\tilde{T}=\int d^4\{k'p'kp\}\,|k';p'\rangle\hat{T}(k',p';k,p;\{\Gamma(n)\})\langle k;p\,| \;, \tag{7.43}$$

where $n=1,2$ refers to the particle labels of the two Dirac particles. The decomposition of $\tilde{T}$ into its projections in each of the 16 ρ-spin channels is introduced through the identity

$$\tilde{T}=\sum_{ijlm}\overline{\mathscr{P}}_j(1)\overline{\mathscr{P}}_m(2)\tilde{T}\mathscr{P}_i(1)\mathscr{P}_l(2)\;, \tag{7.44}$$

where $i,j,l,m=\pm$. The parity and time-reversal properties of the various projections of $\tilde{T}$ follow from those of $\tilde{T}$, Eqs. (3.5) and (3.20), and from the invariance of the projectors under the parity and time-reversal transformations. As we noted in the preceding subsection, these properties are not (in general) the same as those of $\tilde{T}$. However, from the analog of Eq. (7.22),

$$\begin{aligned}\langle\overline{k',j;p',m}\,|\,\tilde{T}\,|\,k,i;p,l\rangle\\ =\langle\overline{k',j;p',m}\,|\,\overline{\mathscr{P}}_j(1)\overline{\mathscr{P}}_m(2)\tilde{T}\mathscr{P}_i(1)\mathscr{P}_l(2)\,|\,k,i;p,l\rangle\;,\end{aligned} \tag{7.45}$$

it is evident that a valid procedure for constructing the

$$\overline{\mathscr{P}}_j(1)\overline{\mathscr{P}}_m(2)\tilde{T}\mathscr{P}_i(1)\mathscr{P}_l(2)$$

is to fit the appropriate matrix elements to an operator of the same form as $\tilde{T}$ [see Eq. (4.7) or (6.19) for the nonidentical and identical particle forms, respectively, and Eqs. (5.15) and (4.20) for the corresponding on-mass-shell limits] and then to attach the corresponding projectors. Better methods are available, however, and we develop these shortly.

In view of the integral representation in Eq. (7.43), the decomposition of Eq. (7.44) yields, for the spinor-space operator $\hat{T}$,

$$\begin{aligned}\hat{T}(k',p';k,p;\{\Gamma(n)\})\\ =\sum_{ijlm}\bar{P}_j(\mathbf{k}')\bar{P}_m(\mathbf{p}')\hat{T}(k',p';k,p;\{\Gamma(n)\})P_i(\mathbf{k})P_l(\mathbf{p})\end{aligned} \tag{7.46a}$$

$$=\sum_{ijlm}\hat{T}^{ml}_{ji}(k',p';k,p;\{\Gamma(n)\})\;. \tag{7.46b}$$

In this and in the following we suppress particle labels since no confusion should arise if our convention of associating momenta $\mathbf{k},\mathbf{k}'$ and $\mathbf{p},\mathbf{p}'$ with particles (1) and (2), respectively, is kept in mind. With Dirac spinor matrix elements of the spinor-space scattering operator defined by

$$M_{ji}^{ml}(k',p';k,p)=\bar{u}(\mathbf{k}',j)\bar{u}(\mathbf{p}',m)\times\hat{T}(k',p';k,p;\{\Gamma(n)\})u(\mathbf{k},i)u(\mathbf{p},l)\ , \tag{7.47}$$

we find the analog of Eq. (7.45)

$$M_{ji}^{ml}(k',p';k,p)=\bar{u}(\mathbf{k}',j)\bar{u}(\mathbf{p}',m)\times\hat{T}_{ji}^{ml}(k',p';k,p;\{\Gamma(n)\})u(\mathbf{k},i)u(\mathbf{p},l)\ . \tag{7.48}$$

The M_{ji}^{ml} are, of course, operators in the (direct-product) Pauli spin space of the two particles. Similarly, combining Eq. (7.48) with the expression (7.5) for $P_i(\mathbf{k})$ and (7.13b) for $\bar{P}_i(\mathbf{k})$ yields the analog of Eq. (7.26),

$$\hat{T}_{ji}^{ml}(k',p';k,p;\{\Gamma(n)\})=\gamma^0(1)u(\mathbf{k}',j)\gamma^0(2)u(\mathbf{p}',m)M_{ji}^{ml}(k',p';k,p)u(\mathbf{k},i)^\dagger u(\mathbf{p},l)^\dagger\ . \tag{7.49}$$

The reconstruction of the scattering operator $\hat{T}$ from knowledge of the barycentric matrix elements M_{ji}^{ml} through Eqs. (7.46b) and (7.49) suffers from the same disadvantages as described previously for the spin-$\frac{1}{2}$-spin-0 analog, Eq. (7.26). The resultant operator is not in Lorentz scalar form.

In order to construct the scattering operator $\hat{T}$ in Lorentz scalar form [and from it $\tilde{T}$ by means of Eq. (7.43)], we employ, instead of the projectors $P_i(\mathbf{k})$ and $\bar{P}_i(\mathbf{k})$, the covariant extensions $\Delta_i(\mathbf{k})$ and $\bar{\Delta}_i(\mathbf{k})$ given by Eqs. (7.31). With these projectors we can rewrite Eqs. (7.4b) for the ρ-spin decomposition of $\hat{T}$ as

$$\hat{T}(k',p';k,p;\{\Gamma(n)\})=\sum_{ijlm}\bar{\Delta}_j(\mathbf{k}')\bar{\Delta}_m(\mathbf{p}')\hat{T}_{ji}^{ml}(k',p';k,p;\{\Gamma(n)\})\Delta_i(\mathbf{k})\Delta_l(\mathbf{p})\ , \tag{7.50}$$

which is the two-particle analog of Eq. (7.32). Given Lorentz scalar operators $\hat{T}_{ji}^{ml}$ which reproduce the relevant matrix elements according to Eqs. (7.47) and (7.48), Eq. (7.50) solves the reconstruction problem. We note that the procedure indicated by Eq. (7.50) differs from the procedure recently employed by Tjon and Wallace[6] to reconstruct the NN scattering operator. They employ not the projectors Δ_i, but the projectors Λ_i of Eqs. (7.9). The associated negative energy basis states are not free particle eigenstates in the full Hilbert space since the employed spinors, identified by Eq. (7.9b), have the opposite momentum to what is needed [see Eq. (7.1b)]. For this method, the $\hat{T}_{ji}^{ml}$ must be consistently defined using the Bjorken and Drell spinors in Eq. (7.48). Thus the supplied matrix elements M_{ji}^{ml} are also to be in this basis, and the necessary transformation must be carried out for all values of the momentum arguments of the M_{ji}^{ml}. To transform from the basis used here to the basis required by the method of Ref. 6, it is only necessary to multiply the projectors Δ_i from the right by Eq. (7.10a) and then multiply the projectors $\bar{\Delta}_i$ from the left by (Eq. (7.10a). The linear coefficients for the change of basis are produced this way. We note that the nondiagonal terms $\Lambda_-(\mathbf{k})\bar{\Delta}_+(\mathbf{k})$ and $\Delta_+(\mathbf{k})\Lambda_-(\mathbf{k})$ are not zero, while the other nondiagonal combinations are zero. The method used here can be viewed as a development of projectors suited to the free-particle eigenstate basis that is usual for the solution output from scattering equations of the Bethe-Salpeter type. The method of Tjon and Wallace can be viewed as based upon a linear transformation of such supplied matrix elements to a basis suited to the available covariant projectors. Especially in the off-mass-shell case we expect the present method to be more economical due to the number of independent scalar arguments of the matrix elements to be dealt with. Moreover, the approach used here results in a particularly advantageous realization in Sec. VIII.

To complete the discussion of the scheme represented by Eq. (7.50), we develop the properties of the invariant operators $\hat{T}_{ji}^{ml}$ that aid in their construction. In particular, we may ask the same questions as in the treatment of the spin-$\frac{1}{2}$-spin-0 case in the preceding subsection: What is the form of the operators which comprise the $\hat{T}_{ji}^{ml}$ in each ρ-spin channel, e.g., must we employ the full operator form of $\hat{T}$? What constraints arise from parity and time-reversal symmetries? Are there simple alternative fitting methods, etc.? To reveal the underlying features of the operators, we proceed as in the preceding subsection and first develop the parity and time-reversal symmetry properties of the M_{ji}^{ml}.

In the case of the parity transformation we make use of the parity constraint upon $\hat{T}$, Eq. (3.10), to rewrite Eq. (7.47) as

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=\bar{u}(\mathbf{k}',j)\bar{u}(\mathbf{p}',m)\gamma^0(1)\gamma^0(2)\hat{T}(\tilde{k}',\tilde{p}';\tilde{k},\tilde{p};\{\Gamma(n)\})\gamma^0(1)\gamma^0(2)u(\mathbf{k},i)u(\mathbf{p},l) \tag{7.51a}$$

$$=ijlm\ \bar{u}(-\mathbf{k}',j)\bar{u}(-\mathbf{p}',m)\hat{T}(\tilde{k},\tilde{p}';\tilde{k},\tilde{p};\{\Gamma(n)\})u(-\mathbf{k},i)u(-\mathbf{p},l)\ , \tag{7.51b}$$

where $i,j,l,m=\pm$ and where we have made explicit the possible linear dependences upon the Pauli σ for each particle. Equation (7.51b) yields, for the parity constraint upon the M_{ji}^{ml},

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=ijlm\ M_{ji}^{ml}(\tilde{k}',\tilde{p}';\tilde{k},\tilde{p};\{\sigma(n)\})\ . \tag{7.52}$$

Thus, M_{ji}^{ml} is even (odd) under the reversal of the three momenta of the particles (the Pauli-space parity operator π) if the number of + (or -) spinors involved is even (odd).

To develop the time-reversal properties of the M_{ji}^{ml} we employ the time-reversal symmetry of $\hat{T}$, Eq. (3.24), in Eq. (7.47) to obtain

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=\bar{u}(\mathbf{k}',j)\bar{u}(\mathbf{p}',m)\gamma_T(1)\gamma_T(2)[\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma(n)\})^\dagger]^*\gamma_T(1)\gamma_T(2)u(\mathbf{k},i)u(\mathbf{p},l)\ , \quad (7.53)$$

where we have used the shorthand $i\gamma^0\gamma^1\gamma^3=\gamma_T$. Upon noting that $\gamma_T=-\gamma^2\gamma^5$, $\gamma^2\gamma^5=\Sigma^2\gamma^0=\sigma^2\gamma^0$, and employing the complex conjugation operator K, we find from Eq. (7.53) that

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=K[\bar{u}(\mathbf{k}',j)^*\sigma^2(1)\gamma^0(1)\bar{u}(\mathbf{p}',m)^*\sigma^2(2)\gamma^0(2)\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma(n)\})^\dagger$$
$$\times\gamma^0(1)\sigma^2(1)u(\mathbf{k},i)^*\gamma^0(2)\sigma^2(2)u(\mathbf{p},l)^*]K \quad (7.54a)$$
$$=\tau[\bar{u}(-\mathbf{k}',j)\gamma^0(1)\bar{u}(-\mathbf{p}',m)\gamma^0(2)\hat{T}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\Gamma(n)\})^\dagger$$
$$\times\gamma^0(1)u(-\mathbf{k},i)\gamma^0(2)u(-\mathbf{p},l)]\tau^{-1}\ , \quad (7.54b)$$

where we have employed the two-particle Pauli-space time-reversal operator

$$\tau=\sigma^2(1)\sigma^2(2)K\ . \quad (7.55)$$

Equation (7.54b) may also be written as

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=\tau[\bar{u}(-\mathbf{k},i)\bar{u}(-\mathbf{p},l)\hat{T}(\tilde{k},\tilde{p};\tilde{k}'\tilde{p}';\{\Gamma(n)\})u(-\mathbf{k}',j)u(-\mathbf{p}',m)]^\dagger\tau^{-1} \quad (7.56a)$$
$$=\tau M_{ij}^{lm}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{\sigma(n)\})^\dagger\tau^{-1}\ . \quad (7.56b)$$

Upon performing the τ operation, Eqs. (7.56b) yield a constraint reminiscent of the form of the usual nonrelativistic result:

$$M_{ji}^{ml}(k',p';k,p;\{\sigma(n)\})=M_{ij}^{lm}(\tilde{k},\tilde{p};\tilde{k}',\tilde{p}';\{-\sigma(n)\})\ . \quad (7.57)$$

Equation (7.57) is the time-reversal (reciprocity) constraint upon the M_{ij}^{lm}. It provides a constraint upon the operator form of the four diagonal elements, M_{ii}^{ll}. For the other 12 (nondiagonal, $i\neq j$ and/or $l\neq m$) elements, Eq. (7.57) determines six pairs of elements such that knowledge of one member of the pair yields the other. Thus there are 10 linearly independent ρ-spin sectors of M.

Given the parity [Eq. (7.52) and time-reversal [Eq. (7.57)] constraints upon the spinor matrix elements M_{ji}^{ml} in the 16 ρ-spin channels, we can now obtain an advantageous representation of the M_{ji}^{ml}. The unit matrix and the three Pauli spin matrices span the two-dimensional Pauli spinor space for each of the particles. In view of the fact that the three linearly independent vectors $\mathbf{q}\times\mathbf{K}$, $\mathbf{q}$, and $\mathbf{K}$ span the three space, we can use the set of rotational invariants E_σ (introduced in the preceding subsection and described in Table VI), 1, $\sigma\cdot\mathbf{n}$, $\sigma\cdot\mathbf{q}$, and $\sigma\cdot\mathbf{K}$, to span the spinor space of each particle, where we have used the notation $\mathbf{n}=\mathbf{q}\times\mathbf{K}$. Any (two-particle) spinor-space dependences of the M_{ji}^{ml} can thus be represented by an expansion in terms which are bilinear products of the operators E_σ for each particle: $E_\sigma(1)\times E_\sigma(2)$. Let us denote this set of 16 operators by E_Σ, $E_\Sigma=E_\sigma(1)\times E_\sigma(2)$. Thus, before the imposition of the parity and time-reversal constraints, each of the M_{ji}^{ml} can be expressed in terms of a linear combination of the E_Σ (note that the resulting number of terms is $16^2=256$). However, it is convenient to first subdivide the set E_Σ according to the parity and

TABLE VII. Decomposition of the set E_Σ of two-particle Pauli mixed scalars into subsets $E_\Sigma^{\pi\tau}$ of specific parity (π) and reciprocity (τ) symmetry. The subset E_Σ^{+-}, for example, consists of the parity even and reciprocity odd constituents of E_Σ. Each of the $E_\Sigma^{\pi\tau}$ is further subdivided into its particle-label exchange symmetric (s) and antisymmetric (a) parts using the notation of Eq. (6.16).

E_Σ^{++}	E_Σ^{+-}	E_Σ^{-+}	E_Σ^{--}
$E_\Sigma^{++}(s)$:	$E_\Sigma^{+-}(s)$:	$E_\Sigma^{-+}(s)$:	$E_\Sigma^{--}(s)$:
1	$[\mathbf{q}\cdot\sigma\,\sigma\cdot\mathbf{K}]^+$	$\frac{1}{2}[\sigma(1)-\sigma(2)]\cdot\mathbf{K}$	$\frac{1}{2}[\sigma(1)-\sigma(2)]\cdot\mathbf{q}$
$\sigma(1)\cdot\mathbf{n}\,\sigma(2)\cdot\mathbf{n}$		$[\mathbf{K}\cdot\sigma\,\sigma\cdot\mathbf{n}]^-$	$[\sigma\cdot\mathbf{q}\,\sigma\cdot\mathbf{n}]^-$
$\sigma(1)\cdot\mathbf{q}\,\sigma(2)\cdot\mathbf{q}$	$E_\Sigma^{+-}(a)$:	$E_\Sigma^{-+}(a)$:	$E_\Sigma^{--}(a)$:
$\sigma(1)\cdot\mathbf{K}\,\sigma(2)\cdot\mathbf{K}$		$\frac{1}{2}[\sigma(1)+\sigma(2)]\cdot\mathbf{K}$	$\frac{1}{2}[\sigma(1)+\sigma(2)]\cdot\mathbf{q}$
$\frac{1}{2}[\sigma(1)+\sigma(2)]\cdot\mathbf{n}$	$[\mathbf{q}\cdot\sigma\,\sigma\cdot\mathbf{K}]^-$	$[\mathbf{K}\cdot\sigma\,\sigma\cdot\mathbf{n}]^+$	$[\sigma\cdot\mathbf{q}\,\sigma\cdot\mathbf{n}]^+$
$E_\Sigma^{++}(a)$:			
$\frac{1}{2}[\sigma(1)-\sigma(2)]\cdot\mathbf{n}$			

time-reversal properties of its constituents and, for subsequent use in the identical particle circumstance, according to the particle-label exchange symmetry of the constituents as well. Thus, we split E_Σ into the four sets $E_\Sigma^{\pi\tau}$, where, for example, E_Σ^{+-} consists of those elements of E_Σ which are parity even and reciprocity odd. The results of this decomposition are summarized in Table VII, where we have also subdivided each of the $E_\Sigma^{\pi\tau}$ into its particle-exchange symmetric $E_\Sigma^{\pi\tau}(s)$ and antisymmetric $E_\Sigma^{\pi\tau}(a)$ parts for later use. The decomposition described in Table VII follows directly from the results of Table VI and the use of the (particle-exchange symmetrization) notation of Eq. (6.16).

Thus, before the imposition of the constraints which arise from the discrete symmetries, the general form of each of the M_{ji}^{ml} may be written

$$M_{ji}^{ml}=\sum_{I_\sigma\in E_\Sigma}^{(16)}(F_\sigma+q\cdot K\,G_\sigma+q\cdot\omega\,H_\sigma)I_\sigma\ , \tag{7.58}$$

where the F_σ, G_σ, and H_σ are arbitrary functions of the time-reversal and parity even momentum-space (Lorentz) scalars: s, t, u, $K\cdot\omega$, $(q\cdot K)^2$, $(q\cdot\omega)^2$, and $q\cdot Kq\cdot\omega$ [see, e.g., Eq. (7.7) or (7.41)]. In Eq. (7.58) we have manifested the possible linear dependences upon the time-reversal odd (but parity even) momentum-space scalars $q\cdot K$ and $q\cdot\omega$ (see Sec. IV and Table I). Because of the character of the parity [Eq. (7.52)] and time-reversal [Eq. (7.57)] constraints, there are three cases to consider.

(1) The four operators M_{ii}^{ll}. In this case, the operators must be both parity and reciprocity even, so that we see from Table VII that Eq. (7.58) reduces to

$$M_{ii}^{ll}=\sum_{I_\sigma\in E_\Sigma^{++}}^{(6)}F_\sigma I_\sigma+\sum_{I_\sigma\in E_\Sigma^{+-}}^{(2)}(q\cdot K\,G_\sigma+q\cdot\omega\,H_\sigma)I_\sigma\ . \tag{7.59a}$$

(2) The four independent operators M_{ji}^{ll} $(i\neq j)$ and M_{ii}^{ml} $(m\neq l)$. In this case the operators must be parity odd; the reciprocity constraint reduces the number of independent operators of this type from eight to four. Making use of Table VII, we obtain the reduction of Eq. (7.58):

$$M_{ji}^{ml}=\sum_{I_\sigma\in E_\Sigma^{-+},E_\Sigma^{--}}^{(8)}(F_\sigma+q\cdot K\,G_\sigma+q\cdot\omega\,H_\sigma)I_\sigma\ . \tag{7.59b}$$

(3) The two independent operators M_{ji}^{ml} $(i\neq j,\ l\neq m)$. In this case the operators must be parity even; the reciprocity constraint reduces the number of independent operators of this type from four to two. Making use of Table VII, we obtain, from Eq. (7.58),

$$M_{ji}^{ml}=\sum_{I_\sigma\in E_\Sigma^{++},E_\Sigma^{+-}}^{(8)}(F_\sigma+q\cdot K\,G_\sigma+q\cdot\omega\,H_\sigma)I_\sigma\ . \tag{7.59c}$$

From Eqs. (7.59) we see that 184 independent true scalar amplitude functions must be specified in the fully off-mass-shell circumstance. This agrees with the number required to specify the Lorentz invariant form of $\hat{T}$ constructed in Eq. (4.7). With one particle on mass shell, $q\cdot K$ and $q\cdot\omega$ are no longer independent (see Sec. IV) and the number of amplitude functions in Eqs. (7.59) is reduced to 128, in agreement with the number needed in the construction of Eq. (5.14). If both particles are on mass shell, then $q\cdot K$ and $q\cdot\omega$ vanish and Eqs. (7.59) require only 72 coefficient functions, in agreement with the number needed in the covariant construction of Eq. (5.15). From these results it is evident that Eqs. (7.59) identify a minimal set of true scalar amplitudes necessary to describe the operator $\hat{T}$. It is also apparent that Eqs. (7.59) identify the minimal number of linearly independent true scalar operators which are necessary in forming an operator $\hat{T}_{ji}^{ml}$ to be fitted to the M_{ji}^{ml} (in each ρ-spin channel). In cases [(1),(2),(3)] the number of invariant (operator) constituents of the operator $\hat{T}_{ji}^{ml}$ which are necessary is [10,24,24] in the fully off-mass-shell circumstance (rather than the full 184 term operator form of $\hat{T}$), and [8,16,16] when one particle is on mass shell (rather than the full 128 term expansion of $\hat{T}$), and only [6,8,8] when both particles are on mass shell (rather than the full 72 term expansion of $\hat{T}$). Thus, Eqs. (7.59) reveal drastic simplifications in the invariant operator forms needed for constructing the $\hat{T}_{ji}^{ml}$ in each of the ρ-spin channels. Equations (7.59) also suggest the manner in which the total number of operators employed in each ρ-spin sector should be apportioned between terms with and without linear dependences upon the time-reversal odd momentum-space scalars, just as in the spin-$\frac{1}{2}$-spin-0 case. Otherwise, no further constraint upon the invariant operator form of the $\hat{T}_{ji}^{ml}$ is present, except, of course, that they should be linearly independent and nonvanishing on the projected space where they are employed. A specific procedure which obviates questions of linear independence and stability is derived in Sec. VIII. This scheme also provides a vehicle for examining the characteristics and behavior of procedures such as those outlined above. If one were to treat the quantities in parentheses in Eqs. (7.59) as single amplitudes that have no particular time-reversal symmetry, then 80 of such amplitudes (eight for each of the 10 independent ρ-spin sectors) are needed to reconstruct $\hat{T}$ off the mass shell.

In order to obtain the restriction of these results to the special circumstance wherein the two spin-$\frac{1}{2}$ particles are indistinguishable, it is first necessary to obtain the permutation symmetries of the Pauli-space operators M_{ji}^{ml}. In view of the particle-exchange symmetry Eq. (6.8) of the full scattering operator $\tilde{T}$, it is evident that the projected operators $\tilde{T}_{ji}^{ml}$ of Eq. (7.44) satisfy

$$E_{12}\tilde{T}_{ji}^{ml}E_{12}=\tilde{T}_{ml}^{ji}\ , \tag{7.60}$$

so that we may expect the implications of particle identity to result in restrictions upon the form of the operators for $(m,l=j,i)$ and relationships among the different operators for $(m,l\neq j,i)$, in general. To deduce the precise statement of the particle-label symmetry obeyed by the M_{ji}^{ml}, we define Pauli spinor matrix elements of these operators by [see Eqs. (2.1)]

$$X_{ji}^{ml}(k',p';k,p)=\chi_{s'}^\dagger(1)\chi_{\sigma'}^\dagger(2)M_{ji}^{ml}(k',p';k,p;\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2))\times\chi_s(1)\chi_\sigma(2)\ . \tag{7.61}$$

If we now employ the particle-label symmetry relation (6.15) obeyed by the spinor-space scattering operator $\hat{T}$ together with the decomposition of the full two-particle exchange operator E_{12}, Eq. (6.23), to obtain

$$\hat{T}(k',p';k,p;\Gamma(1),\Gamma(2)) = E^{\gamma}_{12}\hat{T}(p',k';p,k;\Gamma(1),\Gamma(2))E^{\gamma}_{12} , \quad (7.62)$$

then we may employ this expression for $\hat{T}$ in the definition Eq. (7.47) of the M^{ml}_{ji} to obtain a slightly modified form of that equation. Inserting this result into Eq. (7.61) and making use of Eq. (6.24) for the (full, Dirac) two-particle spinor exchange property of E^{γ}_{12} yields

$$X^{ml}_{ji}(k',p';k,p) = \chi^{\dagger}_{\sigma'}(1)\chi^{\dagger}_{s'}(2)M^{ji}_{ml}(p',k';p,k;\sigma(1),\sigma(2)) \times \chi_{\sigma}(1)\chi_{s}(2) . \quad (7.63)$$

Use of the Pauli-space spin-exchange operator

$$E^{\sigma}_{12} = \frac{1+\sigma(1)\cdot\sigma(2)}{2} \quad (7.64)$$

in Eq. (7.63) yields

$$X^{ml}_{ji}(k',p';k,p) = \chi^{\dagger}_{s'}(1)\chi^{\dagger}_{\sigma'}(2)E^{\sigma}_{12} \times M^{ji}_{ml}(p',k';p,k;\sigma(1),\sigma(2)) \times E^{\sigma}_{12}\chi_{s}(1)\chi_{\sigma}(2) , \quad (7.65)$$

or, making use of the completeness of the σ (to note that the M^{ml}_{ji} contain terms, at most, linear in the σ for each particle) to perform the E^{σ}_{12} operation,

$$X^{ml}_{ji}(k',p';k,p) = \chi^{\dagger}_{s'}(1)\chi^{\dagger}_{\sigma'}(2)M^{ji}_{ml}(p',k';p,k;\sigma(2),\sigma(1)) \times \chi_{s}(1)\chi_{\sigma}(2) . \quad (7.66)$$

Since Eq. (7.66) is valid for arbitrary choices of the s,s',σ,σ' in the spinors, comparison of Eqs. (7.66) and (7.61) yields, for the permutation symmetry of the M^{ml}_{ji},

$$M^{ml}_{ji}(k',p';k,p;\sigma(1),\sigma(2)) = M^{ji}_{ml}(p',k';p,k;\sigma(2),\sigma(1)) . \quad (7.67)$$

As expected, this result provides a constraint upon the form of the M^{ji}_{ml} for $(i,j)=(l,m)$, whereas for $(i,j)\neq(l,m)$ it determines one of the operators M^{ji}_{ml} and M^{ml}_{ji} from the other.

We note that two of the seven momentum-space scalar arguments of the coefficient functions in Eqs. (7.59) ($K\cdot\omega$ and $q\cdot Kq\cdot\omega$; we refer to these as the set MSS2, see Sec. VI A) are particle-label exchange, E_{12}, odd. Thus, in considering Pauli symmetries it is advantageous to divide the functional dependence of the coefficient functions into an arbitrary dependence upon the set of seven (particle-exchange even) variables: s, t, u, $(K\cdot\omega)^2$, $(q\cdot K)^2$, $(q\cdot\omega)^2$, and $q\cdot Kq\cdot\omega K\cdot\omega$ (we refer to these as MSS1, see Sec. VI A) together with possible linear dependences upon the MSS2.

We can now obtain the reduction of Eqs. (7.59) in the identical particle circumstance. We again treat the three cases of Eqs. (7.59) separately. In the following, the arbitrary coefficient functions F_σ, G_σ, and H_σ are functions of the (Lorentz scalar) parity, time-reversal and particle-exchange even variables, MSS1. We also recall from Table II that the time-reversal odd scalars, $q\cdot K$ and $q\cdot\omega$, are even and odd, respectively, under particle-exchange.

(1a) The two operators M^{++}_{++} and M^{--}_{--}. In this case, Eqs. (7.67) requires that the operators be symmetric under particle exchange. We find from Eq. (7.59a) that

$$M^{ii}_{ii} = \sum^{(5)}_{I_\sigma\in E^{++}_{\Sigma}(s)} F_\sigma I_\sigma + \sum^{(1)}_{I_\sigma\in E^{++}_{\Sigma}(a)} (K\cdot\omega\, G_\sigma + q\cdot Kq\cdot\omega\, H_\sigma)I_\sigma + \sum^{(1)}_{I_\sigma\in E^{+-}_{\Sigma}(s)} (q\cdot K\, G_\sigma + q\cdot\omega K\cdot\omega\, H_\sigma)I_\sigma + \sum^{(1)}_{I_\sigma\in E^{+-}_{\Sigma}(a)} (q\cdot KK\cdot\omega\, G_\sigma + q\cdot\omega\, H_\sigma)I_\sigma . \quad (7.68a)$$

(1b) The one independent operator of the pair M^{++}_{--} and M^{--}_{++}; knowledge of one of the pair determines the other via Eq. (7.67). We find

$$M^{ll}_{ii} = \sum^{(6)}_{I_\sigma\in E^{++}_{\Sigma}} (F_\sigma + K\cdot\omega\, G_\sigma + q\cdot Kq\cdot\omega\, H_\sigma)I_\sigma + \sum^{(2)}_{I_\sigma\in E^{+-}_{\Sigma}} (q\cdot K\, G_\sigma + q\cdot KK\cdot\omega\, G'_\sigma + q\cdot\omega\, H_\sigma + q\cdot\omega K\cdot\omega\, H'_\sigma)I_\sigma . \quad (7.68b)$$

(2) The two independent operators M^{ll}_{ji} $(i\neq j)$; the operators M^{ji}_{ll} $(i\neq j)$ are determined from these via Eq. (7.67). We find

$$M^{ll}_{ji} = \sum^{(8)}_{I_\sigma\in E^{\pm}_{\Sigma}} (F_\sigma + K\cdot\omega\, F'_\sigma + q\cdot Kq\cdot\omega\, F''_\sigma + q\cdot K\, G_\sigma + q\cdot KK\cdot\omega\, G'_\sigma + q\cdot\omega\, H_\sigma + q\cdot\omega K\cdot\omega H'_\sigma)I_\sigma . \quad (7.68c)$$

(3a) The one independent operator M^{ij}_{ij} $(i\neq j)$; the operator M^{ji}_{ji} is determined from M^{ij}_{ij} via the time-reversal constraint (7.57). In this case Eq. (7.67) requires particle-label symmetry and from Eq. (7.59c) we find

$$M^{ij}_{ij} = \sum^{(6)}_{I_\sigma\in E^{\pm\pm}_{\Sigma}(s)} (F_\sigma + q\cdot K\, G_\sigma + q\cdot\omega K\cdot\omega\, H_\sigma)I_\sigma + \sum^{(2)}_{I_\sigma\in E^{\pm\pm}_{\Sigma}(a)} (K\cdot\omega\, F_\sigma + q\cdot Kq\cdot\omega F'_\sigma + q\cdot KK\cdot\omega\, G_\sigma + q\cdot\omega\, H_\sigma)I_\sigma . \quad (7.68d)$$

(3b) The one independent operator M_{ji}^{ij} $(i \neq j)$; knowledge of this operator determines the operator M_{ij}^{ji} by either the time-reversal constraint (7.57) or the particle-exchange constraint (7.67). Furthermore, combination of these two constraints yields a constraint upon the form of M_{ji}^{ij}, namely the requirement that M_{ji}^{ij} $(i \neq j)$ be symmetric under the product of the time-reversal and particle-exchange transformations. Employing this requirement, we obtain, from Eq. (7.59c),

$$M_{ji}^{ij} = \sum_{I_\sigma \in E_\Sigma^{++}(s), E_\Sigma^{+-}(a)}^{(6)} (F_\sigma + q\cdot K K\cdot\omega\, G_\sigma + q\cdot\omega\, H_\sigma) I_\sigma + \sum_{I_\sigma \in E_\Sigma^{++}(a), E_\Sigma^{+-}(s)}^{(2)} (K\cdot\omega\, F_\sigma + q\cdot K q\cdot\omega\, F'_\sigma + q\cdot K\, G_\sigma + q\cdot\omega K\cdot\omega\, H_\sigma) I_\sigma \ . \qquad (7.68e)$$

Thus, in the identical-particle case the number of linearly independent ρ-spin sectors of M is reduced from 10 to seven (in the ρ-spin product representation).

It is easily verified from Eqs. (7.68) that the total number of true scalar amplitudes in the off-mass-shell case is 212, while the on-mass-shell circumstance requires only 44. These numbers are the same as the number of amplitudes required by the covariant constructions of Eqs. (4.19) and (4.20), respectively. Thus, it is clear that we have identified the minimal number of terms required to construct acceptable forms for the independent $\hat{T}_{ji}^{ml}$, and thus to reconstruct $\hat{T}$. Equations (7.68) also determine the minimal number of linearly independent true scalar operators which must be employed to properly represent the $\hat{T}_{ji}^{ml}$ in each of the ρ-spin sectors. In cases [(1a),(1b),(2),(3a),(3b)] the number of linearly independent operator constituents of $\hat{T}_{ji}^{ml}$ must be [11,26,56,26,26] in the off-mass-shell circumstance and [5,6,8,6,6] in the on-mass-shell circumstance. These represent the minimal number of true scalar operators for each case. As before, the quantities in parentheses in Eqs. (7.68) could be treated as single amplitudes that are not, in general, scalars with respect to time-reversal or particle label exchange. In the off-mass-shell case it is clear that 56 such amplitudes (eight for each of the seven independent ρ-spin sector) can describe the full operator $\hat{T}$. This makes contact with the number of amplitudes employed in Ref. 6.

We note from Eq. (7.68a) that for M_{++}^{++} the 11 invariant terms that are present off the mass shell reduce to five on the mass shell. This is a relativistic generalization of the well-known circumstance[1] that in the Wolfenstein representation of Galilean-invariant nonrelativistic scattering operators for two identical fermions there are six Pauli invariants off the energy shell which reduce to five on the energy shell. These five on-shell Pauli invariants are given by the set $E_\Sigma^{++}(s)$ that enter the first term of Eq. (7.68a), and are listed in Table VII. Of the six purely off-mass-shell invariants in Eq. (7.68a), only the first term in the third summation has a nonrelativistic counterpart. To see this, we refer to Fig. 2, and note that

$$\omega\cdot K = \sqrt{s}\,(\tau'_0 + \tau_0) \ ,$$

$$\omega\cdot q = \sqrt{s}\,(\tau'_0 - \tau_0) \ ,$$

and

$$q\cdot K = (\tau_0'^2 - \tau_0^2) - \mathbf{q}\cdot\mathbf{K} \ .$$

In a nonrelativistic treatment there is no counterpart to the variables τ'_0 and τ_0. In the nonrelativistic limit $\tau'_0 \to 0$ and $\tau_0 \to 0$, so that the only momentum-space scalar available is $\mathbf{q}\cdot\mathbf{K}$. Thus only the first term of the third summation survives the passage to the nonrelativistic treatment and it becomes the Wolfenstein purely off energy shell invariant given by

$$\mathbf{q}\cdot\mathbf{K}[\sigma(1)\cdot\mathbf{q}\,\sigma(2)\cdot\mathbf{K} + \sigma(2)\cdot\mathbf{q}\,\sigma(1)\cdot\mathbf{K}] \ . \qquad (7.69)$$

The other five off-mass-shell invariants in Eq. (7.68a) are purely relativistic terms. Thus for the nucleon-nucleon scattering operator in the sector where all the Dirac spinors are positive energy spinors, there are as many purely off-shell amplitudes beyond the nonrelativistic treatment as there are on-shell amplitudes that can be determined from data. When nucleon-nucleon scattering takes place in the environment of a nucleus, the purely off-mass-shell two-body amplitudes will, in general, contribute. We note that recently the off-mass-shell character of bound nucleons has been shown to be a very possible explanation of the results from the European Muon Collaboration experiment—known as the EMC effect.[24]

To summarize this section, we have developed a general framework for the reconstruction of scattering operators from barycentric matrix elements for the spin-$\frac{1}{2}$-spin-0, nonidentical spin-$\frac{1}{2}$-spin-$\frac{1}{2}$, and identical spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ systems. In all cases, reconstruction of the full Lorentz scalar scattering operators is achieved in terms of invariant operator forms which are used and fitted in each particular ρ-spin channel. The Pauli-space operator forms of the different ρ-spin projections of the scattering operators

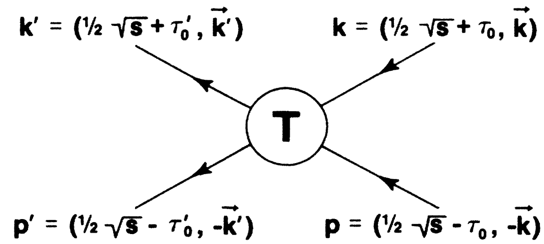

FIG. 2. Schematic representation of the scattering process in the barycentric frame. The four momenta are expressed in terms of the nine independent quantities that exist in that frame for the general off-mass-shell case.

provide a convenient intermediate representation of the known matrix elements; these might be obtained as the first step in a fitting procedure. More importantly, this representation of the operators reveals alternatives and simplifications in the invariant operator forms to be employed in each ρ-spin channel. Our results not only identify the number of operators required in each ρ-spin channel, they also indicate the manner in which these are to be apportioned between terms with and without linear dependences upon certain momentum-space scalars. The outcome is a generalization of the method recently employed on the mass shell to reconstruct the NN scattering operator.[6] Projectors which are consistent with the convenient free-particle basis states are developed in preference to undertaking a change of basis for every supplied matrix element. Considerable flexibility in the reconstruction process, particularly in the choice of the invariant operator forms to be employed in each of the ρ-spin channels, remains. In the next section we develop a specific reconstruction scheme in which there is a unique association between the Pauli spin structure of the ρ-spin sectors and operators in Dirac spinor space. This scheme obviates questions of linear dependence (kinematic singularities) as well as of other instabilities. It also provides a means for assessing the properties of alternative procedures, such as those which follow the general framework detailed above.

VIII. COVARIANT EXTENSION OF MATRIX ELEMENTS

The general framework developed in the preceding section for the reconstruction of the scattering operator from a given complete set of matrix elements is ambiguous in that the form of the ρ-spin projected operators $\hat{T}_{ji}$ and $\hat{T}_{ji}^{ml}$ is not fully determined. Such flexibility allows the possibility of employing a number of different assumed forms for these operators, and thus may permit "convenient" choices. However, a number of important issues are also raised by this circumstance. The linear independence of the chosen set must be guaranteed in order to remove the possibility of the inadvertent introduction of kinematic singularities (see the discussion below) and it is not yet clear how this is to be accomplished. Furthermore, some choices of the set of invariant operator forms to be employed in a given ρ-spin sector are likely to be better adapted for practical use than others. Since the number of available invariant operators is vastly larger (see Sec. VII) than the number needed in a particular ρ-spin sector, it is clear that some choices will inadvertently depend upon large (implicit) cancellations (that can occur when ρ-spin projections are carried out) in order to properly represent the underlying linearly independent "natural" operator forms. This may appreciably affect the accuracy and stability of both the fitting procedure and subsequent approximations. These issues have been somewhat clarified in the preceding section, where the choice of invariants in each ρ-spin sector was subdivided according to certain essential kinematical dependences. In this section we find a clear and complete resolution of these issues which also provides an avenue for investigation of the alternative reconstruction procedures.

As described in the preceding section, the given M matrix elements in each ρ-spin sector are Pauli-space operators defined in one Lorentz frame (here taken to be the barycentric frame) and they must be converted or related to (Dirac-space) Lorentz covariant γ-matrix operator forms in order to obtain the full scattering operator in covariant form. It is necessary to determine how to select a subset of the many available Dirac spinor operators for use in the representation of the M in each of the ρ-spin sectors. A set of Dirac spinor operators which are linearly independent on the full space may, in general, become linearly dependent when projected onto a particular ρ-spin sector. Such induced linear dependences are the source of the so-called kinematic singularities which often arise in the covariant representations of scattering amplitudes.[25] A zero caused by the lack of linear independence of the projected operators can induce a divergence in the associated amplitudes in an "attempt" to maintain linear independence. The occurrence of kinematic singularities thus simply signifies a poor choice of Dirac invariants to be employed in a particular ρ-spin sector. To ensure the absence of such behavior, it is sufficient to find the (unique) direct mapping which connects each of the Pauli-space rotationally invariant operators in each ρ-spin sector with an appropriate Dirac-space Lorentz invariant operator on the full space. We develop such a mapping in this section. This information can then be used to specify the Dirac invariant operator forms which should be selected for use as the operator constituents of $\hat{T}_{ji}$ of Eq. (7.25b) and $\hat{T}_{ji}^{ml}$ of Eq. (7.46) (which are to be fitted to the supplied matrix elements M_{ji} and M_{ji}^{ml}). However, the analysis supplied in this section also constitutes a more direct technique for reconstruction of the covariant form of the scattering operator on the full Dirac space. The projected operators $\hat{T}_{ji}$ and $\hat{T}_{ji}^{ml}$ are shown to be directly obtainable from the *covariant extensions* of the Pauli-space operators M_{ji} and M_{ji}^{ml} into Dirac-space operators. We illustrate the method for the spin-$\frac{1}{2}$-spin-0 system first, and then extend it to the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system.

A. Spin $\frac{1}{2}$-spin 0

From Eqs. (7.23b) and (7.26) we have the scattering operator in the form $\hat{T}=\sum_{ij}\hat{T}_{ji}$, where, in the barycentric frame,

$$\hat{T}_{ji}(k',p';k,p;\{\Gamma\})=\gamma^0 u(\mathbf{k}',j)M_{ji}(k',p';k,p)u(\mathbf{k},i)^{\dagger}\ . \tag{8.1}$$

Note that the projectors onto ρ-spin sectors j and i are effectively built into this expression. We convert this expression to covariant form by representing each spinor in terms of the special Lorentz boost applied to the appropriate spinor for a particle at rest.[14] That is, we employ

$$u(\mathbf{k},\pm)=\left[\frac{m}{E_k}\right]^{1/2}S(\pm\mathbf{k})u(\mathbf{0},\pm)\ , \tag{8.2}$$

where the special boost operator is given by[16]

$$S(\mathbf{k}) = \exp\left[\frac{u}{2}\frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{|\mathbf{k}|}\right] \tag{8.3}$$

$$= \left[\frac{E_k+m}{2m}\right]^{1/2}\left[1+\frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{E_k+m}\right], \tag{8.4}$$

where $\tanh u = |\mathbf{k}|/E_k$. It is convenient to employ the notation $S_i(\mathbf{k})$, where, for each ρ-spin index $i=+$ or $-$, the definition is

$$S_+(\mathbf{k}) = S(\mathbf{k}), \quad S_-(\mathbf{k}) = S(-\mathbf{k}) = S^{-1}(\mathbf{k}) . \tag{8.5}$$

Since $S^\dagger = S$ and $\gamma^0 S\gamma^0 = S^{-1}$, Eq. (8.1) can now be expressed in the form

$$\hat{T}_{ji} = S_j^{-1}(\mathbf{k}')\hat{M}_{ji}(k',p';k,p)S_i(\mathbf{k}) , \tag{8.6}$$

where we have introduced

$$\hat{M}_{ji} = \left[\frac{m^2}{E_{k'}E_k}\right]^{1/2}\gamma^0 u(\mathbf{0},j)M_{ji}(k',p';k,p)u(\mathbf{0},i)^\dagger \tag{8.7}$$

which is a Dirac spinor operator constructed directly out of the Pauli spinor operator M_{ji}. One interpretation of Eq. (8.6) is that, for each ρ-spin sector, the constituent scattering operator may be expressed as the product of two Lorentz boosts and a scattering operator $\hat{M}_{ji}$ for "at-rest" spinors. The first boost converts the incident spinor from its value in the specified frame to its value at rest, then $\hat{M}_{ji}$ describes the scattering for at-rest spinors, and finally the second boost converts the scattered at-rest spinor to its value in the specified frame. The reason that this particular factorization is preferred over the equivalent form given in Eq. (8.1) is the ease with which each of the factors in Eq. (8.6) may be cast into covariant form.

As might have been anticipated for the two-body system under consideration, the ingredients needed to construct the covariant extension[14] of the boost operator $S(\mathbf{k})$ have already been obtained in our construction of covariant projectors in Sec. VII. In fact, if we rewrite Eq. (8.4) in the equivalent form

$$S(\mathbf{k}) = \left[2\left[1+\frac{E_k}{m}\right]\right]^{-1/2}(1+\gamma\cdot\hat{k}\,\gamma^0) , \tag{8.8}$$

where the unit four vector $\hat{k} = (1/m)(E_k,\mathbf{k})$, then it is evident that covariant extensions of the barycentric quantity m/E_k and γ^0 are all that is required. These are given in Eq. (7.30) and in the discussion preceding that equation, respectively. Thus we find that the covariant boost, $B(\mathbf{k})$, is given by

$$B(\mathbf{k}) = \{2[1+R(|\mathbf{k}|)^{-1}]\}^{-1/2}\left[1+\gamma\cdot\hat{k}\frac{\gamma\cdot\omega}{\sqrt{s}}\right], \tag{8.9a}$$

while the covariant extension of the inverse boost $S^{-1}(\mathbf{k})$ is

$$B^{-1}(\mathbf{k}) = \{2[1+R(|-\mathbf{k}|)^{-1}]\}^{-1/2}\left[1+\frac{\gamma\cdot\omega}{\sqrt{s}}\gamma\cdot\hat{k}\right], \tag{8.9b}$$

where the covariant quantity R is as defined in Eq. (7.30).

With the notation $B_i(\mathbf{k})$, $i=+$ or $-$, and the definitions

$$B_+(\mathbf{k}) = B(\mathbf{k}), \quad B_-(\mathbf{k}) = B(-\mathbf{k}) = B^{-1}(\mathbf{k}) , \tag{8.10}$$

we have, in place of Eq. (8.6), the equivalent form

$$\hat{T}_{ji} = B_j^{-1}(\mathbf{k}')\hat{M}_{ji}(k',p';k,p)B_i(\mathbf{k}) . \tag{8.11}$$

If the Dirac spinor operator $\hat{M}_{ji}$, which is defined in the barycentric frame, can now be assigned a covariant extension, then a covariant form for $\hat{T}_{ji}$ (and hence also for the full scattering operator $\hat{T} = \sum_{ij}\hat{T}_{ji}$) is assured. Furthermore, this constructive procedure clearly avoids the introduction of kinematic singularities since it maps well-behaved Pauli-space operators directly to their covariant Dirac-space extensions.

Explicit evaluation of the direct product of the spinors in Eq. (8.7) for $\hat{M}_{ji}$ leads to the results

$$\hat{M}_{++} = h(\gamma^0+1)M_{++}(k',p';k,p) , \tag{8.12a}$$

$$\hat{M}_{+-} = h(\gamma^0+1)\gamma^5 M_{+-}(k',p';k,p) , \tag{8.12b}$$

$$\hat{M}_{-+} = h(\gamma^0-1)\gamma^5 M_{-+}(k',p';k,p) , \tag{8.12c}$$

$$\hat{M}_{--} = h(\gamma^0-1)M_{--}(k',p';k,p) , \tag{8.12d}$$

where

$$h = \frac{1}{2}\left[\frac{m^2}{E_{k'}E_k}\right]^{1/2} \equiv \tfrac{1}{2}[R(|\mathbf{k}'|)R(|\mathbf{k}|)]^{1/2} . \tag{8.13}$$

The Pauli spin structure of the operators M_{ji}, which is given in Eq. (7.42), may now be combined with the γ matrices evident in Eqs. (8.12) in order to produce a γ-matrix representation of the $\hat{M}_{ji}$. In terms of the Dirac spin operator $\boldsymbol{\Sigma} = (\sigma^{23},\sigma^{31},\sigma^{12})$, the results are

$$\hat{M}_{++} = f_1(\gamma^0+1)+f_2(\gamma^0+1)\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K}) , \tag{8.14a}$$

$$\hat{M}_{--} = f_3(\gamma^0-1)+f_4(\gamma^0-1)\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K}) , \tag{8.14b}$$

$$\hat{M}_{+-} = (f_5+f_6 q\cdot K+f_7 q\cdot\omega)(\gamma^0+1)\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{q} + (f_8+f_9 q\cdot K+f_{10}q\cdot\omega)(\gamma^0+1)\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{K} , \tag{8.14c}$$

where the ten scalar amplitudes f_i, $i=1-10$, are related to the amplitudes of Eq. (7.42) by multiplication by the factor h. The Dirac operator $\hat{M}_{-+}$ is not independent, but is given by Eq. (8.12c) with the Pauli operator M_{-+} determined from M_{+-} by the reciprocity relation of Eq. (7.40). Thus we have

$$\hat{M}_{-+} = -(f_5-f_6 q\cdot K-f_7 q\cdot\omega)(\gamma^0-1)\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{q} + (f_8-f_9 q\cdot K-f_{10}q\cdot\omega)(\gamma^0-1)\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{K} . \tag{8.14d}$$

The quantities in Eq. (8.14) are defined in the barycentric frame. It remains only to obtain covariant expressions which have the same value in that frame and then to employ these to produce covariant versions of Eq. (8.11). In Eqs. (8.14) there are eight linearly independent Dirac matrix operators and they have even parity. The eight are 1, γ^0, $\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K})$, $\gamma^0\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K})$, $\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{q}$, $\gamma^0\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{q}$, $\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{K}$, and $\gamma^0\gamma^5\boldsymbol{\Sigma}\cdot\mathbf{K}$. We recall from Table III that there are also eight linearly independent one-body mixed (Lorentz) scalars that have even parity. Hence, in the barycentric frame, there must be a unique linear transformation which maps one of these sets of matrices into the other. We employ this fact to obtain the covariant extension of the $\hat{M}_{ji}$ in terms of the one-body mixed scalars.

The kinematics of the scattering process in the barycentric frame is illustrated in Fig. 2. With τ_0 and τ_0' denoting the relative energies in the initial and final channels, the particular form of the momentum variables can be taken as

$$q=(\tau_0'-\tau_0,\mathbf{q}),\quad K=(\tau_0'+\tau_0,\mathbf{K}),$$
$$\omega=(\sqrt{s},\mathbf{0}),\quad \xi=(0,\sqrt{s}\,\mathbf{q}\times\mathbf{K}). \tag{8.15}$$

The spin structure evident in Eqs. (8.14) originates from a Pauli spin representation of matrix elements and is thus described by scalars formed out of the three vectors $\mathbf{\Sigma}$, $\mathbf{q}$, and $\mathbf{K}$. The energy components of the four momenta are not involved in the formation of these barycentric frame quantities (which are rotational scalars). This separate treatment of energy and three momentum suggests that the four momenta whose values in the barycentric frame are $K'=(0,\mathbf{K})$ and $q'=(0,\mathbf{q})$ are most efficient for describing the covariant extensions of the mixed scalars of Eqs. (8.14). In the barycentric frame these are clearly orthogonal to the total four momentum ω; that is, $K'\cdot\omega=0=q'\cdot\omega$. The covariant extensions of q' and K' which have the specified value in the barycentric frame and which remain orthogonal to ω in all Lorentz frames are

$$q'=q-\hat{\omega}\,\hat{\omega}\cdot q$$

and (8.16)

$$K'=K-\hat{\omega}\,\hat{\omega}\cdot K\,.$$

We may loosely refer to q' and K' as the covariant extensions of the three momenta $\mathbf{q}$ and $\mathbf{K}$. Since $\mathbf{\Sigma}$ transforms as (part of) a tensor under Lorentz transformations, the covariant extension of $\mathbf{\Sigma}\cdot\mathbf{q}\times\mathbf{K}$ is evidently $\sigma{:}q'K'$. In like manner the eight Dirac operators in Eqs. (8.14) are found to have covariant extensions which are linear combinations of the eight even-parity mixed scalars given in Table III. When the momenta q' and K' are used instead of q and K, the correspondence is diagonal. These results are (excluding the unit operator)

$$\gamma^0=\frac{\gamma\cdot\omega}{\sqrt{s}}\,, \tag{8.17a}$$

$$\mathbf{\Sigma}\cdot\mathbf{q}\times\mathbf{K}=\sigma{:}q'K'=\sigma{:}qK-\frac{\omega\cdot K}{s}\sigma{:}q\omega-\frac{\omega\cdot q}{s}\sigma{:}\omega K\,, \tag{8.17b}$$

$$\gamma^0\mathbf{\Sigma}\cdot\mathbf{q}\times\mathbf{K}=\frac{\gamma^5\gamma\cdot\xi}{\sqrt{s}}\,, \tag{8.17c}$$

$$\gamma^0\gamma^5\mathbf{\Sigma}\cdot\mathbf{q}=-\gamma\cdot q'=\frac{\omega\cdot q}{s}\gamma\cdot\omega-\gamma\cdot q\,, \tag{8.17d}$$

$$\gamma^5\mathbf{\Sigma}\cdot\mathbf{q}=+\frac{i}{\sqrt{s}}\sigma{:}\omega q'=+\frac{i}{\sqrt{s}}\sigma{:}\omega q\,, \tag{8.17e}$$

$$\gamma^0\gamma^5\mathbf{\Sigma}\cdot\mathbf{K}=-\gamma\cdot K'=\frac{\omega\cdot K}{s}\gamma\cdot\omega-\gamma\cdot K\,, \tag{8.17f}$$

$$\gamma^5\mathbf{\Sigma}\cdot\mathbf{K}=\frac{i}{\sqrt{s}}\sigma{:}\omega K'=\frac{i}{\sqrt{s}}\sigma{:}\omega K\,. \tag{8.17g}$$

In the above we also show, where necessary, the expansions in terms of the standard mixed scalars of Table III. The extensions of Eqs. (8.17) hold off, as well as on, the mass shell. When the two particles have the same mass and are both on mass shell, the momenta q' and K' become identical to q and K and there is then a one-to-one correspondence with the standard set of mixed scalars which we employ.

Substitution of these expressions into Eqs. (8.14) produces the desired covariant extensions of the spinor operators $\hat{M}_{ji}$. In particular, the results are

$$\hat{M}_{++}=f_1\left[\frac{\gamma\cdot\omega}{\sqrt{s}}+1\right]+f_2\left[\sigma{:}q'K'+\frac{\gamma^5\gamma\cdot\xi}{\sqrt{s}}\right], \tag{8.18a}$$

$$\hat{M}_{--}=f_3\left[\frac{\gamma\cdot\omega}{\sqrt{s}}-1\right]+f_4\left[-\sigma{:}q'K'+\frac{\gamma^5\gamma\cdot\xi}{\sqrt{s}}\right], \tag{8.18b}$$

$$\hat{M}_{+-}=(f_5+f_6q\cdot K+f_7q\cdot\omega)\left[-\gamma\cdot q'-\frac{i}{\sqrt{s}}\sigma{:}q'\omega\right]$$
$$+(f_8+f_9q\cdot K+f_{10}q\cdot\omega)\left[-\gamma\cdot K'+\frac{i}{\sqrt{s}}\sigma{:}\omega K'\right], \tag{8.18c}$$

$$\hat{M}_{-+}=-(f_5-f_6q\cdot K-f_7q\cdot\omega)\left[-\gamma\cdot q'+\frac{i}{\sqrt{s}}\sigma{:}q'\omega\right]$$
$$+(f_8-f_9q\cdot K-f_{10}q\cdot\omega)\left[-\gamma\cdot K'-\frac{i}{\sqrt{s}}\sigma{:}\omega K'\right]. \tag{8.18d}$$

Use of Eq. (8.11) to attach the covariantly extended boosts, followed by summation over all ρ-spin sectors, produces the manifestly covariant form

$$\hat{T}(k',p';k,p;\{\Gamma\})=\sum_{i,j}B_j^{-1}(\mathbf{k}')\hat{M}_{ji}(k',p';k,p)B_i(\mathbf{k})\,, \tag{8.19}$$

for the reconstructed scattering operator in the spin-$\frac{1}{2}$-spin-0 case. This is our final result for this case.

In order to identify the contributions of the various pieces of the right-hand side of Eq. (8.19) to each of the invariant amplitudes F_i, G_i, and $\overline{G}_i$ appearing in the general form for $\hat{T}$ given in Eq. (4.3b), it is only necessary to expand the product of the three factors in Eq. (8.19) in terms of the mixed Lorentz scalars. The required algebra of the mixed scalars is most easily handled through the trace properties of the γ matrices. This tedious process is straightforward and will not be detailed here.

B. Spin $\frac{1}{2}$-spin $\frac{1}{2}$

The spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case is handled as an extension of the spin-$\frac{1}{2}$-spin-0 case treated in the preceding subsection. The introduction into Eq. (7.49) of "at-rest" spinors for each of the two fermions leads to the expression

$$\hat{T}_{ji}^{ml} = B_j^{-1}(\mathbf{k}')B_m^{-1}(\mathbf{p}')\hat{M}_{ji}^{ml}(k',p';k,p)B_i(\mathbf{k})B_l(\mathbf{p}) , \quad (8.20)$$

for the projected scattering operator in the ρ-spin sector (ji,ml). Here, the scattering operator for "at-rest" spinors has been defined as the spinor-space operator analogous to that of Eq. (8.7),

$$\hat{M}_{ji}^{ml} = 4h^2\gamma^0(1)\gamma^0(2)u(\mathbf{0},j)u(\mathbf{0},m)M_{ji}^{ml}u(\mathbf{0},i)^\dagger u(\mathbf{0},l)^\dagger , \quad (8.21)$$

where the M_{ji}^{ml} are the supplied matrix elements of $\hat{T}$ in the barycentric frame and are defined by Eq. (7.47). In obtaining the above expressions we have made use of the covariantly extended boost operators B defined by Eqs. (8.9), together with the notation displayed in Eq. (8.10). We have also employed the covariantly extended quantity h, defined by Eq. (8.13) as the collection of normalization factors introduced by the action of the special boosts as shown in Eq. (8.2). In order to obtain manifestly covariant forms for the $\hat{T}_{ji}^{ml}$ via Eq. (8.20), and hence for $\hat{T}$ via Eq. (7.46b), it is only necessary to cast the $\hat{M}_{ji}^{ml}$ into manifestly covariant form. To do this, we first introduce (for each particle) the spinor direct products

$$\Theta_{ji}(1) = 2\gamma^0(1)u(\mathbf{0},j)u(\mathbf{0},i)^\dagger , \quad (8.22a)$$

$$\Theta_{ml}(2) = 2\gamma^0(2)u(\mathbf{0},m)u(\mathbf{0},l)^\dagger . \quad (8.22b)$$

The particular values of these operators are

$$\Theta_{++} = \gamma^0 + 1, \quad \Theta_{--} = \gamma^0 - 1 , \quad (8.23a)$$

$$\Theta_{+-} = (\gamma^0+1)\gamma^5; \quad \Theta_{-+} = (\gamma^0-1)\gamma^5 , \quad (8.23b)$$

which we have already implicitly employed in Eq. (8.12). In terms of these quantities the Dirac spinor operators $\hat{M}_{ji}^{ml}$ can be written as

$$\hat{M}_{ji}^{ml}(k',p';k,p) = h^2\Theta_{ji}(1)\Theta_{ml}(2) \times M_{ji}^{ml}(k',p';k,p;\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2)) . \quad (8.24)$$

At this point, the general forms of the Pauli-spin operators M need to be made explicit in order to proceed further with the covariant extension of the $\hat{M}$. These have been detailed in Sec. VII. There, each ρ-spin sector of M is expanded in terms of the 16 rotationally invariant operators that can be constructed in the two-particle Pauli-spinor space. The general form of the expansion is given in Eq. (7.58), and the basis set E_Σ of rotationally invariant operators, which are bilinear in the possible Pauli-space mixed scalars for each particle, is given in Table VII. When such an expansion for M is introduced into the right-hand side of Eq. (8.24), the effect of the factors Θ for each particle is to convert each Pauli-space rotational invariant into a Dirac-space rotational invariant, just as in the preceding subsection. We therefore require the covariant extensions of these Dirac-space rotational invariants into Lorentz invariants. Evidently, if we provide this extension for all possible one-particle Pauli invariants, then the results for any two-particle invariant can be obtained by simply forming the appropriate bilinear products. This, in turn, yields a covariant form for the right-hand side of Eq. (8.24) and the invariant operator $\hat{T}$ via Eqs. (8.20).

In the preceding subsection, we have carried out the covariant extension of the set of eight even-parity Dirac operators that are required to describe the M_{ji} of the spin-$\frac{1}{2}$-spin-0 system. For the present system, the eight odd-parity operators for a single Dirac particle will also be required, since an even-parity two-particle operator can be formed from the direct product of odd-parity operators for each particle. These are obtained shortly.

As an illustration, the covariant extension of the operator $\hat{M}$ from Eq. (8.24) will be explicitly described for the ρ-spin sector $(ji,ml) = (++,++)$ in the on-mass-shell limit and for identical particles. The other sectors and the off-mass-shell extensions can be treated in an analogous way. From Eq. (7.68a) and Table VII we see that, for this special case, the Pauli spin structure is given by

$$M_{++}^{++} = F_1 + F_2\boldsymbol{\sigma}_1\cdot\mathbf{n}\,\boldsymbol{\sigma}_2\cdot\mathbf{n} + F_3\boldsymbol{\sigma}_1\cdot\mathbf{q}\,\boldsymbol{\sigma}_2\cdot\mathbf{q} + F_4\boldsymbol{\sigma}_1\cdot\mathbf{K}\,\boldsymbol{\sigma}_2\cdot\mathbf{K} + F_5\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n} , \quad (8.25)$$

where $\mathbf{n} = \mathbf{q}\times\mathbf{K}$, and subscripts on the Pauli spin operators have been used to denote particle labels. Substitution into Eq. (8.24) yields, for the desired Dirac operator, the expression

$$\hat{M}_{++}^{++} = f_1(\gamma_1^0+1)(\gamma_2^0+1) + f_2\mathbf{S}_1\cdot\mathbf{n}\,\mathbf{S}_2\cdot\mathbf{n} + f_3\mathbf{S}_1\cdot\mathbf{q}\,\mathbf{S}_2\cdot\mathbf{q} + f_4\mathbf{S}_1\cdot\mathbf{K}\,\mathbf{S}_2\cdot\mathbf{K} + f_5\tfrac{1}{2}[\mathbf{S}_1(\gamma_2^0+1) + \mathbf{S}_2(\gamma_1^0+1)]\cdot\mathbf{n} , \quad (8.26)$$

where we have defined the Dirac operator

$$\mathbf{S} = (\gamma^0+1)\boldsymbol{\Sigma} , \quad (8.27)$$

and have absorbed the kinematic factor h^2 into the new amplitudes $f_i = h^2F_i$. The covariant extensions of γ^0 and $\mathbf{S}\cdot\mathbf{n}$ follow from the spin-$\frac{1}{2}$-spin-0 results, Eqs. (8.17). We still require the covariant extensions of $\mathbf{S}\cdot\mathbf{q}$ and $\mathbf{S}\cdot\mathbf{K}$. These are two of the set of eight possible parity-odd, Dirac-space rotational scalars made from the available three vectors $\mathbf{q}$, $\mathbf{K}$, and $\boldsymbol{\Sigma}$ in the barycentric frame. The complete set is obtainable from the even-parity set which appears in Eqs. (8.14) by multiplication from the right by γ^5. These eight Dirac operators are γ^5, $\gamma^0\gamma^5$, $\gamma^5\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K})$, $\gamma^0\gamma^5\boldsymbol{\Sigma}\cdot(\mathbf{q}\times\mathbf{K})$, $\gamma^0\boldsymbol{\Sigma}\cdot\mathbf{q}$, $\boldsymbol{\Sigma}\cdot\mathbf{q}$, $\gamma^0\boldsymbol{\Sigma}\cdot\mathbf{K}$, and $\boldsymbol{\Sigma}\cdot\mathbf{K}$. Thus to form the covariant extensions in all possible cases, we need to find the unique linear transformation which relates this set of operators to the set of odd-parity mixed (Lorentz) scalars of Table III when the latter are evaluated in the barycentric frame.

For the same reasons raised previously for the spin-$\frac{1}{2}$-spin-0 case, the results are efficiently expressed when the Lorentz scalars are written in terms of the four momenta q' and K', which are the covariant extensions of the barycentric vectors $(0,\mathbf{q})$ and $(0,\mathbf{K})$ automatically orthogonal to ω. Multiplication of Eqs. (8.17) from the right by γ^5 produces results which are easily cast in terms of our standard odd-parity mixed (Lorentz) scalars for six of the eight operators in question; the results for these six are included in Eqs. (8.28) below. The remaining two operators $\mathbf{\Sigma}\cdot\mathbf{q}$ and $\mathbf{\Sigma}\cdot\mathbf{K}$ are not so easily dealt with. For Lorentz transformations, $\mathbf{\Sigma}$ is (part of) a rank-2 tensor and must be contracted with a rank-2 tensor formed from two four momenta in order to make a mixed scalar. Thus both $\mathbf{q}$ and $\mathbf{K}$ must be expressed as rank-2 tensors. For both particles on the mass shell this is straightforward since $(\mathbf{q},\mathbf{K},\xi)$ are orthogonal, and the required rank-2 tensor forms are $\mathbf{K}\propto\mathbf{q}\times\xi$ and $\mathbf{q}\propto\mathbf{K}\times\xi$. It is not surprising then that the extension of $\mathbf{\Sigma}\cdot\mathbf{q}$ is proportional to $\sigma{:}\xi K'$, and that the extension of $\mathbf{\Sigma}\cdot\mathbf{K}$ is proportional to $\sigma{:}\xi q'$. In the general off-mass-shell case, q' and K' are not orthogonal, and the required rank-2 momentum tensors are more complicated than the above. One would expect to achieve simple forms such as $\sigma{:}\xi K''$ and $\sigma{:}\xi q''$ if the new momentum K'' is defined to be orthogonal to q' and ξ, while the new momentum q'' is defined to be orthogonal to K' and ξ. This expectation is indeed fulfilled, and the sought-after covariant extensions of the full set of terms, other than γ^5, are found to be

$$\gamma^0\gamma^5=-\frac{\gamma^5\gamma\cdot\omega}{\sqrt{s}}\ , \tag{8.28a}$$

$$\gamma^5\mathbf{\Sigma}\cdot\mathbf{q}\times\mathbf{K}=-\frac{i}{s}\sigma{:}\xi\omega\ , \tag{8.28b}$$

$$\gamma^0\gamma^5\mathbf{\Sigma}\cdot\mathbf{q}\times\mathbf{K}=-\frac{1}{\sqrt{s}}\gamma\cdot\xi\ , \tag{8.28c}$$

$$\gamma^0\mathbf{\Sigma}\cdot\mathbf{q}=\gamma^5\gamma\cdot q'\ , \tag{8.28d}$$

$$\mathbf{\Sigma}\cdot\mathbf{q}=-\frac{\sqrt{s}}{\xi\cdot\xi}\sigma{:}\xi K''\ , \tag{8.28e}$$

$$\gamma^0\mathbf{\Sigma}\cdot\mathbf{K}=\gamma^5\gamma\cdot K'\ , \tag{8.28f}$$

$$\mathbf{\Sigma}\cdot\mathbf{K}=-\frac{\sqrt{s}}{\xi\cdot\xi}\sigma{:}\xi q''\ , \tag{8.28g}$$

where q' and K' are defined in Eq. (8.16), and q'' and K'' are defined in terms of them by

$$\begin{aligned} q''&=(q'\cdot K')K'-(K'\cdot K')q'\ ,\\ K''&=(q'\cdot q')K'-(q'\cdot K')q'\ . \end{aligned} \tag{8.29}$$

Note that the orthogonality relations $q''\cdot K'=0=q'\cdot K''$ hold, but that q'' and K'' are not orthogonal to each other. Thus even off the mass shell the set of momenta (q'',K',ω,ξ) provides an orthogonal basis, as does the alternative set (q',K'',ω,ξ). In the limit of equal mass particles on their mass shell, both sets become identical to the basis (q,K,ω,ξ) which we have used throughout this paper to express our standard mixed scalar Dirac operators. When q'', K'', q', and K', are expressed in terms of the momenta q and K, Eqs. (8.28) provide the covariant extensions of all the odd-parity operators that arise from the barycentric frame matrix elements.

We return now to the task of developing the covariant extension of Eq. (8.26) and, in particular, the $\mathbf{S}\cdot\mathbf{q}$ and $\mathbf{S}\cdot\mathbf{K}$ terms. Use of Eqs. (8.27) and (8.28) allows the desired forms to be read off immediately.

Each ρ-spin sector can be treated in this manner, even off the mass shell. The resulting form will always be covariant. Use of these results for the $\hat{M}_{ji}^{ml}$ in Eq. (8.20), together with the covariant boost operators, produces $\hat{T}_{ji}^{ml}$ (and hence also $\hat{T}=\sum_{jiml}\hat{T}_{ji}^{ml}$) in covariant form.

It is worth emphasizing that the mapping of the Pauli-space operators to covariant Dirac-space operators eliminates the possibility of kinematic singularities. Each of the (well-behaved) Pauli-space terms is mapped to a unique covariant extension. Linear independence of the set of invariants is maintained at all stages, thus eliminating the concerns in this regard that arise in the alternative reconstruction procedure described in Sec. VII. Finally, it should be noted that the explicit construction of the (minimal form of the) general operator $\hat{T}$ in terms of the $\hat{T}_{ji}^{ml}$ of Eq. (8.20) is an exercise in γ-matrix trace techniques, just as in the spin-$\frac{1}{2}$-spin-0 case of the preceding subsection.

IX. SUMMARY

In this paper we have systematically derived the general invariant form of the relativistic scattering operators for spin-$\frac{1}{2}$ particles. A complete set of invariant operators in Dirac spinor space is developed for the representation of the scattering operators both on and off the mass shell. The results presented here are the Lorentz-invariant generalizations of the familiar Wolfenstein representations of nonrelativistic Galilean-invariant scattering operators. Both the spin-$\frac{1}{2}$-spin-0 and the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ systems were treated in parallel, with the former used as a prototype for the methods which are extended to the more complex and interesting latter case. The invariant forms of scattering operators developed in this paper should provide a framework for a systematic investigation of those relativistic dynamical mechanisms in nucleon-nucleus scattering which derive from similar features of the NN scattering operator. The general forms obtained are constrained by the successive application of the symmetries of Lorentz invariance, parity conservation, time-reversal symmetry, and, for the fermion-fermion case, the Pauli principle for identical particles. This succession can be broken at any point so that, as a by-product, the form of symmetry breaking mechanisms may be considered in a general, covariant context as a further application. The imposition of isospin symmetry is indicated. For identical fermion-fermion scattering, the standard Fierz transformation between exchange and direct Fermi invariants is generalized to treat the complete set of invariants employed here.

The analysis is facilitated by the development of independent invariants (and corresponding amplitudes) that possess all the imposed symmetries of the system. We refer to such quantities as true scalars. The results for the general off-mass-shell invariant forms are given in terms of 10 true scalar amplitudes in Eqs. (4.3) for the spin-$\frac{1}{2}$-spin-0 case, in terms of 184 true scalar amplitudes in Eq.

(4.7) for the case of two distinguishable spin-$\frac{1}{2}$ particles, and in terms of 212 true scalar amplitudes in Eq. (4.19) for the case of two identical spin-$\frac{1}{2}$ particles. We show that if the individual amplitudes are required to be scalars only under Lorentz transformations, then the above numbers become 6, 80, and 56, respectively. The finding of 56 amplitudes of this type confirms a result from a recent work on the relativistic NN scattering problem. An advantage in dealing with true scalar amplitudes, on the other hand, is that purely off-mass-shell terms enter separately into the representation. The completely on-mass-shell invariant forms are given in terms of six true scalar amplitudes by Eq. (5.13) for the spin-$\frac{1}{2}$-spin-0 case, in terms of 72 true scalar amplitudes by Eq. (5.15) for the case of two distinguishable spin-$\frac{1}{2}$ particles, and in terms of 44 true scalar amplitudes by the first term of Eq. (6.19) for the case of two identical spin-$\frac{1}{2}$ particles.

The numerical input necessary for a complete determination of scattering operators may be taken from the solution of a dynamical equation of the Bethe-Salpeter type with a meson-theoretic interaction model which accurately describes the available two-body data. Such solutions will usually be obtained in the barycentric frame. Because of the considerable complexity arising from he four-component nature of the states, the reconstruction of covariant scattering operators from barycentric amplitudes is a nontrivial task, even conceptually. Thus, we develop a general framework for such a reconstruction. We find that a combined ρ-spin and Pauli-spin analysis is advantageous both practically and conceptually. However, the natural projectors for such an approach, which involve the free-particle Dirac eigenstates that are also invariably used in solutions of dynamical equations of the Bethe-Salpeter type, are not covariant. The usual covariant projectors are not convenient. We utilize kinematical properties of the two-particle scattering system to define covariant extensions of the natural projectors in order to formulate the reconstruction procedure along the desired lines. These covariantly extended projectors on the full Hilbert space are given in Eqs. (7.31), and in terms of them, the reconstructed scattering operator is expressed by Eq. (7.32) for the spin-$\frac{1}{2}$-spin-0 case and by Eq. (7.50) for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ case.

We relate the covariant scattering operators in the full Dirac spinor space and the Pauli spin structure of their matrix elements in each ρ-spin sector. One outcome of this is a specification of the number, type, and kinematical character of the Dirac operator invariants needed to represent the scattering operators on each of the ρ-spin sectors. This determines the manner in which the total number of unknown amplitude functions is distributed among the various ρ-spin sectors. All of this is done in the fully off-mass-shell circumstance as well as in the presence of on-mass-shell constraints. For the spin-$\frac{1}{2}$-spin-0 system there are three independent ρ-spin sectors needed to construct the operator, while for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system there are ten independent sectors in the product ρ-spin representation (seven for identical particles). The Pauli spin representation of the scattering operators in the various ρ-spin sectors thus subdivides the reconstruction process, provides a convenient intermediate representation of the scattering amplitudes, and identifies certain essential kinematical properties which enable one to avoid instabilities. The Pauli spin structure of each independent ρ-spin sector is given by Eqs. (7.42) for the spin-$\frac{1}{2}$-spin-0 case, by Eqs. (7.59) for the system of two distinguishable spin-$\frac{1}{2}$ particles, and by Eqs. (7.68) for the system of two identical spin-$\frac{1}{2}$ particles. The number of independent true scalar amplitudes revealed by this analysis of the ρ-spin sectors agrees with, and therefore confirms, the numbers identified in the analysis on the full Dirac space presented in Secs. IV and VI.

Knowledge of the Pauli spin structure on the full collection of ρ-spin sectors also forms the basis of a specific procedure for direct reconstruction on the full Dirac space. This method employs a covariant extension of the special Lorentz boost operator in a manner similar to that used some decades ago by Stapp in work on a relativistic density matrix formalism for polarization phenomena. In this procedure each member of the set of invariants in Pauli spin space is assigned a unique covariant extension into Dirac spin-space invariant operators. This method not only absorbs the covariant projection operators, but it also avoids any question of the introduction of kinematic singularities by virtue of the fact that it maps Pauli-spin operators directly to (unique) Dirac-space extensions. The covariantly extended special Lorentz boost operator is given by Eqs. (8.9), and the Lorentz covariant extensions of the rotational invariant Pauli-spin operators are given in Eqs. (8.17) and (8.28). For the spin-$\frac{1}{2}$-spin-0 system, the latter are used to explicitly display in Eqs. (8.18) the covariantly extended barycentric frame values of the ρ-spin scattering amplitudes. The final covariant form of the reconstructed scattering operator on the full Dirac space is given by Eq. (8.19). The application of this technique to the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system is described by Eqs. (8.20), (8.24), and (8.28).

The results obtained in this paper provide the superstructure needed for a systematic treatment of relativistic spin-$\frac{1}{2}$-spin-0 scattering in the circumstance that the spin-0 object consists of spin-$\frac{1}{2}$ constituents. This paper is specifically designed with an investigation of the relationship between relativistic nucleon-nucleus and nucleon-nucleon scattering in mind. As an immediate application, a future work from this perspective will address the construction of the relativistic nucleon-nucleus optical potential. Under the assumption that the nuclear ground state matrix element of the NN t matrix is an appropriate first-order optical potential in the relativistic context, the general Lorentz-invariant form of this spin-$\frac{1}{2}$-spin-0 interaction will be related, term by term, to the general form of the NN t matrix. The identification of the dominant contributions is facilitated by knowledge of the general operator forms consistent with symmetry principles. The final steps of this program include the numerical construction of appropriate models of the relativistic NN scattering operator based upon meson theory, their use in determining nucleon-nucleus optical potentials, and, finally, nucleon-nucleus scattering computations and investigations.

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APPENDIX: CHARGE CONJUGATION AND PCT INVARIANCE

Charge conjugation for a Dirac particle is implemented in momentum space by the antiunitary operator

$$C=U_cP_4K=i\gamma^2P_4K\ , \tag{A1a}$$

where the unitary (and Hermitian) operator $U_c=i\gamma^2$, and P_4 is the four-space inversion operator. The operation K, which has been introduced earlier in Eq. (3.13), produces the complex conjugate. The momentum-space operator combination P_4K is just complex conjugation in the position-time representation. For a spin-0 particle, U_c is replaced by 1.

We consider the implications of charge conjugation for the scattering operator in three distinct circumstances: a spin-$\frac{1}{2}$ (Dirac) particle in an external field, spin-$\frac{1}{2}$-spin-0 scattering, and spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ scattering. The charge conjugation operator for the first circumstance is given by Eq. (A1a) with the possibility of the addition of some instruction concerning the external field, while the charge conjugation operators applicable in the other two circumstances are, respectively,

$$C=U_c(1)P_4(1)P_4(2)K=i\gamma^2P_4(1)P_4(2)K \tag{A1b}$$

and

$$C=U_c(1)U_c(2)P_4(1)P_4(2)K$$
$$=[i\gamma^2(1)][i\gamma^2(2)]P_4(1)P_4(2)K\ . \tag{A1c}$$

In all three circumstances the inverse of the free covariant propagator of the system, S^{-1}, is charge conjugate invariant,

$$CS^{-1}C^{-1}=S^{-1}\ . \tag{A2}$$

This may be verified using the inverse propagators for the three systems, which are, respectively,

$$S^{-1}=S_0^{-1}\ , \tag{A3a}$$

$$S^{-1}=S_0^{-1}(1)\Delta_0^{-1}(2)\ , \tag{A3b}$$

and

$$S^{-1}=S_0^{-1}(1)S_0^{-1}(2)\ , \tag{A3c}$$

where the spin-$\frac{1}{2}$ operator S_0^{-1} is

$$S_0^{-1}=\not p-m \tag{A4a}$$

and the spin-0 operator Δ_0^{-1} is

$$\Delta_0^{-1}=p^2-m^2\ . \tag{A4b}$$

Thus the covariant propagators of the system, with the boundary conditions specified by the addition of a vanishingly small positive imaginary part to Eqs. (A4), are, respectively,

$$S(+)=S_0^{(+)}\ , \tag{A5a}$$

$$S(+)=S_0^{(+)}(1)\Delta_0^{(+)}(2)\ , \tag{A5b}$$

and

$$S(+)=S_0^{(+)}(1)S_0^{(+)}(2)\ , \tag{A5c}$$

where the Feynman fermion propagator is

$$S_0^{(+)}=\int d^4p\,|p\rangle(\not p-m+i\epsilon)^{-1}\langle p|\ , \tag{A6}$$

and the boson propagator is

$$\Delta_0^{(+)}=\int d^4p\,|p\rangle(p^2-m^2+i\epsilon)^{-1}\langle p|\ . \tag{A7}$$

Because of the antiunitary nature of C, and Eq. (A2), each of the free propagators of Eq. (A5) satisfies

$$CS(+)C^{-1}=S(-)\ . \tag{A8}$$

The propagators also satisfy

$$S(-)=S(+)^{\ddagger}\ , \tag{A9}$$

where we recall that ‡ denotes a Dirac adjoint $A^{\ddagger}=\Gamma^0A^{\dagger}\Gamma^0$, with $\Gamma^0=\gamma^0(1)$ or $\gamma^0(1)\gamma^0(2)$ according to whether there are one or two Dirac particles. Thus,

$$CS(+)C^{-1}=S(+)^{\ddagger}\ . \tag{A10}$$

If we assume a similar property for the covariant interaction $\hat U$,

$$C\hat UC^{-1}=\hat U^{\ddagger}\ , \tag{A11}$$

then the inverses of the full propagators for the three systems all satisfy

$$C(S^{-1}-\hat U)C^{-1}=S^{-1}-\hat U^{\ddagger}\ . \tag{A12}$$

If, additionally, $\hat U^{\ddagger}=\hat U$, then the inverse propagator is charge conjugation invariant. Note that with $\hat U=\Gamma^0V$, this condition is one of hermiticity: $V=V^{\dagger}$. However, this condition is not necessary.

The transition operator (our convention for the two-particle transition operator is such that the S matrix for the scattering of two positive-energy particles is $1+\tilde T$ rather than $1-i\tilde T$, for example) for each of the three systems satisfies an equation of the Bethe-Salpeter form,[26]

$$\tilde T=\hat U+\hat US(+)\tilde T\ , \tag{A13}$$

whose adjoint equation can be written

$$\tilde T^{\ddagger}=\hat U^{\ddagger}+\hat U^{\ddagger}S(+)^{\ddagger}\tilde T^{\ddagger}\ . \tag{A14}$$

Using Eqs. (A10) and (A11) in Eq. (A13) yields

$$[C\tilde{T}C^{-1}]=\hat{U}^{\ddagger}+\hat{U}^{\ddagger}S(+)^{\ddagger}[C\tilde{T}C^{-1}] . \quad (A15)$$

Comparison of Eqs. (A14) and (A15) yields

$$C\tilde{T}C^{-1}=\tilde{T}^{\ddagger} . \quad (A16)$$

This is the charge conjugation reciprocity relation satisfied by $\hat{T}$. We note that it is form identical to the time-reversal reciprocity relation of Eq. (3.17). In fact, the entire development of Eqs. (A2)–(A16) applies equally well to time-reversal if the charge conjugation operators are replaced by the corresponding time-reversal operators of Sec. III.

When the C operator acts on a free state it reverses the sign of the energy, the three momentum, and the spin of the particle. Because C is antiunitary, matrix elements of Eq. (A16) yield a relation of the schematic form

$$\langle \bar{F} \mid \tilde{T} \mid I \rangle=(-1)^{n_f}\langle \bar{I}_c \mid \tilde{T} \mid F_c \rangle , \quad (A17)$$

where $\mid I_c \rangle$ denotes $C \mid I \rangle$, and n_f is the number of fermions in either I or F. Thus in the example where $\mid I \rangle$ and $\mid F \rangle$ represent negative-energy states, Eq. (A17) relates this scattering amplitude to a positive-energy amplitude of reversed three momentum and spin. With the Feynman interpretation, Eq. (A17) is then a relation between an antiparticle amplitude [left-hand side of Eq. (A17)] and a particle amplitude [on the right-hand side of Eq. (A17)]. This depends only on the assumption that Eq. (A11) holds. In the case of a Dirac particle in an external (Hermitian) electromagnetic field, for example, $\hat{U}=e\boldsymbol{A}$ and Eq. (A1a) yields

$$C(e\boldsymbol{A})C^{-1}=-e\boldsymbol{A} , \quad (A18)$$

so that we must add to C the instruction $A^{\mu}\rightarrow -A^{\mu}$ in order for Eq. (A11) to obtain. Given this, Eqs. (A16) and (A17) then express a relationship between the scattering of an antiparticle (positron) for given momenta and spins in a field A and the scattering of a particle (electron) with the same momenta and spins in the charge conjugate field $(-A)$.

Use in Eq. (A16) of the integral representation expressed by Eq. (2.12) leads to the charge conjugation constraint

$$\hat{T}(q,K,\omega,\xi,\{\Gamma(i)\})=\hat{T}(-q,-K,-\omega,-\xi,\{\Gamma'(i)\}) , \quad (A19)$$

where the C transform of a γ matrix is denoted by

$$\Gamma'=U_c(\Gamma^{\ddagger})^{*}U_c \quad (A20)$$

for each Dirac particle.

The general Lorentz invariant forms for $\hat{T}$ constructed in the text by imposition of the symmetries Π and $\mathscr{T}$ automatically satisfy Eq. (A19). To relate this to PCT symmetry, we use Eq. (A16) in conjugation with Eqs. (3.5) and (3.20) for the symmetries Π and $\mathscr{T}$ to obtain

$$\Theta\tilde{T}\Theta^{-1}=\tilde{T} , \quad (A21)$$

where the unitary operator

$$\Theta=\Pi C\mathscr{T} \quad (A22)$$

is the PCT operator. For a single Dirac particle, this operator is given in momentum-space by

$$\Theta=\Pi C\mathscr{T}=[\gamma^0 P][i\gamma^2 P_4 K][i\gamma^1\gamma^3 PK] = i\gamma^5 P_4 , \quad (A23)$$

and for a spin-0 particle it is

$$\Theta=\Pi C\mathscr{T}=[P][P_4 K][PK] = P_4 . \quad (A24)$$

For a two-particle system, the PCT operator is

$$\Theta=\Theta(1)\Theta(2) . \quad (A25)$$

Thus Eq. (A21) for the PCT symmetry becomes

$$\tilde{T}=[\gamma^5 P_4(1)P_4(2)]\tilde{T}[\gamma^5 P_4(1)P_4(2)] \quad (A26)$$

for the spin-$\frac{1}{2}$-spin-0 system, and

$$\tilde{T}=[\gamma^5(1)P_4(1)][\gamma^5(2)P_4(2)] \times\tilde{T}[\gamma^5(2)P_4(2)][\gamma^5(1)P_4(1)] \quad (A27)$$

for the spin-$\frac{1}{2}$-spin-$\frac{1}{2}$ system. With the integral representation given in Eq. (2.12), the PCT constraint for the spinor-space scattering operator $\hat{T}$ immediately follows from Eqs. (A26) and (A27) and is the result given in the text as Eq. (3.28).

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